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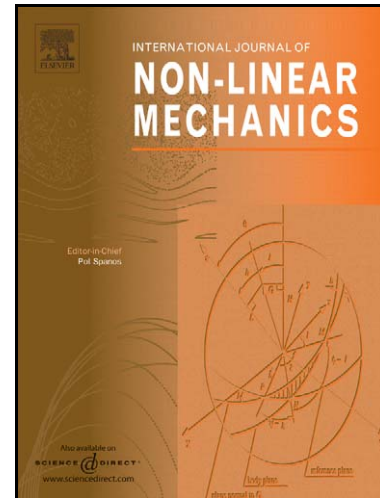
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Stroh formulation for a generally constrained and pre-stressed elastic material

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Abstract

The Stroh formalism is essentially a spatial Hamiltonian formulation and has been recognized to be a powerful tool for solving elasticity problems involving generally anisotropic elastic materials for which conventional methods developed for isotropic materials become intractable. In this paper we develop the Stroh/Hamiltonian formulation for a generally constrained and prestressed elastic material. We derive the corresponding integral representation for the surface-impedance tensor and explain how it can be used, together with a matrix Riccati equation, to calculate the surface-wave speed. The proposed algorithm can deal with any form of constraint, pre-stress, and direction of wave propagation. As an illustration, previously known results are reproduced for surface waves in a pre-stressed incompressible elastic material and an unstressed inextensible fibre-reinforced composite, and an additional example is included analyzing the effects of pre-stress upon surface waves in an inextensible material.

1 Introduction

The Stroh formalism [1, 2] is now well-known in the elasticity community as a powerful tool for dealing with problems associated with general anisotropy. A major application of this formalism is in the study of surface waves in a generally anisotropic elastic material, where it had its first success in establishing the fact that the secular equation for the surface-wave speed is always real irrespective of the constitution of the material [2]. There now exists an elegant surface-wave theory that says that whenever a surface wave exists it is unique and that a surface wave should normally exist except in some very special circumstances; see Barnett and Lothe [3, 4, 5] and Chadwick and Smith [6]. This surface-wave theory also contains many results that are directly applicable to static problems [7].

For instance, the surface-impedance tensor, which plays a pivotal role in the surface-wave theory, also features in many other applications [8]. This tensor has an explicit integral representation [3, 9] and also satisfies a matrix Riccati equation [10, 11, 12]. These results, originally developed for unconstrained elastic materials, have been extended to incompressible elastic materials [8] and edge waves in an anisotropic plate described by the Kirchhoff plate theory [13, 14]. More recently, Fu [15] elaborated on the Hamiltonian interpretation of the Stroh formulation, and paved the way for application of the Stroh formulation to other elasticity problems. As one such application, we derive in this paper the Stroh formulation for a generally constrained, prestressed, elastic material, and use it to investigate the associated surface-wave propagation problem.

In nonlinear elasticity, constrained material models are often used to simplify analysis and highlight specific material properties. Incompressibility, $\lambda_1\lambda_2\lambda_3 = 1$ where the λ 's are the three principal stretches, is the most familiar example. Other forms of kinematic constraints include inextensibility, restricted shear, Bell constraint ($\lambda_1 + \lambda_2 + \lambda_3 = 3$), Ericksen constraint ($\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 3$), and the areal constraint ($\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = 3$). We refer to Destrade and Scott [16] for a discussion of the physical interpretation of these constraints and other related references. Over the past five decades, such constrained material models have frequently been used in the study of acceleration waves, shock waves, and body waves [17-25]. They have also been used in the study of surface waves [26-37]. In this paper, we shall consider a generally constrained and prestressed elastic material. We do not impose any restrictions on the form of the constraint, the prestress or the strain-energy function, and in considering surface waves we do not impose any restriction on the direction of propagation. Our aim is to derive results for the surface-impedance tensor and surface waves that are comparable with those for unconstrained elastic materials. This is achieved using the Stroh formulation and the theoretical framework of Fu [15]. Our strategy is to first derive a Stroh formulation and then obtain an integral representation and a Riccati equation for the surface-impedance tensor. To determine the surface-wave speed, we increase the speed from zero in small steps and at each speed value we use the integral representation to evaluate the surface-impedance tensor and hence its determinant. As soon as the determinant changes sign, we use the associated solution as an initial guess and solve the Riccati equation to find the surface-wave speed precisely. This robust algorithm is underpinned by the same generic properties of the surface-impedance tensor as in the unconstrained case. Also, the existence is built into our search: if a speed is found before the surface-impedance matrix becomes non-Hermitian it must be the unique speed and we may stop the search; otherwise (if a speed still has not been found when the surface-impedance matrix has become non-Hermitian) a subsonic surface wave cannot

exist.

The rest of this paper is divided into four sections as follows. After writing down the governing equations in Section 2, we derive in Section 3 the Stroh formulation for the problem under consideration, and in Section 4 an integral representation and a Riccati equation for the surface-impedance tensor. The integral representation is possible due to the fact that our Stroh formulation is written entirely in terms of tensors, unlike those of Chadwick [31] and Destrade *et al* [36] when the material is incompressible. All degenerate cases are also considered separately. In Section 5 we carry out some example calculations for the surface-wave speed. We first validate our method by reproducing previously known results for surface waves in an incompressible elastic material and a strongly anisotropic fibre-reinforced composite. We then present some new results for surface waves in a strongly anisotropic fibre-reinforced composite that is also pre-stressed.

2 Governing equations

We consider a homogeneous body made of a non-heat-conducting elastic material which is subjected to a single scalar constraint, $C(\mathbf{F}) = 0$, acting on the deformation gradient $\mathbf{F}$. The elastic body is first subjected to a static homogeneous finite deformation and then a further small-amplitude time-dependent deformation. We denote the natural undeformed configuration, the finitely deformed configuration, and the final configuration by B_r , B_e , and B_t , respectively, and choose a common rectangular coordinate system relative to which the coordinates of a representative material particle in B_r , B_e , and B_t are given by X_A , x_i , $\tilde{x}_i$, respectively. We shall denote, for instance, the position vector with coordinates X_A by $\mathbf{X}$. We write

$$\tilde{\mathbf{x}}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}) + \mathbf{u}(\mathbf{x}, t), \quad (2.1)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the small-amplitude displacement from B_e to B_t . From the above equation we obtain

$$F_{iA} \equiv \tilde{x}_{i,A} = \bar{F}_{iA} + u_{i,j} \bar{F}_{jA}, \quad (2.2)$$

and so $\bar{\mathbf{F}}$ and $\mathbf{F}$ are the deformation gradients from B_r to B_e and B_t , respectively. Throughout this paper, $x_{i,A} = \partial x_i / \partial X_A$ and $u_{i,j} = \partial u_i / \partial x_j$ (differentiation with respect to $\tilde{x}_j$ will never arise and so there will be no ambiguity in $u_{i,j}$).

By expanding $C(\mathbf{F}) = 0$ in the neighborhood of $\bar{\mathbf{F}}$ and using the fact that $C(\bar{\mathbf{F}}) = 0$, we obtain

$$\frac{\partial C(\bar{\mathbf{F}})}{\partial \bar{F}_{iA}} \bar{F}_{jA} u_{i,j} = 0, \quad (2.3)$$

where here and throughout this paper we discard terms which are quadratic or of higher order in $\mathbf{u}$.

