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Counting packings of generic subsets in finite groups

Roland Bacher

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*Abstract*¹: A packing of subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a group G is an element (g_1, \dots, g_n) of G^n such that $g_1\mathcal{S}_1, \dots, g_n\mathcal{S}_n$ are disjoint subsets of G . We give a formula for the number of packings if the group G is finite and if the subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ satisfy a genericity condition.

1 Introduction

A (left-)packing of n non-empty subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a group G is an element (g_1, \dots, g_n) of G^n such that the left-translates $g_1\mathcal{S}_1, \dots, g_n\mathcal{S}_n$ of the sets \mathcal{S}_i are disjoint. The sets $\mathcal{S}_1, \dots, \mathcal{S}_n$ are labelled by their indices. In particular, permuting the elements g_1, \dots, g_n of a packing $(g_1, \dots, g_n) \in \mathcal{G}^n$ of $\mathcal{S}_1 = \dots = \mathcal{S}_n$ yields a different packing. Moreover, in the case where \mathcal{S}_1 for example is of the form $\mathcal{S}_1 = H\mathcal{S}_1$ for some subgroup H of G , a packing (g_1, \dots, g_n) gives rise to $\sharp(H)$ distinct packings (g_1h, g_2, \dots, g_n) , $h \in H$.

There is an obvious one-to-one map between packings of $\mathcal{S}_1, \dots, \mathcal{S}_n \subset G$ and packings of $a_1\mathcal{S}_1, \dots, a_n\mathcal{S}_n \subset G$ for every $(a_1, \dots, a_n) \in G^n$.

This paper deals with enumerative properties of left-packings in the case where G is a finite group. Using the involutive antiautomorphism $g \mapsto g^{-1}$, its content can easily be modified in order to deal with right-packings $\mathcal{S}_1g_1, \dots, \mathcal{S}_ng_n$.

In the sequel, we denote by $\alpha(G; \mathcal{S}_1, \dots, \mathcal{S}_n) \leq N^n$ the number of packings of n non-empty subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G with N elements. Computing $\alpha(G; \mathcal{S}_1, \dots, \mathcal{S}_n)$ for arbitrary subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G is probably difficult. There are however easy lower and upper bounds:

Proposition 1.1. *We set $a = \alpha(G; \mathcal{S}_1, \dots, \mathcal{S}_n)$ and $b = \alpha(G; \mathcal{S}_1, \dots, \mathcal{S}_n, \mathcal{S}_{n+1})$ where $\mathcal{S}_1, \dots, \mathcal{S}_{n+1}$ are $(n+1)$ non-empty subsets in a finite group G . We have the inequalities*

$$\left(N - \sharp(\mathcal{S}_{n+1}) \sum_{i=1}^n \sharp(\mathcal{S}_i) \right) a \leq b \leq \left(N - \sum_{i=1}^n \sharp(\mathcal{S}_i) \right) a .$$

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In particular, we have

$$b = \left(N - \sum_{i=1}^n \#(\mathcal{S}_i) \right) a \quad (1)$$

if \mathcal{S}_{n+1} is a singleton.

A family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of n non-empty subsets in a group G with identity element e is *generic* if for every sequence i_1, \dots, i_k of k distinct elements in $\{1, \dots, n\}$ and for every choice of elements $g_{i_j} \in \mathcal{S}_{i_j}^{-1} \mathcal{S}_{i_j} \setminus \{e\}$, we have

$$g_{i_1} g_{i_2} \cdots g_{i_k} \neq e.$$

Otherwise stated, a family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets in a group G is generic if the only solution of the equations $g_{i_1} \cdots g_{i_n} = e$ with $g_{i_j} \in \mathcal{S}_{i_j}^{-1} \mathcal{S}_{i_j}$ for $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ is given by $g_{i_j} = e$ for all j .

Example. Genericity in an additive abelian group G boils down to the fact that the subset $(\mathcal{S}_1 - \mathcal{S}_1) \times \cdots \times (\mathcal{S}_n - \mathcal{S}_n)$ of the group G^n intersects the subgroup $\{(x_1, \dots, x_n) \in G^n \mid \sum_{i=1}^n x_i = 0\}$ of G^n only in the identity element $(0, \dots, 0)$.

A generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets in the additive group \mathbb{Z} with prescribed cardinalities $s_i = \#(\mathcal{S}_i)$ can be constructed by starting with $\mathcal{S}_1 = \{0, \dots, s_1 - 1\}$ and by defining \mathcal{S}_i recursively as $\mathcal{S}_i = \{0, k_i, 2k_i, \dots, (s_i - 1)k_i\}$ where k_i is an arbitrary natural integer strictly larger than $\sum_{j=1}^{i-1} (\max(\mathcal{S}_j) - \min(\mathcal{S}_j)) = \sum_{j=1}^{i-1} (s_j - 1)k_j$. A generic family is thus for example given by the sets $\mathcal{S}_1 = \{0, 1\}, \mathcal{S}_2 = \{0, 2\}, \dots, \mathcal{S}_i = \{0, 2^{i-1}\}, \dots, \mathcal{S}_n = \{0, 2^{n-1}\}$.

Reduction of a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n \subset \mathbb{Z}$ modulo a natural integer N yields a generic family of $\mathbb{Z}/N\mathbb{Z}$ except if N is a divisor of a non-zero integer in the finite set $\{\sum_{i=1}^n (\mathcal{S}_i - \mathcal{S}_i)\}$.

Remark 1.2. The terminology “generic family” is motivated by the following fact: Given n strictly positive natural numbers s_1, \dots, s_n , most uniform random choices of n subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ with $\#(\mathcal{S}_i) = s_i$ (among all $\binom{N}{s_i}$ possible subsets) in a finite group G of order N should yield a generic family if N is large compared to $\sum_{k=2}^n k! \tau_k$ with τ_2, \dots, τ_n defined by $\sum_{k=0}^n \tau_k t^k = \prod_{j=1}^n (1 + s_j(s_j - 1)t)$. Observe that $\sum_{k=2}^n k! \tau_k \leq 2^n n! \prod_{j=1}^n s_j^2$.

The aim of this paper is to describe a universal formula for the number of packings for a generic family of subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G . The number of associated packings depends then only on the cardinalities of G and $\mathcal{S}_1, \dots, \mathcal{S}_n$. A trivial example is the generic family given by n subsets reduced to singletons. The associated number of packings in a finite group with N elements is then easily seen to be given by the polynomial $n! \binom{N}{n} = N(N-1) \cdots (N-n+1) \in \mathbb{Z}[N]$ with coefficients given by Stirling numbers of the first kind.

The study of generic packings in groups is, as far as I am aware, a new addition to the already large set of classical notions of packings. Well-known and well-studied examples are lattice-packings in Euclidean spaces or more generally sphere-packings in metric spaces. Error-correcting codes corresponding to packings of spheres (with respect to the Hamming distance given by the number of distinct coordinates) into \mathbb{F}_q^d are discrete analogues. The associated theories have however a different flavour since one tries to pack a huge (perhaps infinite) number of identical copies of spheres as tightly as possible.

Subsets in generic families are in general all distinct: Repetition destroys genericity except in the case of singletons. Moreover, packings of generic sets have typically very small densities. Generic families are mainly interesting for enumerative properties of the corresponding packings.

This paper is organized as follows: Section 2 contains the main result, Theorem 2.1. It expresses the number of packings of a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group in terms of a formal power series $U = U(x, \sigma_1, \sigma_2, \dots) \in A[[x]]$ with coefficients in the ring $A = \mathbb{Z}[\sigma_1, \sigma_2, \dots]$ of polynomials in elementary symmetric functions $\sigma_1 = \sum_{i=1}^n \#\mathcal{S}_i, \sigma_2 = \sum_{i < j} \#\mathcal{S}_i \#\mathcal{S}_j, \dots$ of $\#\mathcal{S}_1, \dots, \#\mathcal{S}_n$. The series U is given explicitly by Formula (4) and involves combinatorial integers $t_{i,j}(n)$ (defined recursively by Formula (2)) which extend Stirling numbers of the first kind. The first few coefficients of U are given by

