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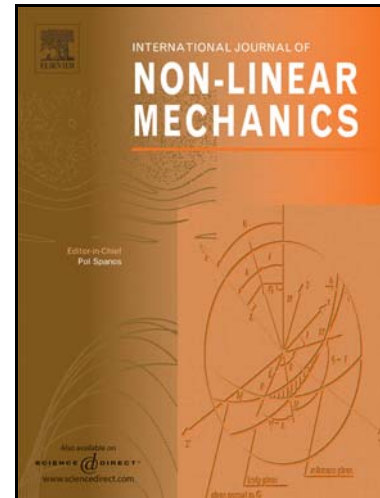
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# Incremental elastic motions superimposed on a finite deformation in the presence of an electromagnetic field

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## Abstract

The equations describing the interaction of an electromagnetic sensitive elastic solid with electric and magnetic fields under finite deformations are summarized, both for time-independent deformations and, in the non-relativistic approximation, time-dependent motions. The equations are given in both Eulerian and Lagrangian form, and the latter are then used to derive the equations governing incremental motions and electromagnetic fields superimposed on a configuration with a known static finite deformation and time-independent electromagnetic field. As a first application the equations are specialized to the quasimagnetostatic approximation and in this context the general equations governing time-harmonic plane-wave disturbances of an initial static configuration are derived. For a prototype model of an incompressible isotropic magnetoelastic solid a specific formula for the acoustic shear wave speed is obtained, which allows results for different relative orientations of the underlying magnetic field and the direction of wave propagation to be compared. The general equations are then used to examine two-dimensional motions, and further expressions for the wave speed are obtained for a general incompressible isotropic magnetoelastic solid.

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## 1 Introduction

In the past 40–50 years the formulation of the governing equations of the electrodynamics of continuous media and their static specializations have re-

ceived considerable attention in the literature. Classic texts include Stratton [1], Becker and Sauter [2], Penfield and Haus [3], Maugin [5], the two-volume work by Eringen and Maugin [6], and Nelson [7], the latter concerned primarily with dielectric materials. See also the monograph by Hutter et al. [8], which is a new edition of a work first published in 1978 [9], and the reviews by Maugin and Eringen [10] and Pao [11]. Equivalence of some of the different theories was established in [9]. The influence of uniform electric and magnetic fields on the propagation of mechanical waves was studied by Dunkin and Eringen [12] in the context of linear elasticity theory, while a general nonlinear theory of the electrodynamics of dielectrics was discussed in detail by Toupin [13], and equations of nonlinear magnetoacoustics were examined by Abd-Alla and Maugin [14]; see also the review by Maugin [15]. This list of references is far from exhaustive and further pointers to the extensive literature can be found therein. A more recent source of references is the set of lecture notes by Trimarco and Maugin [16], and yet more recent contributions include the papers by Fosdick and Tang [17] and Steigmann [18].

While originally developed in Eulerian form the equations (both Maxwell's equations and the continuum balance equations) can be expressed equivalently in Lagrangian form. The Lagrangian forms of the continuum balance equations are well known, but the Lagrangian forms of Maxwell's equations are less well known. They appear to have been used first by McCarthy [19, 20], in the context of acceleration wave propagation, and an independent derivation was given by Lax and Nelson [21]; see also Maugin [5], for example. The Lagrangian equations have an important role, in particular in the context of configurational forces (see, for example, [16], [22, 23]), and have also led, at least in the static context, to rather elegant general formulations of the constitutive laws and the accompanying equilibrium equations for magnetoelastic [24] and electroelastic [25] solids capable of undergoing finite deformations. Such formulations have facilitated the solution of a number of boundary-value problems for both magnetoelasticity and electroelasticity (see, for example, [26, 27, 28]).

In the application of the dynamic equations to particular problems much attention has focused on surface wave propagation. For example, the effect of a magnetic field on the propagation of Rayleigh waves in an elastic half-space has been analyzed by Lee and Its [29] for a linearly elastic perfectly conducting material, and a nonlinear counterpart of this problem was examined by Hefni et al. [30] for a material with finite electrical conductivity. For dielectric and piezoelectric materials, surface waves of different types were studied by Paria [31, 32] and Li [33], for example, based on linear elasticity theory; see also the recent paper by Collet et al. [34] and references therein.

Dispersive piezoelectric surface and interfacial waves have been examined recently by Romeo [35, 36, 37]. Inhomogeneous plane waves in magnetoelastic materials have been studied by Boulanger (see, for example, [38]).

The effect of finite deformation (or initial stress) on the propagation of electroacoustic or magnetoacoustic waves has received relatively little attention in the literature and there are only a few references to hand. Magnetoelastic waves in an infinite conducting medium subject to initial stress were analyzed by Yu and Tang [39], while De and Sengupta [40, 41] discussed Rayleigh, Love and Stoneley waves in an initially stressed conducting magnetoelastic material. The work of Yu and Tang was extended to include thermal effects by Massalas and Tsolakidis [42]. Against this background the purpose of the present paper is to provide a general theoretical framework for the analysis of incremental motions superimposed on a state of finite deformation or motion that is accompanied by an electromagnetic field in order to enable a variety of incremental wave propagation problems to be addressed. This requires an appropriate formulation of the governing equations that can be used as a basis for the derivation of the incremental equations. In the static context of magnetoelasticity the Lagrangian formulation of the equations enabled the incremental equations to be obtained in a relatively simple form [43]; these were then applied to the analysis of surface instability of a half-space. The corresponding analysis for electroelasticity was given in [44]. Here we extend this work to the dynamic context and then, for the quasimagnetoacoustic approximation, we conduct a basic wave propagation analysis for an infinite medium as a first application of the equations.

In Section 2 we summarize the necessary kinematical notation and formulas needed in the subsequent analysis, while Section 3 provides the relevant background equations governing nonlinear magnetostatics and electrostatics, including compact formulations of the constitutive equations for both compressible and incompressible materials. The equations of continuum electrodynamics are then given in Section 4 in both Eulerian and Lagrangian form, together with the boundary/jump conditions. The Lagrangian forms of the equations are then used in Section 5 to derive the incremental governing equations, linearized as appropriate. In Section 6 the incremental equations are specialized for the situation in which the incremental motion is superimposed on a *static* finite deformation accompanied by static magnetic and electric fields. Expressions for the various elastic, electromagnetic and coupling ‘moduli’ tensors are given in general form.

For purposes of illustration attention then focuses on the quasimagneto-static approximation and the set of equations is thereby reduced. These equations are then used in Section 7 to analyze the propagation of plane harmonic waves in an infinite magnetoelastic medium. For an isotropic mag-

netoelastic material the general forms of the constitutive equations are summarized and expressions for the components of the various moduli tensors are given. The wave speed is obtained first for a special prototype model of magnetoelasticity based on the neo-Hookean model commonly used in rubber elasticity. For this model it is shown that the wave speed is greater than the corresponding (transverse) wave speed in the absence of a magnetic field except when the direction of propagation is perpendicular to the underlying magnetic field. As a second illustration we specialize to two-dimensional motions in a principal plane of an underlying deformation that corresponds to pure homogeneous strain with the magnetic field aligned within the plane. Explicit formulas for the in-plane components of the moduli tensors are given in Appendix A. An expression for the wave speed is then derived for an arbitrary direction of propagation within the considered plane. Some concluding remarks are provided in Section 8.

## 2 Continuum kinematics

We consider an electromagnetically sensitive continuum which is located in the fixed *reference configuration*  $\mathcal{B}_0$  in three-dimensional Euclidean space in the absence of mechanical loads and electromagnetic fields. Let  $\mathbf{X}$  be the position vector of a generic material particle in  $\mathcal{B}_0$ . The body is subject to a time-dependent deformation under the combined action of mechanical loads and an electromagnetic field, so that at time  $t$  the particle  $\mathbf{X}$  is located at position  $\mathbf{x}$ , which is given by  $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ , where the vector function  $\boldsymbol{\chi}$  describes the motion (time-dependent deformation) and  $\mathbf{x} \in \mathcal{B}_t$ , where  $\mathcal{B}_t$  is the image of  $\mathcal{B}_0$  under the motion and referred to as the *current configuration*. For each time  $t$ ,  $\boldsymbol{\chi}$  is a one-to-one, onto mapping and must satisfy suitable regularity requirements; in the present paper  $\boldsymbol{\chi}$  and its inverse are assumed to be twice continuously differentiable.

The associated particle velocity and acceleration are denoted  $\mathbf{v}$  and  $\mathbf{a}$ , respectively, and are defined by

$$\mathbf{v} = \mathbf{x}_{,t} \equiv \frac{\partial \boldsymbol{\chi}}{\partial t}(\mathbf{X}, t), \quad \mathbf{a} = \mathbf{v}_{,t} = \mathbf{x}_{,tt} \equiv \frac{\partial^2 \boldsymbol{\chi}}{\partial t^2}(\mathbf{X}, t), \quad (1)$$

where  $_{,t}$  represents the material time derivative, i.e. the time derivative at fixed  $\mathbf{X}$ . This notation will be used henceforth, whereas the notation  $\partial/\partial t$  will be used for the spatial time derivative, i.e. the time derivative at fixed  $\mathbf{x}$ .

In what follows, the notations grad, div, curl are used for the standard differential operators with respect to  $\mathbf{x}$ , while Grad, Div, Curl are the corresponding operators with respect to  $\mathbf{X}$ .

The deformation gradient tensor, denoted  $\mathbf{F}$  is defined by  $\mathbf{F} = \text{Grad} \mathbf{x}$  (in Cartesian components,  $F_{i\alpha} = \partial x_i / \partial X_\alpha$ ) and its determinant by  $J = \det \mathbf{F}$ , which, by convention, is positive. Herein we shall use Greek (Roman) characters for indices associated with the reference (current) configuration. The standard formulas

$$\mathbf{F}_{,t} = \mathbf{\Gamma} \mathbf{F}, \quad (\mathbf{F}^{-1})_{,t} = -\mathbf{F}^{-1} \mathbf{\Gamma}, \quad J_{,t} = J \text{tr} \mathbf{\Gamma} = J \text{div} \mathbf{v}, \quad (2)$$

should be noted (see, for example, [45]) for subsequent use, where  $\mathbf{\Gamma}$  is the velocity gradient, defined by  $\mathbf{\Gamma} = \text{grad} \mathbf{v}$  (in components  $\Gamma_{ij} = \partial v_i / \partial x_j$ ), and  $\text{tr}$  denotes the trace (of a second order tensor).

The kinematic identities

$$\text{Div}(J\mathbf{F}^{-1}) = \mathbf{0}, \quad \text{div}(J^{-1}\mathbf{F}) = \mathbf{0}, \quad \text{Curl}(\mathbf{F}^T) = \mathbf{0}, \quad \text{curl}(\mathbf{F}^{-T}) = \mathbf{0}, \quad (3)$$

are also very useful, where  $^T$  denotes the transpose,  $\mathbf{0}$  the zero vector and  $\mathbf{O}$  the zero second-order tensor. The convention here is that the differentiation operates on the first index of the following tensor; for example, in index notation,  $\text{div}(J^{-1}\mathbf{F}) \equiv \partial(J^{-1}F_{j\alpha})/\partial x_j$  and  $\text{Curl}(\mathbf{F}^T) \equiv \epsilon_{\alpha\beta\gamma} \partial F_{i\gamma} / \partial X_\beta$ , where  $\epsilon_{\alpha\beta\gamma}$  is the alternating symbol. In particular, the formulas (3) provide important switches between Eulerian (spatial) and Lagrangian (material) descriptions. Suppose that  $\mathbf{a} = \mathbf{a}(\mathbf{x}, t)$  is an Eulerian vector (defined in  $\mathcal{B}_t$ ); then, it follows from the identities (3)<sub>1,3</sub> that

$$\text{Div}(J\mathbf{F}^{-1}\mathbf{a}) = J \text{div} \mathbf{a}, \quad \text{Curl}(\mathbf{F}^T \mathbf{a}) = J\mathbf{F}^{-1} \text{curl} \mathbf{a}. \quad (4)$$

Similarly, let  $\mathbf{A} = \mathbf{A}(\mathbf{X}, t)$  be a Lagrangian vector (defined in  $\mathcal{B}_0$ ). Then, by (3)<sub>2,4</sub>,

$$\text{div}(J^{-1}\mathbf{F}\mathbf{A}) = J^{-1} \text{Div} \mathbf{A}, \quad \text{curl}(\mathbf{F}^{-T} \mathbf{A}) = J^{-1} \mathbf{F} \text{Curl} \mathbf{A}. \quad (5)$$

Of course, the divergence identities in equations (4) and (5) are equivalent, as can be seen by taking  $\mathbf{a} = J^{-1}\mathbf{F}\mathbf{A}$ , while the equivalence of the curl identities requires, instead, the connection  $\mathbf{a} = \mathbf{F}^{-T} \mathbf{A}$ .

