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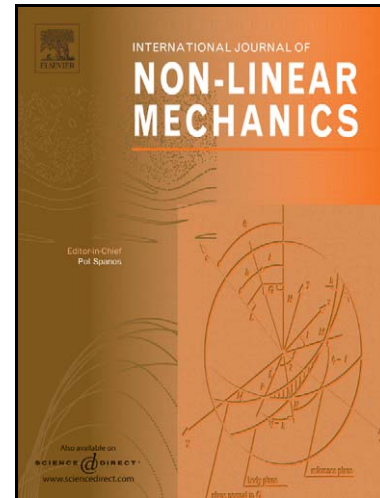
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Generalisations of long wave theories for pre-stressed compressible elastic plates

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Abstract

Generalisations of classical bending and extension are established for pre-stressed compressible elastic plates. In respect of the analogue of extension, the associated quasi-front is shown to be either advancing or receding, contrasting with the classical case. For the generalisation of bending, the long wave limit of the fundamental mode is non-zero; thus, unlike its classical counterpart, an associated quasi-front can therefore exist and is again noted to be either advancing or receding. In both cases appropriate leading order and higher order corrected governing equations are obtained. The ideas are illustrated through investigation of a model problem involving impact edge loading. For the generalised theory of bending, the leading order governing equation for the mid-surface deflection is used to establish the classical equation for wave propagation along an infinite string, with its second order refinement providing a second order correction. Motion within the vicinity of the thickness shear and thickness stretch resonance frequencies is also investigated. Special cases, in which either a stretch resonance and shear resonance frequency are very close, or the speeds of longitudinal and shear waves are very close, are also discussed.

1 Introduction

Lower dimensional plate theories have helped elucidate qualitative features of static and dynamic structural response for many years. For the most part, certainly until relatively recently, these theories were only established within the framework of linear isotropic elasticity. The first attempts to extend these theories to include the influence of pre-stress were in [1], to which the reader is referred for some historical background. In [1] generalisations of classical bending and extension are established for a pre-stressed, incompressible elastic structures. Later, in [2], models for motion close to the cut-off frequencies were derived within the same constitutive framework. These models, to help elucidate two-dimensional motion, were later extended to models for three dimensional

motion in incompressible pre-stressed layers, see [3] and [4] and also to problems involving slightly compressible elastic plates, see [5] and [6]. All of the above studies use the method of long wave asymptotic integration first developed in [7]. Our intention is to extend such studies and investigate the compressible pre-stressed problem. Within this context it is far easier to make direct comparison with the classical theories of bending and extension. Additionally, an interesting case arises in connection with motion near the shear and stretch resonance frequencies when the speeds of the shear and longitudinal waves are close. This is a phenomenon not possible within the classical linear isotropic context without strong convexity being violated, the bulk modulus being negative and Poisson's ratio tending to minus infinity in the limit of the speeds being equal.

This paper is organised as follows. In Section 2 the governing equations are reviewed and the dispersion relation associated with harmonic waves propagating in a layer with zero incremental traction on its faces is established. In Section 3, this relation is first very briefly investigated numerically and then long wave approximations are presented. In contrast to the classical linear isotropic case, the long wave limit of the anti-symmetric fundamental mode, the so-called long wave low frequency limit, is non-zero. The implication is that an associated quasi-front exists. The long wave high frequency region is also investigated, this being within the vicinity of the thickness shear and stretch resonance frequencies.

In Section 4, asymptotic integration is carried out in respect of low frequency long wave motion, providing theories which are analogous to classical bending and extension. In the anti-symmetric motion case, the counterpart of classical bending, the leading order equation for the mid-plane deflection is shown to take the form of the classical wave equation. This second order equation is refined and an associated fourth order equation established. It is essential to use this higher order correction within the vicinity of the quasi-front. These ideas are illustrated through the setting up and solving of a model problem involving impact edge loading. In addition, it is demonstrated that if the normal pre-stress is zero, and the in-plane pre-stress a pure tension, the leading order equation then reduces to that of the classical equation for wave propagation along an infinite string. Within this context, the refined equation for the mid-surface deflection provides a higher order correction for the classical string equation. Asymptotic integration for symmetric motion is also carried out.

In Section 5, asymptotic integration is carried out for motion within the vicinity of either the thickness stretch or thickness shear resonance frequencies. In all cases, governing equations are derived for the long wave amplitudes. In Section 6, the first of two special cases is considered, namely the case in which one of the shear and one of the stretch resonance frequencies are very close. Modifications of the asymptotic integration procedure are made, with series expansions for the displacement components now in powers of the scaled wave number, rather than squares. The second special case, namely that for which the speeds of shear and longitudinal wave propagation coincide, is discussed in Section 7.

2 Governing equations and the dispersion relation

In this section we briefly review the appropriate governing equations and establish the dispersion relations; for further details the reader is referred to [8] and [9]. We consider a homogeneous, isotropic, compressible elastic layer of thickness $2h$ and infinite lateral extent. The layer possesses an initial unstressed configuration \mathcal{B}_u and is subject to a homogeneous static deformation, resulting in the equilibrium pre-stressed state \mathcal{B}_e . A small time-dependent motion $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is then superimposed on \mathcal{B}_e , resulting in the current configuration \mathcal{B}_t . A Cartesian coordinate system $Ox_1x_2x_3$, coincident with the principal axes of deformation in \mathcal{B}_e , is chosen, with Ox_2 normal to the layer's upper surface and origin O located in the mid-plane. The three principal stretches associated with the primary static deformation $\mathcal{B}_u \rightarrow \mathcal{B}_e$ are denoted by λ_1, λ_2 and λ_3 . We utilize a plane-strain assumption, with $u_3 \equiv 0$ and u_1, u_2 independent of x_3 . The governing equations of motion may be derived, see for example [9], in the form

$$\alpha_{11}u_{1,11} + \gamma_2u_{1,22} + \beta u_{2,12} = \rho_e \ddot{u}_1, \quad \gamma_1u_{2,11} + \alpha_{22}u_{2,22} + \beta u_{1,12} = \rho_e \ddot{u}_2, \quad (2.1)$$

within which

$$\alpha_{ij} = \mathcal{A}_{ijjj}, \quad i \in \{1, 2\}, \quad \gamma_1 = \mathcal{A}_{1212}, \quad \gamma_2 = \mathcal{A}_{2121}, \quad \beta = \alpha_{12} + \gamma_2 - \sigma_2, \quad (2.2)$$

with σ_i , $i \in \{1, 2\}$ the principal Cauchy stresses in \mathcal{B}_e , ρ_e the material density in \mathcal{B}_e and \mathcal{A}_{ijkl} components of the fourth-order elasticity tensor. A comma and a dot indicate differentiation with respect to x_1, x_2 and time t , respectively. Linearised measures of incremental traction, with outward unit normals along Ox_1 and Ox_2 in \mathcal{B}_e , have components

$$\begin{aligned} \tau_{1(1)} &= \alpha_{11}u_{1,1} + \alpha_{12}u_{2,2} & \tau_{2(1)} &= (\gamma_2 - \sigma_2)u_{1,2} + \gamma_1u_{2,1}, \\ \tau_{1(2)} &= \gamma_2u_{1,2} + (\gamma_2 - \sigma_2)u_{2,1} & \tau_{2(2)} &= \alpha_{12}u_{1,1} + \alpha_{22}u_{2,2}. \end{aligned} \quad (2.3)$$

Our initial concern is a layer with incrementally traction-free upper and lower boundaries, indicating that $\tau_{1(2)}(\pm h) = \tau_{2(2)}(\pm h) = 0$. Solutions of the equations of motion are sought in the form of the travelling harmonic wave

$$(u_1, u_2) = (A, B)e^{kqx_2}e^{ik(x_1-vt)}, \quad (2.4)$$

where k is the wave number, v is the phase speed and q is to be determined.

Substituting the solutions (2.4) into the equations of motion (2.1), a system of linear homogeneous equations is obtained. This system possesses a non-trivial solution provided

$$\alpha_{22}\gamma_2q^4 + \{\beta^2 - \alpha_{22}(\alpha_{11} - \bar{v}^2) - \gamma_2(\gamma_1 - \bar{v}^2)\}q^2 + (\alpha_{11} - \bar{v}^2)(\gamma_1 - \bar{v}^2) = 0, \quad \bar{v}^2 = \rho_e v^2. \quad (2.5)$$

Solutions for the displacement components u_1 and u_2 may be represented as linear combinations of the four linearly independent functions $\exp(kq_i x_2)$ and $\exp(-kq_i x_2)$, $i \in \{1, 2\}$, where $\pm q_1, \pm q_2$

are the four generally distinct and non-zero roots of (2.5). Substituting these solutions into the traction free boundary conditions, a system of four linear equations is obtainable. This system may be decomposed into two independent systems of two linear equations, corresponding to so-called anti-symmetric and symmetric motion. These two systems provide the two dispersion relations, which are expressible in the forms

$$q_1(\zeta_1 - \zeta_2 q_2^2) \tanh(q_1 \eta) = q_2(\zeta_1 - \zeta_2 q_1^2) \tanh(q_2 \eta) \quad (2.6)$$

and

$$q_1(\zeta_1 - \zeta_2 q_2^2) \tanh(q_2 \eta) = q_2(\zeta_1 - \zeta_2 q_1^2) \tanh(q_1 \eta), \quad (2.7)$$

respectively, where $\eta = kh$ and

$$\zeta_1 = (\alpha_{11} - \bar{v}^2)(\mathcal{E}_a - \bar{v}^2), \quad \zeta_2 = \alpha_{22}(\mathcal{E}_s - \bar{v}^2), \quad (2.8)$$

with

$$\mathcal{E}_a = \gamma_1 - \frac{(\gamma_2 - \sigma_2)^2}{\gamma_2}, \quad \mathcal{E}_s = \alpha_{11} - \frac{\alpha_{12}^2}{\alpha_{22}}. \quad (2.9)$$