The nominal stress tensor $\mathbf{S}$ is given by [38]

$$S_{Ai} = \frac{\partial \hat{W}}{\partial F_{iA}}, \quad \hat{W} = W(\mathbf{F}) + pC(\mathbf{F}), \quad (2.4)$$

where W is the strain-energy function and p is a Lagrangian multiplier which is a function of $\mathbf{x}$ and t but is independent of $\mathbf{F}$. We denote by $\bar{S}_{Ai}$ and $\bar{p}$ the nominal stress and the value of p associated with the deformation from B_r to B_e , and following Fu and Ogden [39] define an incremental stress tensor χ_{ij} through

$$\chi_{ij} = \bar{J}^{-1} \bar{F}_{jA} (S_{Ai} - \bar{S}_{Ai}), \quad (2.5)$$

where $\bar{J} = \det \bar{\mathbf{F}}$. On substituting (2.4) into (2.5) and then expanding the resulting expression around $\bar{\mathbf{F}}$ and $\bar{p}$, we obtain

$$\chi_{ij} = (\mathcal{A}_{jilk} + \mathcal{C}_{jilk}^{(2)}) u_{k,l} + p^* \mathcal{C}_{ji}^{(1)}, \quad (2.6)$$

where $p^* = p - \bar{p}$, and we have defined

$$\mathcal{A}_{jilk} = \bar{J}^{-1} \bar{F}_{jA} \bar{F}_{lB} \frac{\partial^2 W(\bar{\mathbf{F}})}{\partial \bar{F}_{iA} \partial \bar{F}_{kB}}, \quad (2.7)$$

$$\mathcal{C}_{ji}^{(1)} = \bar{J}^{-1} \bar{F}_{jA} \frac{\partial C(\bar{\mathbf{F}})}{\partial \bar{F}_{iA}}, \quad (2.8)$$

$$\mathcal{C}_{jilk}^{(2)} = \bar{p} \bar{J}^{-1} \bar{F}_{jA} \bar{F}_{lB} \frac{\partial^2 C(\bar{\mathbf{F}})}{\partial \bar{F}_{iA} \partial \bar{F}_{kB}}. \quad (2.9)$$

Using the above notation the incremental constraint (2.3) may be written

$$\mathcal{C}_{ji}^{(1)} u_{i,j} = 0. \quad (2.10)$$

We recall that the equation of motion can be written as

$$S_{Ai,A} = \rho_r \ddot{u}_i, \quad (2.11)$$

where a superimposed dot on u_i indicates differentiation by t and ρ_r is the density in B_r . In terms of χ_{ij} this becomes

$$\chi_{ij,j} = \rho \ddot{u}_i, \quad (2.12)$$

where $\rho = \rho_r / \bar{J}$. On substituting (2.6) into (2.12), we obtain

$$(\mathcal{A}_{jilk} + \mathcal{C}_{jilk}^{(2)}) u_{k,lj} + p_{,j}^* \mathcal{C}_{ji}^{(1)} = \rho \ddot{u}_i. \quad (2.13)$$

We assume that the prestressed material under consideration satisfies the strong ellipticity condition. This condition can be derived by considering propagation of body waves of the form

$$u_k = \alpha_k e^{i(\psi - vt)}, \quad p^* = \gamma e^{i(\psi - vt)}, \quad (2.14)$$

where $\psi = \boldsymbol{\beta} \cdot \mathbf{x}$ and v is the wave velocity, and requiring that the wave speed is real for all permissible vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and scalar γ . We also note that we may substitute (2.14) into the general incremental constraint (2.10) to obtain a constraint for the choice of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$,

$$\mathcal{C}_{ji}^{(1)} \alpha_i \beta_j = 0. \quad (2.15)$$

Now we use the incremental equation of motion (2.13) together with (2.14) to obtain

$$(\mathcal{A}_{jilk} + \mathcal{C}_{jilk}^{(2)}) \alpha_k \beta_j \beta_l - i \gamma \beta_j \mathcal{C}_{ji}^{(1)} = \rho \alpha_i v^2. \quad (2.16)$$

Multiplication throughout by α_i and use of (2.15) results in our final expression

$$(\mathcal{A}_{jilk} + \mathcal{C}_{jilk}^{(2)}) \alpha_i \alpha_k \beta_j \beta_l = \rho |\boldsymbol{\alpha}|^2 v^2.$$

Thus, the strong ellipticity condition takes the form

$$(\mathcal{A}_{jilk} + \mathcal{C}_{jilk}^{(2)}) \alpha_i \alpha_k \beta_j \beta_l > 0 \quad \forall \text{ non-zero } \boldsymbol{\alpha}, \boldsymbol{\beta} \text{ such that } \boldsymbol{\beta} \cdot \mathcal{C}^{(1)} \boldsymbol{\alpha} = 0. \quad (2.17)$$

We make a quick note of the expressions for $\mathcal{C}_{ji}^{(1)}$ and $\mathcal{C}_{jilk}^{(2)}$ for some common constraints. The constraint of incompressibility yields

$$\mathcal{C}_{ji}^{(1)} = \delta_{ij}, \quad \mathcal{C}_{jilk}^{(2)} = \bar{p}(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}), \quad (2.18)$$

while the constraint of inextensible fibres in the initial direction $\mathbf{A}$, deforming to $\bar{\mathbf{a}} = \bar{\mathbf{F}}\mathbf{A}$ results in

$$\mathcal{C}_{ji}^{(1)} = 2\bar{J}^{-1}\bar{a}_i\bar{a}_j, \quad \mathcal{C}_{jilk}^{(2)} = 2\bar{p}\bar{J}^{-1}\delta_{ik}\bar{a}_j\bar{a}_l, \quad (2.19)$$

and the Ericksen constraint, $(C(\mathbf{F}) = \text{tr}\mathbf{B} - 3 = 0)$ gives

$$\mathcal{C}_{ji}^{(1)} = 2\bar{J}^{-1}\bar{B}_{ij}, \quad \mathcal{C}_{jilk}^{(2)} = 2\bar{p}\bar{J}^{-1}\delta_{ik}\bar{B}_{jl}, \quad (2.20)$$

where $\bar{\mathbf{B}} = \bar{\mathbf{F}}\bar{\mathbf{F}}^T$. Finally for the case of restricted shear, $(C(\mathbf{F}) = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} - \mathbf{A} \cdot \mathbf{B} = 0)$, where shear is restricted between the initial directions $\mathbf{A}$ and $\mathbf{B}$ which deform respectively to $\bar{\mathbf{a}}$ ($= \bar{\mathbf{F}}\mathbf{A}$) and $\bar{\mathbf{b}}$ ($= \bar{\mathbf{F}}\mathbf{B}$) with $\hat{\mathbf{a}} = \bar{\mathbf{a}}/|\bar{\mathbf{a}}|$ and $\hat{\mathbf{b}} = \bar{\mathbf{b}}/|\bar{\mathbf{b}}|$, we obtain

$$\mathcal{C}_{ji}^{(1)} = \bar{J}^{-1}|\bar{\mathbf{a}}||\bar{\mathbf{b}}|(\hat{a}_i\hat{b}_j + \hat{b}_i\hat{a}_j - (\mathbf{A} \cdot \mathbf{B})(\hat{a}_i\hat{a}_j + \hat{b}_i\hat{b}_j)) \quad (2.21)$$

and

$$\begin{aligned} \mathcal{C}_{jilk}^{(2)} = & \bar{p}\bar{J}^{-1}|\bar{\mathbf{a}}||\bar{\mathbf{b}}|(\delta_{ik}(\hat{a}_j\hat{b}_l + \hat{b}_j\hat{a}_l - (\mathbf{A} \cdot \mathbf{B})(\hat{a}_j\hat{a}_l + \hat{b}_j\hat{b}_l)) \\ & + (\mathbf{A} \cdot \mathbf{B})(\hat{a}_i\hat{a}_j - \hat{b}_i\hat{b}_j)(\hat{a}_k\hat{a}_l - \hat{b}_k\hat{b}_l)). \end{aligned} \quad (2.22)$$

It may be seen that for the case of incompressibility the expression $\mathcal{C}_{jilk}^{(2)}\alpha_i\alpha_k\beta_j\beta_l$, which occurs in the strong ellipticity condition, is 0.

