



**HAL**  
open science

## Degenerate weakly nonlinear elastic plane waves

W. Włodzimierz Domański, Andrew N. Norris

► **To cite this version:**

W. Włodzimierz Domański, Andrew N. Norris. Degenerate weakly nonlinear elastic plane waves. International Journal of Non-Linear Mechanics, 2009, 44 (5), pp.486. 10.1016/j.ijnonlinmec.2008.12.009 . hal-00531856

**HAL Id: hal-00531856**

**<https://hal.science/hal-00531856>**

Submitted on 4 Nov 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Author's Accepted Manuscript

Degenerate weakly nonlinear elastic plane waves

Włodzimierz Domański, Andrew N. Norris

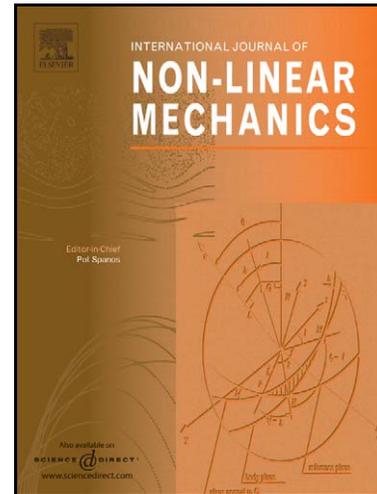
PII: S0020-7462(08)00228-X  
DOI: doi:10.1016/j.ijnonlinmec.2008.12.009  
Reference: NLM 1564

To appear in: *International Journal of Non-Linear Mechanics*

Received date: 12 October 2008  
Revised date: 18 December 2008  
Accepted date: 18 December 2008

Cite this article as: Włodzimierz Domański and Andrew N. Norris, Degenerate weakly nonlinear elastic plane waves, *International Journal of Non-Linear Mechanics* (2009), doi:10.1016/j.ijnonlinmec.2008.12.009

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting galley proof before it is published in its final citable form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



[www.elsevier.com/locate/nlm](http://www.elsevier.com/locate/nlm)

# Degenerate weakly nonlinear elastic plane waves

Włodzimierz Domański\* and Andrew N. Norris†

December 17, 2008

*Dedicated to Philippe Boulanger on the occasion of his sixtieth birthday.*

## Abstract

Weakly nonlinear plane waves are considered in hyperelastic crystals. Evolution equations are derived at a quadratically nonlinear level for the amplitudes of quasi-longitudinal and quasi-transverse waves propagating in arbitrary anisotropic media. The form of the equations obtained depends upon the direction of propagation relative to the crystal axes. A single equation is found for all propagation directions for quasi-longitudinal waves, but a pair of coupled equations occurs for quasi-transverse waves propagating along directions of degeneracy, or acoustic axes. The coupled equations involve four material parameters but they simplify if the wave propagates along an axis of material symmetry. Thus, only two parameters arise for propagation along an axis of two-fold symmetry, and one for a three-fold axis. The transverse wave equations decouple if the axis is four-fold or higher. In the absence of a symmetry axis it is possible that the evolution equations of the quasi-transverse waves decouple if the third order elastic moduli satisfy a certain identity. The theoretical results are illustrated with explicit examples.

## 1 Introduction

We characterize and analyze degenerate weakly nonlinear elastic waves in crystals. The degeneracy considered here arises from the existence of acoustic axes in elastic materials. Acoustic axes are directions for which the phase velocities of at least two waves coincide. For classical elasticity this phenomenon typically occurs for transverse or quasi-transverse waves. In the mathematical literature the coincidence of wave speeds is called a loss of strict hyperbolicity. Conditions for the existence of acoustic axes were first derived by Khatkevich (1962), and a useful review of the topic is given by Fedorov (1968). Recent developments can be found in e.g. Boulanger and Hayes (1998); Mozhaev et al. (2001); Norris (2004). Analysis and properties of nonlinear elastic waves propagating along acoustic axes were discussed in Shuvalov and Radowicz (2001), see also the book of Lyamov (1983).

The existence of acoustic axes, i.e. the loss of strict hyperbolicity, is typically accompanied by the local loss of genuine nonlinearity, that is, vanishing of the scalar product

---

\*Military University of Technology, Faculty of Cybernetics, Institute of Mathematics and Cryptology, Gen. S. Kaliskiego 2, 00-908 Warsaw 49, Poland, domanski.wlodek@gmail.com

†Rutgers University, Department of Mechanical and Aerospace Engineering, 98 Brett Road, Piscataway, NJ 08854-8058, norris@rutgers.edu

of the gradient of the phase velocity with the corresponding polarization vector, all evaluated at the origin. The local loss of genuine nonlinearity implies the presence of weaker than quadratic (e.g. cubic) nonlinearities in the decoupled evolution equations for degenerate weakly nonlinear (quasi-)transverse waves. This happens e.g. for shear elastic plane waves in an isotropic material as well as in a cubic crystal e.g. for  $[1\ 0\ 0]$  or  $[1\ 1\ 0]$  directions (see Domanski (2000a)). However, coupled quadratically nonlinear evolution equations do occur for transverse elastic waves. Although this is not possible in isotropic materials (Goldberg, 1960), such couplings can manifest themselves for special directions in crystals, for instance, for propagation in the  $[1\ 1\ 1]$  direction in a cubic crystal (see Domanski (2000a)). These special directions are the acoustic axes.

In this paper we clarify what kind of coupling is possible according to the type of a symmetry axis and we demonstrate how many constants are needed to describe the coupling of pairs of (quasi-)transverse waves in a particular case. It turns out that we need two constants for 2-fold symmetry axis, and only one constant for 3-fold axis. Moreover we also prove that there cannot be a quadratically nonlinear coupling for shear wave equations if the symmetry axis is 4-fold or 6-fold. In the absence of symmetry, the propagation of shear waves along an acoustic axis depends upon four constants governing the nonlinear terms in the coupled equations. It is possible that a pair of evolution equations decouples if the four constants satisfy certain special relations, which are derived here. We illustrate these general statements with some examples of particular elastic materials. Some of these results appeared previously in an abbreviated form (Domanski and Norris, 2008).

The paper is organized as follows. In Section 2 we present the model of nonlinear elastodynamics and its constitutive assumptions and we demonstrate the reduction of the governing equations to a quasilinear plane wave system. Section 3 contains a presentation of the method of Weakly Nonlinear Geometric Optics (WNGO) and its applications to quasi-longitudinal and quasi-transverse waves. Special attention is devoted to the degenerate case of pairs of quasi-transverse waves propagating along acoustic axes. Coupled evolution equations are derived for two-fold and three-fold symmetry axes. The analysis of the coupled evolution equations is discussed in Section 4 under conditions of symmetry about the propagation direction. Explicit examples which illustrate some of these theoretical results are provided for cubic crystals. Section 5 considers the special case of quasi-transverse waves propagating along an acoustic axis in the absence of material symmetry, and derives a condition both necessary and sufficient that the evolution equations decouple. Some concluding remarks are offered at the end of the paper.

## 2 Preliminaries

### 2.1 Basic equations

The equation of motion of a continuum written in material (Lagrangian) coordinates is, in the absence of body forces,

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \text{Div } \mathbf{T}. \quad (1)$$

Here  $\rho_0$  is the mass density in the reference configuration,  $\mathbf{u}$  is the particle displacement,  $\mathbf{T}$  is the first Piola-Kirchhoff stress tensor, and  $\text{Div}$  denotes the divergence operator with respect to the material coordinates  $\mathbf{X}$ . In the hyperelastic medium there exists a stored energy density per unit volume in the reference configuration, denoted by  $W(\mathbf{F})$ , such that

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{F}}, \quad (2)$$

where the deformation gradient is

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u} \quad (3)$$

with  $\mathbf{I}$  the identity tensor.

