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Edvige Pucci, Giuseppe Saccomandi. Parametric resonance in nonlinear elastodynamics. *International Journal of Non-Linear Mechanics*, 2009, 44 (5), pp.560. 10.1016/j.ijnonlinmec.2008.08.001 . hal-00531851

HAL Id: hal-00531851

<https://hal.science/hal-00531851>

Submitted on 4 Nov 2010

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Author's Accepted Manuscript

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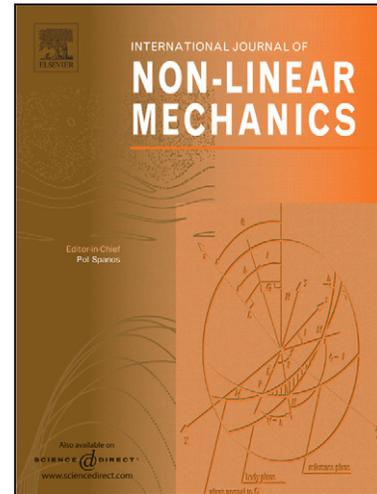
PII: S0020-7462(08)00150-9
DOI: doi:10.1016/j.ijnonlinmec.2008.08.001
Reference: NLM 1511

To appear in: *International Journal of Non-Linear Mechanics*

Received date: 17 June 2008
Revised date: 6 August 2008
Accepted date: 6 August 2008

Cite this article as: Edvige Pucci and Giuseppe Saccomandi, Parametric resonance in nonlinear elastodynamics, *International Journal of Non-Linear Mechanics* (2008), doi:10.1016/j.ijnonlinmec.2008.08.001

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Parametric Resonance in Nonlinear Elastodynamics

Edvige Pucci & Giuseppe Saccomandi
Dipartimento di Ingegneria Industriale,
Università degli Studi di Perugia, 06125 Perugia, Italy
e-mail: epucci@unipg.it; saccomandi@mec.dii.unipg.it

August 20, 2008

Abstract

We show that finite amplitude shearing motions superimposed on an *unsteady* simple extension are admissible in any incompressible isotropic elastic material. We show that the determining equations for these shearing motions admit a general reduction to a system of ordinary differential equations (ODEs) in the remarkable case of generalized circularly polarized transverse waves. When these waves are standing and the underlying unsteady simple extension is composed of a harmonic perturbation of a static stretch it is possible to reduce the determining ODEs to linear or nonlinear Mathieu equations. We use this property for a detailed study of the phenomenon of parametric resonance in nonlinear elastodynamics.

Keywords: Nonlinear elastodynamics, parametric resonance, shearing motions, Mooney-Rivlin materials, Mathieu equation.

1 Introduction

The equations of elastodynamics, under the usual constitutive assumptions, belong to the class of hyperbolic systems. It is known that if a hyperbolic system is nonlinear then smooth solutions of initial-value or initial boundary-value problems do not usually exist globally in time. Singularities will develop, typically after a finite time, even when the initial or boundary data are smooth (Dafermos, 2005). Existence of smooth solutions to the initial-value problem of nonlinear elastodynamics is possible only in very special

situations and only for *bulk waves*. To be more precise non-singular waves are possible:

- in the case of bulk transverse waves in Hadamard materials (or incompressible Mooney-Rivlin materials, see John, 1966);
- in the case of Carroll bulk waves (solutions valid for any material but only for very special initial conditions, see Carroll, 1967);
- when the null condition is satisfied and we consider small initial data (see Domanski, Ogden, 2006).

On the other hand, there is a long list of examples of blow-up of solutions (see, for example, John, 1974).

It is not possible to know a priori if the Carroll waves form the only class of solutions that for special initial conditions may ensure global existence. This is because the method used by Carroll is a semi-inverse method and, as usual, the initial conditions are enforced *a posteriori*. In this framework the result in (Carroll, 1967) is quite remarkable because it shows that progressive finite amplitude transverse circularly-polarized wave motions can propagate in the symmetry direction of any (transversely) isotropic elastic material, which may also be subjected to a uniform *steady* stretching (always in the symmetry direction).

As usual, we let $\mathbf{X} \in k_R(\mathcal{B})$ denote a typical particle belonging to the reference configuration $k_R(\mathcal{B})$ of the body, and let $\mathbf{x} \in k_t(\mathcal{B})$ denote the position occupied by \mathbf{X} at time t . By the motion of the body, we mean a one-to-one mapping χ_{k_R} that assigns to each point \mathbf{X} belonging to $k_R(\mathcal{B})$, a point $\mathbf{x} \in k_t(\mathcal{B})$, at each instant of time:

$$\mathbf{x} = \chi_{k_R}(\mathbf{X}, t). \quad (1.1)$$

In (Rajagopal, 1998) the following class of motions is considered

$$x = \mu(t)X + f(Z, t), \quad y = \mu(t)Y + g(Z, t), \quad z = \lambda(t)Z, \quad (1.2)$$

i.e. two shearing motions in the Z -direction plus a *time dependent* homogeneous biaxial stretch.

Rajagopal (1998) studies the class of motions (1.2) only in the case of incompressible neo-Hookean solids, for which we have $\mu^2(t)\lambda(t) = 1$; here we are able to extend the study of Rajagopal to a general hyperelastic isotropic material. In so doing we not only provide a remarkable and general addition to the few exact solutions for the nonlinear equations of elastodynamics (see for example Carroll, 1967, 1974, 1976, 1978 and the survey by Saccomandi,

2007), but we are able to show explicit examples of instability connected with the phenomenon of *(auto)-parametric resonance*.

The phenomenon we encounter is similar to that pointed out in the classical experiment of Melde (1860). In such an experiment a periodic change in the longitudinal tension of a taut string resonantly parametrically excites transverse waves when the frequency of the change of the tension is close to twice the natural frequency of the transverse waves. The simplicity of this physical system has generated an enormous literature from the experimental (e.g., Hanson et al., 1994) and theoretical side (e.g., Rowland, 1994), starting from the considerations of Lord Rayleigh summarized in his book on acoustics (Strutt, 1945).

The theoretical investigations of the Melde experiment are based on the study of vibrating string equations, but here we are considering bulk waves in an elastic material¹. As in Melde's experiment we find that in the simplest model of incompressible nonlinear elasticity (the neo-Hookean material) the first parametric resonance (designated the 2 : 1 resonance) occurs when the speed of the unsteady perturbation of the axial stretch is twice the speed of propagation of the transverse bulk waves in the material. Working with a full three-dimensional theory and for the general class of hyperelastic isotropic incompressible solids the situation is more complex and intriguing. Now, the 2 : 1 resonance depends on the amount of pre-stretch and the various material properties. Here, using suitable approximations, we are able to give a detailed discussion of the emergence of this resonance as a function of the various parameters involved in the problem.

We recall that for the Carroll waves (Carroll, 1967) the Cartesian components of (1.1) are specified as

$$x = \mu X + f(z, t), \quad y = \mu Y + g(z, t), \quad z = \lambda Z, \quad (1.3)$$

where

$$f(z, t) = A \cos(kz - \varpi t), \quad g(z, t) = A \sin(kz - \varpi t). \quad (1.4)$$

This class of motions represents a homogeneous stretching in the z -direction, with longitudinal and transverse extension ratios λ and μ , respectively, together with a plane transverse wave which propagates in the z -direction. Strictly speaking (1.2) does not contain (1.3) as special case but, as we will show in the last section of the present paper, (1.2) is equivalent to a generalization of (1.3) if the underlying biaxial stretch is time-dependent. Moreover,

¹A string in the ideal world of mathematics is a slender body characterized by the property that it cannot withstand compression or bending and is therefore a very special elastic structure.

we point out that in (Carroll, 1974) the special case of (1.2) where

$$\lambda = \mu \equiv 1, \quad f(Z, t) = \varphi(t) \cos(kZ), \quad g(Z, t) = \varphi(t) \sin(kZ), \quad (1.5)$$

is considered. Therefore, also the results contained in (Carroll, 1974) are generalized by our findings.

The plan of the paper is the following. In the next section we write down the basic equations and deduce the determining equations for the class of shearing motions (1.2). Section 3 is devoted to a discussion of the possible reduction of these determining equations to a system of ordinary differential equations when a suitable generalization of the idea of circularly polarized transverse waves is considered. In section 4 we investigate in detail this generalization in the special case of standing waves. For a Mooney-Rivlin material, assuming the underlying unsteady simple extension in the Z -direction to be an harmonic oscillation superimposed on a static stretch, under some reasonable approximations of mechanical interest, it is possible to compute the amplitude of the standing waves by solving a linear Mathieu equation. This enables us to discuss the emergence of the parametric resonance phenomenon in a clear and direct way. For nonlinear standing waves we give numerical evidence that the nonlinear and non-autonomous equation for the amplitude of the standing waves is stable although the so-called *pseudo-instability* (Kidachi and Onogi, 1997) phenomenon may occur. In this phenomenon the amplitude of the solution is bounded for all time but it grows-up significantly with respect to the initial conditions. In section 5 we show that there is an equivalence between the motion introduced in (Rajagopal, 1998) and the direct generalization to an unsteady biaxial stretch of the class of motions (1.3) containing the Carroll waves (Carroll, 1967). Some concluding remarks are contained in the last section of the paper.