Now suppose that  $\mathbf{a}$  and  $\mathbf{A}$  are indeed connected via  $\mathbf{a} = J^{-1}\mathbf{F}\mathbf{A}$  (the Piola transform). Then, since the material time derivative of  $\mathbf{a}$  is given by

$$\mathbf{a}_{,t} = \frac{\partial \mathbf{a}}{\partial t} + (\mathbf{v} \cdot \text{grad}) \mathbf{a}, \quad (6)$$

we obtain, on taking the material time derivative of  $\mathbf{a} = J^{-1}\mathbf{F}\mathbf{A}$  and using (2)<sub>1,3</sub>,

$$J^{-1}\mathbf{F}\mathbf{A}_{,t} = \frac{\partial \mathbf{a}}{\partial t} - \mathbf{\Gamma} \mathbf{a} + (\text{div} \mathbf{v}) \mathbf{a} + (\mathbf{v} \cdot \text{grad}) \mathbf{a}, \quad (7)$$

which can also be expressed in the form

$$J^{-1}\mathbf{F}\mathbf{A}_{,t} = \frac{\partial \mathbf{a}}{\partial t} - \text{curl}(\mathbf{v} \times \mathbf{a}) + (\text{div} \mathbf{a})\mathbf{v}. \quad (8)$$

The right-hand side of (8) is the Eulerian form of the convected time derivative, which is more simply expressed in Lagrangian form by the left-hand side.

Next we note that by (4)<sub>2</sub> we have

$$\text{curl}(\mathbf{v} \times \mathbf{a}) = J^{-1}\mathbf{F}\text{Curl}[\mathbf{F}^T(\mathbf{v} \times \mathbf{a})]. \quad (9)$$

If we introduce the notation  $\mathbf{V}$  defined by  $\mathbf{v} = \mathbf{F}\mathbf{V}$  then, by standard vector identities, we obtain  $\mathbf{F}^T(\mathbf{v} \times \mathbf{a}) = \mathbf{V} \times \mathbf{A}$  and hence Eq. (8) can be rewritten in the form

$$J\mathbf{F}^{-1}\frac{\partial \mathbf{a}}{\partial t} = \mathbf{A}_{,t} + \text{Curl}(\mathbf{V} \times \mathbf{A}) - (\text{Div} \mathbf{A})\mathbf{V}. \quad (10)$$

We shall make use of these expressions in Section 4.

In the next section we summarize the equations of electroelastostatics and magnetoelastostatics since subsequently we shall consider mechanical motions superimposed on a static configuration subject to either an electrostatic or magnetostatic field, or both.

### 3 Electrostatics, magnetostatics and mechanical equilibrium

#### 3.1 Electrostatics

In Eulerian form the equations of electrostatics are

$$\text{curl} \mathbf{E} = \mathbf{0}, \quad \text{div} \mathbf{D} = \rho_e, \quad (11)$$

where  $\mathbf{E}$  is the electric field,  $\mathbf{D}$  the electric displacement and  $\rho_e$  the distributed electric charge density, defined per unit volume in  $\mathcal{B}_t$ . The associated boundary conditions are

$$\mathbf{n} \times \llbracket \mathbf{E} \rrbracket = \mathbf{0}, \quad \mathbf{n} \cdot \llbracket \mathbf{D} \rrbracket = \sigma_e, \quad (12)$$

where  $\mathbf{n}$  is the unit outward normal to the boundary  $\partial\mathcal{B}_t$  of  $\mathcal{B}_t$  and  $\sigma_e$  is the surface charge density (measured per unit area of  $\partial\mathcal{B}_t$ ). In vacuum  $\mathbf{D}$  and  $\mathbf{E}$  are related via  $\mathbf{D} = \varepsilon_0\mathbf{E}$ , where the constant  $\varepsilon_0$  is the vacuum permittivity. In polarizable material this connection is replaced by  $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ , where  $\mathbf{P}$



is the polarization vector. We shall not need to make use of the polarization in the present paper, but a constitutive law relating  $\mathbf{D}$  and  $\mathbf{E}$  will be needed to describe the material properties.

The Lagrangian counterparts of  $\mathbf{E}$  and  $\mathbf{D}$ , denoted  $\mathbf{E}_l$  and  $\mathbf{D}_l$ , respectively, are defined by

$$\mathbf{E}_l = \mathbf{F}^T \mathbf{E}, \quad \mathbf{D}_l = J \mathbf{F}^{-1} \mathbf{D}, \quad (13)$$

and they satisfy the equations

$$\text{Curl} \mathbf{E}_l = \mathbf{0}, \quad \text{Div} \mathbf{D}_l = \rho_E, \quad (14)$$

where we have introduced the notation  $\rho_E = J \rho_e$  for the charge density per unit reference volume. In terms of these variables the boundary conditions (12) transform into

$$\mathbf{N} \times \llbracket \mathbf{E}_l \rrbracket = \mathbf{0}, \quad \mathbf{N} \cdot \llbracket \mathbf{D}_l \rrbracket = \sigma_E, \quad (15)$$

where  $\sigma_E$  is the surface charge density per unit area of the boundary  $\partial \mathcal{B}_0$  of  $\mathcal{B}_0$ . This requires use of Nanson's formula  $\mathbf{n} da = J \mathbf{F}^{-T} \mathbf{N} dA$ , where  $da$  and  $dA$  are the area elements on  $\partial \mathcal{B}_0$  and  $\partial \mathcal{B}_t$ , respectively. Thus,  $\sigma_E dA = \sigma_e da$ .

There are many different ways to express the constitutive law of a non-linear electroelastic material, using either Eulerian or Lagrangian electric variables. Here we consider the 'total energy density' function of Dorfmann and Ogden [25], as a function of  $\mathbf{F}$  and  $\mathbf{E}_l$ , written  $\Omega(\mathbf{F}, \mathbf{E}_l)$  and defined per unit volume in  $\mathcal{B}_0$ , from which the simple equations

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{D}_l = -\frac{\partial \Omega}{\partial \mathbf{E}_l} \quad (16)$$

are obtained for the 'total nominal stress tensor'  $\mathbf{T}$  and the Lagrangian electric displacement  $\mathbf{D}_l$ . The corresponding Eulerian variables are then

$$\boldsymbol{\tau} = J^{-1} \mathbf{F} \mathbf{T} = J^{-1} \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{D} = J^{-1} \mathbf{F} \mathbf{D}_l = -J^{-1} \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{E}_l}, \quad (17)$$

where  $\boldsymbol{\tau}$  is the 'total Cauchy stress tensor'.

In the absence of mechanical body forces the equation of mechanical equilibrium may be written in either of the equivalent forms

$$\text{Div} \mathbf{T} = \mathbf{0}, \quad \text{div} \boldsymbol{\tau} = \mathbf{0}, \quad (18)$$

and the associated angular equilibrium equations are

$$\mathbf{F} \mathbf{T} = (\mathbf{F} \mathbf{T})^T, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^T \quad (19)$$

provided there are no intrinsic couples, which we assume to be the case here. Note that electric body forces are incorporated in the stress tensor. With  $\mathbf{E}_l$  as the independent electric variable, Eq. (14)<sub>1</sub> is satisfied by introducing a scalar electric potential function, say  $\Phi(\mathbf{X})$ , such that  $\mathbf{E}_l = -\text{Grad}\Phi$ . Then, the governing equations are given in Lagrangian form, for example, by (19)<sub>1</sub> with (16)<sub>1</sub>, together with (14)<sub>2</sub> and (16)<sub>2</sub>.

### 3.2 Magnetostatics

The magnetostatic counterparts of equations (11) are

$$\text{curl}\mathbf{H} = \mathbf{J}, \quad \text{div}\mathbf{B} = 0, \quad (20)$$

where  $\mathbf{H}$  is the magnetic field,  $\mathbf{B}$  the magnetic induction and  $\mathbf{J}$  is the (time-independent) distributed current density (per unit volume in  $\mathcal{B}_t$ ). The associated boundary conditions on  $\partial\mathcal{B}_t$  are

$$\mathbf{n} \times \llbracket \mathbf{H} \rrbracket = \mathbf{K}, \quad \mathbf{n} \cdot \llbracket \mathbf{B} \rrbracket = 0, \quad (21)$$

where  $\mathbf{K}$  is the surface current density (per unit area of  $\partial\mathcal{B}_t$ ). In vacuum we have the connection  $\mathbf{B} = \mu_0\mathbf{H}$ , where  $\mu_0$  is the vacuum permeability constant. In a magnetizable material this is replaced by the relation  $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ , which identifies the material-dependent magnetization vector  $\mathbf{M}$ . As with the polarization this will not be needed in the present paper.

Similarly to the Lagrangian electric variables, the Lagrangian magnetic variables, denoted  $\mathbf{H}_l$  and  $\mathbf{B}_l$ , are defined by

$$\mathbf{H}_l = \mathbf{F}^T\mathbf{H}, \quad \mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B} \quad (22)$$

and satisfy

$$\text{Curl}\mathbf{H}_l = \mathbf{J}_l, \quad \text{Div}\mathbf{B}_l = 0, \quad (23)$$

where we have defined  $\mathbf{J}_l$  as the Lagrangian counterpart of  $\mathbf{J}$  by  $\mathbf{J}_l = J\mathbf{F}^{-1}\mathbf{J}$ .

The associated boundary conditions are

$$\mathbf{N} \times \llbracket \mathbf{H}_l \rrbracket = \mathbf{K}_l, \quad \mathbf{N} \cdot \llbracket \mathbf{B}_l \rrbracket = 0, \quad (24)$$

where  $\mathbf{K}_l = \mathbf{F}^{-1}\mathbf{K}da/dA$  defines the Lagrangian surface current density per unit area of  $\partial\mathcal{B}_0$ .

For the constitutive law we again use the notation  $\Omega$  for the energy density function, per unit volume in  $\mathcal{B}_0$ , but now it is a function of  $\mathbf{F}$  and  $\mathbf{B}_l$ :  $\Omega(\mathbf{F}, \mathbf{B}_l)$  [24]. This yields the Lagrangian relations

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{H}_l = \frac{\partial\Omega}{\partial\mathbf{B}_l}, \quad (25)$$

analogously to (16), and the corresponding Eulerian equations

$$\boldsymbol{\tau} = J^{-1} \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{H} = \mathbf{F}^{-T} \frac{\partial \Omega}{\partial \mathbf{B}_l}. \quad (26)$$

Again the equilibrium equations (18) and (19) are satisfied, while if we take the independent magnetic variable  $\mathbf{B}_l$  to be given by a vector potential  $\mathbf{A}_l(\mathbf{X})$  so that  $\mathbf{B}_l = \text{Curl} \mathbf{A}_l$  and Eq. (23)<sub>2</sub> is satisfied, the equilibrium equation is coupled with (23)<sub>1</sub> and we require  $\mathbf{H}_l$  to be given by the constitutive law (25)<sub>2</sub>.