$$\begin{aligned} & 1 - \sigma_2 x - ((1 - \sigma_1)\sigma_3 + \sigma_4)x^2 \\ & - ((2 - 3\sigma_1 + \sigma_1^2)\sigma_4 + (5 - 3\sigma_1)\sigma_5 + 3\sigma_6)x^3 \\ & - ((6 - 11\sigma_1 + 6\sigma_1^2 - \sigma_1^3)\sigma_5 + (26 - 26\sigma_1 + 6\sigma_2^2)\sigma_6 \\ & + (35 - 15\sigma_1)\sigma_7 + 15\sigma_8)x^4 + \dots \end{aligned}$$

with omitted terms divisible by x^5 .

Section 3 discusses the combinatorics of packings associated to an arbitrary (not necessarily generic) family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets in a group.

In Section 4, we specialize the results of Section 3 by applying them to generic packings. The underlying combinatorics are then simpler and yield a proof of Proposition 2.2, a crucial ingredient for establishing the main result.

Sections 5 and 6 contain the proof of Propositions 2.4 and 2.5 thus completing the proof of Theorem 2.1.

Section 7 uses Theorem 2.1 and its proof for computing the Möbius function of the poset of finite labelled hyperforests.

Section 8 deals with computational aspects and examples.

Section 9 contains a conjectural asymptotic formula for the coefficients of $U(x, 0, -1, -1, -1, \dots)$.

The paper ends with section 10 describing a few experimental observations concerning arithmetical properties of the coefficients of $U(x, 0, -1, -1, -1, \dots)$.

for U given by Formula (4) and for $\sigma_1, \sigma_2, \dots$ defined by

$$\sum_{j=0}^{\infty} \sigma_j t^j = \prod_{k=1}^n (1 + \#(\mathcal{S}_k)t) .$$

Remark that Formula (5) of Theorem 2.1 is polynomial of degree n in N for fixed complex numbers $\sigma_1, \sigma_2, \dots$ such that $\sigma_{n+1} = \sigma_{n+2} = \dots = 0$. Indeed, the coefficient of x^m in $U(x, \sigma_1, \sigma_2, \dots)$ belongs to the ideal generated by $\sigma_{m+1}, \sigma_{m+2}, \dots, \sigma_{2m}$ of $\mathbb{Z}[\sigma_1, \sigma_2, \dots]$ and is thus zero for $m \geq n$ if $\sigma_{n+1} = \sigma_{n+2} = \dots = 0$.

The ingredients for proving Theorem 2.1 are the following four results:

Proposition 2.2. *There exists a series $U \in \mathbb{Z}[[x, \sigma_1, \sigma_2, \dots]]$ such that Formula (5) with $\sigma_1, \sigma_2, \dots$ defined as in Theorem 2.1 gives the number of packings for every generic family of n non-empty subsets in a finite group with N elements.*

Moreover, the coefficient of a non-constant monomial x^m in this series U is of degree at most $2m$ with respect to the grading $\deg \sigma_i = i$ and belongs to the ideal of $\mathbb{Z}[\sigma_1, \sigma_2, \dots]$ generated by $\sigma_{m+1}, \sigma_{m+2}, \dots, \sigma_{2m}$.

The proof of Proposition 2.2 relies on combinatorial properties of intersection graphs encoding non-trivial intersections among subsets $g_1 \mathcal{S}_1, \dots, g_n \mathcal{S}_n$ of a group G .

Proposition 2.3. *A series U as in Proposition 2.2 satisfies the functional equation*

$$(1 - \sigma_1 x)U(x, \sigma_1, \sigma_2, \sigma_3, \dots) = U(x, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \dots) \quad (6)$$

where $\tilde{\sigma}_i = \sigma_{i-1} + \sigma_i$, using the convention $\sigma_0 = 1$.