3 Stroh formulation

We now look for a traveling-wave solution of the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{z}(iy_2)e^{i(y_1-vt)}, \quad p^* = ig(iy_2)e^{i(y_1-vt)}, \quad (3.1)$$

where $y_1 = \mathbf{m} \cdot \mathbf{x}$, $y_2 = \mathbf{n} \cdot \mathbf{x}$, $\mathbf{z}$ and g are functions to be determined, and $\mathbf{m}$ and $\mathbf{n}$ are unit vectors such that $\mathbf{m} \cdot \mathbf{n} = 0$. On substituting these expressions into the incremental equation of motion (2.12), we obtain

$$\mathbf{T}\mathbf{z}'' + (\mathbf{R} + \mathbf{R}^T)\mathbf{z}' + \mathbf{Q}\mathbf{z} + g'\boldsymbol{\theta} + g\boldsymbol{\phi} = \rho v^2 \mathbf{z}, \quad (3.2)$$

where a prime denotes differentiation with respect to the argument iy_2 , and the various vectors and matrices are defined by their components

$$T_{ik} = (\mathcal{A}_{jilk} + \mathcal{C}_{jilk}^{(2)})n_jn_l, \quad R_{ik} = (\mathcal{A}_{jilk} + \mathcal{C}_{jilk}^{(2)})m_jn_l, \quad Q_{ik} = (\mathcal{A}_{jilk} + \mathcal{C}_{jilk}^{(2)})m_jm_l, \quad (3.3)$$

$$\theta_i = \mathcal{C}_{ji}^{(1)}n_j, \quad \phi_i = \mathcal{C}_{ji}^{(1)}m_j. \quad (3.4)$$

We note that $\mathbf{T}$ and $\mathbf{Q}$ are symmetric. The corresponding traction on a surface with normal $\mathbf{n}$ is given by

$$\mathbf{t} = \boldsymbol{\chi}\mathbf{n} = i\mathbf{l}(iy_2)e^{i(y_1-vt)}, \quad (3.5)$$

where

$$\mathbf{l} = \mathbf{T}\mathbf{z}' + \mathbf{R}^T\mathbf{z} + g\boldsymbol{\theta}. \quad (3.6)$$

Our next step requires solving (3.6) for $\mathbf{z}'$. For the unconstrained case, strong ellipticity ensures that $\mathbf{T}$ has an inverse so that an expression for $\mathbf{z}'$ can be obtained immediately. However when there is a constraint this is no longer guaranteed. To resolve this problem, we first rewrite our constraint (2.10) as

$$\mathbf{z}' \cdot \boldsymbol{\theta} + \mathbf{z} \cdot \boldsymbol{\phi} = 0. \quad (3.7)$$

It then follows that

$$(\boldsymbol{\theta} \otimes \boldsymbol{\theta})\mathbf{z}' + (\boldsymbol{\theta} \otimes \boldsymbol{\phi})\mathbf{z} = \mathbf{0}, \quad (3.8)$$

$$(\boldsymbol{\theta} \otimes \boldsymbol{\theta})\mathbf{z}'' + (\boldsymbol{\theta} \otimes \boldsymbol{\phi})\mathbf{z}' = \mathbf{0}, \quad (3.9)$$

$$(\boldsymbol{\phi} \otimes \boldsymbol{\theta})\mathbf{z}' + (\boldsymbol{\phi} \otimes \boldsymbol{\phi})\mathbf{z} = \mathbf{0}. \quad (3.10)$$

Now if we multiply all of the above three equations by an arbitrary positive constant λ and add (3.9) and (3.10) to (3.2) and (3.8) to (3.6) we obtain

$$\hat{\mathbf{T}}\mathbf{z}'' + (\hat{\mathbf{R}} + \hat{\mathbf{R}}^T)\mathbf{z}' + \hat{\mathbf{Q}}\mathbf{z} + g'\boldsymbol{\theta} + g\boldsymbol{\phi} = \rho v^2 \mathbf{z}, \quad (3.11)$$

and

$$\mathbf{l} = \hat{\mathbf{T}}\mathbf{z}' + \hat{\mathbf{R}}^T \mathbf{z} + g\boldsymbol{\theta}, \quad (3.12)$$

where

$$\hat{\mathbf{T}} = \mathbf{T} + \lambda(\boldsymbol{\theta} \otimes \boldsymbol{\theta}), \quad \hat{\mathbf{R}} = \mathbf{R} + \lambda(\boldsymbol{\phi} \otimes \boldsymbol{\theta}), \quad \hat{\mathbf{Q}} = \mathbf{Q} + \lambda(\boldsymbol{\phi} \otimes \boldsymbol{\phi}). \quad (3.13)$$

We now show that we may always choose λ so as to obtain a positive definite $\hat{\mathbf{T}}$ and hence $\hat{\mathbf{T}}^{-1}$ will exist. To this end, we consider

$$\alpha_i \alpha_k \hat{T}_{ik} = (\mathcal{A}_{jilk} + \mathcal{C}_{jilk}^{(2)}) n_j n_l \alpha_i \alpha_k + \lambda (\mathcal{C}_{ji}^{(1)} \alpha_i n_j)^2, \quad (3.14)$$

where $\boldsymbol{\alpha}$ is an arbitrary real vector. If $\mathcal{C}_{ji}^{(1)} \alpha_i n_j = 0$ then the constraint (2.15) is satisfied and we may use the strong ellipticity condition (2.17) to deduce that the expression in (3.14) is always greater than 0 for any choice of $\boldsymbol{\alpha}$. For the other case when $\mathcal{C}_{ji}^{(1)} \alpha_i n_j \neq 0$ then it can be easily seen that for a sufficiently large choice of λ , $\hat{\mathbf{T}}$ will be positive definite. Hence, upon the addition of the arbitrary λ term we may always choose a λ such that $\hat{\mathbf{T}}$ will have an inverse as required.

We now obtain our expression for $\mathbf{z}'$ from (3.12)

$$\mathbf{z}' = \hat{\mathbf{T}}^{-1}(\mathbf{l} - \hat{\mathbf{R}}^T \mathbf{z} - g\boldsymbol{\theta}), \quad (3.15)$$

which after differentiation by y_2 gives

$$\mathbf{z}'' = \hat{\mathbf{T}}^{-1}(\mathbf{l}' - \hat{\mathbf{R}}^T \mathbf{z}' - g'\boldsymbol{\theta}). \quad (3.16)$$

On substituting these two expressions into (3.11) we obtain the following equation for $\mathbf{l}'$ in terms of $\mathbf{l}$ and $\mathbf{z}$:

$$\mathbf{l}' = -\hat{\mathbf{R}}\hat{\mathbf{T}}^{-1}\mathbf{l} + (\hat{\mathbf{R}}\hat{\mathbf{T}}^{-1}\hat{\mathbf{R}}^T - \hat{\mathbf{Q}} + \rho v^2 \mathbf{I})\mathbf{z} + g\hat{\mathbf{R}}\hat{\mathbf{T}}^{-1}\boldsymbol{\theta} - g\boldsymbol{\phi}. \quad (3.17)$$

The Stroh formulation is complete if we can eliminate the incremental Lagrangian multiplier p^* which is now represented by its amplitude g . This can be done using (3.7) together with (3.15) so we may obtain

$$g = \zeta(\mathbf{z} \cdot \boldsymbol{\phi} + \mathbf{l} \cdot \hat{\mathbf{T}}^{-1}\boldsymbol{\theta} - \mathbf{z} \cdot \hat{\mathbf{R}}\hat{\mathbf{T}}^{-1}\boldsymbol{\theta}), \quad (3.18)$$

where

$$\zeta = 1/(\boldsymbol{\theta} \cdot \hat{\mathbf{T}}^{-1} \boldsymbol{\theta}). \quad (3.19)$$