If both electrostatic and magnetostatic fields are present then we may regard  $\Omega$  as a function of both  $\mathbf{E}_l$  and  $\mathbf{B}_l$ , for example, in addition to  $\mathbf{F}$ :  $\Omega(\mathbf{F}, \mathbf{E}_l, \mathbf{B}_l)$ . Then, the relevant Lagrangian forms of the constitutive relations are

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{D}_l = -\frac{\partial \Omega}{\partial \mathbf{E}_l}, \quad \mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l}. \quad (27)$$

### 3.3 Incompressible materials

For an incompressible material we have the constraint

$$J = \det \mathbf{F} = 1 \quad (28)$$

and the expressions for the stresses in equations (16), (17), (25) and (26) require modification in the form

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p \mathbf{F}^{-1}, \quad \boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}} - p \mathbf{I}, \quad (29)$$

where  $p$  is the Lagrange multiplier associated with the constraint (28) and  $\mathbf{I}$  is the identity tensor.

## 4 Continuum electrodynamics

We emphasize that herein we are considering the non-relativistic Galilean approximation of the governing equations of the motion of a continuum, for which the velocity has magnitude that is negligible compared with the speed of light.

### 4.1 Maxwell's equations

The full Maxwell equations of electrodynamics have the (Eulerian) forms

$$\text{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{curl} \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \quad \text{div} \mathbf{D} = \rho_e, \quad \text{div} \mathbf{B} = 0, \quad (30)$$

where  $\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{J}$  and  $\rho_e$  are now in general functions of both  $\mathbf{x}$  and  $t$ . From (30)<sub>2,3</sub> the conservation of charge equation

$$\operatorname{div} \mathbf{J} + \frac{\partial \rho_e}{\partial t} = 0 \quad (31)$$

follows.

Now, using the formula (8) with  $\mathbf{a}$  and  $\mathbf{A}$  replaced by  $\mathbf{B}$  and  $\mathbf{B}_l$ , respectively, and using (30)<sub>4</sub>, we may re-write Eq. (30)<sub>1</sub> as

$$\operatorname{curl}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = -J^{-1} \mathbf{F} \mathbf{B}_{l,t}, \quad (32)$$

and similarly, on use of (30)<sub>3</sub>, Eq. (30)<sub>2</sub> becomes

$$\operatorname{curl}(\mathbf{H} - \mathbf{v} \times \mathbf{D}) = J^{-1} \mathbf{F} \mathbf{D}_{l,t} + \mathbf{J} - \rho_e \mathbf{v}, \quad (33)$$

where  $\mathbf{v}$  is the velocity of a material particle.

## 4.2 Boundary conditions

The boundary (or jump) conditions associated with equations (32), (33) and (47) are, respectively,

$$\mathbf{n} \times \llbracket \mathbf{E} + \mathbf{v} \times \mathbf{B} \rrbracket = \mathbf{0}, \quad (34)$$

$$\mathbf{n} \times \llbracket \mathbf{H} - \mathbf{v} \times \mathbf{D} \rrbracket = \mathbf{K} - \sigma_e \mathbf{v}_s, \quad (35)$$

where  $\mathbf{K}$ , as in (21)<sub>1</sub>, is the surface current density, while  $\mathbf{v}_s$  is the velocity of the material particle at the point in question on the boundary. Note that we are not considering propagating surfaces of discontinuity within the body.

The boundary (jump) conditions associated with Eqs. (30)<sub>3,4</sub> are

$$\mathbf{n} \cdot \llbracket \mathbf{D} \rrbracket = \sigma_e, \quad \mathbf{n} \cdot \llbracket \mathbf{B} \rrbracket = 0, \quad (36)$$

unchanged from (12)<sub>2</sub> and (21)<sub>2</sub>.

## 4.3 Lagrangian formulation

By means of (4)<sub>2</sub> and the formula following from (9), equations (32) and (33) may be re-cast entirely in terms of Lagrangian variables, to give

$$\operatorname{Curl}(\mathbf{E}_l + \mathbf{V} \times \mathbf{B}_l) = -\mathbf{B}_{l,t}, \quad (37)$$

and

$$\operatorname{Curl}(\mathbf{H}_l - \mathbf{V} \times \mathbf{D}_l) = \mathbf{D}_{l,t} + \mathbf{J}_E, \quad (38)$$

respectively, where we have introduced the notation  $\mathbf{J}_E$  defined by

$$\mathbf{J}_E = \mathbf{J}_l - \rho_E \mathbf{V} = J \mathbf{F}^{-1} (\mathbf{J} - \rho_e \mathbf{v}). \quad (39)$$

Equations (37) and (38) are the Lagrangian versions of equations (30)<sub>1,2</sub> and they are coupled with the Lagrangian versions of (30)<sub>3,4</sub>, namely

$$\text{Div} \mathbf{D}_l = \rho_E, \quad \text{Div} \mathbf{B}_l = 0. \quad (40)$$

These Lagrangian forms of the equations have been given in many publications, dating back at least to McCarthy [19, 20]; see also [21].

Note that the charge conservation equation (31) takes on the Lagrangian form

$$\text{Div} \mathbf{J}_E + \rho_{E,t} = 0. \quad (41)$$

The corresponding boundary conditions are obtained by conversion of equations (34)–(36) as

$$\mathbf{N} \times \llbracket \mathbf{E}_l + \mathbf{V} \times \mathbf{B}_l \rrbracket = \mathbf{0}, \quad (42)$$

$$\mathbf{N} \times \llbracket \mathbf{H}_l - \mathbf{V} \times \mathbf{D}_l \rrbracket = \mathbf{K}_l - \sigma_E \mathbf{V}_s, \quad (43)$$

where  $\mathbf{K}_l$  is as defined following (24),  $\sigma_E$  as in (15), and  $\mathbf{V}_s = \mathbf{F}^{-1} \mathbf{v}_s$ .

The boundary (jump) conditions associated with equations (40) are

$$\mathbf{N} \cdot \llbracket \mathbf{D}_l \rrbracket = \sigma_E, \quad \mathbf{N} \cdot \llbracket \mathbf{B}_l \rrbracket = 0. \quad (44)$$

#### 4.4 Equations of motion

The equation governing the motion of a continuous material in the presence of an electromagnetic field can be expressed in many different forms, depending on the particular model used for the electromagnetic force. The choice of this model has been a source of much debate over the years and we do not enter into the controversy here. In any event the equation of motion can in general be written in the form

$$\text{div} \boldsymbol{\sigma} + \rho \mathbf{f} + \mathbf{f}^{\text{em}} = \rho \mathbf{a}, \quad (45)$$

where  $\boldsymbol{\sigma}$  is a Cauchy-like stress tensor,  $\rho$  is the mass density of the material,  $\mathbf{f}$  is the mechanical body force (per unit mass) and  $\mathbf{f}^{\text{em}}$  is the electromagnetic ‘body force’ (per unit volume in  $\mathcal{B}_t$ ). (Recall that  $\mathbf{a}$  denotes the acceleration.) For detailed discussion of the alternative models and the associated governing equations we refer to, for example, Pao [11], Maugin [5] and Hutter et al.

[8]. Unlike its static counterparts  $\mathbf{f}^{\text{em}}$  cannot in general be written as the divergence of a second-order (stress) tensor, but it can be written as

$$\mathbf{f}^{\text{em}} = \text{div} \boldsymbol{\sigma}^{\text{em}} - \frac{\partial \mathbf{G}}{\partial t}, \quad (46)$$

where  $\boldsymbol{\sigma}^{\text{em}}$  is an electromagnetic stress tensor and  $\mathbf{G}$  is an electromagnetic momentum vector (see, for example, Maugin and Eringen [10]), and for certain models  $\mathbf{G} = \mathbf{0}$ .

Since we shall be restricting attention subsequently to the quasimagneto-static approximation we shall not require the most general form of the equation of motion, and it suffices, for simplicity of illustration, to take  $\mathbf{G} = \mathbf{0}$  here and write to (45) in the simple form

$$\text{div} \boldsymbol{\tau} + \rho \mathbf{f} = \rho \mathbf{a}, \quad (47)$$

where  $\boldsymbol{\tau} = \boldsymbol{\sigma} + \boldsymbol{\sigma}^{\text{em}}$  is the total stress tensor, which, as in the static situation, is symmetric in the absence of intrinsic couples. The corresponding Lagrangian form of the equation is

$$\text{Div} \mathbf{T} + \rho_0 \mathbf{f} = \rho_0 \mathbf{a}, \quad (48)$$

where  $\rho_0 = \rho J$  is the mass density in  $\mathcal{B}_0$  and, as for the static situation,  $\mathbf{T}$  may be defined via the connection  $\mathbf{T} = J \mathbf{F}^{-1} \boldsymbol{\tau}$ .

The associated traction boundary conditions are

$$[[\boldsymbol{\tau}]] \mathbf{n} = \mathbf{0}, \quad [[\mathbf{T}^T]] \mathbf{N} = \mathbf{0}, \quad (49)$$

which include both the applied mechanical traction and electromagnetic tractions acting on the exterior boundary of the body.

## 5 Incremental equations

Suppose that the motion  $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$  is known, along with the electromagnetic fields. Now let an incremental motion be superimposed on  $\boldsymbol{\chi}(\mathbf{X}, t)$ , which we denote by  $\dot{\mathbf{x}} = \dot{\boldsymbol{\chi}}(\mathbf{X}, t)$ , where the displacement  $\dot{\mathbf{x}}$  is ‘small’. Here and henceforth a superimposed dot represents an increment in the quantity concerned.

### 5.1 Incremental kinematics

Through the motion  $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$  we may identify  $\dot{\mathbf{x}}$  with its Eulerian equivalent, which we denote by  $\mathbf{u}(\mathbf{x}, t)$ , i.e.  $\mathbf{u} = \dot{\mathbf{x}}$ . Then,  $\mathbf{u}_{,t} = \dot{\mathbf{x}}_{,t} = \dot{\mathbf{v}}$ , where  $\mathbf{v}$

is the particle velocity. Recalling that  $\mathbf{F}_{,t} = \mathbf{\Gamma}\mathbf{F}$  we then obtain the formulas

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \quad \dot{\mathbf{F}}_{,t} = (\text{grad } \dot{\mathbf{v}})\mathbf{F}, \quad \text{grad } \dot{\mathbf{v}} = \dot{\mathbf{\Gamma}} + \mathbf{\Gamma}\mathbf{L} = \mathbf{L}_{,t} + \mathbf{L}\mathbf{\Gamma}, \quad (50)$$

where  $\mathbf{L} = \text{grad } \mathbf{u}$ . Note that it follows from (50)<sub>4</sub> that  $\text{tr}(\dot{\mathbf{\Gamma}}) = \text{tr}(\mathbf{L}_{,t})$ . The formulas in (50) are exact since they involve linear operations in the increment and hence in these cases the increment is not required to be ‘small’. By contrast, the increments

$$\overline{(\mathbf{F}^{-1})} = -\mathbf{F}^{-1}\mathbf{L}, \quad \dot{j} = J\text{div } \mathbf{u} \quad (51)$$

require linearization in the derivatives of  $\mathbf{u}$ .