## 2.2 First order system

It is convenient for our purposes to write the equation of motion as a first order system of partial differential equations. To this aim we introduce the particle velocity

$$\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}, \quad (4)$$

and instead of (1) we write the system of equations

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = \text{Div } \mathbf{T}, \quad (5a)$$

$$\frac{\partial \mathbf{F}}{\partial t} = \nabla \mathbf{v}. \quad (5b)$$

Equation (5b) follows from the comparison of the time derivative of (3) with the space gradient of (4), assuming that the displacement vector  $\mathbf{u}$  is at least twice continuously differentiable. We introduce the following notation for the displacement gradient

$$\mathbf{M} \equiv \nabla \mathbf{u}. \quad (6)$$

Then, using (2) and (3) in the form  $\mathbf{F} = \mathbf{I} + \mathbf{M}$  we can express the system (5) as

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = \text{Div } \frac{\partial W}{\partial \mathbf{M}}, \quad (7a)$$

$$\frac{\partial \mathbf{M}}{\partial t} = \nabla \mathbf{v}. \quad (7b)$$

## 2.3 Constitutive relations

Components of vectors and other quantities will be referred to an orthonormal basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ . The strain energy in an arbitrary material is assumed to have the following expansion for small strains:

$$W = \frac{1}{2!} c_{abcd} E_{ab} E_{cd} + \frac{1}{3!} c_{abcdef} E_{ab} E_{cd} E_{ef} + \dots, \quad (8)$$

where  $E_{ab}$  are the components of the left Cauchy-Green strain tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^t \mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{M} + \mathbf{M}^t + \mathbf{M}^t \mathbf{M}), \quad (9)$$

and the summation convention on repeated subscripts  $a, b, \dots, g, h$  is assumed. The symmetry of the strain implies the relations

$$c_{abcd} = c_{cdab} = c_{bacd}, \quad c_{abcdef} = c_{cdabef} = c_{cdefab} = c_{bacdef}, \quad (10)$$

which imply that there are no more than 21 and 56 independent second and third order moduli, respectively. The second order moduli are assumed to be positive definite in the sense that  $c_{abcd}s_{ab}s_{cd} > 0$  for all non-zero  $\mathbf{s} = \mathbf{s}^t$ .

Hearmon (1953) provides a complete enumeration of the third-order constants for all crystal classes. Equations (2), (3), (8) and (9) together imply that the Piola-Kirchhoff stress is

$$T_{ab} = c_{abcd} M_{cd} + \frac{1}{2} N_{abcdef} M_{cd} M_{ef} + \frac{1}{6} N_{abcdefgh} M_{cd} M_{ef} M_{gh} + \dots, \quad (11)$$

where

$$N_{abcdef} = c_{abcdef} + c_{abdf} \delta_{ce} + c_{bfcd} \delta_{ae} + c_{bdef} \delta_{ac}. \quad (12)$$

Thurston (1984, eq. (38.5)) gives an expression for the higher order coefficients  $N_{abcdefgh}$ . Note that  $N_{abcdef} \neq N_{bacdef}$ , which implies that the non-symmetry of  $\mathbf{T}$  is a second-order effect.

For the remainder of the paper we take  $\rho_0 = 1$  for simplicity.

## 2.4 Plane waves

Plane wave solutions are described by displacement  $\mathbf{u}$  that depends upon a single component of  $\mathbf{X}$ , say  $x = \mathbf{X} \cdot \mathbf{n}$ , where  $\mathbf{n}$  is the direction of propagation. The displacement gradient of (6) reduces to

$$\mathbf{M} = \mathbf{m} \otimes \mathbf{n}, \quad (13)$$

where

$$\mathbf{m} = \frac{\partial \mathbf{u}}{\partial x}, \quad (14)$$

is a displacement gradient vector. Defining the energy function for plane deformation by

$$V(\mathbf{m}) \equiv W(\mathbf{m} \otimes \mathbf{n}), \quad (15)$$

we can write the system of elastodynamic equations (7) for plane waves as

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial^2 V}{\partial \mathbf{m}^2} \cdot \frac{\partial \mathbf{m}}{\partial x}, \quad (16a)$$

$$\frac{\partial \mathbf{m}}{\partial t} = \frac{\partial \mathbf{v}}{\partial x}. \quad (16b)$$

The above system can be expressed in a quasilinear form as

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x} = \mathbf{0}, \quad (17)$$

where

$$\mathbf{w} = \begin{pmatrix} \mathbf{v} \\ \mathbf{m} \end{pmatrix}, \quad \mathbf{A}(\mathbf{w}) = - \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}. \quad (18)$$

The  $3 \times 3$  matrix  $\mathbf{B}$  is

$$\mathbf{B} = \mathbf{\Lambda} + \mathbf{\Psi} \mathbf{m} + \frac{1}{2} \mathbf{\Pi} \mathbf{m} \mathbf{m} + \dots, \quad (19)$$

where  $\mathbf{\Lambda}(\mathbf{n})$ ,  $\mathbf{\Psi}(\mathbf{n})$  and  $\mathbf{\Pi}(\mathbf{n})$  are, in components,

$$\Lambda_{ac} = c_{abcd}n_b n_d, \quad (20a)$$

$$\Psi_{ace} = N_{abcdef}n_b n_d n_f, \quad (20b)$$

$$\Pi_{aceg} = N_{abcdefgh}n_b n_d n_f n_h, \quad (20c)$$

or, in short,

$$B_{ac} = c_{abcd}n_b n_d + N_{abcdef}n_b n_d n_f m_e + \frac{1}{2}N_{abcdefgh}n_b n_d n_f n_h m_e m_g + \dots \quad (21)$$

The positive definite property of the second order moduli implies that  $\mathbf{\Lambda}$ , also known as the Christoffel or acoustical tensor, has spectral form

$$\mathbf{\Lambda} = \alpha_1 \mathbf{k}_1 \otimes \mathbf{k}_1 + \alpha_2 \mathbf{k}_2 \otimes \mathbf{k}_2 + \alpha_3 \mathbf{k}_3 \otimes \mathbf{k}_3, \quad (22)$$

where  $\alpha_j > 0$  and  $\{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}$  is an orthonormal triad of vectors. The six eigenvalues of  $\mathbf{A}(0)$  therefore split into three pairs with opposite signs:

$$\lambda_1 = -\sqrt{\alpha_1} = -\lambda_2, \quad (23a)$$

$$\lambda_3 = -\sqrt{\alpha_2} = -\lambda_4, \quad (23b)$$

$$\lambda_5 = -\sqrt{\alpha_3} = -\lambda_6. \quad (23c)$$

The corresponding right and left eigenvectors of  $\mathbf{A}(0)$  are, respectively,

$$\begin{aligned} \mathbf{r}_{2j-1} &= \begin{pmatrix} -\lambda_{2j-1} \mathbf{k}_j \\ \mathbf{k}_j \end{pmatrix}, & \mathbf{r}_{2j} &= \begin{pmatrix} -\lambda_{2j} \mathbf{k}_j \\ \mathbf{k}_j \end{pmatrix}, \\ & & j &= 1, 2, 3. \end{aligned} \quad (24)$$

$$\mathbf{l}_{2j-1} = \frac{1}{2} \begin{pmatrix} -\lambda_{2j-1}^{-1} \mathbf{k}_j \\ \mathbf{k}_j \end{pmatrix}, \quad \mathbf{l}_{2j} = \frac{1}{2} \begin{pmatrix} -\lambda_{2j}^{-1} \mathbf{k}_j \\ \mathbf{k}_j \end{pmatrix},$$

Note that  $\mathbf{l}_i \cdot \mathbf{r}_j = \delta_{ij}$ .

*Remark 1.* The property  $\alpha_j > 0$  implies that all six eigenvalues  $\lambda_k$ ,  $k = 1, 2, \dots, 6$  of  $\mathbf{A}(0)$  are real. We assume that all the right and left eigenvectors of  $\mathbf{A}(0)$  form a full set of linearly independent eigenvectors. The system is therefore hyperbolic at the origin. This assumption can be expressed in terms of restrictions on the strain energy (e.g. rank-one convexity), but we will not discuss this issue here. We would like to emphasize, however, that we do admit the possibility of coincident eigenvalues, i.e. non-strict hyperbolicity. Moreover, the main objective of this paper is the analysis of the case when pairs of coincident eigenvalues correspond to (quasi-)shear waves. We call such waves *degenerate*.

### 3 WNGO asymptotics

In this section we are interested in deriving the simplest nonlinear evolution equations for the amplitudes of weakly nonlinear elastic plane waves. Using the method of weakly nonlinear geometric optics (WNGO) we will first derive equations for single waves and then for coupled pairs of waves. No attempt is made here to review WNGO, which

has been widely developed and applied in many areas of mechanics and physics since the seminal work by Lax (1957). We refer the interested read to the review of Hunter (1995).