2 Basic equations

The kinematic quantities of interest associated with the motion (1.1) are

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^T, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}. \quad (2.1)$$

When the material is incompressible it can undergo only isochoric motions and thus

$$\det \mathbf{F} = 1. \quad (2.2)$$

Requiring that the material in equilibrium is hyperelastic and isotropic we introduce the strain energy density function $W(I_1, I_2)$, where I_1 and I_2

are the principal invariants of \mathbf{B} or \mathbf{C} . For isochoric deformations

$$I_1 = \text{tr}\mathbf{B} \equiv \text{tr}\mathbf{C}, \quad I_2 = \text{tr}(\mathbf{B}^{-1}) \equiv \text{tr}(\mathbf{C}^{-1}). \quad (2.3)$$

The usual representation formula for the Cauchy stress tensor reads

$$\mathbf{T} = -p\mathbf{I} + 2W_1\mathbf{B} - 2W_2\mathbf{B}^{-1}, \quad (2.4)$$

where p is the indeterminate pressure introduced by the constraint (2.2), and $W_i = \partial W / \partial I_i$, $i = 1, 2$.

The balance equation of linear momentum is

$$\text{div}\mathbf{T} + \rho\mathbf{b} = \rho \frac{d\mathbf{v}}{dt}, \quad (2.5)$$

where ρ is the material density, \mathbf{v} is the velocity and, in what follows, we shall ignore the body force \mathbf{b} .

From (1.2) we compute

$$(F_{ij}) = \begin{pmatrix} \mu & 0 & f_Z \\ 0 & \mu & g_Z \\ 0 & 0 & \lambda \end{pmatrix} \quad (F_{ij}^{-1}) = \begin{pmatrix} \mu^{-1} & 0 & -\mu f_Z \\ 0 & \mu^{-1} & -\mu g_Z \\ 0 & 0 & \lambda^{-1} \end{pmatrix}, \quad (2.6)$$

where a suffix denotes partial differentiation with respect to the considered variable and because of (2.2) $\lambda(t)\mu^2(t) \equiv 1$.

On the other hand,

$$(B_{ij}) = \begin{pmatrix} \mu^2 + f_Z^2 & f_Z g_Z & \lambda f_Z \\ f_Z g_Z & \mu^2 + g_Z^2 & \lambda g_Z \\ \lambda f_Z & \lambda g_Z & \lambda^2 \end{pmatrix}, \quad (2.7)$$

$$(B_{ij}^{-1}) = \begin{pmatrix} \lambda & 0 & -f_Z \\ 0 & \lambda & -g_Z \\ -f_Z & -g_Z & \lambda^{-1}(\mu^2 + f_Z^2 + g_Z^2) \end{pmatrix}.$$

From (2.7) we have

$$I_1 = 2\mu^2 + \lambda^2 + f_Z^2 + g_Z^2 \quad (2.8)$$

and

$$I_2 = 2\lambda^2 + \lambda^{-1}(\mu^2 + f_Z^2 + g_Z^2). \quad (2.9)$$

Using (2.4) we compute the components of the Cauchy stress tensor as

$$\begin{aligned}
T_{11} &= -p + 2(\mu^2 + f_Z^2)W_1 - 2\lambda W_2, \\
T_{22} &= -p + 2W_1(\mu^2 + g_Z^2) - 2\lambda W_2, \\
T_{33} &= -p + 2\lambda^2 W_1 - 2\lambda^{-1}(\mu^2 + f_Z^2 + g_Z^2)W_2, \\
T_{12} &= 2f_Z g_Z W_1, \\
T_{13} &= 2(\lambda W_1 + W_2)f_Z, \\
T_{23} &= 2(\lambda W_1 + W_2)g_Z.
\end{aligned} \tag{2.10}$$

Introducing (2.10) into the balance equations (2.5) we obtain

$$\begin{aligned}
\rho(f_{tt} + \mu_{tt}X) &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}[2(\lambda W_1 + W_2)f_Z] \\
\rho(g_{tt} + \mu_{tt}Y) &= -\frac{\partial p}{\partial y} + \lambda \frac{\partial}{\partial z}[2(\lambda W_1 + W_2)g_Z],
\end{aligned} \tag{2.11}$$

and

$$\rho \lambda_{tt} Z = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial z} \{2\lambda^2 W_1 - 2\lambda^{-1}(\mu^2 f_Z^2 + g_Z^2)W_2\}. \tag{2.12}$$

Using the formulas

$$\begin{aligned}
\frac{\partial}{\partial x} &= \mu^{-1} \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} = \mu^{-1} \frac{\partial}{\partial Y}, \\
\frac{\partial}{\partial z} &= -\mu f_Z \frac{\partial}{\partial X} - \mu g_Z \frac{\partial}{\partial Y} + \mu^2 \frac{\partial}{\partial Z},
\end{aligned} \tag{2.13}$$

and the notations $F = f_Z$, $G = g_Z$ and $Q = 2(W_1 + \lambda^{-1}W_2)$ and eliminating the indeterminate Lagrange multiplier p , we obtain the determining equations for F and G

$$\begin{aligned}
\rho \left(F_{tt} - \frac{\mu_{tt}}{\mu} F \right) &= \frac{\partial^2 (QF)}{\partial Z^2}, \\
\rho \left(G_{tt} - \frac{\mu_{tt}}{\mu} G \right) &= \frac{\partial^2 (QG)}{\partial Z^2},
\end{aligned} \tag{2.14}$$

where $Q = Q(t; F^2 + G^2)$.

The system (2.14) is the generalization to any incompressible, isotropic hyperelastic material of the equations in (Rajagopal, 1998) for neo-Hookean solids. In this special case of constant Q the system (2.14) comprises two *linear* and *uncoupled* partial differential equations for F and G .

3 Special shearing motions

Here we determine a special class of solutions for the general system (2.14) by reducing the partial differential equations to a system of ordinary differential equations via a generalization of the classical method of separation of variables. Obviously, when Q is constant the standard method of separation of variables may be applicable, but here our aim is to consider the general class of incompressible isotropic elastic materials.

To this end, it is convenient to recast (2.14) as a single differential equation by introducing the complex notation $\Lambda(Z, t) = F + iG$ to obtain

$$\Lambda_{tt} - \frac{\mu_{tt}}{\mu} \Lambda = \frac{\partial^2}{\partial Z^2} \left[\frac{Q(t; \Lambda^2)}{\rho} \Lambda \right]. \quad (3.1)$$

We now search for solutions of (3.1) in the form

$$F = \Omega(t) \cos(\theta(Z, t)), \quad G = \Omega(t) \sin(\theta(Z, t)), \quad (3.2)$$

or equivalently

$$\Lambda = \Omega(t) \exp(i\theta(Z, t)). \quad (3.3)$$

Introducing (3.3) in (3.1), we may write the real and imaginary part of the resulting equation as

$$\mathcal{E}_1 := \theta_t^2 - \rho^{-1} Q(t; \Omega^2) \theta_Z^2 - \delta_1(t) = 0, \quad (3.4)$$

$$\mathcal{E}_2 := \theta_{tt} - \rho^{-1} Q(t; \Omega^2) \theta_{ZZ} + \delta_2(t) \theta_t = 0,$$

where $\delta_1 = \Omega_{tt}/\Omega - \mu_{tt}/\mu$ and $\delta_2 = 2\Omega_t/\Omega$.

If in $\mathcal{E}_1 = 0$ we have $\delta_1 = 0$ then we obtain the factorization

$$\left(\theta_t - \sqrt{\rho^{-1} Q} \theta_Z \right) \left(\theta_t + \sqrt{\rho^{-1} Q} \theta_Z \right) = 0. \quad (3.5)$$

Assuming that the Baker-Ericksen inequalities hold we deduce that $Q > 0$ (Beatty, 1987) and therefore (3.5) is meaningful.

By standard methods, we obtain the general integral of (3.5) in the form

$$\theta(Z, t) = H(Z \pm \psi(t)), \quad (3.6)$$

where H is a function to be determined and

$$\psi'^2 = \rho^{-1} Q. \quad (3.7)$$

Introducing (3.6) into $\mathcal{E}_2 = 0$ we obtain

$$\psi'' + \delta_2 \psi' = 0, \quad (3.8)$$

and hence $\Omega^2 \psi' = \text{constant}$, and so, by (3.7),

$$\Omega^4 Q(\lambda(t), \Omega) = \text{constant}. \quad (3.9)$$

Therefore, for a given material and an unsteady stretch $\lambda(t)$ equation (3.9) determines $\Omega(t)$ as a function of $\lambda(t)$. By substitution of the resulting $\Omega = \Omega(\lambda(t))$ in $\delta_1 = 0$ we obtain a second order differential equation in $\lambda = \lambda(t)$ that restricts for the given material the admissible class of motions.