In the case of anti-symmetric motion, u_1 and u_2 are expressible in terms of one arbitrary constant \tilde{A} , yielding

$$\begin{aligned} u_1 &= \{H(q_2) \sinh(q_2 \eta) \sinh(kq_1 x_2) - H(q_1) \sinh(q_1 \eta) \sinh(kq_2 x_2)\} \tilde{A}, \\ u_2 &= \{F(q_1)H(q_2) \sinh(q_2 \eta) \cosh(kq_1 x_2) - F(q_2)H(q_1) \sinh(q_1 \eta) \cosh(kq_2 x_2)\} \tilde{A}, \end{aligned} \quad (2.10)$$

with the exponential factor $e^{ik(x_1 - vt)}$ incorporated into \tilde{A} and $F(q)$, $H(q)$ defined by

$$F(q) = \frac{\alpha_{11} - \bar{v}^2 - \gamma_2 q^2}{\beta i q}, \quad H(q) = \beta(\alpha_{22} i q F(q) - \alpha_{12}). \quad (2.11)$$

In the symmetric case, analogous solutions for u_1 and u_2 may be obtained by replacing \sinh with \cosh and \cosh with \sinh in (2.10). Finally in this section, necessary and sufficient conditions for strong ellipticity can be expressed in the form [9]

$$\alpha_{ii} > 0, \quad \gamma_i > 0, \quad (\alpha_{11}\alpha_{22})^{1/2} + (\gamma_1\gamma_2)^{1/2} \pm \beta > 0, \quad i \in \{1, 2\}. \quad (2.12)$$

3 Analysis of the dispersion relations

The dispersion relations (2.6) and (2.7) were first derived in [10], with a long wave asymptotic analysis later carried out in [11]. This section contains only the essential asymptotic results required in this paper. Our attention is focussed on long wave motion, implying that $\eta \rightarrow 0$. There are two types of asymptotic approximations needed to describe long wave motion, namely low and high-frequency. The modes associated with these types of motion are usually referred to as fundamental modes and harmonics, respectively.

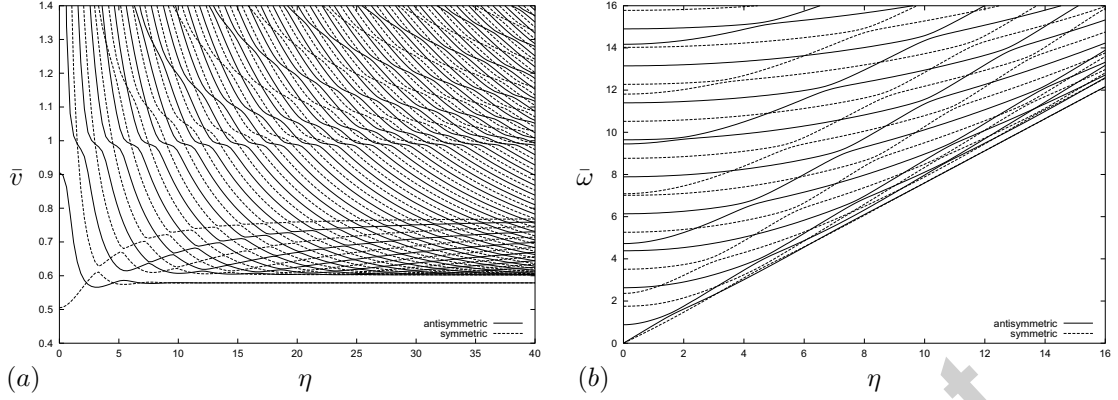


Figure 1: Dispersion relation for a Blatz-ko material; showing (a) Scaled phase speed and (b) scaled frequency against wave number, $\lambda_1 = 2.0$, $\lambda_2 = 0.9$, $\lambda_3 = 0.7$, $\mu = 1.0$, $\sigma_2 = 0.02$.

For numerical illustrations we make use of either a compressible neo-Hookean or Blatz-Ko material. The compressible neo-Hookean material has a strain energy function given by

$$W = \frac{\mu}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\ln(\lambda_1\lambda_2\lambda_3)) + \frac{\kappa}{2} (\lambda_1\lambda_2\lambda_3 - 1)^2, \quad (3.1)$$

within which μ and κ are material parameters. The strain energy function associated with a Blatz-Ko material is assumed in the form

$$W = \frac{\mu}{2} \left(\frac{\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2}{\lambda_1^2\lambda_2^2\lambda_3^2} + 2\lambda_1\lambda_2\lambda_3 - 5 \right), \quad (3.2)$$

with μ a material constant.

In Figure 1, plots of the scaled phase velocity and scaled frequency against η are presented for a Blatz-Ko material. Within Figure 1(a) we observe that there are two finite (non-zero) phase speed limits as $\eta \rightarrow 0$. One of these modes is associated with anti-symmetric motion, the other with symmetric motion. In the classical isotropic case, the analogues of these two modes are associated with classical bending (anti-symmetric) and extension (symmetric). However, in the classical case only the limit of extensional motion is non-zero. This will be shown to have interesting implications. For the harmonics, we observe that $\bar{v} \rightarrow \infty$ as $\eta \rightarrow 0$. In terms of the scaled frequency, we note that as $\eta \rightarrow 0$, for the two fundamental modes $\bar{\omega} \rightarrow 0$, with all other modes (the harmonics) tending to finite non-zero cut-off frequencies; some of these relate to thickness shear resonance, others to thickness stretch.

3.1 Low-frequency motion

Low-frequency motion, associated with the fundamental modes, is characterised by the fact that \bar{v} remains finite as $\eta \rightarrow 0$ but may become zero. Accordingly, within the long wave low frequency regime we may assume that $\tanh(q_n\eta) \sim q_n\eta$, $n \in \{1, 2\}$, with q_1, q_2 either both real or complex

conjugates. The anti-symmetric dispersion relation (2.6) may be used to establish the approximation

$$\bar{v}^2 = \mathcal{F}_a^{(0)} + \mathcal{F}_a^{(2)}\eta^2 + O(\eta^4), \quad \text{where } \mathcal{F}_a^{(0)} = \mathcal{E}_a, \quad \mathcal{F}_a^{(2)} = \frac{(\gamma_2 - \sigma_2)^2}{3\gamma_2^2}(\mathcal{E}_s - \mathcal{E}_a), \quad (3.3)$$

with its symmetric counterpart given by

$$\bar{v}^2 = \mathcal{F}_s^{(0)} + \mathcal{F}_s^{(2)}\eta^2 + O(\eta^4), \quad \text{where } \mathcal{F}_s^{(0)} = \mathcal{E}_s, \quad \mathcal{F}_s^{(2)} = \frac{\alpha_{12}^2}{3\alpha_{22}^2}(\mathcal{E}_a - \mathcal{E}_s). \quad (3.4)$$

3.2 High-frequency motion

For long wave high-frequency motion, i.e. motion within the vicinity of the non-zero cut-off frequencies, $\bar{v}^2 \sim O(\eta^{-2})$ as $\eta \rightarrow 0$. It may be deduced, from equation (2.5), that both q_1^2 and q_2^2 are negative as $\eta \rightarrow 0$, indicating that $q_1 = i\hat{q}_1$ and $q_2 = i\hat{q}_2$, where $\hat{q}_1 \geq 0$ and $\hat{q}_2 \geq 0$ are real. In view of the fact that there is significant overlap between the analysis of the anti-symmetric and symmetric cases, we only give a detailed discussion in respect of the former. Using equation (2.5), we obtain expansions for \hat{q}_1^2 and \hat{q}_2^2 in the form

$$\hat{q}_1^2\eta^2 = \frac{\bar{\omega}^2}{\alpha_{22}} + \frac{\beta^2 - \gamma_1(\gamma_2 - \alpha_{22})}{\alpha_{22}(\gamma_2 - \alpha_{22})}\eta^2 + O(\eta^4), \quad \hat{q}_2^2\eta^2 = \frac{\bar{\omega}^2}{\gamma_2} - \frac{\beta^2 + \alpha_{11}(\gamma_2 - \alpha_{22})}{\gamma_2(\gamma_2 - \alpha_{22})}\eta^2 + O(\eta^4), \quad (3.5)$$

where $\bar{\omega} = \eta\bar{v}$ is a scaled frequency and at this stage we assume $\gamma_2 \neq \alpha_{22}$. The anti-symmetric dispersion relation indicates two possible asymptotic situations: either $\tan(\hat{q}_1\eta) \sim O(1)$ and $\tan(\hat{q}_2\eta) \sim O(\eta^{-2})$; or $\tan(\hat{q}_1\eta) \sim O(\eta^2)$ and $\tan(\hat{q}_2\eta) \sim O(1)$. These requirements are known to correspond to motion in the vicinity of the thickness shear and stretch resonance frequencies, respectively. Further details may be found in [11].