For single waves two cases will be discussed. In considering the system (17) we first choose an eigenvalue  $\lambda_s(\mathbf{w})$  of the matrix  $\mathbf{A}(\mathbf{w})$  in (17) such that

$$\nabla_{\mathbf{w}} \lambda_s(\mathbf{w}) \cdot \mathbf{r}_s|_{\mathbf{w}=\mathbf{0}} \neq 0, \quad (25)$$

where  $\mathbf{r}_s$  is the eigenvector corresponding to  $\lambda_s$ . This assumption, called *genuine nonlinearity*, at the zero constant state is typically satisfied for longitudinal or quasi-longitudinal waves. Using the perturbation method we will derive a nonlinear evolution for such waves in Sec. 3.1. Next, in Sec. 3.2 we also present the simplified evolution equations for single waves which do not satisfy the assumption (25). This is typical for shear or quasi-shear elastic waves. Finally, in Sec. 3.3 we will also derive the evolution equations for the amplitudes of coupled waves that have coincident speeds.

### 3.1 Evolution equations for single (quasi-)longitudinal waves

Let us consider an initial-value problem for a quasilinear hyperbolic system (17):

$$\begin{cases} \frac{\partial \mathbf{w}^\epsilon}{\partial t} + \mathbf{A}(\mathbf{w}^\epsilon) \frac{\partial \mathbf{w}^\epsilon}{\partial x} = \mathbf{0}, \\ \mathbf{w}^\epsilon(0, x) = \epsilon \mathbf{w}_1(0, x, x/\epsilon), \end{cases} \quad (26)$$

where  $\epsilon$  is a small parameter. Note that the high frequency ('fast') variable  $\frac{x}{\epsilon}$  appears in the initial condition. The initial data are presumed to have compact support.

We consider the single wave weakly nonlinear geometrical optics solution to (26)

$$\mathbf{w}^\epsilon(t, x) = \epsilon \sigma_s(t, x, \eta) \mathbf{r}_s + O(\epsilon^2), \quad (27)$$

with an unknown amplitude  $\sigma_s$  and a new independent variable

$$\eta = \epsilon^{-1}(x - \lambda_s t). \quad (28)$$

It is assumed that the eigenvalue  $\lambda_s$  is distinct from the others, and its eigenvector is  $\mathbf{r}_s$ .

*Remark 2.* Instead of using a 'fast' variable, we may equally well use a 'slow' variable  $\epsilon x$  and work with a solution of the form  $w = \bar{w}(x - \lambda t; \epsilon x)$ , which represents a traveling wave modified by nonlinear effects over a large length scale. Using this alternative starting point we could obtain the same results as WNGO.

Introducing the ansatz (27) into (26), using a Taylor expansion of  $\mathbf{A}(\mathbf{w}^\epsilon)$  around  $\mathbf{0}$ , we then apply the method of multiple-scale asymptotics. This relies on treating  $\eta$  as a new independent variable. We sequentially collect terms of like powers in  $\epsilon$  and equating these terms to zero one finds that the solvability condition applied at the  $O(\epsilon)$  level yields a nonlinear evolution equation for the unknown amplitude  $\sigma_s$  (see (Domanski, 2000a) for the details):

$$\frac{\partial \sigma_s}{\partial t} + \lambda_s \frac{\partial \sigma_s}{\partial x} + \frac{1}{2} \Gamma_s \frac{\partial \sigma_s^2}{\partial \eta} = 0, \quad (29)$$

where the coefficient of nonlinearity is

$$\Gamma_s = \mathbf{1}_s \cdot (\nabla_{\mathbf{w}} \mathbf{A}(\mathbf{w}) \mathbf{r}_s \mathbf{r}_s)|_{\mathbf{w}=\mathbf{0}}. \quad (30)$$

One can show that  $\Gamma_s$  is equal to the left hand side of (25), and hence  $\Gamma_s \neq 0$ . An explicit equation for the nonlinearity coefficient follows from eqs. (12) and (17) as

$$\begin{aligned}\Gamma_s &= \frac{1}{2}\lambda_s^{-1}\mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor} \cdot \Psi \mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor} \mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor} \\ &= \frac{1}{2}\lambda_s^{-1}c_{abcdef}n_b n_d n_f k_a^{\lfloor\frac{s+1}{2}\rfloor} k_c^{\lfloor\frac{s+1}{2}\rfloor} k_e^{\lfloor\frac{s+1}{2}\rfloor} + \frac{3}{2}\lambda_s \mathbf{n} \cdot \mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor},\end{aligned}\quad (31)$$

where  $\lfloor \cdot \rfloor$  denotes the floor function, that is, the largest integer less than or equal to a given number. We also use the notation  $\mathbf{k}_j \equiv k_1^j \mathbf{e}_1 + k_2^j \mathbf{e}_2 + k_3^j \mathbf{e}_3$  in the Cartesian basis. The parameter  $\Gamma_s$  is well known in nonlinear elastodynamics - for instance, it governs the growth of elastic acceleration waves (Norris, 1991).

*Example 1.* Consider an isotropic elastic medium for which the strain energy is given by

$$W = \frac{1}{2}(\lambda_L + 2\mu_L)I_{\mathbf{E}}^2 - 2\mu_L II_{\mathbf{E}} + \frac{1}{3}(l_M + 2m_M)I_{\mathbf{E}}^3 - 2m_M I_{\mathbf{E}} II_{\mathbf{E}} + n_M III_{\mathbf{E}}, \quad (32)$$

where  $\lambda_L, \mu_L$  are second order Lamé constants,  $l_M, m_M$  and  $n_M$  are third order Murnaghan constants<sup>1</sup>, and the strain invariants are

$$I_{\mathbf{E}} = \text{tr } \mathbf{E}, \quad II_{\mathbf{E}} = \frac{1}{2}[(\text{tr } \mathbf{E})^2 - \text{tr } \mathbf{E}^2], \quad III_{\mathbf{E}} = \det \mathbf{E}. \quad (33)$$

In this case the two coefficients appearing in the evolution equation (29) are

$$\lambda_s = -\sqrt{\lambda_L + 2\mu_L} \quad \text{and} \quad \Gamma_s = \frac{3(\lambda_L + 2\mu_L) + 2(l_M + 2m_M)}{2\sqrt{\lambda_L + 2\mu_L}}. \quad (34)$$

Hence, we see that in the isotropic elastic medium the coefficients in the evolution equation for longitudinal waves are determined by the two second order and the two third order constants,  $\lambda_L, \mu_L$  and  $l_M, m_M$ , respectively.

### 3.2 Evolution equations for single (quasi-)shear waves

We now present the simplest nonlinear evolution equation for the amplitude  $\sigma_s$  of a single weakly nonlinear (quasi-)shear elastic wave for which  $\Gamma_s = 0$ . The case when  $\Gamma_s = 0$  requires a different scaling from that of (28), one appropriate to cubic nonlinearity as the leading term. The procedure is described in detail by Domanski (2000a). The WNGO solution for the single wave has the same formal expansion as in eq. (27), but now, crucially,  $\eta = \epsilon^{-2}(x - \lambda_s t)$ . The modified asymptotics leads to the following governing equation for  $\sigma_s$ :

$$\frac{\partial \sigma_s}{\partial t} + \lambda_s \frac{\partial \sigma_s}{\partial x} + \frac{1}{3}G_s \frac{\partial \sigma_s^3}{\partial \eta} = 0, \quad (35)$$

with nonlinearity coefficient

$$G_s = \frac{1}{2}\lambda_s^{-1}\mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor} \cdot (3\Psi \mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor} \mathbf{q} + \Pi \mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor} \mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor} \mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor}), \quad (36)$$

where the vector  $\mathbf{q}$  is orthogonal to  $\mathbf{k}_s$  and satisfies

$$(\Lambda - \lambda_s^2 \mathbf{I})\mathbf{q} + \Psi \mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor} \mathbf{k}_{\lfloor\frac{s+1}{2}\rfloor} = \mathbf{0}. \quad (37)$$

<sup>1</sup>Relations between the three Murnaghan constants and alternative triads of third order constants for isotropic solids are listed by Kostek et al. (1993).