We note that due to the positive definiteness of $\hat{\mathbf{T}}$, ζ is well defined for all $\boldsymbol{\theta} \neq \mathbf{0}$. The case when $\boldsymbol{\theta} = \mathbf{0}$ is an exceptional case and will be dealt with later in this paper. Finally with the aid of (3.15) and (3.17) we obtain

$$\mathbf{z}' = (\zeta(\hat{\mathbf{T}}^{-1} \boldsymbol{\theta} \otimes (\hat{\mathbf{R}}\hat{\mathbf{T}}^{-1} \boldsymbol{\theta} - \boldsymbol{\phi})) - \hat{\mathbf{T}}^{-1} \hat{\mathbf{R}}^T) \mathbf{z} + (\hat{\mathbf{T}}^{-1} - \zeta(\hat{\mathbf{T}}^{-1} \boldsymbol{\theta} \otimes \hat{\mathbf{T}}^{-1} \boldsymbol{\theta})) \mathbf{l}, \quad (3.20)$$

$$\begin{aligned} \mathbf{l}' = & (\hat{\mathbf{R}}\hat{\mathbf{T}}^{-1} \hat{\mathbf{R}}^T - \hat{\mathbf{Q}} + \rho v^2 \mathbf{I} - \zeta(\boldsymbol{\phi} - \hat{\mathbf{R}}\hat{\mathbf{T}}^{-1} \boldsymbol{\theta}) \otimes (\boldsymbol{\phi} - \hat{\mathbf{R}}\hat{\mathbf{T}}^{-1} \boldsymbol{\theta})) \mathbf{z} \\ & + (\zeta(\hat{\mathbf{R}}\hat{\mathbf{T}}^{-1} \boldsymbol{\theta} - \boldsymbol{\phi}) \otimes \hat{\mathbf{T}}^{-1} \boldsymbol{\theta} - \hat{\mathbf{R}}\hat{\mathbf{T}}^{-1}) \mathbf{l}. \end{aligned} \quad (3.21)$$

Juxtaposing these two expressions, we obtain the Stroh formulation

$$\boldsymbol{\eta}' = \mathbf{N} \boldsymbol{\eta}, \quad (3.22)$$

where

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{z} \\ \mathbf{l} \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \quad (3.23)$$

and

$$\mathbf{N}_1 = \zeta(\hat{\mathbf{T}}^{-1} \boldsymbol{\theta} \otimes (\hat{\mathbf{R}}\hat{\mathbf{T}}^{-1} \boldsymbol{\theta} - \boldsymbol{\phi})) - \hat{\mathbf{T}}^{-1} \hat{\mathbf{R}}^T, \quad (3.24)$$

$$\mathbf{N}_2 = \hat{\mathbf{T}}^{-1} - \zeta(\hat{\mathbf{T}}^{-1} \boldsymbol{\theta} \otimes \hat{\mathbf{T}}^{-1} \boldsymbol{\theta}) = \mathbf{N}_2^T, \quad (3.25)$$

$$\mathbf{N}_3 = \hat{\mathbf{R}}\hat{\mathbf{T}}^{-1} \hat{\mathbf{R}}^T - \hat{\mathbf{Q}} + \rho v^2 \mathbf{I} - \zeta(\boldsymbol{\phi} - \hat{\mathbf{R}}\hat{\mathbf{T}}^{-1} \boldsymbol{\theta}) \otimes (\boldsymbol{\phi} - \hat{\mathbf{R}}\hat{\mathbf{T}}^{-1} \boldsymbol{\theta}) = \mathbf{N}_3^T. \quad (3.26)$$

Due to the arbitrary nature of our choice of λ we expect that the above components of $\mathbf{N}$ should be independent of λ . This can be easily seen by showing that differentiation of each of the above expressions by λ gives an answer of $\mathbf{0}$. For more details, see Fu [15].

On specializing the above results to the case of incompressibility, we recover the expressions given by Fu [15]. A Stroh formulation for a prestressed incompressible elastic material has previously been derived by Chadwick [31] using a different procedure, but his expressions for $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$ are not in terms of tensors. Expressions in terms of tensors are necessary when we derive the integral expression for the surface-impedance tensor.

4 An integral representation for the surface-impedance matrix

4.1 General case

We first make the following assumptions

$$\boldsymbol{\theta} \neq \mathbf{0}, \boldsymbol{\phi} \neq \mathbf{0} \text{ and } \boldsymbol{\phi} \neq k\boldsymbol{\theta}, \quad (4.1)$$

for any constant k . Constraints that do not satisfy these conditions are special cases and will be dealt with in the next two subsections. We note, however, that the important constraint of inextensibility along a direction $\mathbf{e}$ has $\mathcal{C}^{(1)} = \mathbf{e} \otimes \mathbf{e}$ so that $\boldsymbol{\theta} = (\mathbf{n} \cdot \mathbf{e})\mathbf{e}$, $\boldsymbol{\phi} = (\mathbf{m} \cdot \mathbf{e})\mathbf{e}$, which violates the last condition above.

To fix ideas we now consider our material to occupy the half-space

$$H = \{\mathbf{x} \mid \mathbf{n} \cdot \mathbf{x} \geq 0, -\infty < \mathbf{m} \cdot \mathbf{x} < \infty\}, \quad (4.2)$$

where the unit vectors $\mathbf{n}$ and $\mathbf{m}$ have the same meaning as in the previous section. We also impose the following decay condition on $\mathbf{z}$:

$$\mathbf{z}(iy_2) \rightarrow 0 \text{ as } y_2 \rightarrow \infty. \quad (4.3)$$

The surface-impedance matrix $\mathbf{M}(v)$ is defined such that [40]

$$\mathbf{l}(0) = i\mathbf{M}(v)\mathbf{z}(0), \quad (4.4)$$

and for the case of constant $\bar{\mathbf{F}}$ and $\bar{p}$ which we are considering here this implies, as we shall demonstrate shortly, that

$$\mathbf{l}(iy_2) = i\mathbf{M}(v)\mathbf{z}(iy_2). \quad (4.5)$$

We observe that for a constrained elastic material, the stress tensor can only be determined to within an arbitrary multiple of the reaction stress, and one may then think that the traction $\mathbf{l}(iy_2)$ cannot be determined uniquely. However, the seemingly non-uniqueness is eliminated, for most constraints, by the requirement that $\mathbf{l}(iy_2)$ decays to zero as $y_2 \rightarrow \infty$. In the next subsection, we shall show how to deal with other constraints, typified by that of inextensibility, for which the non-uniqueness does exist.

On substituting (4.5) into the Stroh formulation (3.22) and eliminating $\mathbf{z}'$, we obtain

$$(\mathbf{M}\mathbf{N}_2\mathbf{M} - i\mathbf{M}\mathbf{N}_1 + i\mathbf{N}_1^T\mathbf{M} + \mathbf{N}_3)\mathbf{z} = \mathbf{0}.$$

Assuming that $\mathbf{z}(0)$, and hence $\mathbf{z}(iy)$, can be arbitrarily specified, we then obtain

$$\mathbf{M}\mathbf{N}_2\mathbf{M} - i\mathbf{M}\mathbf{N}_1 + i\mathbf{N}_1^T\mathbf{M} + \mathbf{N}_3 = \mathbf{0}, \quad (4.6)$$

which is a well known Riccati equation and is the identity which needs to be solved to calculate $\mathbf{M}$.

Solutions of (4.6) for $\mathbf{M}$ are not unique. In order to select the desired solution, we define another matrix $\mathbf{E}$ through

$$\mathbf{z}(iy_2) = e^{-\mathbf{E}y_2}\mathbf{z}(0). \quad (4.7)$$

This matrix $\mathbf{E}$ can be expressed in terms of the eigenpairs (p_α, ζ_α) ($\alpha = 1, 2, \dots, 6$) of the Stroh matrix $\mathbf{N}$. To show this, we assume that the eigenvalues are all complex and are ordered such that

$$\text{Im } p_\alpha > 0, \quad p_{\alpha+3} = \bar{p}_\alpha, \quad \alpha = 1, 2, 3.$$

We also write

$$\zeta^{(\alpha)} = \begin{bmatrix} \mathbf{a}^{(\alpha)} \\ \mathbf{b}^{(\alpha)} \end{bmatrix}, \quad \alpha = 1, 2, \dots, 6,$$

and define two matrices $\mathbf{A}$ and $\mathbf{B}$ through

$$\mathbf{A} = [\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}], \quad \mathbf{B} = [\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \mathbf{b}^{(3)}]. \quad (4.8)$$