3.2.1 Thickness shear resonance frequencies.

For motion within the vicinity of the thickness shear resonance frequencies, the appropriate asymptotic expansion for the scaled frequency is given by

$$\bar{\omega}^2 = \mathcal{L}_{sh}^{(0)} + \mathcal{L}_{sh}^{(2)}\eta^2 + O(\eta^4), \quad (3.6)$$

where

$$\mathcal{L}_{sh}^{(0)} = \gamma_2(\Lambda_{sh}^a)^2, \quad \mathcal{L}_{sh}^{(2)} = \frac{\beta^2 + \alpha_{11}(\gamma_2 - \alpha_{22})}{\gamma_2 - \alpha_{22}} + \frac{2\chi\{\alpha_{22}(\gamma_2 - \sigma_2) + \gamma_2\alpha_{12}\}^2}{\gamma_2(\gamma_2 - \alpha_{22})^2\Lambda_{sh}^a \tan(\chi\Lambda_{sh}^a)}, \quad (3.7)$$

and

$$\Lambda_{sh}^a = \left(n - \frac{1}{2}\right)\pi, \quad n = 1, 2, \dots, \quad \chi = \sqrt{\frac{\gamma_2}{\alpha_{22}}}. \quad (3.8)$$

3.2.2 The thickness stretch resonance frequencies

For thickness stretch resonance, the analogue of (3.6) takes the form

$$\bar{\omega}^2 = \mathcal{L}_{st}^{(0)} + \mathcal{L}_{st}^{(2)}\eta^2 + O(\eta^4), \quad (3.9)$$

where

$$\mathcal{L}_{st}^{(0)} = \gamma_2(\Lambda_{st}^a)^2, \quad \mathcal{L}_{st}^{(2)} = \frac{\gamma_1(\gamma_2 - \alpha_{22}) - \beta^2}{\gamma_2 - \alpha_{22}} - \frac{2\{\alpha_{22}(\gamma_2 - \sigma_2) + \gamma_2\alpha_{12}\}^2}{\gamma_2(\gamma_2 - \alpha_{22})^2\Lambda_{st}^a \cot(\Lambda_{st}^a)}, \quad \Lambda_{st}^a = \frac{n\pi}{\chi}. \quad (3.10)$$

3.2.3 Special cases

Consider now the situation when one of the thickness stretch resonance frequencies is close to one of the thickness shear resonance frequencies. We denote these appropriate resonance frequencies by Λ_1 and Λ_2 and therefore suppose that $|\Lambda_1 - \Lambda_2| \ll 1$. It is now the case that $\tan(\chi\Lambda_{sh}^a) \rightarrow 0$ and $\tan(\Lambda_{st}^a) \rightarrow \infty$ and the asymptotic approximations (3.6) and (3.9) are therefore not applicable. We can obtain appropriate asymptotic expansions for \hat{q}_1 and \hat{q}_2 in the forms

$$\hat{q}_1 = \frac{\chi\Lambda_1}{\eta} + \chi\xi_1 + O(\eta), \quad \hat{q}_2 = \frac{\Lambda_2}{\eta} + \xi_2 + O(\eta), \quad (3.11)$$

where the $O(1)$ constants ξ_1 and ξ_2 are to be determined. Substituting these expansions into (2.6), and neglecting higher order terms, it is established that

$$\xi_1\xi_2 = \frac{(\alpha_{22}(\gamma_2 - \sigma_2) + \gamma_2\alpha_{12})^2}{\alpha_{22}^2(\gamma_2 - \alpha_{22})^2\Lambda_1\Lambda_2}, \quad (3.12)$$

with expansions (3.11) and (3.5) then enabling us to conclude that

$$(\bar{\omega}^2 - \gamma_2\Lambda_1^2)(\bar{\omega}^2 - \gamma_2\Lambda_2^2) = \frac{4(\alpha_{22}(\gamma_2 - \sigma_2) + \gamma_2\alpha_{12})^2}{(\gamma_2 - \alpha_{22})^2}\eta^2. \quad (3.13)$$

The asymptotic approximations (3.6) and (3.9) are also not applicable when the coefficients $\mathcal{L}_{sh}^{(2)}$ and $\mathcal{L}_{st}^{(2)}$ become very large, this occurring when α_{22} is close to γ_2 . In passing we note from (2.5) that when $q = 0$ the two associated body waves have speeds given $\bar{v}^2 = \alpha_{22}$ and $\bar{v}^2 = \gamma_2$. These are the speeds of the shear and longitudinal waves that can propagate along Ox_1 . The situation in which $|\alpha_{22} - \gamma_2| \ll 1$ therefore occurs when the speeds of these two waves are very close. As previously mentioned, this situation cannot occur in the classical linear isotropic case. Considering the limit $\alpha_{22} = \gamma_2$, expansions for \hat{q}_1^2 and \hat{q}_2^2 can be derived in the form

$$\hat{q}_i^2\eta^2 = \frac{\bar{\omega}^2}{\gamma_2} \pm \frac{\beta\bar{\omega}}{\gamma_2^{3/2}}\eta + \frac{\beta^2 - \gamma_2(\alpha_{11} + \gamma_1)}{2\gamma_2^2}\eta^2 + O(\eta^3), \quad i = 1, 2. \quad (3.14)$$

Appropriate approximations for $\bar{\omega}^2$ in the vicinities of the thickness shear and thickness stretch resonance frequencies are now readily found in forms similar to (3.6) and (3.9). In this case, (3.7)-(3.10) require some appropriate modifications; with Λ_{sh}^a and Λ_{st}^a unchanged, but with $\mathcal{L}_{sh}^{(2)} = \mathcal{P}_{sh}$ and $\mathcal{L}_{st}^{(2)} = \mathcal{P}_{st}$, where

$$\mathcal{P}_{sh} = \mathcal{E}_s - \mathcal{E}_a + \gamma_1 - \frac{\beta^2}{4\gamma_2}, \quad \mathcal{P}_{st} = \mathcal{E}_a - \mathcal{E}_s + \alpha_{11} - \frac{\beta^2}{4\gamma_2}. \quad (3.15)$$

In the case of symmetric motion, approximations within the vicinities of the thickness shear and stretch resonance frequencies may be obtained in a similar way; for details see [11].

4 Low frequency asymptotic integration

4.1 Anti-symmetric motion

In order to examine long wave, low frequency, anti-symmetric motion, appropriate non-dimensional space and time variables ξ , ζ and τ are introduced in the form

$$x_1 = l\xi, \quad x_2 = l\eta\zeta, \quad t = l\sqrt{\frac{\rho}{\mathcal{E}_a}}\tau, \quad \mathcal{E}_a \neq 0, \quad (4.1)$$

where l is a typical wavelength. We remark that the time scale is that associated with the time taken for a wave with speed $\bar{v} = \sqrt{\mathcal{E}_a}$ to travel the distance l . The relative orders of displacements may be obtained from (2.10), indicating that

$$u_1 \sim \eta\tilde{A}, \quad u_2 \sim \tilde{A},$$

motivating introduction of non-dimensional displacement components u_1^* and u_2^* , defined by

$$u_1 = l\eta u_1^*, \quad u_2 = l u_2^*,$$

with $*$ indicating quantities of the same asymptotic order. The equations of motion (2.1) may now be re-cast in the form

$$\begin{aligned} \gamma_2 u_{1,\zeta\zeta}^* + \beta u_{2,\xi\xi}^* + \eta^2(\alpha_{11} u_{1,\xi\xi}^* - \mathcal{E}_a u_{1,\tau\tau}^*) &= 0, \\ \alpha_{22} u_{2,\zeta\zeta}^* + \eta^2(\gamma_1 u_{2,\xi\xi}^* + \beta u_{1,\xi\zeta}^* - \mathcal{E}_a u_{2,\tau\tau}^*) &= 0, \end{aligned} \quad (4.2)$$

with the boundary conditions given by

$$\begin{aligned} \gamma_2 u_{1,\zeta}^* + (\gamma_2 - \sigma_2) u_{2,\xi}^* &= 0 \quad \text{at } \zeta = \pm 1, \\ \alpha_{22} u_{2,\zeta}^* + \eta^2 \alpha_{12} u_{1,\xi}^* &= 0 \quad \text{at } \zeta = \pm 1. \end{aligned} \quad (4.3)$$

Solutions of equations (4.2), subject to (4.3), are now sought in the series form

$$(u_1^*, u_2^*) = \sum_{l=0}^m \eta^{2l} (u_1^{(2l)}, u_2^{(2l)}) + O(\eta^{2m+2}). \quad (4.4)$$

Substituting the solutions (4.4) into (4.2)-(4.3) yields a hierarchy of governing equations at various orders.

4.1.1 Leading order problem

The leading-order governing equations are readily established in the form

$$\gamma_2 u_{1,\zeta\zeta}^{(0)} + \beta u_{2,\xi\xi}^{(0)} = 0, \quad (4.5)$$

$$\alpha_{22} u_{2,\zeta\zeta}^{(0)} = 0, \quad (4.6)$$

$$\gamma_2 u_{1,\zeta}^{(0)} + (\gamma_2 - \sigma_2) u_{2,\xi}^{(0)} = 0 \quad \text{at } \zeta = \pm 1, \quad (4.7)$$

$$\alpha_{22}u_{2,\zeta}^{(0)} = 0 \quad \text{at} \quad \zeta = \pm 1. \quad (4.8)$$

Using equations (4.6) and (4.8), a leading order solution for the normal displacement u_2 is obtainable in the form $u_2^{(0)} = U_2^{(0,0)}$, where $U_2^{(0,0)}$ is a function of ξ and τ only and therefore does not depend on the thickness variable ζ . Henceforth, a function with a double superscript of the form (k, l) is associated with a function of ξ and τ only. Finally, at leading order, from (4.5) and (4.7) we obtain a solution for $u_1^{(0)}$, given by

$$u_1^{(0)} = \zeta U_1^{(0,1)}(\xi, \tau), \quad U_1^{(0,1)}(\xi, \tau) = -\frac{(\gamma_2 - \sigma_2)}{\gamma_2} U_{2,\xi}^{(0,0)}(\xi, \tau).$$

Solutions of the leading order problem are now defined in terms of one function $U_2^{(0,0)}$. A governing equation for $U_2^{(0,0)}$ will be found by considering the next order problem.