*Example 2.* For an isotropic elastic medium with strain energy given by (32) the two coefficients in the evolution equation (35) are

$$\lambda_s = -\sqrt{\mu_L} \quad \text{and} \quad G_s = \frac{3}{4\lambda_s} \left[ \frac{\lambda_L \mu_L + (\mu_L + m_M)^2}{\lambda_L + \mu_L} \right]. \quad (38)$$

Hence, in the isotropic case, the coefficients in the evolution equation for shear waves are determined by the two second order and only one of the third order constants,  $\lambda_L$ ,  $\mu_L$ , and  $m_M$ , respectively.

*Remark 3.* By applying the method of characteristics we reduce the differentiation  $\frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}$  to the differentiation  $\frac{\partial}{\partial \tau}$  where  $\tau = t - \lambda x$  is a characteristic variable. In this way the partial differential equation

$$\frac{\partial \sigma}{\partial t} + \lambda \frac{\partial \sigma}{\partial x} + \frac{1}{j} \Gamma \frac{\partial \sigma^j}{\partial \eta} = 0, \quad (39)$$

for the function of three independent variables  $\sigma = \sigma(t, x, \eta)$  transforms to the partial differential equation

$$\frac{\partial \tilde{\sigma}}{\partial \tau} + \frac{1}{j} \Gamma \frac{\partial \tilde{\sigma}^j}{\partial \eta} = 0, \quad (40)$$

for the function of two independent variables  $\tilde{\sigma} = \tilde{\sigma}(\tau, \eta)$ . Here  $j = 1, 2, 3, \dots$  is a natural number. When  $j = 2$ , as occurs for the longitudinal wave evolution equation (29), we call the equation (40) the *inviscid Burgers equation*. Similarly,  $j = 3$  for the (quasi-) shear wave in (35), the equation (40) is called the *modified inviscid Burgers equation*.

### 3.3 Degenerate plane waves: acoustic axes

In this section we derive the simplest nonlinear evolution equations for the amplitudes of a pair of weakly nonlinear elastic waves in the case when these waves have coincident wave speeds.

Consider the Christoffel tensor  $\mathbf{\Lambda}$  from (22).

**Definition:** We say that the eigenvalues of  $\mathbf{\Lambda}$  are *degenerate*, if

$$\mathbf{\Lambda} = \alpha_1 (\mathbf{I} - \mathbf{k}_3 \otimes \mathbf{k}_3) + \alpha_3 \mathbf{k}_3 \otimes \mathbf{k}_3, \quad (41)$$

that is, if  $\alpha_1 = \alpha_2$  in (22). In such a situation we say that  $\mathbf{n}$  is an *acoustic axis* (see Khatkevich (1962) or Norris (2004)).

Let  $\{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}$  be an orthonormal triad of vectors and define the left and right eigenvectors of  $\mathbf{A}(\mathbf{0})$  as before, see eq. (24). Then  $\mathbf{A}(\mathbf{0})$  has two pairs of coincident eigenvalues:  $\lambda_s = \lambda_{s+2}$  for  $s = 1$  and  $s = 2$ . We now consider the following ansatz for the initial value problem (26):

$$\mathbf{w}(t, x) = \epsilon \left( \sigma_s(t, x, \eta) \mathbf{r}_s + \sigma_{s+2}(t, x, \eta) \mathbf{r}_{s+2} \right) + O(\epsilon^2), \quad s = 1, 2, \quad (42)$$

with  $\eta = \epsilon^{-1}(x - \lambda_s t)$ , and where  $\sigma_s$  and  $\sigma_{s+2}$  are the unknown amplitudes. Inserting (42) into (26) and applying the multiple scale asymptotic methods described in Section

3.1 (see Domanski (2000a)) we obtain the following pair of coupled evolution equations for the amplitudes  $\sigma_s$  and  $\sigma_{s+2}$ :

$$\frac{\partial \sigma_s}{\partial t} + \lambda_s \frac{\partial \sigma_s}{\partial x} + \frac{1}{2} \left( \Gamma_{s,s}^s \frac{\partial \sigma_s^2}{\partial \eta} + 2\Gamma_{s,s+2}^s \frac{\partial (\sigma_s \sigma_{s+2})}{\partial \eta} + \Gamma_{s+2,s+2}^s \frac{\partial \sigma_{s+2}^2}{\partial \eta} \right) = 0, \quad (43a)$$

$$\frac{\partial \sigma_{s+2}}{\partial t} + \lambda_s \frac{\partial \sigma_{s+2}}{\partial x} + \frac{1}{2} \left( \Gamma_{s,s}^{s+2} \frac{\partial \sigma_s^2}{\partial \eta} + 2\Gamma_{s,s+2}^{s+2} \frac{\partial (\sigma_s \sigma_{s+2})}{\partial \eta} + \Gamma_{s+2,s+2}^{s+2} \frac{\partial \sigma_{s+2}^2}{\partial \eta} \right) = 0, \quad (43b)$$

where the interaction coefficients are, in general,

$$\Gamma_{p,q}^j = \mathbf{l}_j \cdot (\nabla_{\mathbf{w}} \mathbf{A}(\mathbf{w}) \mathbf{r}_p \mathbf{r}_q) |_{\mathbf{w}=\mathbf{0}}. \quad (44)$$

The assumption of hyperelasticity (2) implies that these coefficients have the following symmetry property

$$\Gamma_{p,q}^j = \Gamma_{q,p}^j. \quad (45)$$

Moreover the formulas (24) imply that  $\Gamma_{p,q}^j = -\Gamma_{p,q}^{j+1}$  for  $j = 1, 3, 5$ . In our case we have

$$\Gamma_{p,q}^j = \frac{1}{2} \lambda_j^{-1} \mathbf{k}_{[\frac{j+1}{2}]} \cdot \Psi \mathbf{k}_{[\frac{p+1}{2}]} \mathbf{k}_{[\frac{q+1}{2}]}. \quad (46)$$

Using Cartesian components we can express the interaction coefficients as follows

$$\begin{aligned} \Gamma_{p,q}^j &= \frac{1}{2} \lambda_j^{-1} [c_{abcdef} n_b n_d n_f k_a^{[\frac{j+1}{2}]} k_c^{[\frac{p+1}{2}]} k_e^{[\frac{q+1}{2}]} \\ &\quad + c_{abcd} n_b n_c n_d (k_a^{[\frac{j+1}{2}]} \delta_{pq} + k_a^{[\frac{p+1}{2}]} \delta_{jq} + k_a^{[\frac{q+1}{2}]} \delta_{jp})] \\ &= \frac{1}{2} \lambda_j^{-1} c_{abcdef} n_b n_d n_f k_a^{[\frac{j+1}{2}]} k_c^{[\frac{p+1}{2}]} k_e^{[\frac{q+1}{2}]} \\ &\quad + \frac{1}{2} \lambda_j \mathbf{n} \cdot (\mathbf{k}_{[\frac{j+1}{2}]} \delta_{pq} + \mathbf{k}_{[\frac{p+1}{2}]} \delta_{jq} + \mathbf{k}_{[\frac{q+1}{2}]} \delta_{jp}). \end{aligned} \quad (47)$$

Therefore the coefficients have in our case, in addition to the general property (45), the following indicial symmetries:

$$\Gamma_{p,q}^j = \Gamma_{j,q}^p. \quad (48)$$

This makes them *totally symmetric* under the interchange of indices. In particular, note that

$$\Gamma_{s,s}^{s+2} = \Gamma_{s,s+2}^s = \frac{1}{2} \lambda_s^{-1} c_{abcdef} n_b n_d n_f k_a^{[\frac{s+1}{2}]} k_c^{[\frac{s+1}{2}]} k_e^{[\frac{s+3}{2}]} + \frac{1}{2} \lambda_s \mathbf{n} \cdot \mathbf{k}_{[\frac{s+3}{2}]} \equiv \Gamma_s^{s+2}, \quad (49a)$$

$$\Gamma_{s+2,s+2}^s = \Gamma_{s,s+2}^{s+2} = \frac{1}{2} \lambda_s^{-1} c_{abcdef} n_b n_d n_f k_a^{[\frac{s+1}{2}]} k_c^{[\frac{s+3}{2}]} k_e^{[\frac{s+3}{2}]} + \frac{1}{2} \lambda_s \mathbf{n} \cdot \mathbf{k}_{[\frac{s+1}{2}]} \equiv \Gamma_{s+2}^s. \quad (49b)$$

In summary:

**Lemma 1.** *The evolution equations for the amplitudes of shear waves propagating along an acoustic axis are*

$$\frac{\partial \sigma_s}{\partial t} + \lambda_s \frac{\partial \sigma_s}{\partial x} + \frac{1}{2} \left( \Gamma_s \frac{\partial \sigma_s^2}{\partial \eta} + 2\Gamma_s^{s+2} \frac{\partial (\sigma_s \sigma_{s+2})}{\partial \eta} + \Gamma_{s+2}^s \frac{\partial \sigma_{s+2}^2}{\partial \eta} \right) = 0, \quad (50a)$$

$$\frac{\partial \sigma_{s+2}}{\partial t} + \lambda_s \frac{\partial \sigma_{s+2}}{\partial x} + \frac{1}{2} \left( \Gamma_s^{s+2} \frac{\partial \sigma_s^2}{\partial \eta} + 2\Gamma_{s+2}^s \frac{\partial (\sigma_s \sigma_{s+2})}{\partial \eta} + \Gamma_{s+2}^s \frac{\partial \sigma_{s+2}^2}{\partial \eta} \right) = 0. \quad (50b)$$

The nonlinear terms in the equations involve four coefficients:  $\Gamma_s$  and  $\Gamma_{s+2}$  from eq. (31),  $\Gamma_s^{s+2}$  and  $\Gamma_{s+2}^s$  from eqs. (49), the latter two of which determine the coupling between the equations.

## 4 Simplification along symmetry axes

The previous analysis shows that there are at most four independent coefficients appearing in the nonlinear terms of the shear elastic wave's coupled system (50). We will demonstrate the following:

**Lemma 2.** *The number  $r$  of coefficients in the nonlinear terms in the coupled equations (50) for a pair of shear waves is always reduced if the direction of propagation  $\mathbf{n}$  is a symmetry acoustic axis. Specifically,*

$$\begin{aligned} r = 2 & \text{ for propagation along a 2-fold symmetry acoustic axis: } \Gamma_s = \Gamma_{s+2}^s = 0; \\ r = 1 & \text{ for a 3-fold symmetry acoustic axis: } \Gamma_s = \Gamma_{s+2}^s = 0 \text{ and } \Gamma_{s+2} + \Gamma_s^{s+2} = 0; \\ r = 0 & \text{ for an acoustic axis of 4-fold or higher symmetry: } \Gamma_s = \Gamma_{s+2}^s = \Gamma_{s+2} = \\ & \Gamma_s^{s+2} = 0. \end{aligned}$$

### 4.1 Two-fold axis

We say that the propagation direction is a *two-fold axis of symmetry* if it lies in a plane of symmetry of a monoclinic solid. Let  $\mathbf{e}$  be the normal to the plane of monoclinic symmetry, then  $\mathbf{e}$  also belongs to the plane of degenerate wave vectors. Let  $\mathbf{e}$  coincide with  $\mathbf{e}_1$ , and be parallel to  $\mathbf{k}_{\lfloor \frac{s+1}{2} \rfloor}$ . It then follows that  $\mathbf{n} \cdot \mathbf{k}_{\lfloor \frac{s+1}{2} \rfloor} = 0$ , and from eqs. (31) and (49b), we have

$$\Gamma_s^s \equiv \Gamma_{s,s}^s = \frac{1}{2} \lambda_s^{-1} c_{abcdef} n_b n_d n_f k_a^{\lfloor \frac{s+1}{2} \rfloor} k_c^{\lfloor \frac{s+1}{2} \rfloor} k_e^{\lfloor \frac{s+1}{2} \rfloor}, \quad (51a)$$

$$\Gamma_{s+2}^s \equiv \Gamma_{s+2,s+2}^s = \frac{1}{2} \lambda_s^{-1} c_{abcdef} n_b n_d n_f k_c k_e k_a^{\lfloor \frac{s+1}{2} \rfloor} k_c^{\lfloor \frac{s+3}{2} \rfloor} k_e^{\lfloor \frac{s+3}{2} \rfloor}. \quad (51b)$$

Each term in these expressions involves an element of  $c_{abcdef}$  with the index 1 occurring either once or thrice. But by definition of a plane of symmetry these elements vanish, and hence the two elements  $\Gamma_s$  and  $\Gamma_{s+2}^s$  vanish. The canonical form of the evolution equations is

$$\frac{\partial \sigma_s}{\partial t} + \lambda_s \frac{\partial \sigma_s}{\partial x} + \Gamma_{s+2}^s \frac{\partial(\sigma_s \sigma_{s+2})}{\partial \eta} = 0, \quad (52a)$$

$$\frac{\partial \sigma_{s+2}}{\partial t} + \lambda_s \frac{\partial \sigma_{s+2}}{\partial x} + \frac{1}{2} \left( \Gamma_{s+2}^s \frac{\partial \sigma_s^2}{\partial \eta} + \Gamma_{s+2} \frac{\partial \sigma_{s+2}^2}{\partial \eta} \right) = 0. \quad (52b)$$

### 4.2 Three-fold axis

The generic configuration in this case is propagation along the axis of a trigonal material. This possesses three planes of symmetry each containing the axis of symmetry, and mutually at  $120^\circ$  to one another. Reasoning similarly to the case of the two-fold symmetry axis, we get that  $\Gamma_s = \Gamma_{s+2}^s = 0$ , but here we have moreover that

$$\Gamma_{s+2} + \Gamma_s^{s+2} = 0, \quad (53)$$

see below for details. Therefore the canonical form of the evolution equations for a three-fold symmetry axis is

$$\frac{\partial \sigma_s}{\partial t} + \lambda_s \frac{\partial \sigma_s}{\partial x} - \Gamma_{s+2} \frac{\partial(\sigma_s \sigma_{s+2})}{\partial \eta} = 0, \quad (54a)$$

$$\frac{\partial \sigma_{s+2}}{\partial t} + \lambda_s \frac{\partial \sigma_{s+2}}{\partial x} - \frac{1}{2} \Gamma_{s+2} \left( \frac{\partial \sigma_s^2}{\partial \eta} - \frac{\partial \sigma_{s+2}^2}{\partial \eta} \right) = 0. \quad (54b)$$

Hence only one coefficient  $\Gamma_{s+2}$  characterizes the nonlinear terms in the evolution equations for a pair of degenerate plane waves in the case when the propagation direction is a three-fold symmetry axis.

*Remark 4.* The system (54) can be transformed into the single complex Burgers equation. This equation was studied in the Noelle's Ph.D. thesis (Noelle, 1990). It was shown there that the solutions of the complex Burgers equation become singular in finite time for a large class of initial data. The shocks which develop are of nonclassical type, that is, they do not satisfy Lax's conditions. This is due to the fact that too few characteristics enter the shock front in comparison to compressive shocks which satisfy Lax's condition.

#### 4.2.1 Proof of equation (53)

The basis for the proof rests on the property (48) combined with the three-fold symmetry. Using the same arguments as for the two-fold axis, the three-fold symmetry is associated with three planes of monoclinic symmetry with normals all perpendicular to  $\mathbf{n}$  and since each normal lies in the plane of degenerate wave vectors, it follows that the degenerate wave vectors are orthogonal to the axis and are therefore pure transverse waves.

In order to simplify matters, recall that for propagation along an acoustic axis  $s = 1$  or  $s = 2$  (see eq. (42)), and therefore the coefficients  $\Gamma_{p,q}^j$  may be identified with the elements of a totally symmetric third order tensor of dimension 2. Thus,

$$\mathbf{g} = g_{\alpha\beta\gamma} \mathbf{k}_\alpha \otimes \mathbf{k}_\beta \otimes \mathbf{k}_\gamma, \quad (55)$$

where lower case Greek subscripts take values 1 and 2, and, for instance,

$$g_{\alpha\beta\gamma} = c_{abcdef} n_b n_d n_f k_a^\alpha k_c^\beta k_e^\gamma + c_{abcd} n_b n_c n_d (k_a^\alpha \delta_{\beta\gamma} + k_a^\beta \delta_{\gamma\alpha} + k_a^\gamma \delta_{\alpha\beta}) \quad (56)$$