Proof Equation (6) corresponds to equation (1) if $\sigma_1, \sigma_2, \dots$ are elementary symmetric functions of a finite set of natural integers. The general case follows by remarking that the algebra of symmetric polynomials is a free polynomial algebra on the set of elementary symmetric polynomials. \square

Proposition 2.4. *The series U defined by Formula (4) satisfies the functional equation (6).*

Proposition 2.5. *The functional equation (4) has at most one solution of the form $U = 1 + \dots$ such that the coefficient of a nonconstant monomial x^n is of degree at most $2n$ (with respect to the grading $\deg \sigma_i = i$) and belongs to the ideal generated by $\sigma_{n+1}, \sigma_{n+2}, \dots, \sigma_{2n}$ in $\mathbb{Z}[\sigma_1, \sigma_2, \dots]$.*

Proof of Theorem 2.1 Proposition (2.2) ensures the existence of a series enumerating packings of generic families in finite groups. This series coincides with the series given by Formula (4) by Propositions 2.3, 2.4 and 2.5. \square

Remark 2.6. *Iterating identity (6) n times we have*

$$U(x, \sigma_1, \sigma_2, \dots) \prod_{j=0}^{n-1} (1 - (\sigma_1 + j)x) = U(x, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \dots)$$

where

$$\tilde{\sigma}_k = \sum_{j=0}^{\min(k,n)} \binom{n}{j} \sigma_{k-j} .$$

A particular case is the specialization

$$U\left(x, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \dots\right) = \prod_{j=1}^{n-1} (1 - jx)$$

associated to generic families $\mathcal{S}_1, \dots, \mathcal{S}_n$ given by n singletons.

3 Combinatorics of packings for arbitrary families $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets in a group G

3.1 Proof of Proposition 1.1

A packing of $\mathcal{S}_1, \dots, \mathcal{S}_n$ given by $(g_1, \dots, g_n) \in G^n$ extends to a packing $(g_1, \dots, g_n, g_{n+1}) \in G^{n+1}$ of $\mathcal{S}_1, \dots, \mathcal{S}_{n+1}$ if and only if $g_{n+1} \in G \setminus (\cup_{i=1}^n g_i \mathcal{S}_i (\mathcal{S}_{n+1})^{-1})$ where $\mathcal{S}^{-1} = \{g^{-1} \mid g \in \mathcal{S}\}$. Since $g_i \mathcal{S}_i (\mathcal{S}_{n+1})^{-1}$ contains at most $\#\mathcal{S}_{n+1} \#\mathcal{S}_i$ elements, we have the first inequality.

Considering a fixed element $h \in \mathcal{S}_{n+1}$ we have the inequality

$$\#\left(\cup_{i=1}^n g_i \mathcal{S}_i (\mathcal{S}_{n+1})^{-1}\right) \geq \#\left(\cup_{i=1}^n g_i \mathcal{S}_i h^{-1}\right) = \#\left(\cup_{i=1}^n g_i \mathcal{S}_i\right) .$$

For a packing (g_1, \dots, g_n) , we have

$$\#\left(\cup_{i=1}^n g_i \mathcal{S}_i\right) = \sum_{i=1}^n \#\mathcal{S}_i$$

showing the second inequality.

All inequalities are sharp if $\#\mathcal{S}_{n+1} = 1$. This proves equality (1). \square

3.2 Intersection graphs

We fix a group G and a family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of n non-empty subsets in G . Given an element $\mathbf{g} = (g_1, \dots, g_n)$ of G^n , we consider the corresponding *intersection graph* $\mathcal{I}(\mathbf{g})$ with vertices $1, \dots, n$ and edges $\{i, j\}$ between distinct vertices i, j if $g_i \mathcal{S}_i \cap g_j \mathcal{S}_j \neq \emptyset$ in G . Observe that $\mathbf{g} = (g_1, \dots, g_n)$ in G^n defines a packing if and only if $\mathcal{I}(\mathbf{g})$ is the trivial graph with n isolated vertices.

Given a finite simple graph Γ with vertices $1, \dots, n$ and edges $E(\Gamma)$, we consider the set

$$\mathcal{R}_\Gamma = \{(g_1, \dots, g_n) \in G^n \mid g_i \mathcal{S}_i \cap g_j \mathcal{S}_j \neq \emptyset \text{ for every } \{i, j\} \in E(\Gamma)\} .$$

An element \mathbf{g} in G^n belongs thus to \mathcal{R}_Γ if and only if Γ is a subgraph of the intersection graph $\mathcal{I}(\mathbf{g})$.