With  $\mathbf{V} = \mathbf{F}^{-1}\mathbf{v}$  we then obtain

$$\dot{\mathbf{V}} = \mathbf{F}^{-1}(\dot{\mathbf{v}} - \mathbf{L}\mathbf{v}), \quad (52)$$

while the material time derivative of (51)<sub>2</sub> leads to

$$\dot{j}_{,t} = J\text{div } \dot{\mathbf{v}} + J(\text{div } \mathbf{u})(\text{div } \mathbf{v}) - J\text{tr}(\mathbf{\Gamma}\mathbf{L}). \quad (53)$$

Note that for an incompressible material the linearization yields

$$\text{div } \mathbf{u} = 0, \quad \text{div } \dot{\mathbf{v}} = \text{tr}(\mathbf{\Gamma}\mathbf{L}). \quad (54)$$

By taking the increments of equations (37), (38) and (40) we obtain the incremental forms of Maxwell’s equations. These are

$$\text{Curl}(\dot{\mathbf{E}}_l + \mathbf{V} \times \dot{\mathbf{B}}_l + \dot{\mathbf{V}} \times \mathbf{B}_l) = -\dot{\mathbf{B}}_{l,t}, \quad (55)$$

$$\text{Curl}(\dot{\mathbf{H}}_l - \mathbf{V} \times \dot{\mathbf{D}}_l - \dot{\mathbf{V}} \times \mathbf{D}_l) = \dot{\mathbf{D}}_{l,t} + \dot{\mathbf{J}}_E, \quad (56)$$

$$\text{Div } \dot{\mathbf{B}}_l = 0, \quad \text{Div } \dot{\mathbf{D}}_l = \dot{\rho}_E. \quad (57)$$

The incremental form of the charge conservation equation follows as

$$\text{Div } \dot{\mathbf{J}}_E + \dot{\rho}_{E,t} = 0. \quad (58)$$

Similarly, by forming the increment of Eq. (48), we obtain the incremental equation of motion

$$\text{Div } \dot{\mathbf{T}} + \rho_0 \dot{\mathbf{f}} = \rho_0 \mathbf{u}_{,tt}. \quad (59)$$

We now collect together the relations (13), (17)<sub>1</sub> and (22) in the form

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\mathbf{T}, \quad \mathbf{B} = J^{-1}\mathbf{F}\mathbf{B}_l, \quad \mathbf{D} = J^{-1}\mathbf{F}\mathbf{D}_l, \quad \mathbf{E} = \mathbf{F}^{-\text{T}}\mathbf{E}_l, \quad \mathbf{H} = \mathbf{F}^{-\text{T}}\mathbf{H}_l, \quad (60)$$

which can be thought of as ‘push forward’ versions of the Lagrangian variables  $\mathbf{T}, \mathbf{B}_l, \mathbf{D}_l, \mathbf{E}_l, \mathbf{H}_l$ , respectively. The corresponding push forward versions of the increments  $\dot{\mathbf{T}}, \dot{\mathbf{B}}_l, \dot{\mathbf{D}}_l, \dot{\mathbf{E}}_l, \dot{\mathbf{H}}_l$  are then defined by

$$\dot{\mathbf{T}}_0 = J^{-1} \mathbf{F} \dot{\mathbf{T}}, \quad \dot{\mathbf{B}}_{l0} = J^{-1} \mathbf{F} \dot{\mathbf{B}}_l, \quad \dot{\mathbf{D}}_{l0} = J^{-1} \mathbf{F} \dot{\mathbf{D}}_l, \quad \dot{\mathbf{E}}_{l0} = \mathbf{F}^{-T} \dot{\mathbf{E}}_l, \quad \dot{\mathbf{H}}_{l0} = \mathbf{F}^{-T} \dot{\mathbf{H}}_l, \quad (61)$$

where the subscript 0 is used to indicate the push forward operation. Equally,  $\dot{\mathbf{T}}_0$  can be thought of as the value of  $\dot{\mathbf{T}}$  when the reference configuration is updated from  $\mathcal{B}_0$  to  $\mathcal{B}_l$  after the increments are formed, and similarly for the other terms in (61). We also note that  $\dot{\mathbf{F}}_0 = \dot{\mathbf{F}} \mathbf{F}^{-1} = \mathbf{L}$ .

The updated versions of equations (55), (56), (57) and (59) are

$$\text{curl}(\dot{\mathbf{E}}_{l0} + \mathbf{v} \times \dot{\mathbf{B}}_{l0} + \dot{\mathbf{v}} \times \mathbf{B}) = -\dot{\mathbf{B}}_{l,t0}, \quad (62)$$

$$\text{curl}(\dot{\mathbf{H}}_{l0} - \mathbf{v} \times \dot{\mathbf{D}}_{l0} - \dot{\mathbf{v}} \times \mathbf{D}) = \dot{\mathbf{D}}_{l,t0} + \dot{\mathbf{J}}_{E0}, \quad (63)$$

$$\text{div} \dot{\mathbf{B}}_{l0} = 0, \quad \text{div} \dot{\mathbf{D}}_{l0} = \dot{\rho}_{E0}, \quad (64)$$

and

$$\text{div} \dot{\mathbf{T}}_0 + \rho \dot{\mathbf{f}} = \rho \mathbf{u}_{,tt}, \quad (65)$$

respectively. For an incompressible material the connections (54) hold.

Note that in Eq. (62) the term  $\dot{\mathbf{B}}_{l,t0}$  is in general different from  $\dot{\mathbf{B}}_{l0,t}$ , i.e. updating (pushing forward) and material time differentiation do not commute. Indeed, it is easy to show that

$$\dot{\mathbf{B}}_{l0,t} - \dot{\mathbf{B}}_{l,t0} = \Gamma \dot{\mathbf{B}}_{l0} - (\text{div} \mathbf{v}) \mathbf{B}, \quad (66)$$

with a similar formula pertaining to  $\mathbf{D}$  in (63). Note that the right-hand side of (66) vanishes if the underlying configuration  $\mathcal{B}_t$  is purely static, so that, in particular,  $\mathbf{v} = \mathbf{0}, \Gamma = \mathbf{0}$  and  $J_{,t} = 0$ , and the simplifications

$$\text{grad} \dot{\mathbf{v}} = \dot{\mathbf{\Gamma}}, \quad \dot{J}_{,t} = J \text{div} \dot{\mathbf{v}}, \quad \dot{\mathbf{v}} = \mathbf{F} \dot{\mathbf{V}} \quad (67)$$

follow from (50), (52) and (53). This is the special case on which we focus henceforth in the present paper.

## 6 Incremental motions superimposed on an underlying static configuration

Henceforth we consider the specialization in which the underlying fields and deformation are purely static, so that  $\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}$  are independent of  $t$ , while  $\mathbf{v} = \mathbf{0}, \Gamma = \mathbf{0}$ . We also assume that there are no distributed charges or



currents in this initial configuration, so that  $\rho_e = 0$ ,  $\mathbf{J} = \mathbf{0}$  and, from (39),  $\dot{\mathbf{J}}_{E0} = \dot{\mathbf{J}}_{I0}$ . We take  $\mathbf{E}_l$  and  $\mathbf{B}_l$  to be the independent electromagnetic variables and the energy function as  $\Omega(\mathbf{F}, \mathbf{E}_l, \mathbf{B}_l)$ , so that the constitutive relations (27) hold for the underlying deformation.

Equations (55)–(59) are exact, i.e. they don't require the increments to be 'small'. However, to take the incremental problem further it is necessary to linearize the constitutive laws for  $\dot{\mathbf{H}}_l$ ,  $\dot{\mathbf{D}}_l$  and  $\dot{\mathbf{T}}$  that are needed in Eqs. (56), (57)<sub>2</sub> and (59). On taking the increments of equations (27) we obtain the linearizations

$$\dot{\mathbf{T}} = \mathcal{A}\dot{\mathbf{F}} + \mathbb{B}\dot{\mathbf{E}}_l + \mathbb{C}\dot{\mathbf{B}}_l, \quad (68)$$

$$\dot{\mathbf{D}}_l = -\mathbb{B}^T\dot{\mathbf{F}} - \mathbf{A}\dot{\mathbf{E}}_l - \mathbf{B}\dot{\mathbf{B}}_l, \quad (69)$$

$$\dot{\mathbf{H}}_l = \mathbb{C}^T\dot{\mathbf{F}} + \mathbf{B}^T\dot{\mathbf{E}}_l + \mathbf{C}\dot{\mathbf{B}}_l, \quad (70)$$

where  $\mathcal{A}$  is a fourth-order tensor (the conventional 'elasticity tensor'),  $\mathbb{B}$  and  $\mathbb{C}$  are third-order tensors coupling the mechanical and electromagnetic material responses, and  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  are second-order electric and magnetic 'moduli tensors'. These, and (where appropriate) their transposes, are defined by

$$\mathcal{A} = \frac{\partial^2 \Omega}{\partial \mathbf{F} \partial \mathbf{F}}, \quad \mathbb{B} = \frac{\partial^2 \Omega}{\partial \mathbf{F} \partial \mathbf{E}_l}, \quad \mathbb{B}^T = \frac{\partial^2 \Omega}{\partial \mathbf{E}_l \partial \mathbf{F}}, \quad \mathbb{C} = \frac{\partial^2 \Omega}{\partial \mathbf{F} \partial \mathbf{B}_l}, \quad \mathbb{C}^T = \frac{\partial^2 \Omega}{\partial \mathbf{B}_l \partial \mathbf{F}}, \quad (71)$$

and

$$\mathbf{A} = \frac{\partial^2 \Omega}{\partial \mathbf{E}_l \partial \mathbf{E}_l}, \quad \mathbf{B} = \frac{\partial^2 \Omega}{\partial \mathbf{E}_l \partial \mathbf{B}_l}, \quad \mathbf{B}^T = \frac{\partial^2 \Omega}{\partial \mathbf{B}_l \partial \mathbf{E}_l}, \quad \mathbf{C} = \frac{\partial^2 \Omega}{\partial \mathbf{B}_l \partial \mathbf{B}_l}. \quad (72)$$

In component form we use the representations

$$\mathcal{A}_{\alpha i \beta j} = \frac{\partial^2 \Omega}{\partial F_{i\alpha} \partial F_{j\beta}}, \quad \mathbb{B}_{\alpha i | \beta} = \frac{\partial^2 \Omega}{\partial F_{i\alpha} \partial E_{l\beta}}, \quad \mathbb{C}_{\alpha i | \beta} = \frac{\partial^2 \Omega}{\partial F_{i\alpha} \partial B_{l\beta}}, \quad (73)$$

and

$$\mathbf{A}_{\alpha\beta} = \frac{\partial^2 \Omega}{\partial E_{l\alpha} \partial E_{l\beta}}, \quad \mathbf{B}_{\alpha\beta} = \frac{\partial^2 \Omega}{\partial E_{l\alpha} \partial B_{l\beta}}, \quad \mathbf{C}_{\alpha\beta} = \frac{\partial^2 \Omega}{\partial B_{l\alpha} \partial B_{l\beta}}. \quad (74)$$

Assuming that the component partial derivatives commute, we have the symmetries

$$\mathcal{A}_{\alpha i \beta j} = \mathcal{A}_{\beta j \alpha i}, \quad \mathbb{B}_{\alpha i | \beta} = \mathbb{B}_{\beta | \alpha i}, \quad \mathbb{C}_{\alpha i | \beta} = \mathbb{C}_{\beta | \alpha i}, \quad (75)$$

and

$$\mathbf{A}_{\alpha\beta} = \mathbf{A}_{\beta\alpha}, \quad \mathbf{B}_{\alpha\beta} = \mathbf{B}_{\beta\alpha}, \quad \mathbf{C}_{\alpha\beta} = \mathbf{C}_{\beta\alpha}. \quad (76)$$

However, special care is needed in using the symmetries in (75)<sub>2,3</sub>, and for this reason we have separated the pair of indices that always go together

from the single index. Equally, care is needed in interpreting the symmetry in (76)<sub>2</sub> since  $\mathbf{B}$  is a two-point tensor, and while (76)<sub>2</sub> holds  $\mathbf{B}$  itself is not symmetric since its two ‘legs’ operate on different vector spaces. Thus, it is not meaningful to equate  $\mathbf{B}^T$  and  $\mathbf{B}$ . Note, by contrast, that  $\mathbf{A}$  and  $\mathbf{C}$  are symmetric, while  $\mathcal{A}$  possesses only the major symmetry indicated in (75)<sub>1</sub>.

The products in (68)–(70) are defined (in component form) as

$$(\mathcal{A}\dot{\mathbf{F}})_{\alpha i} = \mathcal{A}_{\alpha i \beta j} \dot{F}_{j\beta}, \quad (\mathbb{B}\dot{\mathbf{E}}_l)_{\alpha i} = \mathbb{B}_{\alpha i | \beta} \dot{E}_{l\beta}, \quad (\mathbb{B}^T \dot{\mathbf{F}})_{\alpha} = \mathbb{B}_{\alpha | \beta j} \dot{F}_{j\beta}, \quad (77)$$

and

$$(\mathbf{A}\dot{\mathbf{E}}_l)_{\alpha} = A_{\alpha\beta} \dot{E}_{l\beta}, \quad (\mathbf{B}\dot{\mathbf{B}}_l)_{\alpha} = B_{\alpha\beta} \dot{B}_{l\beta}, \quad (\mathbf{B}^T \dot{\mathbf{E}}_l)_{\alpha} = B_{\beta\alpha} \dot{E}_{l\beta}, \quad (78)$$

and similarly for  $\mathbb{C}$  and  $\mathbf{C}$ .