Assuming further that $\mathbf{A}$ is invertible, we then have

$$\begin{aligned} \mathbf{z}(iy_2) &= \sum_{k=1}^3 c_k \mathbf{a}^{(k)} e^{ip_k y_2} = \mathbf{A} \langle e^{ip y_2} \rangle \mathbf{c} \\ &= \mathbf{A} \langle e^{ip y_2} \rangle \mathbf{A}^{-1} \mathbf{z}(0) = e^{i\mathbf{A} \langle p \rangle \mathbf{A}^{-1} y_2} \mathbf{z}(0), \end{aligned}$$

and

$$\begin{aligned} \mathbf{l}(iy_2) &= \sum_{k=1}^3 c_k \mathbf{b}^{(k)} e^{ip_k y_2} = \mathbf{B} \langle e^{ip y_2} \rangle \mathbf{c} \\ &= \mathbf{B} \mathbf{A}^{-1} \mathbf{z}(iy_2), \end{aligned}$$

where

$$\langle e^{ip y_2} \rangle = \text{diag} \{ e^{ip_1 y_2}, e^{ip_2 y_2}, e^{ip_3 y_2} \}, \quad \langle p \rangle = \text{diag} \{ p_1, p_2, p_3 \}.$$

It then follows that

$$\mathbf{E} = -i\mathbf{A} \langle p \rangle \mathbf{A}^{-1}, \quad \mathbf{M} = -i\mathbf{B} \mathbf{A}^{-1}, \quad (4.9)$$

and the validity of (4.7) is thus guaranteed by the invertibility of $\mathbf{A}$. The above derivation also establishes the fact that if $\mathbf{l}(0) = i\mathbf{M}(v)\mathbf{z}(0)$, then $\mathbf{l}(iy_2) = i\mathbf{M}(v)\mathbf{z}(iy_2)$ for all $y_2 > 0$.

From (3.22), (4.5) and (4.7) we may obtain the following relation between $\mathbf{E}$ and $\mathbf{M}$:

$$\mathbf{E} = -i\mathbf{N}_1 + \mathbf{N}_2 \mathbf{M}. \quad (4.10)$$

Thus, the solution of $\mathbf{M}$ which we are looking for must be such that all the eigenvalues of $\mathbf{E}$ computed from (4.10) have positive real parts.

To derive an integral representation for $\mathbf{M}$, we introduce a new notation where we define

$$\mathbf{T}_\omega = \cos^2 \omega \hat{\mathbf{T}} - \sin \omega \cos \omega (\hat{\mathbf{R}} + \hat{\mathbf{R}}^T) + \sin^2 \omega \hat{\mathbf{Q}}_v, \quad (4.11)$$

$$\mathbf{R}_\omega = \cos^2 \omega \hat{\mathbf{R}} - \sin^2 \omega \hat{\mathbf{R}}^T + \sin \omega \cos \omega (\hat{\mathbf{T}} - \hat{\mathbf{Q}}_v), \quad (4.12)$$

$$\mathbf{Q}_\omega = \cos^2 \omega \hat{\mathbf{Q}}_v + \sin \omega \cos \omega (\hat{\mathbf{R}} + \hat{\mathbf{R}}^T) + \sin^2 \omega \hat{\mathbf{T}}, \quad (4.13)$$

$$\boldsymbol{\theta}_\omega = \cos \omega \boldsymbol{\theta} - \sin \omega \boldsymbol{\phi}, \quad (4.14)$$

$$\boldsymbol{\phi}_\omega = \cos \omega \boldsymbol{\phi} + \sin \omega \boldsymbol{\theta}, \quad (4.15)$$

where $\hat{\mathbf{Q}}_v = \hat{\mathbf{Q}} - \rho v^2 \mathbf{I}$. We note that these expressions reduce to their original values upon the choice of $\omega = 0$. We similarly define $\mathbf{N}_{1\omega}, \mathbf{N}_{2\omega}, \mathbf{N}_{3\omega}$ as

$$\begin{aligned} \mathbf{N}_{1\omega} &= \zeta_\omega (\hat{\mathbf{T}}_\omega^{-1} \boldsymbol{\theta}_\omega \otimes (\hat{\mathbf{R}}_\omega \hat{\mathbf{T}}_\omega^{-1} \boldsymbol{\theta}_\omega - \boldsymbol{\phi}_\omega)) - \hat{\mathbf{T}}_\omega^{-1} \hat{\mathbf{R}}_\omega^T, \\ \mathbf{N}_{2\omega} &= \hat{\mathbf{T}}_\omega^{-1} - \zeta_\omega (\hat{\mathbf{T}}_\omega^{-1} \boldsymbol{\theta}_\omega \otimes \hat{\mathbf{T}}_\omega^{-1} \boldsymbol{\theta}_\omega), \\ \mathbf{N}_{3\omega} &= \hat{\mathbf{R}}_\omega \hat{\mathbf{T}}_\omega^{-1} \hat{\mathbf{R}}_\omega^T - \hat{\mathbf{Q}}_\omega - \zeta_\omega (\boldsymbol{\phi}_\omega - \hat{\mathbf{R}}_\omega \hat{\mathbf{T}}_\omega^{-1} \boldsymbol{\theta}_\omega) \otimes (\boldsymbol{\phi}_\omega - \hat{\mathbf{R}}_\omega \hat{\mathbf{T}}_\omega^{-1} \boldsymbol{\theta}_\omega), \end{aligned} \quad (4.16)$$

$\mathbf{E}_\omega$ as

$$\mathbf{E}_\omega = -i\mathbf{N}_{1\omega} + \mathbf{N}_{2\omega} \mathbf{M}_\omega, \quad (4.17)$$

and $\mathbf{M}_\omega$ through

$$\mathbf{M}_\omega \mathbf{N}_{2\omega} \mathbf{M}_\omega - i\mathbf{M}_\omega \mathbf{N}_{1\omega} + i\mathbf{N}_{1\omega}^T \mathbf{M}_\omega + \mathbf{N}_{3\omega} = \mathbf{0}. \quad (4.18)$$

By following the standard analysis as detailed in Mielke and Fu [9], we may show that $\mathbf{M}_\omega$ is independent of ω and is Hermitian, $\mathbf{M}'_\omega(v)$ is negative definite, and that

$$\int_0^\pi \mathbf{E}_\omega d\omega = \pi \mathbf{I}. \quad (4.19)$$

On integrating (4.17) from 0 to π , we then obtain

$$\mathbf{M} = \left(\int_0^\pi \mathbf{N}_{2\omega} d\omega \right)^{-1} \left(i \int_0^\pi \mathbf{N}_{1\omega} d\omega + \pi \mathbf{I} \right). \quad (4.20)$$

4.2 Case when $\boldsymbol{\phi} = k\boldsymbol{\theta}$ and $\boldsymbol{\theta} \neq \mathbf{0}$

We now consider the case when $\boldsymbol{\phi} = k\boldsymbol{\theta}$ for a particular real constant k . If this is true then our previous derivation fails because our $\boldsymbol{\theta}_\omega = \mathbf{0}$ for a certain choice of ω between 0 and π , and hence our ζ_ω is not well defined. We note that the important constraint of inextensibility belongs to this case. Another indicator that the analysis in the previous

subsection is not valid for an inextensible material is provided by the fact that in this case $\mathbf{A}$ is not invertible. It was shown by Whitworth and Chadwick [34] and Captain and Chadwick [35] that for an inextensible material one of the columns of $\mathbf{A}$ is identically zero. We now show that this is in fact true for all constraints having $\phi = k\theta$.

The eigenvalues p_α and eigenvectors $\mathbf{a}^{(\alpha)}$ ($\alpha = 1, 2, 3$) can be obtained by substituting

$$\mathbf{z} = \mathbf{a}e^{ipy_2}, \quad g = \gamma e^{ipy_2},$$

into the equation of motion (3.2) and the constraint equation (3.7). We have

$$\{p^2\mathbf{T} + p(\mathbf{R} + \mathbf{R}^T) + \mathbf{Q} - \rho v^2\mathbf{I}\}\mathbf{a} + (p + k)\gamma\theta = \mathbf{0}, \quad (4.21)$$

$$(p + k)\mathbf{a} \cdot \theta = 0. \quad (4.22)$$

It is then immediately clear that one eigen solution is given by

$$p = -k, \quad \mathbf{a} = \mathbf{0}, \quad \gamma \text{ is indeterminate.} \quad (4.23)$$

This eigen solution will not produce any displacement but will produce a non-zero reaction traction, in the direction of θ , that does not decay away from the surface. This is very similar to the case of inextensible materials examined by Whitworth and Chadwick [34] and Captain and Chadwick [35].