In this case the relationship with the nonlinearity coefficients  $\Gamma_{p,q}^j$  is defined by

$$\Gamma_{p,q}^j \equiv \frac{1}{2\lambda_j} g_{[i+1][p+1][q+1]}, \quad (57)$$

where  $j, p, q$  take values 1, 2, 3, 4. The totally symmetric property means that  $g_{\alpha\beta\gamma}$  is unchanged under any permutation of the three indices, and hence there are at most four independent elements. Consider the change of basis

$$\mathbf{k}'_1 = \cos \theta \mathbf{k}_1 + \sin \theta \mathbf{k}_2, \quad \mathbf{k}'_2 = -\sin \theta \mathbf{k}_1 + \cos \theta \mathbf{k}_2, \quad (58)$$

and define

$$g_{\alpha\beta\gamma}(\theta) = \mathbf{g} \mathbf{k}'_\alpha \mathbf{k}'_\beta \mathbf{k}'_\gamma, \quad (59)$$

then

$$\begin{pmatrix} g_{111}(\theta) \\ g_{222}(\theta) \\ g_{112}(\theta) \\ g_{122}(\theta) \end{pmatrix} = \begin{bmatrix} c^3 & s^3 & 3c^2s & 3cs^2 \\ -s^3 & c^3 & 3cs^2 & -3c^2s \\ -c^2s & cs^2 & c^3 - 2cs^2 & 2c^2s - s^3 \\ cs^2 & c^2s & s^3 - 2c^2s & c^3 - 2cs^2 \end{bmatrix} \begin{pmatrix} g_{111}(0) \\ g_{222}(0) \\ g_{112}(0) \\ g_{122}(0) \end{pmatrix}, \quad (60)$$

where  $c = \cos \theta$ ,  $s = \sin \theta$ . We note the property  $g_{\alpha\beta\gamma}(\theta) = -g_{\alpha\beta\gamma}(\theta + \pi)$  for each element  $g_{\alpha\beta\gamma}(\theta)$ .

Let  $\mathbf{e}_1$  be the normal to one of the three planes, then the coefficients  $g_{111}(0)$  and  $g_{122}(0)$  vanish. Similarly,  $g_{111}(\pm \frac{2}{3}\pi)$  and  $g_{122}(\pm \frac{2}{3}\pi)$  must vanish. Using (60) with  $g_{111}(0) = g_{122}(0) = 0$ , we find that

$$g_{111}(\pm \frac{2}{3}\pi) = 3g_{122}(\pm \frac{2}{3}\pi) = \pm \frac{3\sqrt{3}}{8} (g_{112}(0) + g_{222}(0)), \quad (61)$$

and hence the 3-fold symmetry requires that

$$g_{112}(0) + g_{222}(0) = 0. \quad (62)$$

This is precisely eq. (53). In summary, the evolution equations remain coupled, but are characterized by a single nonlinearity parameter and take the form (54).

### 4.3 Four-fold axis

If the axis is one of 4-fold symmetry, then the same arguments as above imply that the degenerate wave-vectors are orthogonal to the axis, and are thus pure transverse waves. The same arguments also imply that the elements of  $\mathbf{\Gamma}$  associated with the second plane of symmetry must also vanish, and hence all four are zero. Thus,  $\mathbf{\Gamma} = 0$  and there is no coupling.

The same reasoning applies to axes of higher symmetry, since all such axes are equivalent to an axis of transverse isotropy. Therefore the canonical form of the evolution equations for a symmetry axis with four-fold or higher symmetry is

$$\frac{\partial \sigma_s}{\partial t} + \lambda_s \frac{\partial \sigma_s}{\partial x} = 0, \quad (63a)$$

$$\frac{\partial \sigma_{s+2}}{\partial t} + \lambda_s \frac{\partial \sigma_{s+2}}{\partial x} = 0. \quad (63b)$$

### 4.4 Applications to cubic crystals

Let us consider a cubic crystal of class  $m\bar{3}m$  in which the strain energy  $W$  is defined by three second order and six third order elastic constants (Domanski, 2000a,b, 2008), (see also the Appendix):

$$W = W(c_{11}, c_{12}, c_{44}, c_{111}, c_{112}, c_{144}, c_{123}, c_{166}, c_{456}). \quad (64)$$

We focus on three particular directions of plane waves' propagation:  $[100]$ ,  $[110]$  and  $[111]$ .

*Example 3.* Consider first the case of propagation direction  $\mathbf{n} = [100]$ . This direction is along a four-fold symmetry acoustic axis. The shear wave speeds are given by

$$\lambda_1 = \lambda_3 = -\sqrt{c_{44}} = -\lambda_2 = -\lambda_4. \quad (65)$$

Hence, the eigenvalues  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  and  $\lambda_4$  are coupled but the second order nonlinearity coefficient  $\Gamma_s$  vanishes for  $s = 1, 2, 3, 4$ . Therefore there is no quadratically nonlinear coupling in the evolution equations for shear waves. One can show that propagation of the shear plane waves is described by the evolution equation with cubic nonlinearity, eq. (35) (Domanski, 2000a). The coefficients at cubic nonlinearity in this equation are as follows:

$$G_s = -\frac{3}{4\lambda_s} \left[ \frac{2(c_{11} + c_{166})^2 + c_{44}(c_{11} + c_{166}) + c_{166}(c_{44} + c_{166})}{c_{11} - c_{44}} \right], \quad s = 1, 2, 3, 4. \quad (66)$$

*Example 4.* Consider now the case of propagation direction  $\mathbf{n} = \frac{1}{\sqrt{2}}[110]$ . This direction is along a two-fold symmetry axis and is *not* an acoustic axis. The shear wave speeds are given by

$$\lambda_1 = -\sqrt{\frac{c_{11} - c_{12}}{2}} = -\lambda_2, \quad \lambda_3 = -\sqrt{c_{44}} = -\lambda_4. \quad (67)$$

Hence, the eigenvalues  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  and  $\lambda_4$  are distinct and the second order nonlinearity coefficient  $\Gamma_s$  vanishes for  $s = 1, 2, 3, 4$ . There is no quadratic coupling between the shear waves, and furthermore one can show that propagation of the shear plane waves is described by the evolution equation with cubic nonlinearity, eq. (35) (Domanski, 2000a). The coefficients in this equation are:

$$G_s = -\frac{3}{16\lambda_s} \left[ \frac{[c_{11} - c_{12} + \frac{1}{2}(c_{111} - c_{112})]^2 + 2(c_{11} - c_{12})(c_{12} + c_{44})}{c_{12} + c_{44}} \right], \quad s = 1, 2; \quad (68)$$

$$G_s = -\frac{3}{16\lambda_s} \left[ \frac{(c_{11} - c_{12} + c_{144} + c_{166} + 2c_{456})^2 + 2(c_{11} - c_{12})(c_{12} + c_{44})}{c_{12} + c_{44}} \right], \quad s = 3, 4. \quad (69)$$

*Example 5.* Finally let us consider the case of propagation along an axis of three-fold symmetry,  $\mathbf{n} = \frac{1}{\sqrt{3}}[111]$ , which is also an acoustic axis. In this case the shear wave amplitudes are coupled and are described by the coupled evolution equations (54) with (see the Appendix)

$$\lambda_1 = \lambda_3 = -\sqrt{\frac{c_{11} - c_{12} + c_{44}}{3}} = -\lambda_2 = -\lambda_4, \quad (70a)$$

$$\Gamma_s = \Gamma_1 = 0 = \Gamma_2, \quad (70b)$$

$$\Gamma_{s+2} = \Gamma_3 = \frac{1}{18\sqrt{2}\lambda_1} [c_{111} + 2c_{123} - 2c_{456} - 3(c_{112} - c_{144} + c_{166})] = -\Gamma_4. \quad (70c)$$

## 5 Degenerate transverse waves in the absence of symmetry

### 5.1 The principal result

In this Section we consider the general case of quasi-transverse degenerate wave vectors. No symmetry is assumed. We examine the possibility that the coupled nonlinear wave equations for the two amplitudes decouple, and derive a general condition that is both necessary and sufficient for this to occur.