For an incompressible material Eq. (68) is replaced by

$$\dot{\mathbf{T}} = \mathcal{A}\dot{\mathbf{F}} + \mathbb{B}\dot{\mathbf{E}}_l + \mathbb{C}\dot{\mathbf{B}}_l - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \quad (79)$$

but (69) and (70) are unchanged except that  $\det \mathbf{F} \equiv 1$ .

The incremental constitutive equations (68)–(70) can be written in push-forward form as

$$\dot{\mathbf{T}}_0 = \mathcal{A}_0 \mathbf{L} + \mathbb{B}_0 \dot{\mathbf{E}}_{l0} + \mathbb{C}_0 \dot{\mathbf{B}}_{l0}, \quad (80)$$

$$\dot{\mathbf{D}}_{l0} = -\mathbb{B}_0^T \mathbf{L} - \mathbf{A}_0 \dot{\mathbf{E}}_{l0} - \mathbf{B}_0 \dot{\mathbf{B}}_{l0}, \quad (81)$$

$$\dot{\mathbf{H}}_{l0} = \mathbb{C}_0^T \mathbf{L} + \mathbf{B}_0^T \dot{\mathbf{E}}_{l0} + \mathbf{C}_0 \dot{\mathbf{B}}_{l0}, \quad (82)$$

and the incompressible counterpart of (80) is

$$\dot{\mathbf{T}}_0 = \mathcal{A}_0 \mathbf{L} + \mathbb{B}_0 \dot{\mathbf{E}}_{l0} + \mathbb{C}_0 \dot{\mathbf{B}}_{l0} - \dot{p}\mathbf{I} + p\mathbf{L}, \quad (83)$$

which is taken in conjunction with  $\operatorname{div} \mathbf{u} = 0$ .

By combining (68)–(70) and (61)–(82) and then comparing coefficients of the independent quantities  $\mathbf{L}$ ,  $\dot{\mathbf{E}}_l$ ,  $\dot{\mathbf{B}}_l$ , we obtain the connections between the various ‘moduli’ tensors and their push-forward counterparts. The formulas for the third- and fourth-order tensors, in component form, are

$$\mathcal{A}_{0ipjq} = \mathcal{A}_{0jqip} = J^{-1} F_{i\alpha} F_{j\beta} \mathcal{A}_{\alpha i \beta j} = J^{-1} F_{i\alpha} F_{j\beta} \mathcal{A}_{\beta j \alpha i}, \quad (84)$$

$$\mathbb{B}_{0ij|k} = \mathbb{B}_{0k|ij} = J^{-1} F_{i\alpha} F_{k\beta} \mathbb{B}_{\alpha j | \beta} = J^{-1} F_{i\alpha} F_{k\beta} \mathbb{B}_{\beta | \alpha j}, \quad (85)$$

$$\mathbb{C}_{0ij|k} = \mathbb{C}_{0k|ij} = F_{i\alpha} F_{\beta k}^{-1} \mathbb{C}_{\alpha j | \beta} = F_{i\alpha} F_{\beta k}^{-1} \mathbb{C}_{\beta | \alpha j}. \quad (86)$$

Note, in particular, that (84) has exactly the same form as in pure elasticity theory.

The second-order connections may be put in compact tensor form as

$$\mathbf{A}_0 = J^{-1}\mathbf{F}\mathbf{A}\mathbf{F}^T, \quad \mathbf{C}_0 = J\mathbf{F}^{-T}\mathbf{C}\mathbf{F}^{-1}, \quad \mathbf{B}_0 = \mathbf{F}\mathbf{B}\mathbf{F}^{-1}, \quad \mathbf{B}_0^T = \mathbf{F}^{-T}\mathbf{B}^T\mathbf{F}^T. \quad (87)$$

Further relations follow from the incremental form of the rotational balance equation (77)<sub>1</sub>, which, when expressed in push-forward form, gives

$$\dot{\mathbf{T}}_0 + \mathbf{L}\boldsymbol{\tau} = \dot{\mathbf{T}}_0^T + \boldsymbol{\tau}\mathbf{L}^T. \quad (88)$$

Substitution from (80) then yields

$$\mathcal{A}_0\mathbf{L} + \mathbf{L}\boldsymbol{\tau} = (\mathcal{A}_0\mathbf{L})^T + \boldsymbol{\tau}\mathbf{L}^T, \quad \mathbb{B}_0\dot{\mathbf{E}}_{l0} = (\mathbb{B}_0\dot{\mathbf{E}}_{l0})^T, \quad \mathbb{C}_0\dot{\mathbf{B}}_{l0} = (\mathbb{C}_0\dot{\mathbf{B}}_{l0})^T, \quad (89)$$

from which we obtain

$$\mathcal{A}_{0ipjq} + \delta_{iq}\tau_{jp} = \mathcal{A}_{0pijq} + \delta_{pq}\tau_{ij}, \quad \mathbb{B}_{0ij|k} = \mathbb{B}_{0jz|k}, \quad \mathbb{C}_{0ij|k} = \mathbb{C}_{0ji|k}, \quad (90)$$

which are additional to the symmetries already mentioned. For magnetoelasticity the first and third of these were obtained by Otténio et al. [43], wherein different notation was used for  $\mathbb{C}_{0ij|k}$ . For electroelasticity the first and second were given by Dorfmann and Ogden [44], with different notation for  $\mathbb{B}_{0ij|k}$ .

For an incompressible material, equation (90)<sub>1</sub> is modified to

$$\mathcal{A}_{0ipjq} + \delta_{iq}(\tau_{jp} + p\delta_{jp}) = \mathcal{A}_{0pijq} + \delta_{pq}(\tau_{ij} + p\delta_{ij}), \quad (91)$$

as in elasticity theory [46], where  $p$  is the Lagrange multiplier appearing in Eq. (29).

Since the underlying configuration is purely static, equations (62) and (63) reduce slightly to

$$\text{curl}(\dot{\mathbf{E}}_{l0} + \dot{\mathbf{v}} \times \mathbf{B}) = -\dot{\mathbf{B}}_{l0,t}, \quad (92)$$

$$\text{curl}(\dot{\mathbf{H}}_{l0} - \dot{\mathbf{v}} \times \mathbf{D}) = \dot{\mathbf{D}}_{l0,t} + \dot{\mathbf{J}}_{l0}, \quad (93)$$

respectively, while, if there are no mechanical body forces (which we now assume to be the case), Eq. (65) becomes

$$\text{div}\dot{\mathbf{T}}_0 = \rho\mathbf{u}_{,tt}, \quad (94)$$

and the remaining equations (64) are unchanged. Note that for the considered specialization we have  $\dot{\mathbf{v}} = \mathbf{F}\dot{\mathbf{V}}$  and  $\text{div}\dot{\mathbf{v}} = \text{Div}\dot{\mathbf{V}}$ .

These equations are useful for a variety of specializations, some of which will be considered elsewhere. Here, for simplicity of illustration of the effect of finite deformations, we consider the specialization to the quasimagnetostatic approximation.

## 6.1 The quasimagnetostatic specialization

In the quasimagnetostatic approximation the electric displacement vector  $\mathbf{D}$  can be discarded. Then we set  $\mathbf{E} = \mathbf{D} = \mathbf{0}$  but allow the possibility that  $\dot{\mathbf{E}}_l \neq \mathbf{0}$ . The governing equations that remain are

$$\operatorname{curl}(\dot{\mathbf{E}}_{l0} + \dot{\mathbf{v}} \times \mathbf{B}) = -\dot{\mathbf{B}}_{l0,t}, \quad \operatorname{curl}\dot{\mathbf{H}}_{l0} = \dot{\mathbf{J}}_{l0}, \quad \operatorname{div}\dot{\mathbf{B}}_{l0} = 0, \quad (95)$$

together with (94) and  $\operatorname{div}\mathbf{u} = 0$ , while the incremental constitutive equations (82) and (83) reduce to

$$\dot{\mathbf{H}}_{l0} = \mathbb{C}_0^T \mathbf{L} + \mathbf{C}_0 \dot{\mathbf{B}}_{l0}, \quad \dot{\mathbf{T}}_0 = \mathcal{A}_0 \mathbf{L} + \mathbb{C}_0 \dot{\mathbf{B}}_{l0} - \dot{p} \mathbf{I} + p \mathbf{L}. \quad (96)$$

We shall also need temporarily a constitutive equation for  $\dot{\mathbf{J}}_{l0}$ , which, for definiteness, we obtain from the standard form of Ohm's law, namely  $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ , where the conductivity  $\sigma$  is here taken to be a scalar (material) constant, although more generally it can be a (second-order) tensor. The Lagrangian form of this law is simply obtained as  $\mathbf{J}_l = \sigma J \mathbf{c}^{-1}(\mathbf{E}_l + \mathbf{V} \times \mathbf{B}_l)$ , where  $\mathbf{c} = \mathbf{F}^T \mathbf{F}$  is the right Cauchy-Green deformation tensor. The increment of this form of Ohm's law is

$$\dot{\mathbf{J}}_l = (\operatorname{div}\mathbf{u})\mathbf{J}_l - \mathbf{c}^{-1} \dot{\mathbf{c}} \mathbf{J}_l + \sigma J \mathbf{c}^{-1}(\dot{\mathbf{E}}_l + \dot{\mathbf{V}} \times \mathbf{B}_l + \mathbf{V} \times \dot{\mathbf{B}}_l). \quad (97)$$

On pushing this forward to the current configuration and setting  $\mathbf{J} = \mathbf{v} = \mathbf{0}$  we obtain finally

$$\dot{\mathbf{J}}_{l0} = \sigma(\dot{\mathbf{E}}_{l0} + \dot{\mathbf{v}} \times \mathbf{B}). \quad (98)$$

Using this together with (95)<sub>2</sub> in (95)<sub>1</sub> we eliminate the electric field increment and obtain

$$\operatorname{curl}\operatorname{curl}\dot{\mathbf{H}}_{l0} = -\sigma\dot{\mathbf{B}}_{l0,t}, \quad (99)$$

which is coupled with (94) via (96).

On substituting for the constitutive equations and using the incompressibility condition we obtain the equations

$$\operatorname{div}(\mathcal{A}_0 \mathbf{L} + \mathbb{C}_0 \dot{\mathbf{B}}_{l0}) - \operatorname{grad}\dot{p} = \rho \mathbf{u}_{,tt}, \quad (100)$$

$$\operatorname{curl}\operatorname{curl}(\mathbb{C}_0^T \mathbf{L} + \mathbf{C}_0 \dot{\mathbf{B}}_{l0}) = -\sigma \dot{\mathbf{B}}_{l0,t}, \quad (101)$$

which, together with

$$\operatorname{div}\mathbf{u} = 0, \quad \operatorname{div}\dot{\mathbf{B}}_{l0} = 0, \quad (102)$$

govern the variables  $\mathbf{u}$  and  $\dot{\mathbf{B}}_{l0}$ . Note that  $\dot{p}$  can be eliminated by taking the curl of Eq. (100).

The unknowns are the six components of  $\mathbf{u}$  and  $\dot{\mathbf{B}}_{l0}$  and  $\dot{p}$ . Note that it follows from (101) that  $(\operatorname{div}\dot{\mathbf{B}}_{l0})_{,t} = 0$ , and hence, since  $\operatorname{div}\mathbf{B} = 0$  can

be taken as the initial condition for this equation, we have (102)<sub>2</sub> as a consequence. Thus, there are just seven independent equations governing the seven unknowns in the general three-dimensional situation.

For a non-conducting material  $\sigma = 0$  and Eq. (101) is replaced by

$$\text{curl}(\mathbb{C}_0^T \mathbf{L} + \mathbf{C}_0 \dot{\mathbf{B}}_{l0}) = \mathbf{0}. \quad (103)$$

while the form of Eq. (100) is unaffected.