We expect the other eigenvalues to be complex. Equation (4.22) then implies that $\mathbf{a} \cdot \theta = 0$ for the other associated eigenvectors. As a result, since the general solution for $\mathbf{z}$ is a linear combination of the eigen solutions, we have

$$\mathbf{z} \cdot \theta = 0. \quad (4.24)$$

This suggests that we only need to consider displacement and traction in the plane perpendicular to the vector θ .

Recalling our Stroh formulation (3.23) and breaking it up into two parts we obtain

$$\mathbf{z}' = \mathbf{N}_1\mathbf{z} + \mathbf{N}_2\mathbf{l}, \quad \mathbf{l}' = \mathbf{N}_3\mathbf{z} + \mathbf{N}_1^T\mathbf{l}. \quad (4.25)$$

We now introduce a projection tensor $\mathbf{P}$ given by

$$\mathbf{P} = \mathbf{I} - \frac{\theta \otimes \theta}{\theta \cdot \theta}. \quad (4.26)$$

We note that $\mathbf{P}$ satisfies the usual projection tensor properties

$$\mathbf{P}^2 = \mathbf{P}, \quad \mathbf{P} = \mathbf{P}^T, \quad \text{and} \quad \det \mathbf{P} = 0.$$

It can then be shown that the reduced Stroh formulation for the projected displacement and traction vectors

$$\hat{\mathbf{z}} = \mathbf{P}\mathbf{z}, \quad \hat{\mathbf{l}} = \mathbf{P}\mathbf{l} \quad (4.27)$$

is given by

$$\hat{\mathbf{z}}' = \hat{\mathbf{N}}_1 \hat{\mathbf{z}} + \hat{\mathbf{N}}_2 \hat{\mathbf{l}}, \quad \hat{\mathbf{l}}' = \hat{\mathbf{N}}_3 \hat{\mathbf{z}} + \hat{\mathbf{N}}_1^T \hat{\mathbf{l}}, \quad (4.28)$$

where

$$\hat{\mathbf{N}}_1 = \mathbf{P}\mathbf{N}_1\mathbf{P}, \quad \hat{\mathbf{N}}_2 = \mathbf{P}\mathbf{N}_2\mathbf{P}, \quad \hat{\mathbf{N}}_3 = \mathbf{P}\mathbf{N}_3\mathbf{P}. \quad (4.29)$$

In addition we define the appropriate surface-impedance tensor $\hat{\mathbf{M}}$ through

$$\hat{\mathbf{l}} = i\hat{\mathbf{M}}\hat{\mathbf{z}}. \quad (4.30)$$

Relative to a coordinate system in which the 3-axis is in the direction of $\boldsymbol{\theta}$, the third components of $\hat{\mathbf{z}}$ and $\hat{\mathbf{l}}$ are zero, and $\hat{\mathbf{M}}$ then must necessarily have the partitioned form

$$\hat{\mathbf{M}} = \begin{bmatrix} M_{11} & 0_{2 \times 1} \\ 0_{1 \times 2} & M_{22} \end{bmatrix},$$

where M_{11} is a 2×2 matrix, M_{22} is a scalar and $0_{2 \times 1}$ and $0_{1 \times 2}$ are two zero matrices with the indicated order. We note that M_{22} is indeterminate, that is, $\hat{\mathbf{M}}$ is only determined to within an arbitrary multiple of $\boldsymbol{\theta} \otimes \boldsymbol{\theta}$. Furthermore, it can be verified that

$$\mathbf{P}\hat{\mathbf{M}} = \hat{\mathbf{M}}\mathbf{P} = \begin{bmatrix} M_{11} & 0_{2 \times 1} \\ 0_{1 \times 2} & 0 \end{bmatrix}. \quad (4.31)$$

It then follows that

$$\hat{\mathbf{M}} = \mathbf{P}\hat{\mathbf{M}} + M_{22} \frac{\boldsymbol{\theta} \otimes \boldsymbol{\theta}}{\boldsymbol{\theta} \cdot \boldsymbol{\theta}}, \quad (4.32)$$

and

$$\det \hat{\mathbf{M}} = M_{22} \det(M_{11}) = M_{22} \det \left(\mathbf{P}\hat{\mathbf{M}} + \frac{\boldsymbol{\theta} \otimes \boldsymbol{\theta}}{\boldsymbol{\theta} \cdot \boldsymbol{\theta}} \right). \quad (4.33)$$

We now show that it is again possible to obtain a Riccati equation from the above formulation. Substituting (4.28₁) and (4.30) into (4.28₂) allows us to write

$$(\hat{\mathbf{M}}\hat{\mathbf{N}}_2\hat{\mathbf{M}} - i\hat{\mathbf{M}}\hat{\mathbf{N}}_1 + i\hat{\mathbf{N}}_1^T\hat{\mathbf{M}} + \hat{\mathbf{N}}_3)\hat{\mathbf{z}} = \mathbf{0}. \quad (4.34)$$

Again assuming that $\hat{\mathbf{z}}$ may be arbitrarily specified (subject to $\boldsymbol{\theta} \cdot \hat{\mathbf{z}} = 0$) we obtain the Riccati equation

$$\hat{\mathbf{M}}\hat{\mathbf{N}}_2\hat{\mathbf{M}} - i\hat{\mathbf{M}}\hat{\mathbf{N}}_1 + i\hat{\mathbf{N}}_1^T\hat{\mathbf{M}} + \hat{\mathbf{N}}_3 = \mathbf{0}, \quad (4.35)$$

which is the counterpart of (4.6) for the current case. We define $\hat{\mathbf{E}}$, the counterpart of $\mathbf{E}$, through

$$\hat{\mathbf{E}} = -i\hat{\mathbf{N}}_1 + \hat{\mathbf{N}}_2\hat{\mathbf{M}} = \mathbf{P}\mathbf{E}\mathbf{P}. \quad (4.36)$$

We also define

$$\hat{\mathbf{N}}_{1\omega} = \mathbf{P}\mathbf{N}_{1\omega}\mathbf{P}, \quad \hat{\mathbf{N}}_{2\omega} = \mathbf{P}\mathbf{N}_{2\omega}\mathbf{P}, \quad \hat{\mathbf{N}}_{3\omega} = \mathbf{P}\mathbf{N}_{3\omega}\mathbf{P}, \quad (4.37)$$

and

$$\hat{\mathbf{E}}_\omega = \mathbf{P}\mathbf{E}_\omega\mathbf{P} = -i\hat{\mathbf{N}}_{1\omega} + \hat{\mathbf{N}}_{2\omega}\hat{\mathbf{M}}_\omega, \quad (4.38)$$

where we note that we do not apply the ω transformation to the $\boldsymbol{\theta}$ terms in $\mathbf{P}$. It is again possible to show that $\hat{\mathbf{M}}_\omega$ is Hermitian and is independent of ω , and that

$$\int_0^\pi \hat{\mathbf{E}}_\omega d\omega = \pi\mathbf{P}. \quad (4.39)$$

Integrating (4.38) over ω between 0 and π we obtain

$$\pi\mathbf{P} = -i \int_0^\pi \hat{\mathbf{N}}_{1\omega} d\omega + \int_0^\pi \hat{\mathbf{N}}_{2\omega} d\omega \hat{\mathbf{M}}. \quad (4.40)$$

We see that due to its definition our $\hat{\mathbf{N}}_{2\omega}$ is singular and hence (4.40) cannot be solved straightaway to find $\hat{\mathbf{M}}$ as before. We note however that (4.40) may be rewritten

$$\left(\int_0^\pi \hat{\mathbf{N}}_{2\omega} d\omega + \kappa_1(\mathbf{I} - \mathbf{P}) \right) \mathbf{P}\hat{\mathbf{M}} = i \int_0^\pi \hat{\mathbf{N}}_{1\omega} d\omega + \pi\mathbf{P}, \quad (4.41)$$

and so we may obtain an expression for $\mathbf{P}\hat{\mathbf{M}}$

$$\mathbf{P}\hat{\mathbf{M}} = \left(\int_0^\pi \hat{\mathbf{N}}_{2\omega} d\omega + \kappa_1(\mathbf{I} - \mathbf{P}) \right)^{-1} \left(i \int_0^\pi \hat{\mathbf{N}}_{1\omega} d\omega + \pi\mathbf{P} \right), \quad (4.42)$$

providing κ_1 is not chosen to be 0. We note here that the introduction of the projection tensor $\mathbf{P}$ for this case has effectively reduced the problem from 3D to 2D, and $\mathbf{P}\hat{\mathbf{M}}$ only contains four unknowns; see the discussion between (4.30) and (4.31).