The main result is the following:

**Lemma 3.** *A pair of quasi-transverse waves propagating along an acoustic axis are decoupled if and only if the nonlinearity coefficients satisfy the identity*

$$\Gamma_s \Gamma_{s+2}^s + \Gamma_{s+2} \Gamma_s^{s+2} - (\Gamma_{s+2}^s)^2 - (\Gamma_s^{s+2})^2 = 0. \quad (71)$$

*If this condition is met then there is a coordinate transformation for which the coupling terms disappear and the degenerate transverse waves satisfy separate but different evolution equations:*

$$\frac{\partial \sigma_s}{\partial t} + \lambda_s \frac{\partial \sigma_s}{\partial x} + \frac{1}{2} \Gamma_s' \frac{\partial \sigma_s^2}{\partial \eta} = 0, \quad (72a)$$

$$\frac{\partial \sigma_{s+2}}{\partial t} + \lambda_s \frac{\partial \sigma_{s+2}}{\partial x} + \frac{1}{2} \Gamma_{s+2}' \frac{\partial \sigma_{s+2}^2}{\partial \eta} = 0. \quad (72b)$$

*Remark 5.* The condition (71) can be expressed in terms of the 2-dimensional third order symmetric tensor  $\mathbf{g}$  as (see eq. (84))

$$g_{\alpha\beta\gamma}g_{\alpha\beta\gamma} - g_{\alpha\beta\beta}g_{\alpha\gamma\gamma} = 0. \quad (73)$$

Based upon eq. (56) this may be interpreted as a specific constraint on the third order moduli involving the direction  $\mathbf{n}$  of the acoustic axis. It also depends upon the second order moduli through the common eigenvalue, but this dependence disappears when the degenerate waves are purely transverse, in which case (71) is strictly a relation between the third order moduli.

## 5.2 Derivation of Lemma 3

We use the property seen previously in Section (4) that the coefficients  $\Gamma_{p,q}^j$  are the elements of a third order totally symmetric tensor in two dimensions,  $\mathbf{g}$  of eq. (55). Let  $\lambda_1 = \lambda_2 = \lambda$ . The coupled wave equations can be expressed in terms of a 2-vector for the displacement gradient vector of eq. (14):  $\mathbf{m} = \sigma_s \mathbf{k}_s + \sigma_{s+2} \mathbf{k}_{s+2}$ , or  $\mathbf{m} = m_1 \mathbf{e}_1 + m_2 \mathbf{e}_2$ , where  $\mathbf{e}_1, \mathbf{e}_2$  are orthonormal in the plane of  $\mathbf{k}_s, \mathbf{k}_{s+2}$ :

$$\begin{aligned} \frac{\partial m_1}{\partial t} + \lambda_1 \frac{\partial m_1}{\partial x} + \frac{1}{2} \frac{\partial}{\partial \eta} (g_{111} m_1^2 + 2g_{112} m_1 m_2 + g_{122} m_2^2) &= 0, \\ \frac{\partial m_2}{\partial t} + \lambda_2 \frac{\partial m_2}{\partial x} + \frac{1}{2} \frac{\partial}{\partial \eta} (g_{112} m_1^2 + 2g_{122} m_1 m_2 + g_{222} m_2^2) &= 0, \end{aligned}$$

or

$$\frac{\partial \mathbf{m}}{\partial t} + \lambda \frac{\partial \mathbf{m}}{\partial x} + \frac{1}{2} \frac{\partial}{\partial \eta} (\mathbf{gmm}) = \mathbf{0}. \quad (74)$$

The coupling between  $m_1$  and  $m_2$  vanishes iff the two elements  $g_{112}$  and  $g_{122}$  are simultaneously zero. Even if these are non-zero there might exist a coordinate transformation in which the transformed quantities vanish. It is this possibility that we seek.

Thus, we consider the possibility that there is some angle of rotation  $\theta$  such that  $g_{112}(\theta) = 0$  and  $g_{122}(\theta) = 0$  where  $g_{\alpha\beta\gamma}(\theta)$  are defined by (59). These two conditions are simultaneously satisfied if the 2-vector  $\mathbf{g}\mathbf{e}'_1\mathbf{e}'_2$  vanishes. Under the rotation (58) we have

$$\mathbf{g}\mathbf{e}'_1\mathbf{e}'_2 = \mathbf{a} \cos 2\theta + \mathbf{b} \sin 2\theta, \quad (75)$$

where

$$\mathbf{a} = \mathbf{g}\mathbf{e}_1\mathbf{e}_2, \quad \mathbf{b} = \frac{1}{2}(\mathbf{g}\mathbf{e}_2\mathbf{e}_2 - \mathbf{g}\mathbf{e}_1\mathbf{e}_1). \quad (76)$$

The form of (75) indicates that the vector  $\mathbf{g}\mathbf{e}'_1\mathbf{e}'_2$  can be zero if and only if  $\mathbf{a}$  and  $\mathbf{b}$  are parallel. Thus, the general condition for no coupling is

$$\text{no coupling} \quad \Leftrightarrow \quad \mathbf{a} \times \mathbf{b} = \mathbf{0}, \quad (77)$$

which is equivalent to

$$\text{no coupling} \quad \Leftrightarrow \quad \mu \equiv g_{112}^2 + g_{122}^2 - g_{112}g_{222} - g_{122}g_{111} = 0. \quad (78)$$

This is precisely the result of eq. (71) in Lemma 3. We will show next that the quantity  $\mu$  of eq. (78) is an invariant, independent of the coordinates used.

### 5.3 Tensor properties of the nonlinearity coefficients

The third order tensor  $\mathbf{g}$  is 2-dimensional and totally symmetric, that is, the Cartesian components are unchanged under permutation of the indices. Jerphagnon (1970) considered the general form of third order tensors in 3-dimensions, and based on these results we may partition  $\mathbf{g}$  as follows<sup>2</sup>

$$\mathbf{g} = \mathbf{g}^{(1)} + \mathbf{g}^{(3)}, \quad (79)$$

where

$$g_{\alpha\beta\gamma}^{(1)} = \frac{1}{4}(t_\alpha\delta_{\beta\gamma} + t_\beta\delta_{\gamma\alpha} + t_\gamma\delta_{\alpha\beta}), \quad t_\alpha = g_{\alpha\beta\beta}. \quad (80)$$

$\mathbf{g}^{(1)}$  is pseudovector (Jerphagnon, 1970) with two independent components,  $(g_{122} + g_{111})$  and  $(g_{112} + g_{222})$ .  $\mathbf{g}^{(3)}$ , which also has two independent elements,  $(3g_{122} - g_{111})$  and  $(3g_{112} - g_{222})$ , may be called a dimer. More importantly, it is harmonic, i.e.  $g_{\alpha\beta\beta}^{(3)} = 0$ . Hence, the quadratic invariant  $g_{\alpha\beta\gamma}g_{\alpha\beta\gamma}$  can be expressed

$$g_{\alpha\beta\gamma}g_{\alpha\beta\gamma} = \gamma_1 + \gamma_3, \quad (81)$$

where  $\gamma_1$  and  $\gamma_2$  are the quadratic invariants of the constituent tensors:

$$\gamma_1 = g_{\alpha\beta\gamma}^{(1)}g_{\alpha\beta\gamma}^{(1)} = \frac{3}{4}(g_{111} + g_{122})^2 + \frac{3}{4}(g_{112} + g_{222})^2, \quad (82)$$

$$\gamma_3 = g_{\alpha\beta\gamma}^{(3)}g_{\alpha\beta\gamma}^{(3)} = \frac{1}{4}(3g_{122} - g_{111})^2 + \frac{1}{4}(3g_{112} - g_{222})^2. \quad (83)$$

Comparing these with eq. (78) indicates that  $\mu = \frac{1}{2}\gamma_3 - \frac{1}{6}\gamma_1$ , and hence the no coupling condition can be expressed in terms of invariants as  $\gamma_1 = 3\gamma_3$ . Alternatively, noting that  $\gamma_1 = \frac{3}{4}g_{\alpha\beta\beta}g_{\alpha\beta\beta}$ , we deduce  $\mu = \frac{1}{2}g_{\alpha\beta\gamma}g_{\alpha\beta\gamma} - \frac{1}{2}g_{\alpha\beta\beta}g_{\alpha\beta\beta}$ , and hence

$$\text{no coupling} \Leftrightarrow g_{\alpha\beta\gamma}g_{\alpha\beta\gamma} = g_{\alpha\beta\beta}g_{\alpha\beta\beta}. \quad (84)$$

*Example 6.* We check the condition eq. (71) in two cases of acoustic axes:  $[100]$  and  $[111]$  for the cubic crystal considered earlier. It is easy to see that the condition (71) is satisfied and the evolution equations for shear waves are decoupled for the  $[100]$  acoustic axis. However, for the  $[111]$  axis, the left hand side of eq. (71) is equal to  $-2\Gamma_{s+2} \neq 0$ , so the condition (71) is not satisfied. Therefore there is quadratically nonlinear coupling in the evolution equations for shear waves in this case (see eqs. (54) and (70)).