When $\delta_1 \neq 0$, the compatibility problem for the overdetermined system (3.4) in the unknown $\theta(Z, t)$ is considered by taking into account also the derivatives of \mathcal{E}_1 with respect t and Z . The system

$$(\mathcal{E}_1)_t = 0, \quad (\mathcal{E}_1)_Z = 0, \quad \mathcal{E}_2 = 0, \quad (3.10)$$

allows us to obtain explicitly the second derivatives, θ_{ZZ} , θ_{Zt} , θ_{tt} and to impose the compatibility condition for the system (3.4) by cross differentiation. In so doing, a long but straightforward computation allows us to find that $\theta(Z, t) = kZ + \phi(t)$, where k is a constant and ϕ is an unknown function. Therefore, the overdetermined system (3.4) reduces to the (*determined*) system of ordinary differential equations

$$\Omega'' = \Omega \left[\phi'^2 + \frac{\mu''}{\mu} - k^2 \frac{Q}{\rho} \right], \quad \phi'' \Omega + 2\Omega' \phi' = 0, \quad (3.11)$$

in the unknowns $\Omega(t)$ and $\phi(t)$. This system, when $\mu = \lambda \equiv 1$, has been derived in (Carroll, 1974).

By a simple integration of the second equation in (3.11) we obtain $\phi' = k_1/\Omega^2$, where k_1 is an arbitrary constant, and the amplitude $\Omega(t)$ is the solution of the nonlinear and non-autonomous ordinary differential equation

$$\Omega'' = \Omega \left[\frac{\mu''}{\mu} - k^2 \frac{Q}{\rho} \right] + \frac{k_1^2}{\Omega^3}. \quad (3.12)$$

The solutions of equation (3.11) when $\phi' \neq 0$ are an important case; indeed, the Carroll waves (1.3), (1.4) are solutions of this system when μ is constant, $\Omega(t) = \Omega_0$ and $\phi(t) = \varpi t$.

When $\theta(Z, t) = kZ + \phi(t)$, from (3.2) we deduce in (1.2)

$$f(Z, t) = -\frac{\Omega(t)}{k} \sin(kZ + \phi(t)), \quad g(Z, t) = \frac{\Omega(t)}{k} \cos(kZ + \phi(t)) \quad (3.13)$$

and therefore the solutions investigated in this section may be considered as a generalization of the classical circularly polarized shear waves. Indeed, when (3.13) is in force, the shearing motions in (1.2) are of equal amplitude and they differ in phase by $\pi/2$.

In the following we restrict attention only to the case $k_1 = 0$ and therefore $\phi(t) \equiv \text{constant}$; then the solutions (3.2) may be considered as *standing waves*. The equation (3.12) may be linear for special choices of the constitutive equations. When $k_1 \neq 0$ the equation (3.12) is always nonlinear. This is an interesting case related to the Pinney-Ermakov equation (Pinney, 1950).

4 Standing waves

4.1 Linear waves

We start by considering the Mooney-Rivlin strain-energy density W defined by

$$2W(I_1, I_2) = \mu_0 \left(\frac{1}{2} + \beta \right) (I_1 - 3) + \mu_0 \left(\frac{1}{2} - \beta \right) (I_2 - 3), \quad (4.1)$$

where $\mu_0 > 0$ is the infinitesimal shear modulus and β is a constant satisfying $-1/2 \leq \beta \leq 1/2$. When $\beta = 1/2$ we recover the neo-Hookean material and the results in (Rajagopal, 1998).

Now

$$Q = \mu_0 \left[\frac{1}{2} + \beta + \lambda^{-1} \left(\frac{1}{2} - \beta \right) \right], \quad (4.2)$$

and therefore the system (2.14) is once again *linear* (and decoupled). Now, equation (3.12) reduces to the linear ODE

$$\Omega_{tt} - \left\{ \frac{\mu_{tt}}{\mu} - \frac{\mu_0}{\rho} \left[\frac{1}{2} + \beta + \lambda^{-1} \left(\frac{1}{2} - \beta \right) \right] k^2 \right\} \Omega = 0. \quad (4.3)$$

Let us consider a periodic oscillation superimposed on a static longitudinal stretch,

$$\lambda(t) = \lambda_S + \epsilon \cos(\omega t), \quad (4.4)$$

where we assume $\epsilon > 0$. Because $\lambda(t) > 0$ we must have $\lambda_S - \epsilon > 0$. The functional form (4.4) is of obvious mechanical interest.

We now introduce the dimensionless time $\tau = \omega t$ and normalize (4.4) as

$$\lambda^* \equiv \frac{\lambda}{\lambda_S} = 1 + \epsilon^* \cos(\tau), \quad (4.5)$$

where $\epsilon^* \equiv \epsilon/\lambda_S \in [0, 1[$. Moreover, considering that the squared speed of a transverse shear wave in an unstressed Mooney-Rivlin material is $v_T^2 = \mu_0/\rho$,

we introduce the dimensionless ratio between the squares of the two velocities v_T and ω/k , i.e.

$$v_R^2 = v_T^2 \frac{k^2}{\omega^2},$$

and rewrite (4.3) as

$$\Omega_{\tau\tau} - \left\{ \frac{\mu_{\tau\tau}}{\mu} - v_R^2 \left[\frac{1}{2} + \beta + (\lambda_S \lambda^*)^{-1} \left(\frac{1}{2} - \beta \right) \right] \right\} \Omega = 0. \quad (4.6)$$

Introducing (4.5) in (4.6) we obtain Hill's equation (Stoker, 1950)

$$\begin{aligned} \Omega_{\tau\tau} - \left\{ \frac{\epsilon^*(\epsilon^* \sin^2(\tau) + 2 \cos(\tau) + 2\epsilon^*)}{(2\epsilon^* \cos(\tau) + 2)^2} \right. \\ \left. - v_R^2 \left[\frac{1}{2} + \beta + \frac{(\frac{1}{2} - \beta) \lambda_S^{-1}}{1 + \epsilon^* \cos(\tau)} \right] \right\} \Omega = 0. \end{aligned} \quad (4.7)$$

If $\epsilon^* \ll 1$ equation (4.7) may be approximated by Mathieu's equation (Stoker, 1950)

$$\begin{aligned} \Omega_{\tau\tau} + \left\{ v_R^2 \left[\frac{1}{2} + \beta + \left(\frac{1}{2} - \beta \right) \lambda_S^{-1} \right] \right. \\ \left. - [(1 - 2\beta) v_R^2 \lambda_S^{-1} + 1] \frac{\epsilon^*}{2} \cos(\tau) \right\} \Omega = 0. \end{aligned} \quad (4.8)$$

For the neo-Hookean material ($\beta = 1/2$) we obtain the Hill equation

$$\Omega_{\tau\tau} + \left\{ v_R^2 - \frac{\epsilon^*(\epsilon^* \sin^2(\tau) + 2 \cos(\tau) + 2\epsilon^*)}{(2\epsilon^* \cos(\tau) + 2)^2} \right\} \Omega = 0, \quad (4.9)$$

and the related Mathieu equation

$$\Omega_{\tau\tau} + \left\{ v_R^2 - \frac{\epsilon^*}{2} \cos(\tau) \right\} \Omega = 0. \quad (4.10)$$

The approximation of the Hill equation with the Mathieu equation allows us to obtain in a simple and direct way some analytical results. Indeed, the classical analysis for the Mathieu equation $\Omega_{\tau\tau} + [\delta + \epsilon \cos(\tau)]\Omega = 0$ predicts the first instability (2 : 1 resonance) for any $\epsilon > 0$ when $\delta = 1/4$ (Stoker,

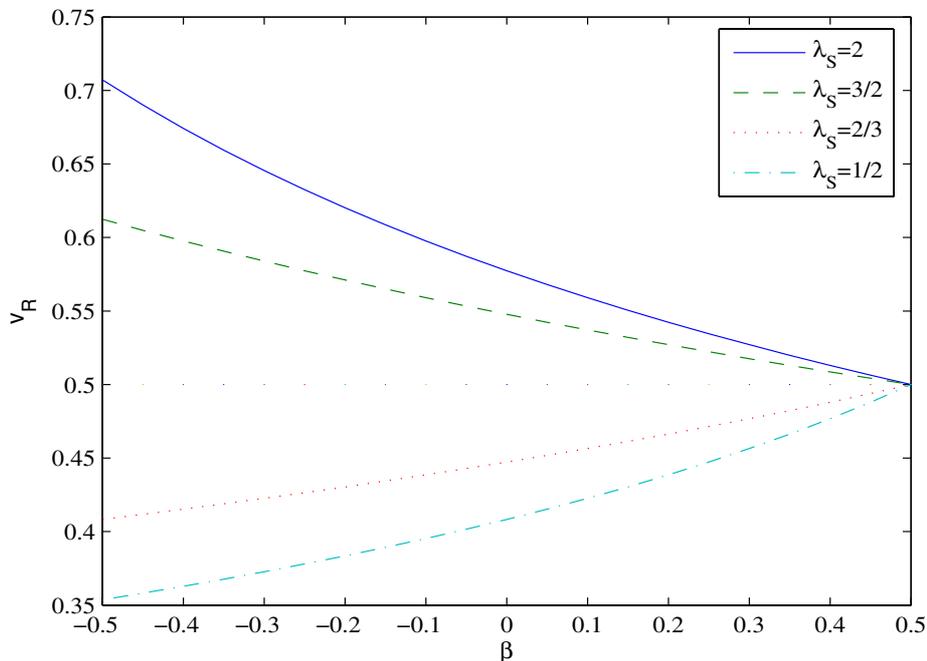


Figure 1: The critical value of v_R at the 2 : 1 resonance as a function of the parameter β for different λ_S .