4.3 Case when $\boldsymbol{\theta} = \mathbf{0}$

For the case when $\boldsymbol{\theta} = \mathbf{0}$ we can no longer use the Stroh Formulation derived in Section 3 as ζ is no longer well defined. In this case equations (3.2), (3.6) and (3.7) reduce to

$$\mathbf{T}\mathbf{z}'' + (\mathbf{R} + \mathbf{R}^T)\mathbf{z}' + \mathbf{Q}_v\mathbf{z} + g\boldsymbol{\phi} = \mathbf{0}, \quad (4.43)$$

$$\mathbf{T}\mathbf{z}' + \mathbf{R}^T\mathbf{z} - \mathbf{l} = \mathbf{0}, \quad (4.44)$$

$$\mathbf{z} \cdot \boldsymbol{\phi} = 0, \quad (4.45)$$

where $\mathbf{Q}_v = \mathbf{Q} - \rho v^2 \mathbf{I}$. We assume that $\boldsymbol{\phi} \neq \mathbf{0}$, for otherwise the constraint becomes impotent and the problem can be treated in the same way as for an unconstrained material. The similarity between (4.45) and (4.24) suggests that in this case it is appropriate to project displacement and traction to the plane perpendicular to $\boldsymbol{\phi}$. We thus introduce the projection tensor

$$\mathbf{P} = \mathbf{I} - \frac{\boldsymbol{\phi} \otimes \boldsymbol{\phi}}{\boldsymbol{\phi} \cdot \boldsymbol{\phi}}, \quad (4.46)$$

and define

$$\hat{\mathbf{z}} = \mathbf{P}\mathbf{z} = \mathbf{z}, \quad \hat{\mathbf{l}} = \mathbf{P}\mathbf{l}. \quad (4.47)$$

On pre-multiplying (4.44) by our projection tensor, we obtain

$$\hat{\mathbf{l}} = \mathbf{P}(\mathbf{T}\hat{\mathbf{z}}' + \mathbf{R}^T \hat{\mathbf{z}}) = \mathbf{P}\mathbf{T}\mathbf{P}\hat{\mathbf{z}}' + \mathbf{P}\mathbf{R}^T \mathbf{P}\hat{\mathbf{z}}. \quad (4.48)$$

It is necessary to rearrange the above equation for $\hat{\mathbf{z}}'$. Clearly $\mathbf{P}\mathbf{T}\mathbf{P}$ is a singular matrix so no inverse exists. However, as this is pre-multiplying $\hat{\mathbf{z}}'$ it is sufficient to find a matrix, $\hat{\mathbf{T}}^{-1}$ say, such that

$$\hat{\mathbf{T}}^{-1} \mathbf{P}\mathbf{T}\mathbf{P} = \mathbf{P}. \quad (4.49)$$

Matrix theory says that such a matrix can be guaranteed to exist providing $\mathbf{T}$ is positive definite and symmetric, which can be shown to be true if the strong ellipticity condition holds. It is then possible to write

$$\hat{\mathbf{T}}^{-1} = (\mathbf{P}\mathbf{T}\mathbf{P} + \mathbf{I} - \mathbf{P})^{-1} + \mathbf{P} - \mathbf{I}, \quad (4.50)$$

where we note that $\hat{\mathbf{T}}^{-1}$ obeys

$$\hat{\mathbf{T}}^{-1} = \mathbf{P}\hat{\mathbf{T}}^{-1}\mathbf{P}. \quad (4.51)$$

On pre-multiplying (4.48) by $\hat{\mathbf{T}}^{-1}$, we obtain

$$\hat{\mathbf{z}}' = \hat{\mathbf{T}}^{-1} \hat{\mathbf{l}} - \hat{\mathbf{T}}^{-1} \mathbf{P}\mathbf{R}^T \mathbf{P}\hat{\mathbf{z}}. \quad (4.52)$$

Now applying the projection tensor to (4.43) we obtain

$$\mathbf{P}\mathbf{T}\hat{\mathbf{z}}'' + \mathbf{P}(\mathbf{R} + \mathbf{R}^T)\hat{\mathbf{z}}' + \mathbf{P}\mathbf{Q}_v \hat{\mathbf{z}} = \mathbf{0}, \quad (4.53)$$

which with the use of (4.52) gives

$$\hat{\mathbf{l}}' = \mathbf{P}(\mathbf{R}\hat{\mathbf{T}}^{-1}\mathbf{R}^T - \mathbf{Q}_v)\hat{\mathbf{z}} - \mathbf{P}\mathbf{R}\hat{\mathbf{T}}^{-1}\hat{\mathbf{l}}. \quad (4.54)$$

We can now obtain expressions in the form of the Stroh formulation

$$\hat{\mathbf{z}}' = \mathbf{N}_1 \hat{\mathbf{z}} + \mathbf{N}_2 \hat{\mathbf{l}}, \quad \hat{\mathbf{l}}' = \mathbf{N}_3 \hat{\mathbf{z}} + \mathbf{N}_1^T \hat{\mathbf{l}}, \quad (4.55)$$

where

$$N_1 = -\mathbf{P}\hat{\mathbf{T}}^{-1}\mathbf{R}^T\mathbf{P}, \quad N_2 = \mathbf{P}\hat{\mathbf{T}}^{-1}\mathbf{P}, \quad N_3 = \mathbf{P}(\mathbf{R}\hat{\mathbf{T}}^{-1}\mathbf{R}^T - \mathbf{Q}_v)\mathbf{P}. \quad (4.56)$$

The surface-impedance matrix $\hat{\mathbf{M}}$ may now be defined as in the previous subsection and the same procedure can be used to derive an integral representation and a Riccati equation for $\hat{\mathbf{M}}$. The relevant results can be written down by adapting the corresponding expressions from the previous subsection, but we do not write them out here for the sake of brevity.

5 Numerical results

5.1 Comparison with established results

A free surface wave must satisfy the traction-free boundary condition and so recalling (4.4) we see that for a non-trivial solution our $\mathbf{M}$ must satisfy

$$\det \mathbf{M} = 0. \quad (5.1)$$

The $\mathbf{M}$ has an integral representation given by (4.20) and satisfies the Riccati equation (4.6).

For the case when ϕ and θ are linearly dependent, we project the displacement and traction vectors onto the plane perpendicular to θ ; the reaction force in the θ -direction is indeterminate. The corresponding surface-impedance matrix $\hat{\mathbf{M}}$ is defined by (4.30) and the surface-wave speed is determined by $\det \hat{\mathbf{M}} = 0$, which, in view of (4.33), may be replaced by

$$\det(\mathbf{P}\hat{\mathbf{M}} + \mathbf{I} - \mathbf{P}) = 0. \quad (5.2)$$

The $\mathbf{P}\hat{\mathbf{M}}$ has an integral representation given by (4.42), and satisfies the Riccati equation (4.35).

Only in very special cases can the Riccati equation be solved analytically and the integral representation evaluated exactly. In the general case, we use the following strategy to find the surface-wave speed: We increase the speed from zero in small steps and at each speed value we use the integral representation to evaluate the surface-impedance tensor and hence its determinant. As soon as the determinant changes sign, we use the associated solution as an initial guess and solve the Riccati equation to find the surface-wave speed precisely. This robust algorithm has been tested on the edge-wave problem [14] and was found to work very well.

As an example, we first consider surface waves in a sheared and stretched incompressible isotropic Gent material. This problem has previously been considered by Destrade and Ogden [32]. This type of constraint belongs to our general case for which (5.1) applies and we are able to reproduce all of the numerical results presented in Destrade and Ogden [32]. We also note that when the $x_3 = 0$ plane is a plane of symmetry, an even easier method for computing the surface-wave speed is available [41, 42].