## 6 Concluding remarks

Starting from a formulation of the governing equations as a first order system of quasi-linear equations, we have derived the general form of the amplitude evolution equations for weakly nonlinear plane wave propagation. The major new results concern the form of the evolution equations for the degenerate conditions associated with propagation along acoustic axes, summarized in Lemma 1 and eqs. (50). The quasi-transverse wave amplitudes are coupled at the quadratically nonlinear level, with at most four interaction coefficients. The number of coefficients reduces in the presence of symmetry, with the

<sup>2</sup>General third order tensors include additional terms  $\mathbf{g}^{(0)}$  and  $\mathbf{g}^{(2)}$  which vanish for a symmetric tensor (Jerphagnon, 1970).

precise number determined by Lemma 2. For instance, the coupling in the presence of three-fold symmetry about the propagation direction depends on a single interaction coefficient, with the canonical form of the coupled equations given by eqs. (54). The nonlinear coupling disappears if the acoustic axis has four-fold or higher symmetry. The isotropic solid is of course the most obvious example, but the results presented here show that similar decoupling can be expected in the presence of anisotropy. We have also shown that it is possible for the coupling to vanish even when the acoustic axis is not a symmetry axis. The condition, defined by Lemma 3, requires that the interaction coefficients satisfy a unique relation. Taken together, the variety of results presented here shed light on the nature of the equations governing nonlinear wave propagation in elastic crystals.

## Acknowledgment

W.D. acknowledges partial support from the Polish State Committee grant No. 0 T00A 014 29.

## A Appendix: Cubic crystals

In a cubic crystal with cube axes  $\mathbf{e}_j$ ,  $j = 1, 2, 3$ , we have (see Domanski (2008))

$$\begin{aligned} c_{abcd}E_{ab}E_{cd} &= c_{11}(I_1^2 - 2I_2) + 2c_{12}I_2 + 4c_{44}I_3, \\ c_{abcdef}E_{ab}E_{cd}E_{ef} &= c_{111}(I_1^3 - 3I_1I_2 + 3I_4) + 3c_{112}(I_1I_2 - 3I_4) \\ &\quad + 12c_{144}(I_1I_3 - I_6) + 6c_{123}I_4 + 48c_{456}I_5 + 12c_{166}I_6, \end{aligned}$$

where  $\mathbf{E}^t = \mathbf{E}$  and

$$\begin{aligned} I_1 &= E_{11} + E_{22} + E_{33}, \\ I_2 &= E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11}, \\ I_3 &= E_{12}^2 + E_{23}^2 + E_{31}^2, \\ I_4 &= E_{11}E_{22}E_{33}, \\ I_5 &= E_{12}E_{23}E_{31}, \\ I_6 &= (E_{11} + E_{22})E_{12}^2 + (E_{22} + E_{33})E_{23}^2 + (E_{33} + E_{11})E_{31}^2. \end{aligned}$$

The four cube diagonals are axes of trigonal or 3-fold symmetry, and are acoustic axes. Consider propagation along the cube diagonal acoustic axis  $\mathbf{n} = [1, 1, 1]/\sqrt{3}$ , and assume  $s = 1$ . The parameters  $\Gamma_s$  and  $\Gamma_{s+2}$  follow by taking  $\mathbf{k}_1$  orthogonal to one of the 3 symmetry planes, e.g.  $\mathbf{k}_1 = [1, -1, 0]/\sqrt{2}$  and  $\mathbf{k}_2 = [-1, -1, 2]/\sqrt{6}$ . The coefficient  $\Gamma_1$  is then obtained by using

$$\mathbf{E} = \frac{1}{2}(\mathbf{n} \otimes \mathbf{k}_1 + \mathbf{k}_1 \otimes \mathbf{n}) = \frac{1}{2\sqrt{6}} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Therefore,  $I_1 = I_4 = I_5 = I_6 = 0$  implying  $c_{abcdef}E_{ab}E_{cd}E_{ef} = 0$  and hence  $\Gamma_s = 0$ , as expected for a two-fold symmetry axis. The coefficient  $\Gamma_3 = \Gamma_{s+2}$  follows from eq. (31)

with

$$\mathbf{E} = \frac{1}{2}(\mathbf{n} \otimes \mathbf{k}_2 + \mathbf{k}_2 \otimes \mathbf{n}) = \frac{1}{6\sqrt{2}} \begin{bmatrix} -2 & -2 & 1 \\ -2 & -2 & 1 \\ 1 & 1 & 4 \end{bmatrix},$$

leading to

$$c_{abcdef}E_{ab}E_{cd}E_{ef} = \frac{1}{9\sqrt{2}}[c_{111} + 2c_{123} - 2c_{456} - 3(c_{112} - c_{144} + c_{166})].$$

## References

- Ph. Boulanger and M. Hayes. Acoustic axes for elastic waves in crystals: theory and applications. *Proc. R. Soc. A*, 454:2323–2346, 1998.
- W. Domanski. Weakly nonlinear elastic plane waves in a cubic crystal. *Contemp. Math.*, 255:45–61, 2000a.
- W. Domanski. Propagation and interaction of finite amplitude waves in a cubic crystal. In W. Lauterborn and T. Kurz, editors, *Nonlinear Acoustics at the Turn of the Millenium*, volume 524 of *AIP Conference Proceedings*, pages 249–252. AIP, 2000b.
- W. Domanski. Propagation and interaction of weakly nonlinear elastic plane waves in a cubic crystal. *Wave Motion*, 45(3):337–349, January 2008. doi: <http://dx.doi.org/10.1016/j.wavemoti.2007.07.011>.
- W. Domanski and A. N. Norris. Nonlinear evolution equations for degenerate transverse waves in anisotropic elastic solids. In B. Enflo, C. M. Hedberg, and L. Kari, editors, *Nonlinear Acoustics - Fundamentals and Applications, 18th International Symposium on Nonlinear Acoustics*, volume 1022 of *AIP Conference Proceedings*, pages 259–262. AIP, 2008.
- F. I. Fedorov. *Theory of Elastic Waves in Crystals*. Plenum Press, New York, 1968.
- Z. A. Goldberg. On self-interaction of plane longitudinal and transverse waves. *Sov. Phys. Acoust.*, 6:307–310, 1960.
- R. F. S. Hearmon. 'Third-order' elastic coefficients. *Acta Cryst.*, 6:331–340, 1953.
- J. K. Hunter. Asymptotic equations for nonlinear hyperbolic waves. *Surveys Appl. Math.*, 2:167–276, 1995.
- J. Jerphagnon. Invariants of the third-rank cartesian tensor: Optical nonlinear susceptibilities. *Phys. Rev. B*, 2(4):1091–1098, 1970. doi: 10.1103/PhysRevB.2.1091.
- A. G. Khatkevich. The acoustic axis in crystals. *Sov. Phys. Crystallogr.*, 7:601–604, 1962.
- S. Kostek, B. K. Sinha, and A. N. Norris. Third-order elastic constants for an inviscid fluid. *J. Acoust. Soc. Am.*, 94(5):3014–3017, 1993.
- P. D. Lax. Asymptotic solutions of oscillating initial value problems. *Duke. Math. J.*, 24: 627–646, 1957.

- V. S. Lyamov. *Polarization Effects and Anisotropic Waves in Crystals*. MGU, Moscow, 1983.
- V. G. Mozhaev, F. Bosia, and M. Weihnacht. Oblique acoustic axes in trigonal crystals. *J. Comp. Acoust.*, 9(3):1147–1161, 2001.
- S. Noelle. *Cauchy Problems for the Complex Burgers Equation in One and Two Space Dimensions*. PhD thesis, Courant Institute, New York University, 1990.
- A. N. Norris. Acoustic axes in elasticity. *Wave Motion*, 40:315–328, 2004.
- A. N. Norris. Symmetry conditions for third order elastic moduli and implications in nonlinear wave theory. *J. Elasticity*, 25:247–257, 1991.
- A. L. Shuvalov and A. Radowicz. Properties of nonlinear elastic waves propagating along acoustic axes. *Crystallography Reports*, 46:813–819, 2001.
- R. N. Thurston. Waves in solids. In C. Truesdell, editor, *Mechanics of Solids*, volume 4, pages 109–308, Berlin, 1984. Springer-Verlag.