1950). This means that, for equation (4.8), the 2 : 1 resonance occurs for any $\epsilon^* > 0$ when

$$v_R^2 = \frac{1}{4} \left[\frac{1}{2} + \beta + \left(\frac{1}{2} - \beta \right) \lambda_S^{-1} \right]^{-1}. \quad (4.11)$$

For the neo-Hookean material (4.11) gives $v_R^2 = 1/4$, i.e. $2v_T = \omega/k$ as in the Melde experiment. It is also important to observe that, when (4.11) holds, then also the Hill equation (4.7) is unstable for any value of ϵ^* .

In figure 1 we plot, for different values of λ_S , v_R versus β at the 2 : 1 resonance. For all values of λ_S the curves collapse to 1/2 when $\beta = 1/2$ (this because the neo-Hookean case is independent of the static amount of pre-stretch). In extension ($\lambda_S > 1$) the critical value of v_R is a monotone decreasing function of β whereas in compression ($\lambda_S < 1$) it is monotone increasing. The case $\lambda_S = 1$ is equivalent to the neo-Hookean case.

It must be clear that the Mathieu approximation is meaningful only for small ϵ^* , whereas, for the critical value v_R both the Hill and Mathieu equations predict the instability for all $\epsilon^* > 0$, and as we depart from this critical

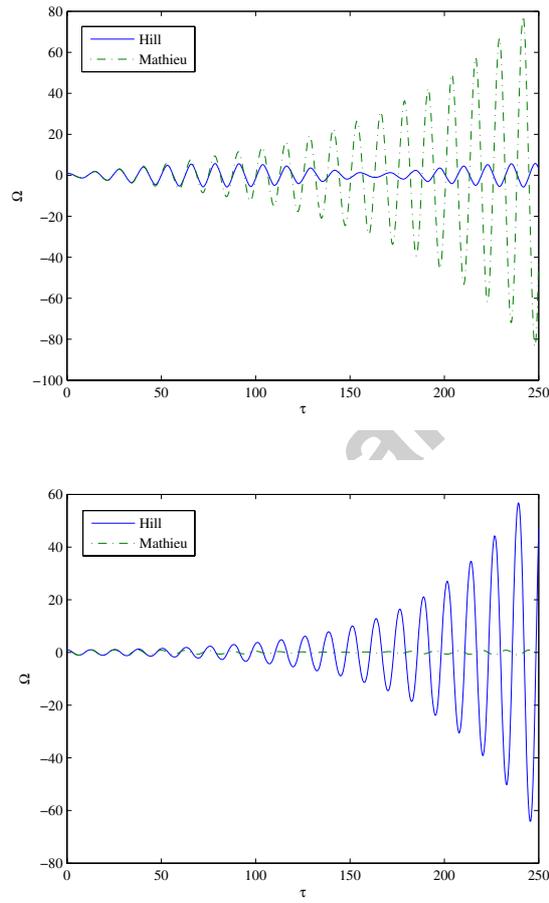


Figure 2: Examples of the limit of the Mathieu approximation for $\epsilon^* = 0.2$ and initial conditions $\Omega(0) = 1$, $\Omega_\tau(0) = 0$ in the neo-Hookean case. In the top figure $v_R^2 = 0.2$ and in the bottom figure $v_R^2 = 0.3$.

value and increase ϵ^* the Mathieu approximation fails. To illustrate this we consider the neo-Hookean case, i.e. equations (4.9) and (4.10). For example, if we pick v_R smaller than $1/2$ and $\epsilon^* = 0.2$ we find that the solution of the Mathieu equation is unstable, whereas the solution of the Hill equation is stable. On the other hand, for a $v_R \geq 1/2$ and $\epsilon^* = 0.2$ the situation is reversed. In figure 2 the two situations are illustrated via two specific numerical examples. The numerical stability (instability) of the Hill equation considered in these example may be confirmed by the standard Floquet theory (Stoker, 1950).

For $\beta \neq 1/2$ the analysis of the problem is more complex because several parameters enter the picture. To understand what is going on, we consider the classical Mathieu tongues (Stoker, 1950) emanating from the $2 : 1$ resonance at order $O(\epsilon)$. In this case the tongues dividing the stable from the unstable region are the two lines² $\delta = 1/4 \pm \epsilon/2$. Let us denote by α the angle between these two lines with v_R^2 given by (4.11). As shown in figure 3 this angle does not depend on λ_S for $\beta = 1/2$ (neo-Hookean materials) or $\beta = -1/2$. For other β increasing the pre-stretch decreases the angle α . This is because a certain amount of extension helps to stabilize the amplitude of the oscillations. On the other hand, the materials with a positive parameter β are more stable than a material with a negative β . In this sense the neo-Hookean material is the most stable within the family of Mooney-Rivlin strain energies (4.1).

From the analysis of this sub-section we recognize that, although the use of a Mathieu approximation of the Hill equation may be misleading (see figure 2), the possibility of using a simple analytical procedure to investigate a differential equation is a fundamental tool for understanding what is happening and to address possible numerical experiments. Clearly, all this is more important when a long list of parameters characterize the problem under scrutiny.

REMARK: We point out that the Mooney-Rivlin material (4.1) is the *only* one for which the determining equation for the amplitude $\Omega(t)$ is linear. Indeed, a necessary condition, when $\phi = 0$, for equation (3.11) to be linear in $\Omega(t)$ is $W_1 + \lambda^{-1}W_2 \equiv \varphi(\lambda)$, where φ is an arbitrary function of its argument. Indeed, in this case we have $Q = Q(\lambda)$ instead of $Q = Q(\Omega^2, \lambda)$. Because, by (2.8) and (2.9), we have $\partial W / \partial(\Omega^2) = W_1 + \lambda^{-1}W_2$ our assumption requires $W = \varphi(\lambda)\Omega^2 + \hat{\varphi}(\lambda)$. A strain-energy function which is linear in Ω^2 must be linear in the principal invariants I_1 and I_2 .

²We are using the standard form of Mathieu's equation.

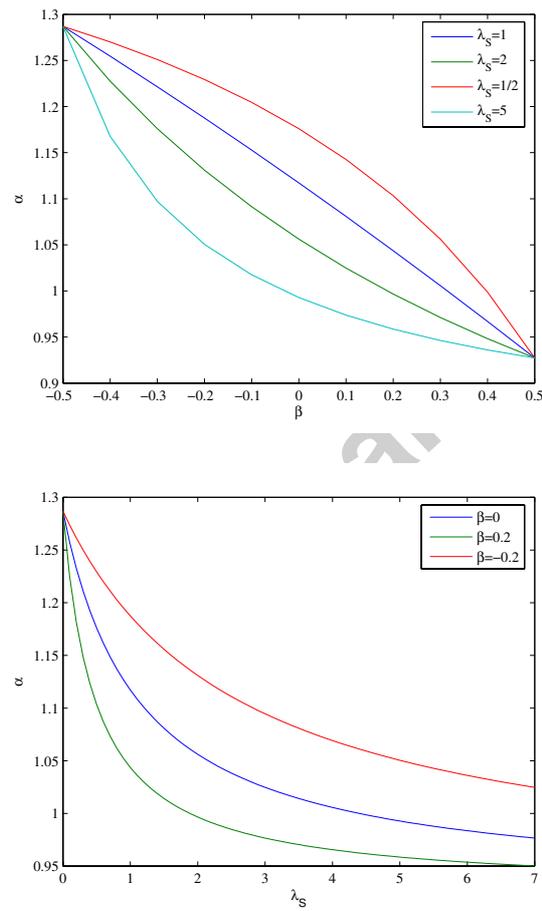


Figure 3: The angle α between the two tongues emanating from the 2 : 1 resonance. Top figure: α as a function of the parameter β for various λ_S . Bottom figure: α as a function of λ_S for various β .