Finally in this section, we take the underlying deformation to be homogeneous and the field vectors  $\mathbf{H}$  and  $\mathbf{B}$  to be uniform. Then, the tensors  $\mathcal{A}_0, \mathbb{C}_0, \mathbf{C}_0$  are constant, as is  $p$ . Equations (100) and (101) may then be written in (rectangular Cartesian) component form as

$$\mathcal{A}_{0minj} u_{j,mn} + \mathbb{C}_{0mi|n} w_{n,m} - \dot{p}_{,i} = \rho u_{i,tt}, \quad (104)$$

and

$$\mathbb{C}_{0j|mn} u_{n,mij} - \mathbb{C}_{0i|mn} u_{n,mjj} + \mathbf{C}_{0jm} w_{m,ij} - \mathbf{C}_{0im} w_{m,jj} = -\sigma w_{i,t}, \quad (105)$$

respectively, where, for convenience, we have set  $\dot{\mathbf{B}}_{l0} = \mathbf{w}$ , and subscripts  $i, j, m$  following a comma indicate partial differentiation (recall, though, that  $_{,t}$  signifies the material time derivative). In component form equations (102) become

$$u_{i,i} = 0, \quad w_{i,i} = 0. \quad (106)$$

## 7 Incremental plane harmonic waves

As a first application of the equations we consider the case of (infinitesimal) harmonic plane wave propagation in an infinite medium subject to an underlying homogeneous deformation and a uniform magnetic field. Let  $\mathbf{n}$  be a unit vector defining the direction of propagation of the wave,  $v$  be the wave speed and  $k$  the wave number. In complex representation we write

$$\mathbf{u} = \mathbf{m} e^{ik(\mathbf{n} \cdot \mathbf{x} - vt)}, \quad \mathbf{w} = ik\mathbf{q} e^{ik(\mathbf{n} \cdot \mathbf{x} - vt)}, \quad \dot{p} = ikP e^{ik(\mathbf{n} \cdot \mathbf{x} - vt)}, \quad (107)$$

where  $\mathbf{m}$  and  $\mathbf{q}$  are constant vectors and  $P$  is a constant. The factors  $ik$  are included in the latter two expressions because the different derivatives then lead to cancellation of the factors involving  $k$  in the left-hand sides of (100) and (101). It follows immediately from (102) that

$$\mathbf{m} \cdot \mathbf{n} = 0, \quad \mathbf{q} \cdot \mathbf{n} = 0. \quad (108)$$

We also note that

$$\operatorname{curl} \mathbf{u} = ik\mathbf{n} \times \mathbf{u}, \quad \operatorname{grad} \mathbf{u} = ik\mathbf{u} \otimes \mathbf{n}, \quad \operatorname{grad} p = ikp\mathbf{n}, \quad (109)$$

with corresponding expressions for  $\mathbf{w}$ .

These expressions are then used in (100) and (101) to give

$$\mathbf{Q}(\mathbf{n})\mathbf{m} + \mathbf{R}^T(\mathbf{n})\mathbf{q} - P\mathbf{n} = \rho v^2 \mathbf{m}, \quad (110)$$

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{s}) = -i\sigma^* \mathbf{q}, \quad (111)$$

where  $\sigma^* = \sigma v/k$  and we have introduced the notations defined by

$$[\mathbf{Q}(\mathbf{n})]_{ij} = \mathcal{A}_{0piqj} n_p n_q, \quad [\mathbf{R}(\mathbf{n})]_{ij} = \mathbb{C}_{0i|jp} n_p, \quad (112)$$

and

$$\mathbf{s} = \mathbf{R}(\mathbf{n})\mathbf{m} + \mathbf{C}_0 \mathbf{q}. \quad (113)$$

Note that  $\mathbf{Q}(\mathbf{n})$  is symmetric; in the context of pure elasticity theory this is known as the *acoustic tensor*. On the other hand,  $\mathbf{R}(\mathbf{n})$  is not in general symmetric. We shall refer to this as the *magnetoelastic coupling tensor* for plane waves.

## 7.1 Application to a non-conducting material

Here we consider the material to be non-conducting so that there is no dissipation and, without loss of generality,  $\mathbf{m}$ ,  $\mathbf{n}$  and  $\mathbf{q}$  may be taken as real vectors and  $k$  as a real wave number. For a non-conducting material the system of equations (110) and (111) is reduced to

$$\mathbf{Q}(\mathbf{n})\mathbf{m} + \mathbf{R}^T(\mathbf{n})\mathbf{q} - P\mathbf{n} = \rho v^2 \mathbf{m}, \quad \mathbf{n} \times \mathbf{s} = \mathbf{0}, \quad (114)$$

the latter of which may also be obtained directly from (103). From (114)<sub>2</sub> we deduce that

$$\mathbf{s} \equiv \mathbf{R}(\mathbf{n})\mathbf{m} + \mathbf{C}_0 \mathbf{q} = A\mathbf{n}, \quad (115)$$

for some scalar constant  $A$ . Use of (108) and (113) then yields

$$[\mathbf{R}(\mathbf{n})\mathbf{m}] \cdot \mathbf{q} = -(\mathbf{C}_0 \mathbf{q}) \cdot \mathbf{q}, \quad [\mathbf{R}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} = -(\mathbf{C}_0 \mathbf{q}) \cdot \mathbf{m}, \quad (116)$$

and, from (114)<sub>1</sub>,

$$[\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{q} + [\mathbf{R}(\mathbf{n})\mathbf{q}] \cdot \mathbf{q} = 0, \quad (117)$$

while  $P$  and  $A$  are given by

$$P = [\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{n} + [\mathbf{R}(\mathbf{n})\mathbf{n}] \cdot \mathbf{q}, \quad A = [\mathbf{R}(\mathbf{n})\mathbf{m}] \cdot \mathbf{n} + (\mathbf{C}_0 \mathbf{q}) \cdot \mathbf{n}. \quad (118)$$

It follows, in particular, that

$$\rho v^2 \mathbf{m} \cdot \mathbf{m} = [\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} - (\mathbf{C}_0 \mathbf{q}) \cdot \mathbf{q}. \quad (119)$$

In the absence of a magnetic field this reduces to  $\rho v^2 = [\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m}$ , which gives the wave speed  $v$  for a purely elastic material for any direction  $\mathbf{n}$  for which such a wave exists. If  $\mathbf{C}_0$  is positive definite then whether the wave speed is reduced by the presence of a magnetic field depends on how  $\mathbf{Q}(\mathbf{n})$  depends on  $\mathbf{B}$ . We shall give specific examples in what follows. Of course, the existence of a wave for a given  $\mathbf{n}$  is not necessarily guaranteed in general.

## 7.2 Isotropic magnetoelasticity

We now specialize the constitutive law to that of an isotropic magnetoelastic material. Magneto-sensitive elastomers have an isotropic response if magnetic particles are distributed within the material during the curing process in the absence of an applied magnetic field, leading to an essentially random particle distribution. When a magnetic field is applied to the material so formed it generates a preferred direction in the material, and this enables the energy function to be expressed in terms of invariants which are rather similar to those arising in the case of an elastic transversely isotropic material. More particularly, for an isotropic material, the energy function  $\Omega$  can be taken to depend on the Lagrangian variables  $\mathbf{c}$  and  $\mathbf{B}_l \otimes \mathbf{B}_l$ , thus guaranteeing both objectivity and independence of the sense of  $\mathbf{B}_l$ . The result is that  $\Omega$  depends on the six invariants

$$\begin{aligned} I_1 &= \text{tr } \mathbf{c}, & I_2 &= \frac{1}{2} [(\text{tr } \mathbf{c})^2 - \text{tr}(\mathbf{c}^2)], & I_3 &= \det \mathbf{c} = J^2, \\ I_4 &= \mathbf{B}_l \cdot \mathbf{B}_l, & I_5 &= (\mathbf{c}\mathbf{B}_l) \cdot \mathbf{B}_l, & I_6 &= (\mathbf{c}^2 \mathbf{B}_l) \cdot \mathbf{B}_l, \end{aligned} \quad (120)$$

which reduces to the five  $I_1, I_2, I_4, I_5,$  and  $I_6$  for an incompressible material, for which  $I_3 \equiv 1$ . The total stress tensor  $\boldsymbol{\tau}$  is then expressed as

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\Omega_1 \mathbf{b} + 2\Omega_2 (I_1 \mathbf{b} - \mathbf{b}^2) + 2\Omega_5 \mathbf{B} \otimes \mathbf{B} + 2\Omega_6 (\mathbf{B} \otimes \mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B} \otimes \mathbf{B}), \quad (121)$$

where  $\Omega_m = \partial\Omega/\partial I_m$ ,  $m = 1, 2, 4, 5, 6$ , and  $\mathbf{b} = \mathbf{F}\mathbf{F}^T$  is the left Cauchy-Green tensor.

The magnetic field vector  $\mathbf{H}$  is found from (26)<sub>2</sub> as

$$\mathbf{H} = 2(\Omega_4 \mathbf{b}^{-1} \mathbf{B} + \Omega_5 \mathbf{B} + \Omega_6 \mathbf{b}\mathbf{B}). \quad (122)$$

Referred to the principal axes of  $\mathbf{b}$ , which then has the diagonal representation  $(\lambda_1^2, \lambda_2^2, \lambda_3^2)$ , where  $\lambda_1, \lambda_2, \lambda_3$  are the principal stretches, equations

(121) and (122) have component forms

$$\tau_{ii} = -p + 2\lambda_i^2\Omega_1 + 2\lambda_i^2(\lambda_j^2 + \lambda_k^2)\Omega_2 + 2\Omega_5 B_i^2 + 4\lambda_i^2\Omega_6 B_i^2, \quad (123)$$

$$\tau_{ij} = 2[\Omega_5 + (\lambda_i^2 + \lambda_j^2)\Omega_6]B_i B_j, \quad (124)$$

where  $(i, j, k)$  is a permutation of  $(1, 2, 3)$ , and

$$H_i = 2(\lambda_i^{-2} + \Omega_5 + \lambda_i^2\Omega_6)B_i, \quad i = 1, 2, 3. \quad (125)$$

We remark that if a magnetic field is applied during the curing process then this ‘freezes-in’ a preferred direction in the undeformed configuration of the material and this generates additional invariants. The theory required for this case is being treated in a separate paper.

For an unconstrained isotropic material, the expressions (73)<sub>1,3</sub> and (74)<sub>3</sub> can be expanded in the forms

$$\mathcal{A}_{\alpha i \beta j} = \sum_{m=1, m \neq 4}^6 \sum_{n=1, n \neq 4}^6 \Omega_{mn} \frac{\partial I_n}{\partial F_{i\alpha}} \frac{\partial I_m}{\partial F_{j\beta}} + \sum_{n=1, n \neq 4}^6 \Omega_n \frac{\partial^2 I_n}{\partial F_{i\alpha} \partial F_{j\beta}}, \quad (126)$$

$$\mathbb{C}_{\alpha i | \beta} = \sum_{m=4}^6 \sum_{n=1, n \neq 4}^6 \Omega_{mn} \frac{\partial I_m}{\partial B_{l_\beta}} \frac{\partial I_n}{\partial F_{i\alpha}} + \sum_{n=5}^6 \Omega_n \frac{\partial^2 I_n}{\partial F_{i\alpha} \partial B_{l_\beta}}, \quad (127)$$

$$\mathbb{C}_{\alpha\beta} = \sum_{m=4}^6 \sum_{n=4}^6 \Omega_{mn} \frac{\partial I_m}{\partial B_{l_\alpha}} \frac{\partial I_n}{\partial B_{l_\beta}} + \sum_{n=4}^6 \Omega_n \frac{\partial^2 I_n}{\partial B_{l_\alpha} \partial B_{l_\beta}}, \quad (128)$$

where  $\Omega_{mn} = \partial^2 \Omega / \partial I_m \partial I_n$ . Explicit expressions for the derivatives of the invariants appearing above are listed in the appendix of [43] and are not repeated here. For an incompressible material these expressions remain in force except that  $I_3 \equiv 1$  and the derivatives of  $\Omega$  with respect to  $I_3$  are omitted in the summations. Corresponding expressions for  $\mathcal{A}_{0piqj}$ ,  $\mathbb{C}_{0ij|k}$ ,  $\mathbb{C}_{0ij}$  are obtained from equations (84)–(86). Since these expressions are rather long in the general case they are not given here, but we shall need some of their specializations, and these are given in Appendix A.