4.1.2 Second order problem

The second order problem is given by

$$\gamma_2 u_{1,\zeta\zeta}^{(2)} + \beta u_{2,\xi\zeta}^{(2)} = -\alpha_{11} u_{1,\xi\xi}^{(0)} + \mathcal{E}_a u_{1,\tau\tau}^{(0)}, \quad (4.9)$$

$$\alpha_{22} u_{2,\zeta\zeta}^{(2)} = -\gamma_1 u_{2,\xi\xi}^{(0)} - \beta u_{1,\xi\zeta}^{(0)} + \mathcal{E}_a u_{2,\tau\tau}^{(0)}, \quad (4.10)$$

$$\gamma_2 u_{1,\zeta}^{(2)} + (\gamma_2 - \sigma_2) u_{2,\xi}^{(2)} = 0, \quad (4.11)$$

$$\alpha_{22} u_{2,\zeta}^{(2)} = -\alpha_{12} u_{1,\xi}^{(0)}. \quad (4.12)$$

Using (4.10) it is possible to establish that

$$u_2^{(2)} = U_2^{(2,0)} + \zeta^2 U_2^{(2,2)}, \quad \text{where} \quad U_2^{(2,2)} = \frac{\alpha_{12}(\gamma_2 - \sigma_2)}{2\alpha_{22}\gamma_2} U_{2,\xi\xi}^{(0,0)}.$$

If use is now made of equation (4.12), a governing equation for $U_2^{(0,0)}$ is obtained, given by

$$U_{2,\xi\xi}^{(0,0)} - U_{2,\tau\tau}^{(0,0)} = 0. \quad (4.13)$$

Finally, utilizing equations (4.9) and (4.11) we arrive at

$$u_1^{(2)} = \zeta U_1^{(2,1)} + \zeta^3 U_1^{(2,3)},$$

within which

$$U_1^{(2,1)} = \frac{(\gamma_2 - \sigma_2)}{2\gamma_2^2} (\mathcal{E}_a - \mathcal{E}_s) U_{2,\xi\xi\xi}^{(0,0)} - \frac{(\gamma_2 - \sigma_2)}{\gamma_2} U_{2,\xi}^{(2,0)},$$

$$U_1^{(2,3)} = \frac{(\gamma_2 - \sigma_2)}{6\gamma_2^2} (\alpha_{11} - \mathcal{E}_a) U_{2,\xi\xi\xi}^{(0,0)}.$$

Solutions of the second order problem are now defined in terms of $U_2^{(0,0)}$ and $U_2^{(2,0)}$. A governing equation for $U_2^{(2,0)}$ can not be derived without resorting to the next order problem.

4.1.3 Third order problem

Our purpose at third order is to derive an equation for $U_2^{(2,0)}$, thus completing the second-order solution. To accomplish this it is only necessary to consider the second equation of motion

$$\alpha_{22}u_{2,\zeta\zeta}^{(4)} = -\gamma_1u_{2,\xi\xi}^{(2)} - \beta u_{1,\xi\zeta}^{(2)} + \mathcal{E}_a u_{2,\tau\tau}^{(2)}, \quad (4.14)$$

subject to the boundary condition

$$\alpha_{22}u_{2,\zeta}^{(4)} = -\alpha_{12}u_{1,\xi}^{(2)} \quad \text{at} \quad \zeta = \pm 1. \quad (4.15)$$

Employing equation (4.14), it is possible to obtain a solution for $u_2^{(4)}$ in the form

$$u_2^{(4)} = U_2^{(4,0)} + \zeta^2 U_2^{(4,2)} + \zeta^4 U_2^{(4,4)},$$

where

$$U_2^{(4,2)} = \frac{\beta(\gamma_2 - \sigma_2)}{4\alpha_{22}\gamma_2^2} (\mathcal{E}_s - \mathcal{E}_a) U_{2,\xi\xi\xi\xi}^{(0,0)} + \frac{1}{2\alpha_{22}} \left(\left\{ \frac{\alpha_{12}(\gamma_2 - \sigma_2)}{\gamma_2} - \mathcal{E}_a \right\} U_{2,\xi\xi}^{(2,0)} + \mathcal{E}_a U_{2,\tau\tau}^{(2,0)} \right),$$

$$U_2^{(4,4)} = \frac{(\gamma_2 - \sigma_2)}{24\alpha_{22}^2\gamma_2^2} \{ \alpha_{22}\beta(\alpha_{11} - \mathcal{E}_a) - \alpha_{12}(\gamma_2 - \sigma_2)^2 \} U_{2,\xi\xi\xi\xi}^{(0,0)},$$

with an equation for the higher order correction $U_2^{(4,0)}$ only obtainable at even higher order. The boundary condition (4.15), together with (4.13), may now be used to establish a governing equation for $U_2^{(2,0)}$, yielding

$$\mathcal{F}_a^{(0)}(U_{2,\xi\xi}^{(2,0)} - U_{2,\tau\tau}^{(2,0)}) - \mathcal{F}_a^{(2)}U_{2,\xi\xi\xi\xi}^{(0,0)} = 0. \quad (4.16)$$

At $\zeta = 0$ the only non-zero displacement is u_2^* , the mid-plane deflection. We now introduce a function $u^{[2m]}$, defined as

$$u^{[2m]} = u_2^*|_{\zeta=0} = \sum_{l=0}^m U_2^{(2l,0)} \eta^{2l} + O(\eta^{2m+2}), \quad (4.17)$$

allowing equations (4.13) and (4.16) to be combined into a single equation for $u^{[2]}$

$$\mathcal{F}_a^{(0)}(u_{,\xi\xi}^{[2]} - u_{,\tau\tau}^{[2]}) - \eta^2 \mathcal{F}_a^{(2)} u_{,\xi\xi\xi\xi}^{[2]} = 0. \quad (4.18)$$

In terms of the original variables, equation (4.18) takes the form

$$\mathcal{F}_a^{(0)} \frac{\partial^2 u^{[2]}}{\partial x_1^2} - \rho \frac{\partial^2 u^{[2]}}{\partial t^2} - h^2 \mathcal{F}_a^{(2)} \frac{\partial^4 u^{[2]}}{\partial x_1^4} = 0. \quad (4.19)$$

We remark that the asymptotic expansion (3.3) may be obtained by substituting the travelling wave solution $u^{[2]} = \tilde{u} e^{ik(x_1 - vt)}$ into the equation (4.19). This indicates that this governing equation is consistent with the asymptotic analysis of the anti-symmetric dispersion relation.

4.1.4 Analogy with waves on a string

We now consider the scenario when the normal pre-stress σ_2 is zero and its in-plane counterpart is σ_1 is a pure tension T . In this case, using the fact that $\gamma_2 - \sigma_2 = \gamma_1 - \sigma_1$, equation (4.13), expressed in terms of x_1 and t , takes the form of the classical wave equation for an infinite elastic string, namely

$$\frac{T}{\rho_e} \frac{\partial^2 u^{[0]}}{\partial x_1^2} - \frac{\partial^2 u^{[0]}}{\partial t^2} = 0. \quad (4.20)$$

In this case, (4.19) may be interpreted as a refined string equation, expressible in the form

$$\frac{T}{\rho_e} \frac{\partial^2 u^{[2]}}{\partial x_1^2} - \frac{\partial^2 u^{[2]}}{\partial t^2} + h^2 \left(\frac{T - \mathcal{E}_s}{3\rho_e} \right) \frac{\partial^4 u^{[2]}}{\partial x_1^4} = 0. \quad (4.21)$$

4.1.5 Model problem

The governing equation (4.18) is a singularly perturbed equation, with the fourth order derivative smoothing the solution associated with the second-order equation (4.13). This phenomenon is known as a quasi-front. To illustrate this phenomenon, within the framework of the above theory, we consider a problem of a semi-infinite plate subject to impact edge loading. To construct the boundary condition for the governing equation we should estimate the relative orders of incremental surface traction on a plane parallel to the edge, defined in (2.3)_{1,2}. Substituting the leading order solutions $u_1^{(0)}$ and $u_2^{(0)}$ into (2.3)_{1,2}, we establish that $\tau_{1(1)}^* \sim O(\eta)$, $\tau_{2(1)}^* \sim O(1)$. Using the asymptotically leading traction, we may construct the boundary condition for the governing equation (4.18) in the form

$$T(\xi, \tau) = \tau_{2(1)}^* = \mathcal{F}_a^{(0)} u_{,\xi}^{[0]}. \quad (4.22)$$

The impact edge loading for our model problem is taken in the form

$$T(0, \tau) = MH(\tau), \quad \text{at } \xi = 0. \quad (4.23)$$

In passing we remark that recently a systematic theory for initial value problems in linear isotropic structures has been established, see [12].

Let us introduce new coordinates, defined by

$$\xi_f = \tau - \xi, \quad \xi_s = \tau + \xi, \quad (4.24)$$

these being the characteristics coordinates associated with (4.13). In the vicinity of the quasi-front, the traction variation with respect to ξ_s is very slow, but fast with respect to ξ_f . We remark that characteristic coordinates were used to construct self similar solutions for quasi-fronts in [13]. Therefore, after the coordinate transformation (4.24) we need only retain the fourth order derivative with respect to ξ_f in the governing equation, which after integration with respect to ξ_f is given by

$$\frac{\partial^3 T}{\partial \xi_f^3} + \frac{4\mathcal{F}_a^{(0)}}{\mathcal{F}_a^{(2)}} \eta^{-2} \frac{\partial T}{\partial \xi_s} = 0. \quad (4.25)$$

Introducing d as the width of the region of quasi-front smoothing, and comparing orders of terms in equation (4.25), we deduce that $d \sim \eta^{2/3} \xi^{1/3}$. Using the variable

$$\hat{\mu} = - \left(\frac{4\mathcal{F}_a^{(0)}}{3\mathcal{F}_a^{(2)}} \right)^{1/3} \eta^{-2/3} \xi_s^{-1/3} \xi_f, \quad (4.26)$$

the governing equation for T may be expressed in the form of the Airy equation

$$\frac{\partial^3 T}{\partial \hat{\mu}^3} + \hat{\mu} \frac{\partial T}{\partial \hat{\mu}} = 0. \quad (4.27)$$