4.2 Nonlinear waves

In the linear case the phenomena of stability or instability depend only on the parameters describing the physics of the phenomena. The situation is quite different when we consider the nonlinear case, because now the initial conditions play an important role. To examine this situation, let us consider the strain-energy density function W given by

$$2W = \mu_0(I_1 - 3) + \frac{\mu_{NL}}{2}(I_1 - 3)^2, \quad (4.12)$$

where $\mu_0 > 0$ is the infinitesimal shear modulus (as in the neo-Hookean material) and μ_{NL} is a constitutive parameter³.

Now

$$Q = \mu_0 + \mu_{NL}(2\mu^2 + \lambda^2 + \Omega^2 - 3) \quad (4.13)$$

and the determining differential equation for the amplitude of the shearing motions is the nonlinear equation

$$\begin{aligned} \Omega_{\tau\tau} - \left\{ \frac{\mu_{\tau\tau}}{\mu} - \frac{k^2}{\rho\omega^2} [\mu_0 + \mu_{NL}(2\lambda_S^{-1}\lambda^{*-1} + \lambda_S^2\lambda^{*2} - 3)] \right\} \Omega \\ - \frac{\mu_{NL}k^2}{\rho\omega^2}\Omega^3 = 0. \end{aligned} \quad (4.14)$$

By assuming $\epsilon^* \ll 1$ we obtain the approximate equation

$$\begin{aligned} \Omega_{\tau\tau} + \left\{ v_R^2 + v_{Rnl}^2 I_{S1} + \left[v_{Rnl}^2 (2\lambda_S^2 - 2\lambda_S^{-1}) - \frac{1}{2} \right] \epsilon^* \cos(\tau) \right\} \Omega \\ + v_{Rnl}^2 \Omega^3 = 0, \end{aligned} \quad (4.15)$$

where

$$v_R^2 = \frac{\mu_0 k^2}{\rho\omega^2}, \quad v_{Rnl}^2 = \frac{\mu_{NL} k^2}{\rho\omega^2}, \quad I_{S1} = 2\lambda_S^{-1} + \lambda_S^2 - 3.$$

It is convenient to rewrite equation (4.15) by introducing the notation

$$\Omega = \frac{\hat{\Omega}}{2v_{Rnl}}, \quad \tau = 2\hat{\tau}, \quad (4.16)$$

and

$$1 + \hat{\delta} = 4(v_R^2 + v_{Rnl}^2 I_{S1}), \quad \epsilon^* = \frac{1}{2} \frac{\hat{\epsilon}}{1 - 2v_{Rnl}^2(2\lambda_S^2 - 2\lambda_S^{-1})}, \quad (4.17)$$

³When the Baker-Ericksen inequality is in force we must have $\mu_{NL} > 0$

so that

$$\hat{\Omega}_{\hat{\tau}\hat{\tau}} + \left[1 + \hat{\delta} \mp \hat{\epsilon} \cos(2\hat{\tau})\right] \hat{\Omega} + \hat{\Omega}^3 = 0. \quad (4.18)$$

Equation (4.18) is a nonlinear Mathieu equation⁴.

To investigate the behavior of the solutions of equation (4.18) it is usual to restrict the analysis to a neighborhood of the origin by considering $\hat{\Omega} = \sqrt{\hat{\epsilon}}\tilde{\Omega}$, i.e.

$$\tilde{\Omega}_{\tau\tau} + \left[1 + \hat{\delta} \mp \hat{\epsilon}\right] \tilde{\Omega} + \hat{\epsilon}\tilde{\Omega}^3 = 0, \quad (4.19)$$

and to use classical perturbation methods (see for example Kidachi and Onogi, 1997).

For $\hat{\epsilon} = 0$ equation (4.18) is autonomous and it is possible to compute its exact solution in terms of elliptic functions. In (Zounes and Rand, 2002) on the basis of this fact a perturbation expansion in $\hat{\epsilon}$, valid also for large amplitudes, has been considered. We mention that the method of Zounes and Rand is applicable only because we know the exact solution of the unperturbed equation and this is a happenstance.

For more complex strain-energy functions than (4.12) the corresponding nonlinear Hill equation contains nonlinearities that may be non-polynomial. For example, this is the case for the Gent model

$$2W = -\mu_0 J_m \ln\left(1 - \frac{I_1 - 3}{J_m}\right), \quad (4.20)$$

where $\mu_0 > 0$ is the infinitesimal shear modulus and $J_m > 1$ is a limiting chain extensibility parameter. Now, since for (4.20) we have

$$Q = \frac{\mu_0 J_m}{J_m - (I_1 - 3)}, \quad (4.21)$$

equation (3.12) (with $k_{1-} = 0$) gives

$$\Omega_{\tau\tau} - \left\{ \frac{\mu_{\tau\tau}}{\mu} - \frac{k^2}{\rho\omega^2} \left[\frac{\mu_0 J_m}{J_m - (2\lambda_S^{-1}\lambda^{*-1} + \lambda_S^2\lambda^{*2} + \Omega^2 - 3)} \right] \right\} \Omega = 0. \quad (4.22)$$

Assuming ϵ^* small we have

$$\Omega_{\tau\tau} + \left\{ \frac{\lambda_S v_R^2}{\Gamma - \lambda_S \Omega^2} + \epsilon^* \cos(\tau) \left[\frac{2(\lambda_S^2 - 1)\lambda_S v_R^2}{(\Gamma - \lambda_S \Omega^2)^2} - \frac{1}{2} \right] \right\} \Omega = 0, \quad (4.23)$$

where $\Gamma = \lambda_S(J_m + 3) - \lambda_S^3 - 2$. It is not possible to perform a direct perturbation in ϵ^* as in (Zounes and Rand, 2002) to study (4.23) because

⁴The \mp in (4.18) has been introduced to set $\hat{\epsilon}$ for any choice of λ_S

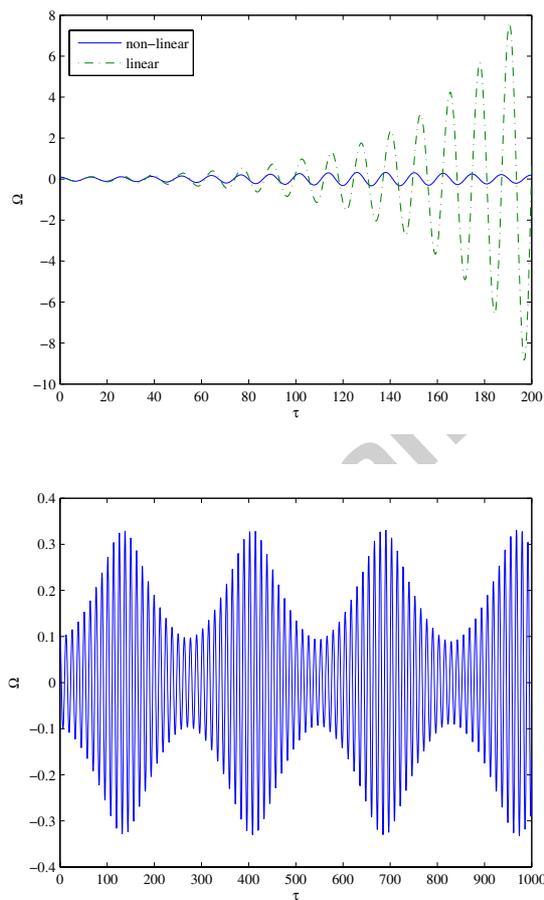


Figure 4: Solutions of (4.15) for $\epsilon^* = 0.1$, $\lambda_S = 1$ and initial conditions $\Omega(0) = 0.1$, $\Omega_\tau(0) = 0$. For the linear case $v_R^2 = 0.24$, $v_{Rnl}^2 = 0$, for the nonlinear case $v_R^2 = 0.24$, $v_{Rnl}^2 = 0.5$. In the bottom figure the solution of the nonlinear equation is computed for a longer time.

we do not know the exact solution of the nonlinear unperturbed equation ($\epsilon^* = 0$). On the other hand, if we consider $\Omega = \sqrt{\epsilon^* \Gamma} \hat{\Omega} / (\lambda_S v_r)$, at order $O(\epsilon^*)$, we obtain

$$\hat{\Omega}_{\tau\tau} + \left\{ \frac{\lambda_S v_R^2}{\Gamma} + \left[\frac{2(\lambda_S^2 - 1)\lambda_S v_R^2}{\Gamma^2} - \frac{1}{2} \right] \epsilon^* \cos(\tau) \right\} \hat{\Omega} + \epsilon^* \hat{\Omega}^3 = 0, \quad (4.24)$$

which may be recast in the form (4.18) by renaming the various parameters.