### 7.3 Example: a prototype magnetoelastic solid

To illustrate the above results we consider a simple prototype model of a magnetoelastic solid based on the neo-Hookean model from rubber elasticity. This has energy function  $\Omega$  given by

$$\Omega = \frac{1}{2}\mu(0)(I_1 - 3) + \frac{1}{2}(\kappa I_4 + \nu I_5), \quad (129)$$



where  $I_1 = \text{tr } \mathbf{c}$  is the first principal invariant of  $\mathbf{c}$ ,  $I_4 = \mathbf{B}_l \cdot \mathbf{B}_l$ ,  $I_5 = \mathbf{B}_l \cdot (\mathbf{c}\mathbf{B}_l)$ ,  $\mu(0)$  is the shear modulus of the material in the absence of a magnetic field, and  $\kappa$  and  $\nu$  are positive material constants with the dimension of  $\mu_0^{-1}$  that couple the mechanical and magnetic effects. Note the distinction between  $\mu(0)$  and  $\mu_0$ , the latter, we recall, being the magnetic permeability of free space. In slightly different notation this model was used in Otténio et al. [43], wherein the interpretation of the coupling constants was discussed.

It follows from (121) and (122) that

$$\boldsymbol{\tau} = -p\mathbf{I} + \mu(0)\mathbf{b} + \nu\mathbf{B} \otimes \mathbf{B} \quad (130)$$

and

$$\mathbf{H} = \kappa\mathbf{b}^{-1}\mathbf{B} + \nu\mathbf{B}. \quad (131)$$

From the definitions (112) we also obtain

$$\mathbf{Q}(\mathbf{n}) = [\mu(0)(\mathbf{b}\mathbf{n}) \cdot \mathbf{n} + \nu(\mathbf{B} \cdot \mathbf{n})^2]\mathbf{I}, \quad (132)$$

$$\mathbf{R}(\mathbf{n}) = \nu[(\mathbf{B} \cdot \mathbf{n})\mathbf{I} + \mathbf{n} \otimes \mathbf{B}], \quad (133)$$

while  $\mathbf{C}_0$  is given by

$$\mathbf{C}_0 = \kappa\mathbf{b}^{-1} + \nu\mathbf{I}. \quad (134)$$

Equations (114) and (115) specialize to

$$[\mu(0)(\mathbf{b}\mathbf{n}) \cdot \mathbf{n} + \nu(\mathbf{B} \cdot \mathbf{n})^2 - \rho v^2]\mathbf{m} + \nu(\mathbf{B} \cdot \mathbf{n})\mathbf{q} = P\mathbf{n}, \quad (135)$$

and

$$\nu[(\mathbf{B} \cdot \mathbf{n})\mathbf{m} + (\mathbf{B} \cdot \mathbf{m})\mathbf{n}] + \kappa\mathbf{b}^{-1}\mathbf{q} + \nu\mathbf{q} = A\mathbf{n}, \quad (136)$$

from which we deduce with the help of (108) that  $A = \kappa(\mathbf{b}^{-1}\mathbf{q}) \cdot \mathbf{n} + \nu(\mathbf{B} \cdot \mathbf{m})$  and  $P = 0$ .

Assuming for the moment that  $\mathbf{B} \cdot \mathbf{n} \neq 0$  we see that  $\mathbf{q}$  and  $\mathbf{m}$  must be parallel (or antiparallel) and it follows after a little manipulation that

$$\rho v^2 = \mu(0)(\mathbf{b}\mathbf{n}) \cdot \mathbf{n} + \frac{\kappa\nu}{\nu + \kappa(\mathbf{b}^{-1}\hat{\mathbf{m}}) \cdot \hat{\mathbf{m}}}(\mathbf{B} \cdot \mathbf{n})^2(\mathbf{b}^{-1}\hat{\mathbf{m}}) \cdot \hat{\mathbf{m}}, \quad (137)$$

where  $\hat{\mathbf{m}} = \mathbf{m}/|\mathbf{m}|$ . If  $\kappa > 0$  and  $\nu > 0$  then the squared wave speed is increased by the presence of the magnetic field compared with that in the absence of the field. Exceptionally, if  $\mathbf{B}$  is perpendicular to the direction of propagation, i.e.  $\mathbf{B} \cdot \mathbf{n} = 0$ , then the wave speed is unaffected by the magnetic field. This case requires that  $\mathbf{q} = \mathbf{0}$ , so that there is no disturbance to the underlying magnetic field by the (transverse) mechanical wave.

If the underlying magnetic field is supported (by appropriate stresses) in the absence of deformation (so that  $\mathbf{b} = \mathbf{I}$ ) then (137) simplifies to

$$\rho v^2 = \mu(0) + \frac{\kappa\nu}{\kappa + \nu}(\mathbf{B} \cdot \mathbf{n})^2. \quad (138)$$

## 7.4 Two-dimensional specialization

We now restrict attention to two-dimensional increments, in the  $(X_1, X_2)$  plane. Let the underlying deformation correspond to a uniform stretch  $\lambda_3$  in the  $X_3$  direction and focus on the  $(X_1, X_2)$  plane, with principal stretches  $\lambda_1$  and  $\lambda_2$ . Let  $\mathbf{B}$  have components  $(B_1, B_2, 0)$ . Then, we take  $\mathbf{u}$  and  $\mathbf{w}$  to have components  $(u_1, u_2, 0)$  and  $(w_1, w_2, 0)$ , respectively, with  $u_1, u_2, w_1, w_2$  functions of  $x_1, x_2, t$  only. By (106), we are able to introduce scalar functions  $\psi(x_1, x_2, t)$  and  $\phi(x_1, x_2, t)$  such that

$$u_1 = \psi, \quad u_2 = -\psi, \quad w_1 = \phi, \quad w_2 = -\phi. \quad (139)$$

After some manipulations it can then be shown that, for  $i = 1, 2$ , equations (105) can be written

$$\Phi_{,2} = \sigma\phi_{,2t}, \quad \Phi_{,1} = \sigma\phi_{,1t}, \quad (140)$$

where

$$\Phi = a\psi_{,111} + b\psi_{,112} + c\psi_{,122} + d\psi_{,222} + f\phi_{,11} - 2g\phi_{,12} + h\phi_{,22}, \quad (141)$$

in which the notations  $a, b, c, d, f, g, h$  for the coefficients are defined by

$$a = \mathbb{C}_{02|12}, \quad b = \mathbb{C}_{02|22} - \mathbb{C}_{02|11} - \mathbb{C}_{01|12}, \quad c = \mathbb{C}_{01|11} - \mathbb{C}_{01|22} - \mathbb{C}_{02|12}, \quad (142)$$

$$d = \mathbb{C}_{01|12}, \quad f = \mathbb{C}_{022}, \quad g = \mathbb{C}_{012}, \quad h = \mathbb{C}_{011}. \quad (143)$$

It follows from (140) that the two non-trivial components of Eq. (105) are essentially the same since (140) can be integrated to give

$$\Phi = \sigma\phi_{,t} \quad (144)$$

by absorbing an arbitrary function of  $t$  into the definition of  $\phi$ .

Again after some manipulations, followed by use of (140), Eq. (104) can be arranged in the form

$$\begin{aligned} & \alpha\psi_{,1111} + \delta\psi_{,1112} + 2\beta\psi_{,1122} + \epsilon\psi_{,1222} + \gamma\psi_{,2222} + a\phi_{,111} + b\phi_{,112} \\ & + c\phi_{,122} + d\phi_{,222} = \rho(\psi_{,11} + \psi_{,22})_{,tt} - \sigma B_1\phi_{,1t} - \sigma B_2\phi_{,2t}, \end{aligned} \quad (145)$$

and (144) becomes explicitly

$$a\psi_{,111} + b\psi_{,112} + c\psi_{,122} + d\psi_{,222} + f\phi_{,11} - 2g\phi_{,12} + h\phi_{,22} = \sigma\phi_{,t}, \quad (146)$$

where we have introduced the notations

$$\alpha = \mathcal{A}_{01212}, \quad 2\beta = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{02112}, \quad \gamma = \mathcal{A}_{02121}, \quad (147)$$

$$\delta = 2(\mathcal{A}_{01222} - \mathcal{A}_{01211}), \quad \epsilon = 2(\mathcal{A}_{01121} - \mathcal{A}_{02221}), \quad (148)$$

for the remaining constant coefficients appearing above. Explicit expressions for the individual components in (142), (143), (147) and (148), referred to the principal axes of  $\mathbf{b}$ , are given in Appendix A.

For an isotropic material, for a pure homogeneous strain aligned with the coordinate axes, with  $B_1 = 0$ , and for the purely static situation, equations (145) and (146) reduce to equations obtained by Otténio et al. [43], in which case  $\delta = \epsilon = a = c = g = 0$ . The notation used here is different from that in the latter paper.

We now specialize once more to the case  $\sigma = 0$ . Now, if either  $B_1 = 0$  or  $B_2 = 0$  then it is easy to see from Appendix A that  $\delta = \epsilon = g = 0$ , and we assume hereon that this is the case. Equations (145) and (146) then reduce to

$$\begin{aligned} \alpha\psi_{,1111} + 2\beta\psi_{,1122} + \gamma\psi_{,2222} + a\phi_{,111} + b\phi_{,112} \\ + c\phi_{,122} + d\phi_{,222} = \rho(\psi_{,11} + \psi_{,22})_{,tt}, \end{aligned} \quad (149)$$

and

$$a\psi_{,111} + b\psi_{,112} + c\psi_{,122} + d\psi_{,222} + f\phi_{,11} + h\phi_{,22} = 0. \quad (150)$$

Note that  $a = c = 0$  if  $B_1 = 0$  and  $b = d = 0$  if  $B_2 = 0$ .

Consider again harmonic plane waves, this time in the form

$$\psi = \psi_0 e^{ik(\mathbf{n}\cdot\mathbf{x}-vt)}, \quad \phi = \phi_0 e^{ik(\mathbf{n}\cdot\mathbf{x}-vt)}, \quad (151)$$

where  $\psi_0$  and  $\phi_0$  are constants and  $\mathbf{n}$  and  $\mathbf{x}$  are two-dimensional vectors, in the (1, 2) plane. Substitution into equations (149) and (150) yields

$$(\alpha n_1^4 + 2\beta n_1^2 n_2^2 + \gamma n_2^4 - \rho v^2)\psi_0 + (an_1^3 + bn_1^2 n_2 + cn_1 n_2^2 + dn_2^3)\phi_0 = 0, \quad (152)$$

$$(an_1^3 + bn_1^2 n_2 + cn_1 n_2^2 + dn_2^3)\psi_0 + (fn_1^2 + hn_2^2)\phi_0 = 0. \quad (153)$$

For a non-trivial solution for  $\psi_0$  and  $\phi_0$  we set the determinant of coefficients to zero and obtain

$$\rho v^2 = \alpha n_1^4 + 2\beta n_1^2 n_2^2 + \gamma n_2^4 - \frac{(an_1^3 + bn_1^2 n_2 + cn_1 n_2^2 + dn_2^3)^2}{fn_1^2 + hn_2^2}, \quad (154)$$

which is the *propagation condition* that determines the wave speed for any direction of propagation  $\mathbf{n} = (n_1, n_2, 0)$ .

Suppose that the direction of propagation is along one of the principal axes; specifically, we take  $n_1 = 1, n_2 = 0$ . Then, since we are restricting to plane strain,  $m_1 = q_1 = 0$ . Then (154) simplifies to

$$\rho v^2 = \alpha - a^2/f. \quad (155)$$

If  $B_1 = 0$  then  $a = 0$  and  $\rho v^2 = \alpha$ , where

$$\alpha = \mathcal{A}_{01212} = 2\lambda_1^2(\Omega_1 + \lambda_3^2\Omega_2) + 2\lambda_1^2\Omega_6 B_2^2. \quad (156)$$

Thus, the wave speed is increased (decreased) by a magnetic field transverse to the direction of propagation compared with the purely elastic case if  $\Omega_6 > 0$  ( $< 0$ ). Moreover,  $\phi_0 = 0$ , so the mechanical wave is not accompanied by a disturbance in the underlying magnetic field. If  $\Omega_6 = 0$  then the wave speed is unaffected by the magnetic field, as for the prototype model discussed previously.