The appropriate bounded solution of equation (4.27) may now be obtained in the following form

$$\frac{\partial T}{\partial \hat{\mu}} = C_1 \text{Ai}(-\hat{\mu}) \quad \text{and} \quad T(\hat{\mu}) = C_2 - C_1 \int_0^{\hat{\mu}} \text{Ai}(-z) dz, \quad (4.28)$$

with Ai denoting the Airy function. The constants C_1 and C_2 may be found from the boundary conditions $T(0, -\infty) = 0$ and $T(0, \infty) = M$, resulting in the solution

$$\hat{T}(\hat{\mu}) = \frac{M}{3} + M \int_0^{\hat{\mu}} \text{Ai}(-z) dz, \quad (4.29)$$

or in terms of the variables ξ and τ , defined in (4.1), as

$$T(\xi, \tau) = \frac{M}{3} + M \int_0^{z_0} \text{Ai}(-z) dz, \quad z_0 = - \left(\frac{4\mathcal{F}_a^{(0)}}{3\mathcal{F}_a^{(2)}} \right)^{1/3} \eta^{-2/3} (\tau - \xi)(\tau + \xi)^{-1/3}. \quad (4.30)$$

In Figure 2, plots of the scaled traction T/M against ξ are presented for two cases in which $\mathcal{F}_a^{(2)}$ is either positive or negative. In the case when $\mathcal{F}_a^{(2)} > 0$ the wavefront is receding, with the front advancing if $\mathcal{F}_a^{(2)} < 0$. We remark that in the classical isotropic case there is no quasi-front associated with anti-symmetric motion, with that associated with extensional motion always necessarily receding.

4.2 Symmetric motion

In this section, using methods similar to those for anti-symmetric motion, a model for long-wave low-frequency symmetric motion is derived. Appropriate scaled variables are given by

$$x_1 = l\xi, \quad x_2 = l\eta\zeta, \quad t = l\sqrt{\frac{\rho}{\mathcal{E}_s}}\tau, \quad \mathcal{E}_s \neq 0, \quad (4.31)$$

with the relative orders of displacements indicating that $u_1 \sim u_2/\eta$ and motivating the re-scalings

$$u_1 = lu_1^*, \quad u_2 = l\eta u_2^*. \quad (4.32)$$

The governing equations associated with long-wave low-frequency symmetric motion are given by

$$\begin{aligned} \gamma_2 u_{1,\zeta\zeta}^{(2m)} + \alpha_{11} u_{1,\xi\xi}^{(2m-2)} - \mathcal{E}_s u_{1,\tau\tau}^{(2m-2)} + \beta u_{2,\xi\xi}^{(2m-2)} &= 0, \\ \beta u_{1,\xi\xi}^{(2m)} + \alpha_{22} u_{2,\zeta\zeta}^{(2m)} + \gamma_1 u_{2,\xi\xi}^{(2m-2)} - \mathcal{E}_s u_{2,\tau\tau}^{(2m-2)} &= 0, \end{aligned} \quad (4.33)$$

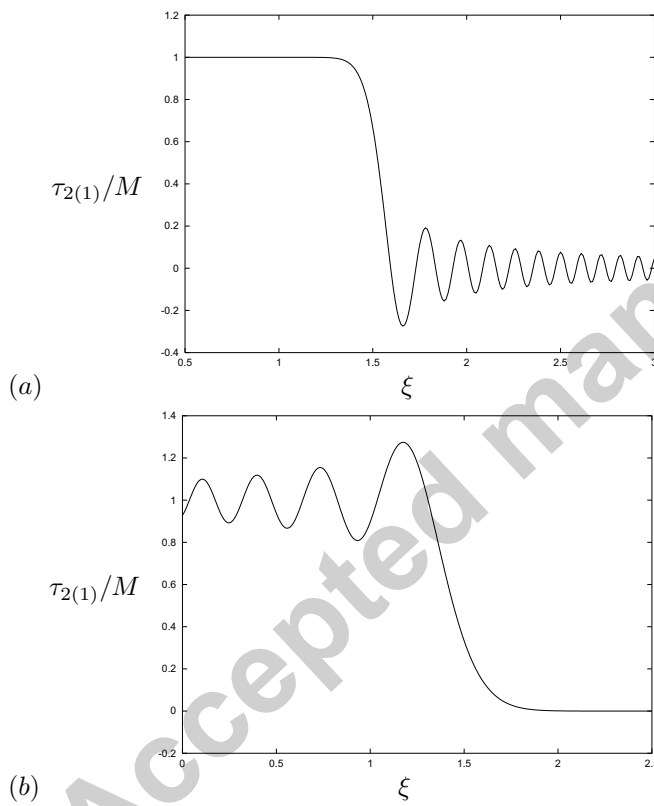


Figure 2: Quasi-fronts for antisymmetric motion with $\tau = 1.5$, $\eta = 0.1$ for (a) the compressible neo-Hookean material $\lambda_1 = 1.7, \lambda_2 = 2.0, \lambda_3 = 1.6, \mu = 1.0, \kappa = 0.1$, indicating $\mathcal{F}_a^{(2)} > 0$; (b) Blatz-Ko material $\lambda_1 = 1.2, \lambda_2 = 1.7, \lambda_3 = 0.72, \mu = 1.0$, indicating $\mathcal{F}_a^{(2)} < 0$.

subject to the boundary conditions

$$\begin{aligned}\gamma_2 u_{1,\zeta}^{(2m)} + (\gamma_2 - \sigma_2) u_{2,\xi}^{(2m-2)} &= 0 \quad \text{at } \zeta = \pm 1, \\ \alpha_{12} u_{1,\xi}^{(2m)} + \alpha_{22} u_{2,\zeta}^{(2m)} &= 0 \quad \text{at } \zeta = \pm 1.\end{aligned}\tag{4.34}$$

4.2.1 Leading order problem

The leading order problem is governed by the following equations and boundary conditions:

$$\gamma_2 u_{1,\zeta\zeta}^{(0)} = 0, \quad \alpha_{22} u_{2,\zeta\zeta}^{(0)} + \beta u_{1,\xi\xi}^{(0)} = 0,\tag{4.35}$$

$$\gamma_2 u_{1,\zeta}^{(0)} = 0 \quad \text{and} \quad \alpha_{12} u_{1,\xi}^{(0)} + \alpha_{22} u_{2,\zeta}^{(0)} = 0 \quad \text{at } \zeta = \pm 1.\tag{4.36}$$

This problem has solution

$$u_1^{(0)} = U_1^{(0,0)}, \quad u_2^{(0)} = \zeta U_2^{(0,1)}, \quad U_2^{(0,1)} = -\frac{\alpha_{12}}{\alpha_{22}} U_{1,\xi}^{(0,0)},\tag{4.37}$$

with an equation for $U_1^{(0,0)}$ not obtainable without resorting to the second order problem.

4.2.2 Second order problem

The second order problem is given by

$$\gamma_2 u_{1,\zeta\zeta}^{(2)} = -\alpha_{11} u_{1,\xi\xi}^{(0)} + \mathcal{E}_s u_{1,\tau\tau}^{(0)} - \beta u_{2,\xi\xi}^{(0)},\tag{4.38}$$

$$\alpha_{22} u_{2,\zeta\zeta}^{(2)} + \beta u_{1,\xi\xi}^{(2)} = -\gamma_1 u_{2,\xi\xi}^{(0)} + \mathcal{E}_s u_{2,\tau\tau}^{(0)},\tag{4.39}$$

$$\gamma_2 u_{1,\zeta}^{(2)} = -(\gamma_2 - \sigma_2) u_{2,\xi}^{(0)} \quad \text{at } \zeta = \pm 1,\tag{4.40}$$

$$\alpha_{12} u_{1,\xi}^{(2)} + \alpha_{22} u_{2,\zeta}^{(2)} = 0 \quad \text{at } \zeta = \pm 1,\tag{4.41}$$

with solution

$$u_1^{(2)} = U_1^{(2,0)} + \zeta^2 U_1^{(2,2)}, \quad u_2^{(2)} = \zeta U_2^{(2,1)} + \zeta^3 U_2^{(2,3)},\tag{4.42}$$

where

$$U_1^{(2,2)} = \frac{(\gamma_2 - \sigma_2)\alpha_{12}}{2\alpha_{22}\gamma_2} U_{1,\xi\xi}^{(0,0)}, \quad U_2^{(2,3)} = \frac{\alpha_{12}}{6\alpha_{22}^2} \left(\mathcal{E}_a - \mathcal{E}_s - \frac{\alpha_{12}(\gamma_2 - \sigma_2)}{\gamma_2} \right) U_{1,\xi\xi\xi}^{(0,0)},$$

$$U_2^{(2,1)} = \frac{\alpha_{12}}{2\alpha_{22}^2} (\mathcal{E}_s - \mathcal{E}_a) U_{1,\xi\xi\xi}^{(0,0)} - \frac{\alpha_{12}}{\alpha_{22}} U_{1,\xi}^{(2,0)},$$

and a governing equation for $U_1^{(0,0)}$ given by

$$U_{1,\xi\xi}^{(0,0)} - U_{1,\tau\tau}^{(0,0)} = 0.\tag{4.43}$$

To complete the second order problem, we need to derive an equation for the function $U_1^{(2,0)}$; this will be done from the third order problem.