For any strain energy such that $Q(\Omega^2, t) = Q_0(t) + Q_1(t)\Omega^2 + \dots$ is valid with $Q_1(t) \neq 0$ in the neighborhood of the origin the determining equation for the amplitude is equivalent, up to $O(\epsilon^*)$, to equation (4.18), therefore we shall restrict our attention to the perturbation approach as in (Kidachi and Onogi, 1997).

A general result about the nonlinear case is that the nonlinearity may stabilize the *linear* unstable motion. For example, when $\delta = 1/4$ in the linear case for all the values of ϵ the corresponding motions are unstable, whereas all the motions of the nonlinear equation are bounded. This situation is justified in (Rand and Zounes, 2002):

From a mechanical point of view, motion under nonlinear resonance is prevented from becoming unbounded: as the amplitude of motion increases, the frequency increases, and the system ceased to be in resonance.

Nevertheless, in the nonlinear case, there may be observed a relevant growth of the amplitude of the solution. Although the solution is bounded, it may be that, for some values of the parameters and of the initial conditions the amplitude evolves in time in such a way that relevant peaks may be reached. The values of the parameters and the initial data for which this phenomenon is encountered may be teased out by performing the classical multiple scale perturbation method. To this end, because there is an interval centered on $\lambda_S = 1$ such that for any v_{Rnl} we have $\hat{\epsilon} > 0$ we restrict our attention to this case and to simplify the notation we drop the hats.

We consider a fast time $\tau_0 \equiv \tau$ and a slow time $\tau_1 \equiv \epsilon\tau$ such that

$$\Omega(\tau) = \sqrt{\epsilon} (\Omega_0(\tau_0, \tau_1) + \epsilon \Omega_1(\tau_0, \tau_1) + \dots), \quad \delta = \epsilon \delta_1 + \dots \quad (4.25)$$

On introducing (4.25) into (4.18) standard computations allow us to find

$$\Omega_0(\tau_0, \tau_1) = 2A(\tau_1) \cos(\tau_0) + 2B(\tau_0) \sin(\tau_0). \quad (4.26)$$

To remove secular terms $A(\tau_1)$ and $B(\tau_1)$ must be determined from the non-linear but autonomous system

$$\begin{aligned} A_{\tau_1} &= -\frac{1}{2}B \left[\frac{1}{2} + \delta_1 + 3(A^2 + B^2) \right], \\ B_{\tau_1} &= \frac{1}{2}A \left[-\frac{1}{2} + \delta_1 + 3(A^2 + B^2) \right]. \end{aligned} \quad (4.27)$$

To obtain information on the behavior of the solutions of (4.18), in a neighbour of the origin, we perform a phase plane analysis of the system (4.27). We start by noting that for $\delta_1 < -1/2$ and $\delta_1 > 1/2$ the origin $(A, B) = (0, 0)$ is a center, whereas for $-1/2 < \delta_1 < 1/2$ the origin is a saddle point. Because

$$\Omega(0) = 2\sqrt{\epsilon}(A(0) + O(\epsilon)), \quad \Omega_\tau(0) = 2\sqrt{\epsilon}(B(0) + O(\epsilon)), \quad (4.28)$$

all the information collected from (4.27) may be read directly in terms of the amplitude of the shearing motion.

For $\delta_1 < -1/2$ there are equilibria of (4.27): the origin, the saddle points $(0, \pm\sqrt{(1/2 - \delta_1)/3})$ and the centers $(\pm\sqrt{(-1/2 - \delta_1)/3}, 0)$. It is therefore interesting to study what happens to the solutions of (4.18) when we move from the center $(A, B) = (0, 0)$ to the saddle point $(A, B) = (0, \sqrt{(1/2 - \delta_1)/3})$. This means that we start from a solution corresponding to initial conditions $\Omega(0), \Omega_\tau(0)$ near to $(0, 0)$ and we increase the initial value of the kinetic energy.

As in (Kidachi and Onogi, 1997) we consider the total energy of the linear autonomous equation, i.e. $\mathcal{E}(\tau) := (\Omega_\tau^2 + \Omega^2)/2$, and the function

$$E(\tau) = \log_{10} \{ \mathcal{E}(\tau)/\mathcal{E}(0) \}. \quad (4.29)$$

At time τ , $\mathcal{E}(\tau)$ is balanced by the power expended by the $O(\epsilon)$ -forces in (4.19) in the interval $[0, \tau]$. Therefore, if the quantity $E(\tau)$ computed from the solutions of (4.19) is positive it means that the terms of order $O(\epsilon)$ *pump* energy into the system, otherwise energy is *absorbed*.

In figure 5 (top) we show a solution usually characterized as *stable*. For any τ the amplitude oscillates in the interval $(-0.2, 0.2)$ and the quantity $E(\tau)$ is negative. On the other hand in the bottom plot we show a *pseudo-unstable* solution because we are nearer to the saddle point of the system (4.27). The range of the oscillations of the amplitude after a very short time (less than $\tau = 50$) is $(-0.2, 0.2)$, then we move to the range $(-0.5, 0.5)$. The solutions are bounded in both cases (as can be checked by considering that their orbits in the phase-plane of (4.27) are closed), but a small difference

in the initial conditions determines an appreciable growth of the amplitude of the shearing motions and this is because, as clearly shown by the time evolution of $E(\tau)$, the $O(\epsilon)$ -forces are pumping energy into the system.

In the last figure 6 we show what happens around the origin $(A, B) = (0, 0)$ when $\delta_1 = -0.02$ and therefore the origin is a saddle point and when $\delta_1 = 0.06$ and the origin is a center. In the first numerical experiment we record a pseudo-unstable solution. We start with $\Omega(0) = 0.01, \Omega_\tau(0) = 0$ and then $\Omega(\tau)$ oscillates in the range $(-0.4, 0.4)$. On the other hand, in the second numerical experiment the initial value is the bound for $|\Omega(\tau)|$ for all τ .

5 An equivalent class of motions

The strict generalization of (1.3) to the case of a dynamical biaxial stretch reads

$$x = \mu(t)X + \tilde{f}(z, t), \quad y = \mu(t)Y + \tilde{g}(z, t), \quad z = \lambda(t)Z, \quad (5.1)$$

where again $\mu^2(t)\lambda(t) = 1$.

From (5.1) it follows that the components of the gradient of the deformation \mathbf{F} and its inverse have the forms

$$(\tilde{F}_{ij}) = \begin{pmatrix} \mu & 0 & \lambda\tilde{f}_z \\ 0 & \mu & \lambda\tilde{g}_z \\ 0 & 0 & \lambda \end{pmatrix}, \quad (\tilde{F}_{ij}^{-1}) = \begin{pmatrix} \mu^{-1} & 0 & -\mu^{-1}\tilde{f}_z \\ 0 & \mu^{-1} & -\mu^{-1}\tilde{g}_z \\ 0 & 0 & \lambda^{-1} \end{pmatrix}. \quad (5.2)$$

We point out that the gradient of the deformation (5.1) may be decomposed as $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_2\tilde{\mathbf{F}}_1$, where

$$(\tilde{F}_{1,ij}) = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad (\tilde{F}_{2ij}) = \begin{pmatrix} 1 & 0 & \tilde{f}_z \\ 0 & 1 & \tilde{g}_z \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.3)$$

From this decomposition it is clear what we mean by declaring that (5.1) is the *superposition* of two principal transverse shearing motions on an isochoric biaxial stretch motion. A similar decomposition is not possible for (1.2). We point out that if we substitute \tilde{f}_z and \tilde{g}_z with $\lambda^{-1}\tilde{f}_Z$ and $\lambda^{-1}\tilde{g}_Z$ the tensor $\tilde{\mathbf{F}}$ looks similar to \mathbf{F} but we have to be careful to evaluate the dependence $\tilde{f}(\lambda(t)Z, t)$ and $\tilde{g}(\lambda(t)Z, t)$. Despite all this, we show that the motions (1.2) and (5.1) are indeed equivalent.