We denote by \mathcal{E}_Γ the equivalence classes of \mathcal{R}_Γ defined by $(g_1, \dots, g_n) \sim (h_1, \dots, h_n)$ if $g_i h_i^{-1} = g_j h_j^{-1}$ for every edge $\{i, j\}$ of Γ . Two elements $\mathbf{g} = (g_1, \dots, g_n)$ and $\mathbf{h} = (h_1, \dots, h_n)$ of \mathcal{R}_Γ represent thus the same equivalence class of \mathcal{E}_Γ if and only if the function $i \mapsto g_i h_i^{-1}$ is constant on (vertices of) connected components.

Proposition 3.1. *Suppose that G is a finite group with N elements. We have then*

$$\#(\mathcal{R}_\Gamma) = \#(\mathcal{E}_\Gamma) N^{c(\Gamma)}$$

where $c(\Gamma)$ denotes the number of connected components of Γ .

Proof We set $c = c(\Gamma)$ and we denote the connected components of Γ by $\Gamma_1, \dots, \Gamma_c$. We get a free action of G^c on \mathcal{R}_Γ by considering

$$(a_1, \dots, a_c) \cdot (g_1, \dots, g_n) \mapsto (a_{\gamma(1)}^{-1} g_1, \dots, a_{\gamma(n)}^{-1} g_n)$$

where $\gamma(i) \in \{1, \dots, c\}$ is defined by the inclusion of the vertex i in the $\gamma(i)$ -th connected component $\Gamma_{\gamma(i)}$ of Γ . Orbits in \mathcal{R}_Γ of this action are thus in one-to-one correspondence with equivalence classes of \mathcal{E}_Γ . \square

Remark 3.2. *The set \mathcal{E}_Γ associated to a graph Γ with c connected components contains at most $(\max_i \#(\mathcal{S}_i))^{2n-2c}$ distinct equivalence classes.*

3.3 Möbius inversion

Proposition 3.3. *The number $\alpha = \alpha(G; \mathcal{S}_1, \dots, \mathcal{S}_n)$ of packings of a family $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G with N elements is given by*

$$\alpha = \sum_{\Gamma \in \mathcal{B}} (-1)^{e(\Gamma)} \#(\mathcal{E}_\Gamma) N^{c(\Gamma)}$$

where the sum is over the Boolean poset \mathcal{B} of all $2^{\binom{n}{2}}$ simple graphs with vertices $1, \dots, n$ and where $e(\Gamma) = \#(E(\Gamma))$, respectively $c(\Gamma)$, denotes the number of edges, respectively connected components, of a graph $\Gamma \in \mathcal{B}$.

Proof Proposition 3.1 shows that it is enough to prove the equality

$$\alpha = \sum_{\Gamma \in \mathcal{B}} (-1)^{e(\Gamma)} \#(\mathcal{R}_\Gamma) .$$

An element $\mathbf{g} = (g_1, \dots, g_n) \in G^n$ defines a packing if and only if its intersection graph $\mathcal{I}(\mathbf{g})$ is trivial. It provides thus a contribution of 1 to α in this case since it is only involved as an element of \mathcal{R}_Γ if Γ is the trivial graph with isolated vertices $1, \dots, n$ and no edges.

An element $\mathbf{g} = (g_1, \dots, g_n) \in G^n$ with non-trivial intersection graph $\mathcal{I}(\mathbf{g})$ containing $e \geq 1$ edges yields a contribution of 0 to α since contributions coming from the 2^{e-1} subgraphs of $\mathcal{I}(\mathbf{g})$ containing an even number of edges cancel out with contributions associated to the 2^{e-1} subgraphs having an odd number of edges. \square

Remark 3.4. *Introducing*

$$\alpha_\Gamma = \{\mathbf{g} \in G^n \mid \mathcal{I}(\mathbf{g}) = \Gamma\},$$

we have $\alpha = \alpha_T$ where T denotes the trivial graph with n isolated vertices $1, \dots, n$. The above proof of Proposition 3.3 computes α by applying Möbius inversion (more precisely, its dual form, see Proposition 3.7.2 of [4])

$$\alpha = \sum_{\Gamma \in \mathcal{B}} \mu(\Gamma) \sharp(\mathcal{R}_\Gamma) \quad (7)$$

(with $\mu(\Gamma) = (-1)^{e(\Gamma)}$ denoting the Möbius function of the Boolean lattice \mathcal{B} of all simple graphs on $1, \dots, n$) to the numbers

$$\sharp(\mathcal{R}_\Gamma) = \sum_{\Gamma' \supset \Gamma} \alpha_{\Gamma'}$$

given by Proposition 3.1.