4.2.3 Third order problem

The governing equation for $U_1^{(2,0)}$ can be derived using first the equation of motion

$$\gamma_2 u_{1,\zeta\zeta}^{(4)} = -\alpha_{11} u_{1,\xi\xi}^{(2)} + \mathcal{E}_s u_{1,\tau\tau}^{(2)} - \beta u_{2,\xi\zeta}^{(2)}, \quad (4.44)$$

together with the appropriate boundary condition, namely

$$\gamma_2 u_{1,\zeta}^{(4)} = -(\gamma_2 - \sigma_2) u_{2,\xi}^{(2)} \quad \text{at} \quad \zeta = \pm 1. \quad (4.45)$$

After some similar manipulations to those carried out in the anti-symmetric case, we arrive at the following equation connecting $U^{(2,0)}$ and $U^{(0,0)}$

$$\mathcal{F}_s^{(0)}(U_{1,\xi\xi}^{(2,0)} - U_{1,\tau\tau}^{(2,0)}) - \mathcal{F}_s^{(2)} U_{1,\xi\xi\xi\xi}^{(0,0)} = 0, \quad (4.46)$$

which, on introduction of

$$v^{[2m]} = u_1^*|_{\zeta=0} = \sum_{l=0}^m U_1^{(2l,0)} \eta^{2l} + O(\eta^{2m+2}), \quad (4.47)$$

results in an analogue of (4.18), namely

$$\mathcal{F}_s^{(0)}(v_{,\xi\xi}^{[2]} - v_{,\tau\tau}^{[2]}) - \eta^2 \mathcal{F}_s^{(2)} v_{,\xi\xi\xi\xi}^{[2]} = 0. \quad (4.48)$$

If we now re-cast the above equation in terms of original variables x_1 and t , it may be expressed as

$$\mathcal{F}_s^{(0)} \frac{\partial^2 v^{[2]}}{\partial x_1^2} - \rho \frac{\partial^2 v^{[2]}}{\partial t^2} - h^2 \mathcal{F}_s^{(2)} \frac{\partial^4 v^{[2]}}{\partial x_1^4} = 0. \quad (4.49)$$

The asymptotic expansion (3.4) of the symmetric dispersion relation (2.7) may be obtained from the equation (4.49), again confirming asymptotic consistency.

5 High-frequency asymptotic integration

We will now consider high-frequency, anti-symmetric motion in the vicinity of the thickness shear and thickness stretch resonance frequencies. In view of the great similarity between both the anti-symmetric and symmetric cases, we only consider anti-symmetric motion.

5.1 Shear resonance

Using the approximation (3.4), it is possible to establish that the appropriate relative orders of displacement components are indicated by

$$u_1 \sim \eta^{-1} \tilde{A}, \quad u_2 \sim \tilde{A}. \quad (5.1)$$

Thus, non-dimensional displacement components are introduced in the form

$$u_1 = l u_1^*, \quad u_2 = l \eta u_2^*, \quad (5.2)$$

with appropriate scaled spatial and time variables given by

$$x_1 = l\xi, \quad x_2 = l\eta\zeta, \quad t = l\eta\sqrt{\frac{\rho_e}{\gamma_2}}\tau. \quad (5.3)$$

We note the different time-scale to that employed in the low frequency case; this is motivated by the typical wave speed being given by $\bar{v} = \sqrt{\gamma_2}/\eta$. For motion within the vicinity of the thickness shear resonance frequencies, we also assume that

$$u_{1,\tau\tau} + (\Lambda_{sh}^a)^2 u_1 \sim \eta^2 u_1, \quad u_{2,\tau\tau} + (\Lambda_{sh}^a)^2 u_2 \sim \eta^2 u_2. \quad (5.4)$$

In terms of the new variables, the boundary value problem may be written as

$$\begin{aligned} \alpha_{11}\eta^2 u_{1,\xi\xi}^* + \beta\eta^2 u_{2,\zeta\xi}^* + \gamma_2 u_{1,\zeta\xi}^* + \gamma_2(\Lambda_{sh}^a)^2 u_1^* - \gamma_2((\Lambda_{sh}^a)^2 u_1^* + u_{1,\tau\tau}^*) &= 0, \\ \gamma_1\eta^2 u_{2,\xi\xi}^* + \beta u_{1,\zeta\xi}^* + \alpha_{22} u_{2,\zeta\xi}^* + \gamma_2(\Lambda_{sh}^a)^2 u_2^* - \gamma_2((\Lambda_{sh}^a)^2 u_2^* + u_{2,\tau\tau}^*) &= 0, \\ \gamma_2 u_{1,\zeta}^* + (\gamma_2 - \sigma_2)\eta^2 u_{2,\xi}^* &= 0 \quad \text{at } \zeta = \pm 1, \\ \alpha_{12} u_{1,\xi}^* + \alpha_{22} u_{2,\zeta}^* &= 0 \quad \text{at } \zeta = \pm 1, \end{aligned}$$

with solutions of these equations again sought in the series form (4.4).

5.1.1 Leading order problem

The leading order problem may be expressed as

$$\gamma_2 u_{1,\zeta\xi}^{(0)} + \gamma_2(\Lambda_{sh}^a)^2 u_1^{(0)} = 0, \quad (5.5)$$

$$\alpha_{22} u_{2,\zeta\xi}^{(0)} + \beta u_{1,\zeta\xi}^{(0)} + \gamma_2(\Lambda_{sh}^a)^2 u_2^{(0)} = 0, \quad (5.6)$$

$$\gamma_2 u_{1,\zeta}^{(0)} = 0 \quad \text{and} \quad \alpha_{12} u_{1,\xi}^{(0)} + \alpha_{22} u_{2,\zeta}^{(0)} = 0 \quad \text{at } \zeta = \pm 1, \quad (5.7)$$

the solution of which is given by

$$u_1^{(0)} = U_1^{(0,0)} \sin(\Lambda_{sh}^a \zeta), \quad u_2^{(0)} = v_2^{(0,0)} \cos(\chi \Lambda_{sh}^a \zeta) + V_2^{(0,0)} \cos(\Lambda_{sh}^a \zeta), \quad (5.8)$$

where

$$V_2^{(0,0)} = \frac{\beta U_{1,\xi}^{(0,0)}}{(\alpha_{22} - \gamma_2)\Lambda_{sh}^a}, \quad v_2^{(0,0)} = \frac{U_{1,\xi}^{(0,0)}(\alpha_{12}\gamma_2 + \alpha_{22}(\gamma_2 - \sigma_2)) \sin(\Lambda_{sh}^a)}{\alpha_{22}(\gamma_2 - \alpha_{22})\chi\Lambda_{sh}^a \sin(\chi\Lambda_{sh}^a)}, \quad (5.9)$$

with an equation for $U_1^{(0,0)}$ obtainable from the next order problem.

5.1.2 Second order problem

The governing equation for $U_1^{(0,0)}$ may be found from the following equation of motion and boundary conditions

$$\gamma_2 u_{1,\zeta\xi}^{(2)} + \gamma_2(\Lambda_{sh}^a)^2 u_1^{(2)} = \gamma_2 \eta^{-2} ((\Lambda_{sh}^a)^2 u_1^{(0)} + u_{1,\tau\tau}^{(0)}) - \alpha_{11} u_{1,\xi\xi}^{(0)} - \beta u_{2,\zeta\xi}^{(0)}, \quad (5.10)$$

$$\gamma_2 u_{1,\zeta}^{(2)} + (\gamma_2 - \sigma_2) u_{2,\xi}^{(0)} = 0 \quad \text{at} \quad \zeta = \pm 1. \quad (5.11)$$

To begin, a solution for $u_1^{(2)}$ is readily obtainable and may be written in the form

$$u_1^{(2)} = U_1^{(2,0)} \sin(\Lambda_{sh}^a \zeta) + u_1^{(2,0)} \sin(\chi \Lambda_{sh}^a \zeta) + V_1^{(2,1)} \zeta \cos(\Lambda_{sh}^a \zeta), \quad (5.12)$$

where

$$u_1^{(2,0)} = \frac{-U_{1,\xi\xi}^{(0,0)} \beta (\alpha_{12} \gamma_2 + \alpha_{22} (\gamma_2 - \sigma_2)) \sin(\Lambda_{sh}^a)}{\alpha_{22} (\gamma_2 - \alpha_{22})^2 \chi^2 (\Lambda_{sh}^a)^2 \sin(\chi \Lambda_{sh}^a)},$$

$$V_1^{(2,1)} = -\frac{1}{2\gamma_2 \Lambda_{sh}^a} \left(\eta^{-2} \gamma_2 \{ (\Lambda_{sh}^a)^2 U_1^{(0,0)} + U_{1,\tau\tau}^{(0,0)} \} + \frac{\beta^2 - \alpha_{11} (\alpha_{22} - \gamma_2)}{\alpha_{22} - \gamma_2} U_{1,\xi\xi}^{(0,0)} \right).$$

The governing equation for $U_1^{(2,0)}$ may only be derived from the next order problem; however, an equation for $U_1^{(0,0)}$ is now obtainable in the form

$$\gamma_2 \eta^{-2} \{ (\Lambda_{sh}^a)^2 U_1^{(0,0)} + U_{1,\tau\tau}^{(0,0)} \} - \mathcal{L}_{sh}^{(2)} U_{1,\xi\xi}^{(0,0)} = 0. \quad (5.13)$$

Equation (5.13) may be re-cast in terms of x_1 and t , yielding

$$\frac{1}{h^2} \mathcal{L}_{sh}^{(0)} U_1^{(0,0)} + \rho_e \frac{\partial^2 U_1^{(0,0)}}{\partial t^2} - \mathcal{L}_{sh}^{(2)} \frac{\partial^2 U_1^{(0,0)}}{\partial x_1^2} = 0. \quad (5.14)$$

5.2 Stretch resonance

The relative orders of displacement components are in this case given by

$$u_1 \sim \eta \tilde{A}, \quad u_2 \sim \tilde{A}, \quad (5.15)$$

the analogues of (5.2), (5.3) and (5.4) then being

$$u_1 = \eta l u_1^*, \quad u_2 = l u_2^*, \quad (5.16)$$

$$x_1 = l \xi, \quad x_2 = l \eta \zeta, \quad t = l \eta \sqrt{\frac{\rho_e}{\alpha_{22}}} \tau. \quad (5.17)$$

$$u_{1,\tau\tau} + (\chi \Lambda_{st}^a)^2 u_1 \sim \eta^2 u_1, \quad u_{2,\tau\tau} + (\chi \Lambda_{st}^a)^2 u_2 \sim \eta^2 u_2. \quad (5.18)$$