The components of the left Cauchy-Green tensor and its inverse associated

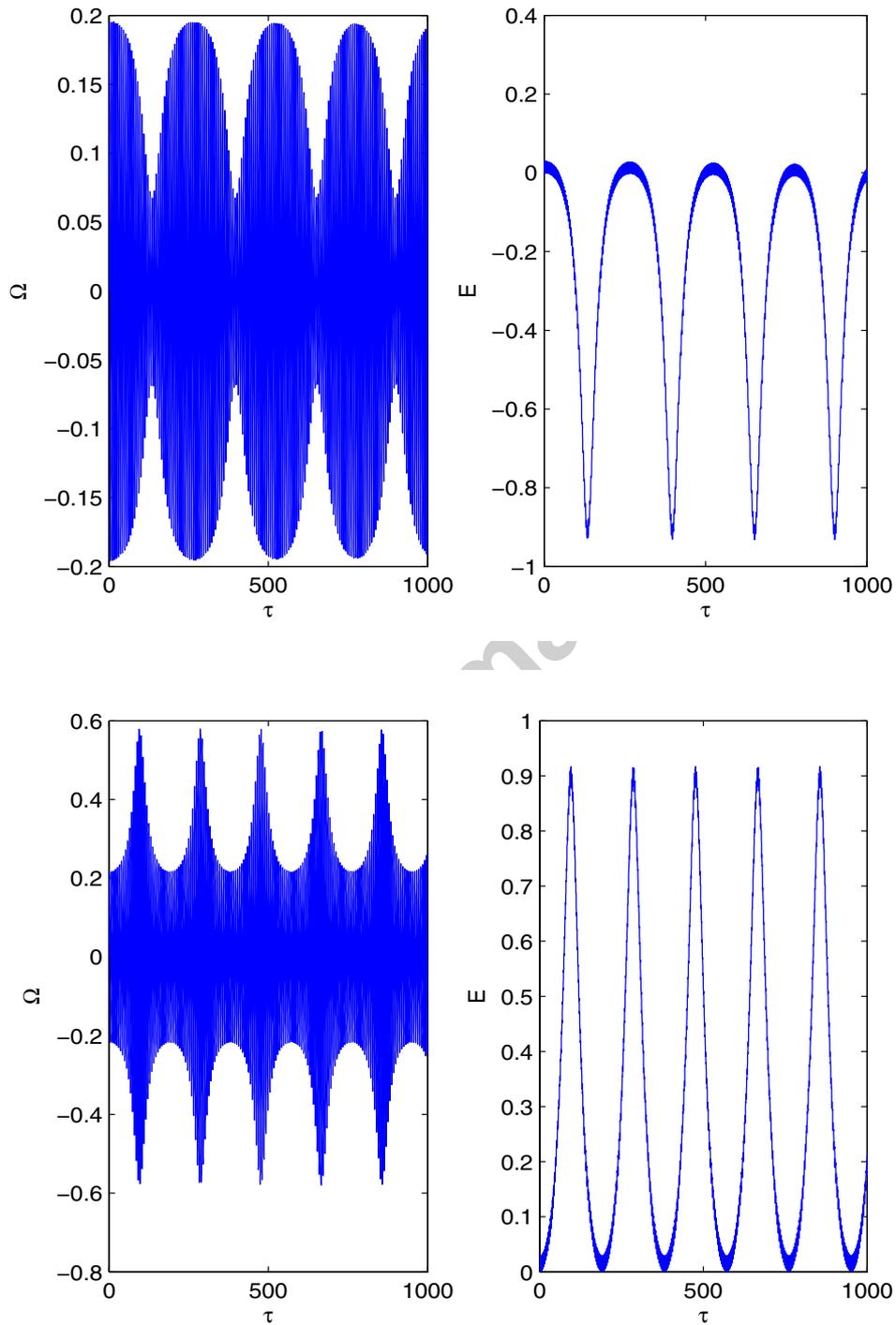


Figure 5: Solutions of equation (4.18) and the corresponding evolution of the quantity $E(\tau)$ for $\epsilon^* = 0.1$ and $\delta = -0.08$ and initial conditions $\Omega(0) = 0$, $\Omega_\tau(0) = 0.19$ (top) and $\Omega(0) = 0$, $\Omega_\tau(0) = 0.21$ (bottom).

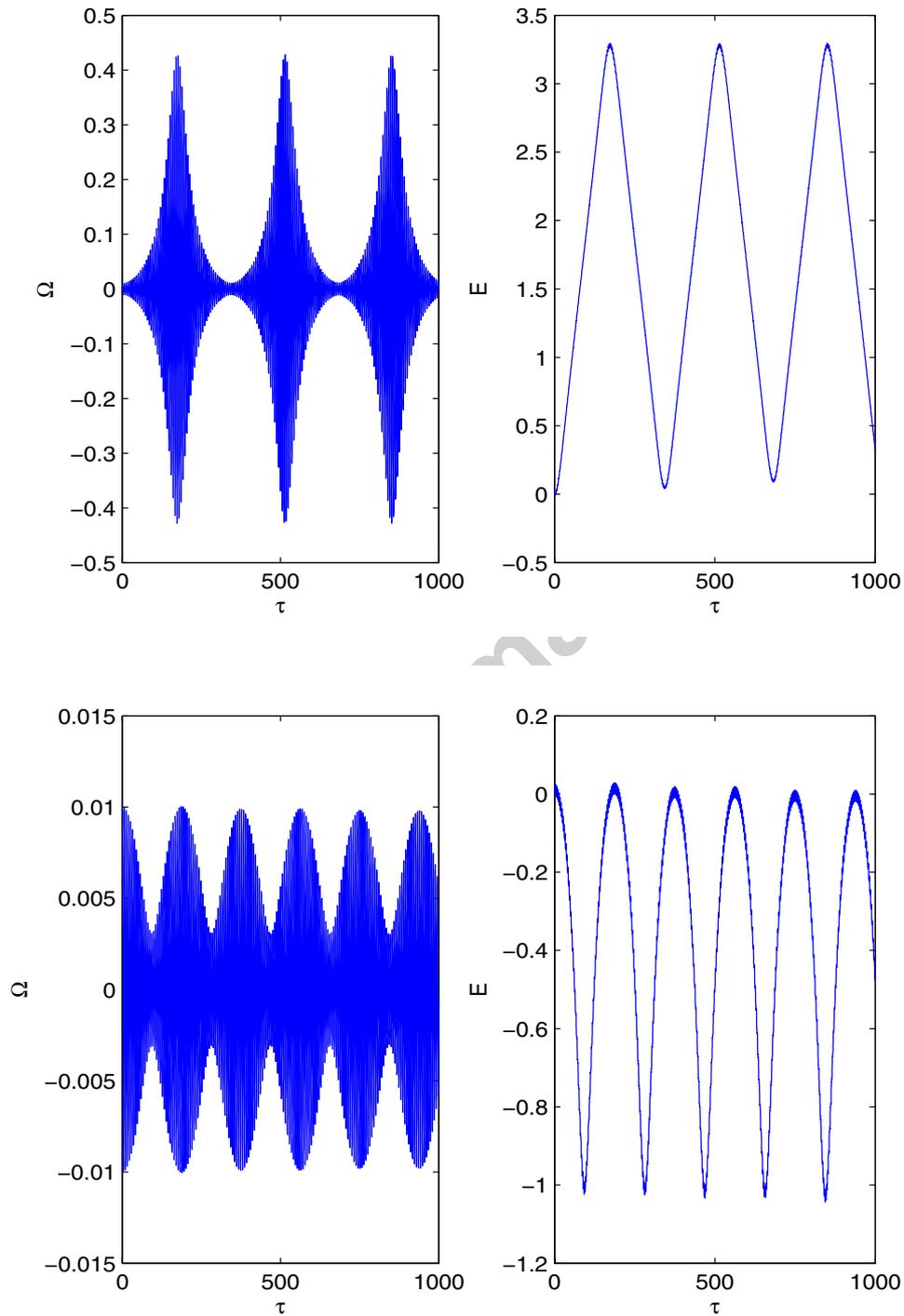


Figure 6: Solutions of equation (4.18) and the corresponding evolution of the quantity $E(\tau)$ for $\epsilon = 0.1$ and $\delta = -0.02$ (top) and $\delta = 0.06$ (bottom). The initial conditions are $\Omega(0) = 0.01$, $\Omega_{\tau}(0) = 0$ in each case.

with the motion (5.1) are

$$\begin{aligned} (\tilde{B}_{ij}) &= \begin{pmatrix} \mu^2 + \lambda^2 \tilde{f}_z^2 & \lambda^2 \tilde{f}_z \tilde{g}_z & \lambda^2 \tilde{f}_z \\ \lambda^2 \tilde{f}_z \tilde{g}_z & \mu^2 + \lambda^2 \tilde{g}_z^2 & \lambda^2 \tilde{g}_z \\ \lambda^2 \tilde{f}_z & \lambda^2 \tilde{g}_z & \lambda^2 \end{pmatrix}, \\ (\tilde{B}_{ij}^{-1}) &= \begin{pmatrix} \lambda & 0 & -\lambda \tilde{f}_z \\ 0 & \lambda & -\lambda \tilde{g}_z \\ -\lambda \tilde{f}_z & -\lambda \tilde{g}_z & \mu^4 + \lambda (\tilde{f}_z^2 + \tilde{g}_z^2) \end{pmatrix}. \end{aligned} \quad (5.4)$$