4 Proof of Proposition 2.2: Combinatorics of generic packings

A *hypergraph* consists of a set \mathcal{V} of vertices and of a set of hyperedges where a hyperedge is a subset of \mathcal{V} containing at least 2 vertices. Two vertices are *adjacent* if they belong to a common hyperedge. A *path* is a sequence of consecutively adjacent vertices. A hypertree is *connected* if any pair of vertices can be joined by a path. A *cycle* is a closed path involving only distinct vertices. A *hyperforest* is a hypergraph with distinct hyperedges intersecting in at most a common vertex and with every cycle contained in a hyperedge. A *hypertree* is a connected hyperforest.

The *primal graph* of a hypergraph with vertices \mathcal{V} is the ordinary graph with vertices \mathcal{V} and ordinary edges encoding adjacency in the hypergraph. An ordinary graph Γ is the primal graph of a hyperforest if and only if every cycle and every edge of Γ is contained in a unique maximal complete subgraph. Maximal complete subgraphs of such a graph Γ are in one-to-one correspondence with hyperedges of the associated hypertree. Primal

graphs of hyperforests are often called *block-graphs* or *cordal and diamond-free graphs*. In the sequel, we identify generally hyperforests with their primal graphs.

Lemma 4.1. *The intersection $g_i\mathcal{S}_i \cap g_j\mathcal{S}_j$ associated to an edge $\{i, j\}$ in an intersection graph $\mathcal{I}(\mathbf{g})$ is reduced to a unique element if $\mathcal{S}_1, \dots, \mathcal{S}_n$ is a generic family of G .*

Proof Otherwise there exist two distinct elements $a_i, b_i \in \mathcal{S}_i$ and two distinct elements $a_j, b_j \in \mathcal{S}_j$ such that $g_i a_i = g_j b_j$ and $g_j a_j = g_i b_i$. This shows

$$g_i a_i b_j^{-1} g_j^{-1} g_j a_j b_i^{-1} g_i^{-1} = e$$

and implies the relation $b_i^{-1} a_i b_j^{-1} a_j = e$ with $b_i^{-1} a_i \in \mathcal{S}_i^{-1} \mathcal{S}_i \setminus \{e\}$ and $b_j^{-1} a_j \in \mathcal{S}_j^{-1} \mathcal{S}_j \setminus \{e\}$ in contradiction with genericity of the family $\mathcal{S}_1, \dots, \mathcal{S}_n$. \square

Proposition 4.2. *Intersection graphs of generic families are (primal graphs of) hyperforests.*

Proof Consider k cyclically consecutive vertices $i_1, i_2, \dots, i_{k-1}, i_k, i_{k+1} = i_1$ in an intersection graph $\mathcal{I}(\mathbf{g})$ of a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n \subset G$. Lemma 4.1 implies the existence of unique elements $a_{i_j} \in \mathcal{S}_{i_j}$ and $b_{i_{j+1}} \in \mathcal{S}_{i_{j+1}}$ such that $g_{i_j} a_{i_j} = g_{i_{j+1}} b_{i_{j+1}}$ for every edge $\{i_j, i_{j+1}\}$ of C . We get thus the relation

$$g_{i_1} a_{i_1} (g_{i_2} b_{i_2})^{-1} g_{i_2} a_{i_2} (g_{i_3} b_{i_3})^{-1} \cdots g_{i_k} a_{i_k} (g_{i_1} b_{i_1})^{-1} = e$$

which is conjugate to the relation

$$(b_{i_1}^{-1} a_{i_1}) (b_{i_2}^{-1} a_{i_2}) \cdots (b_{i_k}^{-1} a_{i_k}) = e .$$