5.2.1 Leading order problem

The leading order governing equations take the form

$$\gamma_2 u_{1,\zeta\zeta}^{(0)} + \beta u_{2,\zeta\zeta}^{(0)} + \gamma_2 (\Lambda_{st}^a)^2 u_1^{(0)} = 0, \quad \alpha_{22} u_{2,\zeta\zeta}^{(0)} + \gamma_2 (\Lambda_{st}^s)^2 u_2^{(0)} = 0, \quad (5.19)$$

$$\gamma_2 u_{1,\zeta}^{(0)} + (\gamma_2 - \sigma_2) u_{2,\xi}^{(0)} = 0, \quad \alpha_{22} u_{2,\zeta}^{(0)} = 0 \quad \text{at} \quad \zeta = \pm 1, \quad (5.20)$$

with solution

$$u_1^{(0)} = U_1^{(0,0)} \sin(\Lambda_{st}^a \zeta) + u_1^{(0,0)} \sin(\chi \Lambda_{st}^a \zeta), \quad u_2^{(0)} = v_2^{(0,0)} \cos(\chi \Lambda_{st}^s \zeta), \quad (5.21)$$

where

$$u_1^{(0,0)} = \frac{\beta v_{2,\xi}^{(0,0)}}{\chi \Lambda_{st}^a (\alpha_{22} - \gamma_2)}, \quad U_1^{(0,0)} = \frac{v_{2,\xi}^{(0,0)} (\alpha_{12} \gamma_2 + \alpha_{22} (\gamma_2 - \sigma_2)) \cos(\chi \Lambda_{st}^a)}{\gamma_2 (\gamma_2 - \alpha_{22}) \Lambda_{st}^a \cos(\Lambda_{st}^a)}. \quad (5.22)$$

The leading order solution has been obtained in term of $v_2^{(0,0)}$, for which an equation can be obtained by considering the next order problem.

5.2.2 Second order problem

At second order, our intention is merely to derive an equation for $v_2^{(0,0)}$; accordingly, we need only consider the equation of motion

$$\alpha_{22} u_{2,\zeta\zeta}^{(2)} + \gamma_2 (\Lambda_{st}^s)^2 u_2^{(2)} = \eta^{-2} \alpha_{22} (\chi^2 (\Lambda_{st}^s)^2 u_2^{(0)} + u_{2,\tau\tau}^{(0)}) - \gamma_1 u_{2,\xi\xi}^{(0)} - \beta u_{1,\zeta\xi}^{(0)}, \quad (5.23)$$

subject to

$$\alpha_{22} u_{2,\zeta}^{(2)} + \alpha_{12} u_{1,\xi}^{(0)} = 0 \quad \text{at} \quad \zeta = \pm 1. \quad (5.24)$$

It is now possible to establish that

$$u_2^{(2)} = v_2^{(2,0)} \cos(\chi \Lambda_{st}^a \zeta) + V_2^{(2,0)} \cos(\Lambda_{st}^a \zeta) + u_2^{(2,1)} \zeta \sin(\chi \Lambda_{st}^a \zeta), \quad (5.25)$$

where

$$V_2^{(2,0)} = \frac{-v_{2,\xi\xi}^{(0,0)} \beta (\alpha_{12} \gamma_2 + \alpha_{22} (\gamma_2 - \sigma_2)) \cos(\chi \Lambda_{st}^a)}{\gamma_2 (\gamma_2 - \alpha_{22})^2 (\Lambda_{st}^a)^2 \cos(\Lambda_{st}^a)},$$

$$u_2^{(2,1)} = \frac{1}{2\alpha_{22} \chi \Lambda_{st}^a} \left(\eta^{-2} \alpha_{22} \{ \chi^2 (\Lambda_{st}^a)^2 v_2^{(0,0)} + v_{2,\tau\tau}^{(0,0)} \} + \frac{\beta^2 + \gamma_1 (\alpha_{22} - \gamma_2)}{\gamma_2 - \alpha_{22}} v_{2,\xi\xi}^{(0,0)} \right),$$

with the governing equation for $v_2^{(0,0)}$ taking the form

$$\alpha_{22} \eta^{-2} \{ \chi^2 (\Lambda_{st}^a)^2 v_2^{(0,0)} + v_{2,\tau\tau}^{(0,0)} \} - \mathcal{L}_{st}^{(2)} v_{2,\xi\xi}^{(0,0)} = 0. \quad (5.26)$$

Equation (5.26) may be expressed in terms of x_1 and t as

$$\frac{1}{h^2} \mathcal{L}_{st}^{(0)} v_2^{(0,0)} + \rho_e \frac{\partial^2 v_2^{(0,0)}}{\partial t^2} - \mathcal{L}_{st}^{(2)} \frac{\partial^2 v_2^{(0,0)}}{\partial x_1^2} = 0. \quad (5.27)$$

6 Special case $|\Lambda_1 - \Lambda_2| \ll 1$

Consider now the special case for which one of the stretch resonance frequencies is close to one of the shear resonance frequencies. Previously established results of the asymptotic analysis of the dispersion relation, together with equations (2.10), enable us to deduce that in this case $u_1 \sim u_2$. Thus, introducing non-dimensional variables

$$x_1 = l\xi, \quad x_2 = l\eta\zeta, \quad t = l\eta \sqrt{\frac{\rho_e}{\gamma_2}} \tau \quad (6.1)$$

and

$$u_1 = l\eta u_1^*, \quad u_2 = l\eta u_2^*, \quad (6.2)$$

we note that

$$u_{p,\tau\tau} + \Lambda_p^2 u_p \sim \eta u_p, \quad p = 1, 2. \quad (6.3)$$

The equations of motion (2.1), subject to traction free boundary conditions, are now expressible as

$$\begin{aligned} \gamma_2 u_{1,\zeta\zeta}^* + \gamma_2 \Lambda_1^2 u_1^* + \beta \eta u_{2,\xi\xi}^* - \gamma_2 (u_{1,\tau\tau}^* + \Lambda_1^2 u_1^*) + \alpha_{11} \eta^2 u_{1,\xi\xi}^* &= 0, \\ \alpha_{22} u_{2,\zeta\zeta}^* + \gamma_2 \Lambda_2^2 u_2^* + \beta \eta u_{1,\xi\xi}^* - \gamma_2 (u_{2,\tau\tau}^* + \Lambda_2^2 u_2^*) + \gamma_1 \eta^2 u_{2,\xi\xi}^* &= 0, \\ \gamma_2 u_{1,\zeta}^* + (\gamma_2 - \sigma_2) \eta u_{2,\xi}^* &= 0 \quad \text{at } \zeta = \pm 1, \\ \alpha_{22} u_{2,\zeta}^* + \alpha_{12} \eta u_{1,\xi}^* &= 0 \quad \text{at } \zeta = \pm 1. \end{aligned} \quad (6.4)$$

The solution of the above system may be presented in the series form

$$(u_1^*, u_2^*) = \sum_{l=0}^m \eta^l (u_1^{(l)}, u_2^{(l)}) + O(\eta^{m+1}). \quad (6.5)$$

Substituting (6.5) into the system (6.4), the following hierarchical system of governing equations, and associated boundary conditions, is obtained

$$\begin{aligned} \gamma_2 u_{1,\zeta\zeta}^{(m)} + \gamma_2 \Lambda_1^2 u_1^{(m)} + \beta u_{2,\xi\xi}^{(m-1)} - \gamma_2 \eta^{-1} (u_{1,\tau\tau}^{(m-1)} + \Lambda_1^2 u_1^{(m-1)}) + \alpha_{11} u_{1,\xi\xi}^{(m-2)} &= 0, \\ \alpha_{22} u_{2,\zeta\zeta}^{(m)} + \gamma_2 \Lambda_2^2 u_2^{(m)} + \beta u_{1,\xi\xi}^{(m-1)} - \gamma_2 \eta^{-1} (u_{2,\tau\tau}^{(m-1)} + \Lambda_2^2 u_2^{(m-1)}) + \gamma_1 u_{2,\xi\xi}^{(m-2)} &= 0, \\ \gamma_2 u_{1,\zeta}^{(m)} + (\gamma_2 - \sigma_2) u_{2,\xi}^{(m-1)} &= 0 \quad \text{at } \zeta = \pm 1, \\ \alpha_{22} u_{2,\zeta}^{(m)} + \alpha_{12} u_{1,\xi}^{(m-1)} &= 0 \quad \text{at } \zeta = \pm 1. \end{aligned} \quad (6.6)$$

6.1 Leading order problem

The leading order problem yields

$$\begin{aligned} u_{1,\zeta\zeta}^{(0)} + \Lambda_1^2 u_1^{(0)} &= 0, \quad u_{2,\zeta\zeta}^{(0)} + (\chi \Lambda_2)^2 u_2^{(0)} = 0, \\ u_{1,\zeta}^{(0)} = u_{2,\zeta}^{(0)} &= 0 \quad \text{at } \zeta = \pm 1, \end{aligned} \quad (6.7)$$

with solution

$$u_1^{(0)} = u_1^{(0,0)} \sin(\Lambda_1 \zeta), \quad u_2^{(0)} = V_2^{(0,0)} \cos(\chi \Lambda_2 \zeta), \quad (6.8)$$

where the two functions $u_1^{(0,0)}$ and $V_2^{(0,0)}$ may be found from the next order problem.