From (5.4) we have

$$\tilde{I}_1 = 2\mu^2 + \lambda^2 (1 + \tilde{f}_z^2 + \tilde{g}_z^2) \quad (5.5)$$

and

$$\tilde{I}_2 = \mu^4 + \lambda (2 + \tilde{f}_z^2 + \tilde{g}_z^2) \equiv 2\lambda + \lambda^{-1} (\mu^2 + \lambda^2 (\tilde{f}_z^2 + \tilde{g}_z^2)). \quad (5.6)$$

The components of the Cauchy stress tensor are given by

$$\begin{aligned} T_{11} &= -p + 2 (\mu^2 + \lambda^2 \tilde{f}_z^2) W_1 - 2\lambda W_2, \\ T_{22} &= -p + 2W_1 (\mu^2 + \lambda^2 \tilde{g}_z^2) - 2\lambda W_2, \\ T_{33} &= -p + 2\lambda^2 W_1 - 2 \left[\mu^4 + \lambda (\tilde{f}_z^2 + \tilde{g}_z^2) \right] W_2, \\ T_{12} &= 2\lambda^2 \tilde{f}_z \tilde{g}_z W_1, \\ T_{13} &= 2 (\lambda W_1 + W_2) \lambda \tilde{f}_z, \\ T_{23} &= 2 (\lambda W_1 + W_2) \lambda \tilde{g}_z. \end{aligned} \quad (5.7)$$

Introducing (5.7) into the representation formula and into the balance equation (2.5) we obtain

$$\begin{aligned} \rho \left[\tilde{f}_{tt} + \mu_{tt} X + \tilde{f}_{zz} \left(\frac{\lambda_t}{\lambda} \right)^2 z^2 + \left(2\tilde{f}_{zt} \frac{\lambda_t}{\lambda} + \tilde{f}_z \frac{\lambda_{tt}}{\lambda} \right) z \right] &= \\ &- \frac{\partial p}{\partial x} + \frac{\partial}{\partial z} \left\{ 2 (\lambda W_1 + W_2) \lambda \tilde{f}_z \right\}, \\ \rho \left[\tilde{g}_{tt} + \mu_{tt} Y + \tilde{g}_{zz} \left(\frac{\lambda_t}{\lambda} \right)^2 z^2 + \left(2\tilde{g}_{zt} \frac{\lambda_t}{\lambda} + \tilde{g}_z \frac{\lambda_{tt}}{\lambda} \right) z \right] &= \\ &- \frac{\partial p}{\partial y} + \frac{\partial}{\partial z} \left\{ 2 (\lambda W_1 + W_2) \lambda \tilde{g}_z \right\}, \end{aligned} \quad (5.8)$$

and

$$\rho \frac{\lambda_{tt}}{\lambda} z = \frac{\partial}{\partial z} \{2\lambda^2 W_1 - 2(\mu^4 + \lambda(f_z^2 + g_z^2)) W_2\} - \frac{\partial p}{\partial z}. \quad (5.9)$$

Now we observe that

$$\tilde{f}(z, t) \equiv \tilde{f}(\lambda Z, t) \equiv \hat{f}(Z, t), \quad (5.10)$$

and therefore the following transformation rules are in force:

$$\frac{\partial \hat{f}}{\partial t} \equiv \frac{\partial \tilde{f}}{\partial z} \frac{\lambda_t}{\lambda} z + \frac{\partial \tilde{f}}{\partial t}, \quad \frac{\partial \hat{f}}{\partial z} \equiv \frac{\partial \tilde{f}}{\partial z} \lambda, \quad (5.11)$$

$$\frac{\partial^2 \hat{f}}{\partial t^2} \equiv \frac{\partial^2 \tilde{f}}{\partial z^2} \frac{\lambda_t^2}{\lambda^2} z^2 + 2 \frac{\partial^2 \tilde{f}}{\partial z \partial t} \frac{\lambda_t}{\lambda} z + \frac{\partial \tilde{f}}{\partial z} \frac{\lambda_{tt}}{\lambda} z + \frac{\partial^2 \tilde{f}}{\partial t^2}.$$

Similar relations are obviously obtained for the derivatives of \hat{g} .

Using in (5.8) the relations (5.11) and the corresponding relations for the derivatives of \hat{g} , we rewrite these equations as

$$\begin{aligned} \rho[\hat{f}_{tt} + \mu_{tt} X] &= -\frac{\partial \hat{p}}{\partial x} + \frac{\partial}{\partial z} [2(\lambda \hat{W}_1 + \hat{W}_2) \hat{f}_z], \\ \rho[\hat{g}_{tt} + \mu_{tt} Y] &= -\frac{\partial \hat{p}}{\partial y} + \lambda \frac{\partial}{\partial z} [2(\lambda \hat{W}_1 + \hat{W}_2) \hat{g}_z], \end{aligned} \quad (5.12)$$

where $\hat{W} = W(\hat{I}_1, \hat{I}_2)$ and

$$\hat{I}_1 = 2\mu^2 + \lambda^2 + \hat{f}_Z^2 + \hat{g}_Z, \quad \hat{I}_2 = 2\lambda + \lambda^{-1}(\mu^2 + \hat{f}_Z^2 + \hat{g}_Z^2). \quad (5.13)$$

Now, it is clear that if $f(Z, t)$ and $g(Z, t)$ are chosen so that the motion (1.2) of the equations (2.11) then

$$f(Z, t) = \hat{f}(Z, t) = \hat{f}\left(\frac{z}{\lambda}, t\right), \quad g(Z, t) = \hat{g}(Z, t) = \hat{g}\left(\frac{z}{\lambda}, t\right) \quad (5.14)$$

determine two functions $\tilde{f}(z, t)$ and $\tilde{g}(z, t)$ such that the motion (5.1) is a solution of the equations (5.11).

The converse is also true: if $\tilde{f}(z, t)$ and $\tilde{g}(z, t)$ are such that the motion (5.1) is a solution of the balance equations the transformation chains

$$\begin{aligned} \tilde{f}(z, t) &= \tilde{f}(\lambda Z, t) = \hat{f}(Z, t) = f(Z, t), \\ \tilde{g}(z, t) &= \tilde{g}(\lambda Z, t) = \hat{g}(Z, t) = g(Z, t), \end{aligned} \quad (5.15)$$

deliver two functions $f(Z, t)$ and $g(Z, t)$ such that the motion (1.2) is a solution of the balance equations.

Therefore the class of motions (1.2) studied in (Rajagopal, 1998) is equivalent to the generalization (5.1) of the solutions proposed in (Carroll, 1967). For example, if we have a solution of the system (3.11) the corresponding functions

$$\tilde{F}(z, t) = \Omega(t) \cos\left(\frac{k}{\lambda}z + \psi(t)\right), \quad \tilde{G}(z, t) = \Omega(t) \sin\left(\frac{k}{\lambda}z + \psi(t)\right)$$

allow us to compute the functions $\tilde{f}(z, t)$ and $\tilde{g}(z, t)$ such that the motion (5.1) is a solution of the corresponding balance equations.

6 Concluding remarks

The shearing motions considered here are interesting for at least two reasons.

First of all there is a class of solutions that may be determined by reducing the balance equations to the system of ordinary differential equations (3.11). This system of ordinary differential equations is non-autonomous and nonlinear, but for some choices of the axial stretch $\lambda(t)$ it is possible to find some exact solutions. In such a way new exact solutions of the equations of nonlinear elastodynamics may be added to the few reported in the literature, but these solutions seem to be quite *artificial*.

The second reason why the solutions reported here are interesting is that they may be used to build up examples of parametric resonance in the framework of nonlinear elastodynamics, and this is exactly the aspect we have chosen to investigate. By choosing the axial stretch in a significant way from the mechanical point of view we have shown that the system of equations (3.11) in the standing wave case may be reduced to a nonlinear or linear Mathieu equation. In the linear case the classical treatment of Mathieu's equation has allowed a detailed discussion of the 2 : 1 resonance case. It is interesting to point out that the same parametric resonance determined in the Melde experiment may be reproduced if we consider the special case of the neo-Hookean material. In the nonlinear case, at least for small ϵ^* and small initial data, we find other explicit examples of global existence in nonlinear elastodynamics. Clearly, more sophisticated analyses are possible for obtaining in this direction more definitive results, but the analyses we have reported are sufficient to give some interesting and mechanically meaningful examples. It is important to point out that these examples of global existence satisfy the same class of initial data as the Carroll waves for the displacement, but a slightly more general class of initial data for the velocities.

Several open problems related to the topic of the present research may be discussed. First of all, a detailed study of traveling waves; then, the possibility of extending the class of solutions determined here to the nonlinear theory of viscoelasticity of differential and integral type. Moreover, because the equations for the propagation of transverse waves are similar to the equations governing the motion of nonlinear elastic strings (see Destrade and Saccomandi, 2004), it would be interesting to investigate if a Melde experiment with a rubber cord is able to display in the real world some of the theoretical predictions of the present research.

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