We next consider a transversely isotropic elastic material which is inextensible in the preferred direction. Such a constraint is very often used to model strongly anisotropic fibre-reinforced composites. Surface-wave propagation in such a constrained elastic material has previously been examined by Whitworth and Chadwick [34] and Captain and Chadwick [35]. If we assume that the fibre direction $\mathbf{e}$ is given by

$$\mathbf{e} = m \mathbf{m} + n \mathbf{n} + l \mathbf{l}, \quad \text{where } \mathbf{l} = \mathbf{m} \wedge \mathbf{n}, \quad (5.3)$$

then Captain and Chadwick's [35] main results may be stated as follows: (i) If $l \neq 0$ and $n \neq 0$, a unique (normal) surface wave exists; (ii) as $l \rightarrow 0$ with $n \neq 0$ the surface wave found in (i) degenerates continuously into a (one-component) shear body wave which satisfies the traction free boundary condition (known as an exceptional body wave); (iii) if $n = 0$ and $m = 0$ the constraint is impotent; (iv) if $n = 0$ and $m \neq 0$ only the projection of traction onto the plane perpendicular to $\mathbf{e}$ is required to vanish and a unique surface wave always exists. As remarked earlier, the constraint of inextensibility belongs to our special case $\phi = k\theta$. With the use of (5.2), we are able to reproduce all of the numerical results in Captain and Chadwick [35] although we do not distinguish the four cases individually. This is not surprising since although no projection was used by Captain and Chadwick [35] for the Cases (i)-(iii) above, they used $\det \mathbf{B} = 0$ to find the surface-wave speed, where $\mathbf{B}$, when referred to a coordinate system in which the 3-direction is aligned with the fibre direction, takes the form

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

Thus, with $b_{33} \neq 0$ we have $\det \mathbf{B} = \det \hat{\mathbf{B}}$ where $\hat{\mathbf{B}}$ is the submatrix formed from b_{11}, b_{12}, b_{21} and b_{22} . It can be seen that $\det \hat{\mathbf{B}}$ is simply a multiple of our $\det \hat{\mathbf{M}}$.

5.2 Surface waves in a pre-stressed inextensible material

We now demonstrate the flexibility and ease of use of the method outlined in this paper by allowing for a pre-stress in the problem of Captain and Chadwick [35]. Thus, we shall

be concerned with surface waves propagating in a pre-stressed transversely isotropic fibre-reinforced composite which is inextensible in the fibre direction. We consider the simplest case when the pre-stress takes the form of a uniaxial tension or compression in the fibre direction, but our program can cope with any form of pre-stress. Using the same notation as in the previous subsection, we study two separate cases, one with $l = 0$ and one with $l \neq 0$. For both examples we use the generalized Mooney-Rivlin strain-energy function

$$W = \alpha(\bar{I}_1 - 3) + \beta(\bar{I}_2 - 3) + \frac{1}{2}\kappa(J - 1)^2, \quad (5.4)$$

where α, β, κ are material constants,

$$J = \det \mathbf{F}, \quad \bar{I}_1 = J^{-2/3}I_1, \quad \bar{I}_2 = J^{-4/3}I_2,$$

I_1 and I_2 being the first two principal invariants of $\mathbf{F}\mathbf{F}^T$. This is a special case of the more general strain-energy function $W = \hat{W}(\bar{I}_1, \bar{I}_2) + U(J)$ that has been examined by Doll and Schweizerhof [43]. It can easily be shown that with W given by (5.4) and the pre-stress as assumed above, the strong ellipticity condition (2.17) is satisfied if and only if $\bar{p} > -1$, where $\bar{p}$ has the same meaning as in Section 2 and in the present case is half of the actual tension in the fibre direction. In our calculations, we take $\alpha = 1, \beta = 0, \kappa = 1$, and the surface-wave speed is determined with the use of (5.2).

In Figure 1, we have shown variation of ρv^2 with respect to $\bar{p}$ for the two cases

$$(a) \ m = 1/\sqrt{3}, \quad n = 1/\sqrt{2}, \quad l = 1/\sqrt{6}, \quad \text{and} \quad (b) \ m = 1/\sqrt{2}, \quad n = 1/\sqrt{2}, \quad l = 0.$$

For case (a) we see that a unique surface wave with a non-zero speed exists in the range of $\bar{p}$ over which strong ellipticity is satisfied. Thus, in this special case a standing surface wave with $v = 0$ cannot exist when strong ellipticity is satisfied and the half space is always stable with respect to surface-wave type perturbations.

Corresponding to case (b), the inextensible fibres lie in the plane spanned by $\mathbf{m}$ and $\mathbf{n}$. As in the unstressed case discussed by Captain and Chadwick [35], the surface wave is again an exceptional body wave. Thus, it is not surprising that the speed of this wave vanishes at $\bar{p} = -1$, the value of $\bar{p}$ at which strong ellipticity is first violated.

5.3 Analysis of restricted shear

As a further demonstration that the three cases discussed in the previous section cover all possible constraint choices we consider the constraint of restricted shear. Justification for this choice may be given by study of the classification system introduced by Pipkin [44]. By his analysis isotropic constraints such as incompressibility are considered to have

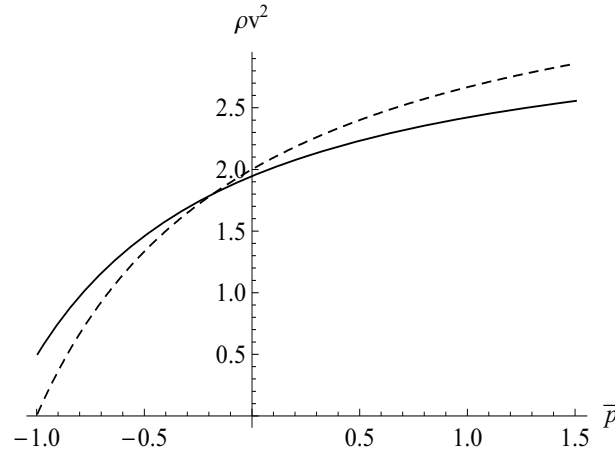


Figure 1: Variation of the surface-wave speed with respect to the pre-stress for the cases $m = 1/\sqrt{3}, n = 1/\sqrt{2}, l = 1/\sqrt{6}$ (solid line) and $m = 1/\sqrt{2}, n = 1/\sqrt{2}, l = 0$ (dashed line).

dimensions 3, whereas constraints acting only in one direction such as inextensibility are said to have dimensions 1. In his classification restricted shear has dimensions 2 as it acts within a plane of the body. In our analysis we have used method §4.1 for incompressibility and §4.2 and §4.3 for inextensibility. It is therefore of interest to see which method we need to use for restricted shear and whether the methods we have defined above are adequate. With the use of (2.21) and (3.4), we see that $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ are linearly independent for most choices of $\mathbf{m}$ and $\mathbf{n}$ and for these cases our theory above dictates that we use the general method §4.1 to obtain a solution. There are however cases for certain specific choices of $\mathbf{m}$ and $\mathbf{n}$ such that $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ are no longer linearly independent and here we must use the methods §4.2 and §4.3. This was tested for various strain energy functions and it is indeed possible to obtain degenerate surface wave solutions for the constraint of restricted shear using the method outlined above although numerical results are not included here for brevity. We note that in the current literature little work has been carried out in studying surface waves under the constraint of restricted shear, the only notable contribution being, to the authors' knowledge, the work by Whitworth and Chadwick [37]. Their work differs from that in the current article though as they did not consider pre-stress and they also assumed that the directions within which shear was restricted were perpendicular, $\mathbf{A} \cdot \mathbf{B} = 0$. Furthermore it is not possible to compare results obtained here with [37] as the constraint of incompressibility was also assumed.

Through further analysis of known constraints it appears that all problems involving isotropic constraints, i.e. those of dimensions 3, may be solved using the first method §4.1 only and that problems involving a non-isotropic constraint, i.e. dimensions 1 or 2, must

be solved using §4.2 and §4.3 together with §4.1.

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