6.2 Second order problem

The second order problem is given by

$$\gamma_2 u_{1,\zeta\zeta}^{(1)} + \gamma_2 \Lambda_1^2 u_1^{(1)} = \gamma_2 \eta^{-1} (u_{1,\tau\tau}^{(0)} + \Lambda_1^2 u_1^{(0)}) - \beta u_{2,\xi\xi}^{(0)} \quad (6.9)$$

$$\alpha_{22} u_{2,\zeta\zeta}^{(1)} + \gamma_2 \Lambda_2^2 u_2^{(1)} = \gamma_2 \eta^{-1} (u_{2,\tau\tau}^{(0)} + \Lambda_2^2 u_2^{(0)}) - \beta u_{1,\xi\xi}^{(0)}, \quad (6.10)$$

$$\gamma_2 u_{1,\zeta}^{(1)} + (\gamma_2 - \sigma_2) u_{2,\xi}^{(0)} = 0, \quad \alpha_{22} u_{2,\zeta}^{(1)} + \alpha_{12} u_{1,\xi}^{(0)} = 0 \quad \text{at} \quad \zeta = \pm 1. \quad (6.11)$$

Solving equation (6.9) we arrive at a solution for $u_1^{(1)}$, given by

$$u_1^{(1)} = u_1^{(1,0)} \sin(\Lambda_1 \zeta) + U_1^{(1,0)} \sin(\chi \Lambda_2 \zeta) + v_1^{(1,1)} \zeta \cos(\Lambda_1 \zeta), \quad (6.12)$$

for which

$$U_1^{(1,0)} = \frac{\beta \Lambda_2 V_{2,\xi}^{(0,0)}}{\chi(\alpha_{22} \Lambda_1^2 - \gamma_2 \Lambda_2^2)}, \quad v_1^{(1,1)} = -\frac{1}{2\Lambda_1} \eta^{-1} (u_{1,\tau\tau}^{(0,0)} + \Lambda_1^2 u_1^{(0,0)}). \quad (6.13)$$

From equation (6.10) a solution for $u_2^{(1)}$ may be obtained, indicating that

$$u_2^{(1)} = V_2^{(1,0)} \cos(\chi \Lambda_2 \zeta) + v_2^{(1,0)} \cos(\Lambda_1 \zeta) + U_2^{(1,1)} \zeta \sin(\chi \Lambda_2 \zeta), \quad (6.14)$$

where

$$v_2^{(1,0)} = \frac{\beta \Lambda_1 u_{1,\xi}^{(0,0)}}{\alpha_{22} \Lambda_1^2 - \gamma_2 \Lambda_2^2}, \quad U_2^{(1,1)} = \frac{\chi}{2\Lambda_2} \eta^{-1} (V_{2,\tau\tau}^{(0,0)} + \Lambda_2^2 V_2^{(0,0)}). \quad (6.15)$$

Using the boundary conditions (6.11), a system of two equations for the two unknowns $u_1^{(0,0)}$ and $V_2^{(0,0)}$ is obtained. From this system an equation for either $u_1^{(0,0)}$ or $V_2^{(0,0)}$ may be derived in the form

$$\gamma_2^2 \eta^{-2} \{u_{1,\tau\tau\tau\tau}^{(0,0)} + u_{1,\tau\tau}^{(0,0)} (\Lambda_1^2 + \Lambda_2^2) + \Lambda_1^2 \Lambda_2^2 u_1^{(0,0)}\} + 4 \frac{\{\gamma_2 \alpha_{12} \Lambda_2^2 + \alpha_{22} (\gamma_2 - \sigma_2) \Lambda_1^2\}^2}{(\alpha_{22} \Lambda_1^2 - \gamma_2 \Lambda_2^2)^2} u_{1,\xi\xi}^{(0,0)} = 0. \quad (6.16)$$

7 Special case $\alpha_{22} = \gamma_2$

Asymptotic models for this case may be obtained after substituting $\alpha_{22} = \gamma_2$ into the previously derived models for shear and stretch resonances.

7.1 Shear resonance

7.1.1 Leading order

The leading order solutions in this case are given

$$u_1^{(0)} = U_1^{(0,0)} \sin(\Lambda_{sh}^a \zeta), \quad u_2^{(0)} = V_2^{(0,0)} \cos(\Lambda_{sh}^a \zeta) + U_2^{(0,1)} \zeta \sin(\Lambda_{sh}^a \zeta),$$

where

$$V_2^{(0,0)} = \frac{U_{1,\xi}^{(0,0)} (2\alpha_{12} - \beta)}{2\gamma_2 \Lambda_{sh}^a}, \quad U_2^{(0,1)} = -\frac{\beta U_{1,\xi}^{(0,0)}}{2\gamma_2},$$

with an equation for the long wave amplitude $U_1^{(0,0)}$ only obtainable from the next order.

7.1.2 Second order

The solution for $u_1^{(2)}$ is given by

$$u_1^{(2)} = U_1^{(2,0)} \sin(\Lambda_{sh}^a \zeta) + V_1^{(2,1)} \zeta \cos(\Lambda_{sh}^a \zeta) + U_1^{(2,2)} \zeta^2 \sin(\Lambda_{sh}^a \zeta),$$

where

$$V_1^{(2,1)} = \frac{1}{2\gamma_2 \Lambda_{sh}^a} \left(\left\{ \frac{\beta^2}{4\gamma_2} + \mathcal{E}_s - \frac{\alpha_{12}(\gamma_2 - \sigma_2)}{\gamma_2} \right\} U_{1,\xi\xi}^{(0,0)} - \eta^{-2} \gamma_2 \{ (\Lambda_{sh}^a)^2 U_1^{(0,0)} + U_{1,\tau\tau}^{(0,0)} \} \right),$$

$$U_1^{(2,2)} = \frac{\beta^2 U_{1,\xi\xi}^{(0,0)}}{8\gamma_2^2}.$$

An equation for $U_1^{(0,0)}$ may also be found in the form

$$\gamma_2 \eta^{-2} \{ (\Lambda_{sh}^a)^2 U_1^{(0,0)} + U_{1,\tau\tau}^{(0,0)} \} - \mathcal{P}_{sh}^a U_{1,\xi\xi}^{(0,0)} = 0.$$

7.2 Stretch resonance

7.2.1 Leading order

The solution of the leading order problem is provided by

$$u_1^{(0)} = U_1^{(0,0)} \sin(\Lambda_{st}^a \zeta) + V_1^{(0,1)} \zeta \cos(\Lambda_{st}^a \zeta), \quad u_2^{(0)} = V_2^{(0,0)} \cos(\Lambda_{st}^a \zeta),$$

where

$$U_1^{(0,0)} = \frac{V_{2,\xi}^{(0,0)} (\beta - 2(\gamma_2 - \sigma_2))}{2\gamma_2 \Lambda_{st}^a}, \quad V_1^{(0,1)} = -\frac{\beta V_{2,\xi}^{(0,0)}}{2\gamma_2}.$$

7.2.2 Second order

In order to find a governing equation for $V_2^{(0,0)}$, a solution for $u_2^{(2)}$ must first be found. This is readily obtainable and takes the form

$$u_2^{(2)} = V_2^{(2,0)} \cos(\Lambda_{st}^a \zeta) + U_2^{(2,1)} \zeta \sin(\Lambda_{st}^a \zeta) + V_2^{(2,2)} \zeta^2 \cos(\Lambda_{st}^a \zeta),$$

where

$$U_2^{(2,1)} = \frac{1}{2\gamma_2 \Lambda_{st}^a} \left(\left\{ -\frac{\beta^2}{4\gamma_2} - \mathcal{E}_a + \frac{\alpha_{12}(\gamma_2 - \sigma_2)}{\gamma_2} \right\} V_{2,\xi\xi}^{(0,0)} + \eta^{-2} \gamma_2 \{ (\Lambda_{st}^a)^2 V_2^{(0,0)} + V_{2,\tau\tau}^{(0,0)} \} \right),$$

$$V_2^{(2,2)} = \frac{\beta^2 V_{2,\xi\xi}^{(0,0)}}{8\gamma_2^2},$$

with an equation for $V_2^{(0,0)}$ given by

$$\gamma_2 \eta^{-2} \{ (\Lambda_{st}^a)^2 V_2^{(0,0)} + V_{2,\tau\tau}^{(0,0)} \} - \mathcal{P}_{st}^a V_{2,\xi\xi}^{(0,0)} = 0.$$

In the case of symmetric motions, the relative orders of the displacement components are identical to those for anti-symmetric motion. The same scaled variables can therefore again be utilised. The

appropriate symmetric long wave high frequency models may be obtained from their anti-symmetric counterparts by replacing Λ_{sh}^a and Λ_{st}^a with Λ_{sh}^s and Λ_{st}^s , respectively, where

$$\Lambda_{sh}^s = n\pi, \quad \Lambda_{st}^s = \left(n - \frac{1}{2}\right) \frac{\pi}{\chi}.$$

8 Some concluding remarks

In this paper we have established models for long wave motion in a pre-stressed layer composed of compressible elastic material. These models relate to both low and high frequency motion. In respect of low frequency motion the models are a generalisation of the classical theories of plate bending and plate extension. In the analogue of plate extension, the theory is similar to its classical counterpart, apart from the fact that, dependent on material parameters and pre-stress, the associated quasi-front may now be either receding or advancing, the classical quasi-front being necessarily receding. In the anti-symmetric case, the counterpart of bending, in contrast to the classical case a quasi-front is shown to exist. This front may again be receding or advancing but must necessarily be of differing type to the extensional front. Models for high frequency motion are also obtained, these being for motion within the vicinity of the thickness stretch and thickness shear resonance frequencies. The paper concludes with discussion of some special cases. The first of these is when one of the thickness stretch and thickness shear resonance frequencies coincide, or are very close. The second, concerns the case in which the speed of propagation of the shear and longitudinal waves is the same or very close. This latter case is not possible within the linear isotropic context.

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