If, instead,  $B_2 = 0$  then

$$a = 2[\Omega_5 + (\lambda_1^2 + \lambda_2^2)\Omega_6]B_1, \quad f = 2(\lambda_2^{-2}\Omega_4 + \Omega_5 + \lambda_2^2\Omega_6), \quad (157)$$

and

$$\alpha = \mathcal{A}_{01212} = 2\lambda_1^2(\Omega_1 + \lambda_3^2\Omega_2) + 2[\Omega_5 + (2\lambda_1^2 + \lambda_2^2)\Omega_6]B_1^2, \quad (158)$$

and the wave speed depends on  $B_1$  and the constitutive functions as well as the underlying deformation in a fairly complicated way. For the prototype model (129) equation (155) then reduces to

$$\rho v^2 = \mu(0)\lambda_1^2 + \frac{\nu\kappa}{\kappa + \nu\lambda_2^2}B_1^2. \quad (159)$$

## 8 Concluding remarks

In this paper we have examined the equations that couple the nonlinear elastic and electromagnetic responses of a solid continuum in the non-relativistic limit and derived the (linearized) incremental equations for small motions superimposed on a known existing motion or finite static deformation. These equations will allow a number of interesting problems to be tackled. Here we have illustrated their application by considering the simple situation of incremental plane wave propagation in an infinite medium. To be addressed in subsequent works will be, *inter alia*, surface and interfacial magnetoacoustic and electroacoustic wave propagation in the presence of a finite elastic deformation.

## Appendix A. In-plane components of $\mathcal{A}_0$ , $\mathbb{C}_0$ and $\mathbf{C}_0$

The in-plane components of  $\mathcal{A}_0$  with respect to the principal axes of  $\mathbf{b}$  are given by

$$\begin{aligned}
\mathcal{A}_{01111} &= 2\lambda_1^2[\Omega_1 + (\lambda_2^2 + \lambda_3^2)\Omega_2] + 4\lambda_1^4[\Omega_{11} + 2(\lambda_2^2 + \lambda_3^2)\Omega_{12} \\
&\quad + (\lambda_2^2 + \lambda_3^2)^2\Omega_{22}] + 2(\Omega_5 + 6\lambda_1^2\Omega_6)B_1^2 \\
&\quad + 8\lambda_1^2[\Omega_{15} + (\lambda_2^2 + \lambda_3^2)\Omega_{25} + 2\lambda_1^2\Omega_{16} + 2\lambda_1^2(\lambda_2^2 + \lambda_3^2)\Omega_{26}]B_1^2 \\
&\quad + 4(\Omega_{55} + 4\lambda_1^2\Omega_{56} + 4\lambda_1^4\Omega_{66})B_1^4, \\
\mathcal{A}_{01122} &= 4\lambda_1^2\lambda_2^2[\Omega_2 + \Omega_{11} + (\lambda_1^2 + \lambda_2^2 + 2\lambda_3^2)\Omega_{12} \\
&\quad + (\lambda_1^2 + \lambda_3^2)(\lambda_2^2 + \lambda_3^2)\Omega_{22}] + 4(\Omega_{15} + \lambda_3^2\Omega_{25})(\lambda_1^2B_2^2 + \lambda_2^2B_1^2) \\
&\quad + 4\lambda_1^2\lambda_2^2(\Omega_{25} + 2\lambda_3^2\Omega_{26} + 2\Omega_{16})(B_1^2 + B_2^2) \\
&\quad + 8\lambda_1^2\lambda_2^2\Omega_{26}(\lambda_1^2B_1^2 + \lambda_2^2B_2^2) \\
&\quad + 4[\Omega_{55} + 2(\lambda_1^2 + \lambda_2^2)\Omega_{56} + 4\lambda_1^2\lambda_2^2\Omega_{66}]B_1^2B_2^2, \\
\mathcal{A}_{01212} &= 2\lambda_1^2(\Omega_1 + \lambda_3^2\Omega_2) + 2[\Omega_5 + (2\lambda_1^2 + \lambda_2^2)\Omega_6]B_1^2 + 2\lambda_1^2\Omega_6B_2^2 \\
&\quad + 4[\Omega_{55} + 2(\lambda_1^2 + \lambda_2^2)\Omega_{56} + (\lambda_1^2 + \lambda_2^2)^2\Omega_{66}]B_1^2B_2^2, \\
\mathcal{A}_{01221} &= -2\lambda_1^2\lambda_2^2\Omega_2 + 2\Omega_6(\lambda_1^2B_2^2 + \lambda_2^2B_1^2) \\
&\quad + 4[\Omega_{55} + 2(\lambda_1^2 + \lambda_2^2)\Omega_{56} + (\lambda_1^2 + \lambda_2^2)^2\Omega_{66}]B_1^2B_2^2, \\
\mathcal{A}_{01112} &= 4\lambda_1^2\Omega_6B_1B_2 + 4\lambda_1^2[\Omega_{15} + (\lambda_2^2 + \lambda_3^2)\Omega_{25} \\
&\quad + (\lambda_1^2 + \lambda_2^2)\Omega_{16} + (\lambda_1^2 + \lambda_2^2)(\lambda_2^2 + \lambda_3^2)\Omega_{26}]B_1B_2 \\
&\quad + 4[\Omega_{55} + (3\lambda_1^2 + \lambda_2^2)\Omega_{56} + 2\lambda_1^2(\lambda_1^2 + \lambda_2^2)\Omega_{66}]B_1^3B_2, \\
\mathcal{A}_{01121} &= 2\Omega_5B_1B_2 + 2(3\lambda_1^2 + \lambda_2^2)\Omega_6B_1B_2 + 4\lambda_1^2[\Omega_{15} + (\lambda_2^2 + \lambda_3^2)\Omega_{25} \\
&\quad + (\lambda_1^2 + \lambda_2^2)\Omega_{16} + (\lambda_1^2 + \lambda_2^2)(\lambda_2^2 + \lambda_3^2)\Omega_{26}]B_1B_2 \\
&\quad + 4[\Omega_{55} + (3\lambda_1^2 + \lambda_2^2)\Omega_{56} + 2\lambda_1^2(\lambda_1^2 + \lambda_2^2)\Omega_{66}]B_1^3B_2, \tag{160}
\end{aligned}$$

together with  $\mathcal{A}_{02222}$ ,  $\mathcal{A}_{02121}$ ,  $\mathcal{A}_{02212}$ ,  $\mathcal{A}_{02221}$ , which are obtained from the above by interchanging the roles of  $\lambda_1$  and  $\lambda_2$  and of  $B_1$  and  $B_2$  jointly.

We note, in particular, the connections

$$\mathcal{A}_{01212} - \mathcal{A}_{01221} = \tau_{11} + p, \quad \mathcal{A}_{02121} - \mathcal{A}_{01221} = \tau_{22} + p, \tag{161}$$

$$\mathcal{A}_{01121} - \mathcal{A}_{01112} = \mathcal{A}_{02212} - \mathcal{A}_{02221} = \tau_{12} = 2[\Omega_5 + (\lambda_1^2 + \lambda_2^2)\Omega_6]B_1B_2. \tag{162}$$

In pure elasticity theory only the components  $\mathcal{A}_{01111}$ ,  $\mathcal{A}_{01122} = \mathcal{A}_{02211}$ ,  $\mathcal{A}_{02222}$ ,  $\mathcal{A}_{01212}$ ,  $\mathcal{A}_{02121}$  and  $\mathcal{A}_{01221} = \mathcal{A}_{02112}$  arise. In the present context, the remaining components  $\mathcal{A}_{01121}$ ,  $\mathcal{A}_{01112}$ ,  $\mathcal{A}_{02221}$ ,  $\mathcal{A}_{02212}$  all vanish if either  $B_1 = 0$  or  $B_2 = 0$ .

On the same basis the components of  $\mathbb{C}_0$  are given by

$$\begin{aligned} \mathbb{C}_{01|11} = & 4(\Omega_5 + 2\lambda_1^2\Omega_6)B_1 + 4[\Omega_{14} + \lambda_1^2\Omega_{15} + \lambda_1^4\Omega_{16} \\ & + (\lambda_2^2 + \lambda_3^2)\Omega_{24} + \lambda_1^2(\lambda_2^2 + \lambda_3^2)\Omega_{25} + \lambda_1^4(\lambda_2^2 + \lambda_3^2)\Omega_{26}]B_1 \\ & + 4(\lambda_1^{-2}\Omega_{45} + \Omega_{55} + 2\Omega_{46} + 3\lambda_1^2\Omega_{56} + 2\lambda_1^4\Omega_{66})B_1^3, \end{aligned} \quad (163)$$

$$\begin{aligned} \mathbb{C}_{02|11} = & 4\lambda_1^2[\lambda_2^{-2}\Omega_{14} + \Omega_{15} + \lambda_2^2\Omega_{16} + (\lambda_2^2 + \lambda_3^2)(\lambda_2^{-2}\Omega_{24} \\ & + \Omega_{25} + \lambda_2^2\Omega_{26})]B_2 + 4[\lambda_2^{-2}\Omega_{45} + \Omega_{55} \\ & + 2\lambda_1^2\lambda_2^{-2}\Omega_{46} + (2\lambda_1^2 + \lambda_2^2)\Omega_{56} + 2\lambda_1^2\lambda_2^2\Omega_{66}]B_1^2B_2, \end{aligned} \quad (164)$$

$$\begin{aligned} \mathbb{C}_{01|12} = & 2[\Omega_5 + (\lambda_1^2 + \lambda_2^2)\Omega_6]B_2 + 4[\lambda_1^{-2}\Omega_{45} + \Omega_{55} + \lambda_1^{-2}(\lambda_1^2 + \lambda_2^2)\Omega_{46} \\ & + (2\lambda_1^2 + \lambda_2^2)\Omega_{56} + \lambda_1^2(\lambda_1^2 + \lambda_2^2)\Omega_{66}]B_1^2B_2, \end{aligned} \quad (165)$$

together with  $\mathbb{C}_{01|22}$ ,  $\mathbb{C}_{02|22}$ ,  $\mathbb{C}_{02|12}$ , again obtained by interchanging the roles of  $\lambda_1$  and  $\lambda_2$  and  $B_1$  and  $B_2$ .

Finally, the in-plane components of  $\mathbf{C}_0$  are given by

$$\begin{aligned} \mathbb{C}_{011} = & 2(\lambda_1^{-2}\Omega_4 + \Omega_5 + \lambda_1^2\Omega_6) + 4(\lambda_1^{-4}\Omega_{44} + 2\lambda_1^{-2}\Omega_{45} + \Omega_{55} \\ & + 2\Omega_{46} + 2\lambda_1^2\Omega_{56} + \lambda_1^4\Omega_{66})B_1^2, \end{aligned} \quad (166)$$

$$\begin{aligned} \mathbb{C}_{022} = & 2(\lambda_2^{-2}\Omega_4 + \Omega_5 + \lambda_2^2\Omega_6) + 4(\lambda_2^{-4}\Omega_{44} + 2\lambda_2^{-2}\Omega_{45} + \Omega_{55} \\ & + 2\Omega_{46} + 2\lambda_2^2\Omega_{56} + \lambda_2^4\Omega_{66})B_2^2, \end{aligned} \quad (167)$$

$$\begin{aligned} \mathbb{C}_{012} = & 4[\lambda_1^{-2}\lambda_2^{-2}\Omega_{44} + (\lambda_1^{-2} + \lambda_2^{-2})\Omega_{45} + \Omega_{55} + (\lambda_1^{-2}\lambda_2^2 + \lambda_1^2\lambda_2^{-2})\Omega_{46} \\ & + (\lambda_1^2 + \lambda_2^2)\Omega_{56} + \lambda_1^2\lambda_2^2\Omega_{66}]B_1B_2. \end{aligned} \quad (168)$$

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