Genericity of the family $\mathcal{S}_1, \dots, \mathcal{S}_n$ implies $a_{i_j} = b_{i_j}$ for all j . The sets $g_{i_j} \mathcal{S}_{i_j}$ intersect thus in the common element $g_{i_1} a_{i_1} = \cdots = g_{i_k} a_{i_k}$ (which is the unique common element of pairwise distinct sets in $\{g_{i_1} \mathcal{S}_{i_1}, \dots, g_{i_k} \mathcal{S}_{i_k}\}$ by Lemma 4.1). All elements i_1, \dots, i_k of $\mathcal{I}(\mathbf{g})$ are thus adjacent vertices contained in a common maximal complete subgraph of $\mathcal{I}(\mathbf{g})$.

Suppose now that an edge $\{i, j\}$ belongs to two distinct maximal complete subgraphs K and K' of $\mathcal{I}(\mathbf{g})$. Maximality of K and K' implies the existence of vertices $k \in K \setminus K'$ and $k' \in K' \setminus K$. Thus we get triplets of mutually adjacent vertices $i, j, k \subset K$ and $i, j, k' \subset K'$. Lemma 4.1 shows that $g_i \mathcal{S}_i \cap g_j \mathcal{S}_j$ is reduced to a unique element a . We have thus $g_i \mathcal{S}_i \cap g_j \mathcal{S}_j \cap g_k \mathcal{S}_k = \{a\} \subset K$. Similarly, we get $a \in g_{k'} \mathcal{S}_{k'}$. This implies $k' \in K$ in contradiction with $k' \in K' \setminus K$.

Distinct maximal complete subgraphs of $\mathcal{I}(\mathbf{g})$ intersect thus at most in a common vertex and every cycle of $\mathcal{I}(\mathbf{g})$ is contained in a unique maximal complete subgraph of $\mathcal{I}(\mathbf{g})$. This implies that $\mathcal{I}(\mathbf{g})$ is (the primal graph of) a hyperforest. \square

For the sake of concision, we identify in the sequel such an intersection graph $\mathcal{I}(\mathbf{g})$ with the corresponding hyperforest.

Applying the proof of Proposition 4.2 to a Hamiltonian cycle visiting all vertices of a hyperedge $\{i_1, \dots, i_k\}$ in an intersection graph $\mathcal{I}(\mathbf{g})$ associated to a generic family we get the following result:

Proposition 4.3. *Given a hyperedge $\{i_1, \dots, i_k\}$ in the intersection graph $\mathcal{I}(\mathbf{g})$ of a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$, there exists a unique element $a \in G$ such that $g_{i_l} \mathcal{S}_{i_l} \cap g_{i_m} \mathcal{S}_{i_m} = \{a\}$ for every pair of distinct vertices i_l, i_m in $\{i_1, \dots, i_k\}$.*

Proposition 4.4. *Let Γ be a hyperforest with vertices $1, \dots, n$ indexing the subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ of a generic family in a group G . Defining the equivalence relation \mathcal{E}_F of a hyperforest F as in Section 3.2, we have*

$$\#(\mathcal{E}_F) = \prod_{j=1}^n (\#\mathcal{S}_j)^{\deg_F(j)}$$

with $\deg_F(j)$ denoting the degree of j defined as the number of distinct hyperedges containing the vertex j .

Proof Let $e = \{i_1, \dots, i_k\}$ be a hyperedge of an intersection graph $\mathcal{I}(\mathbf{g})$. Since $\bigcap_{j=1}^k g_{i_j} \mathcal{S}_{i_j}$ is reduced to a unique element $a_e \in G$, we get a map $\mu_e : \{i_1, \dots, i_k\} \rightarrow G$ such that $\mu_e(i_j) \in \mathcal{S}_{i_j}$ by setting $\mu_e(i_j) = g_{i_j}^{-1} a_e$. This map depends only on the equivalence class in $\mathcal{E}_{\mathcal{I}(\mathbf{g})}$ of $\mathcal{I}(\mathbf{g})$ and the set of all such maps determines the equivalence class of $\mathcal{I}(\mathbf{g})$ in \mathcal{E}_F for any hyperforest F contained in $\mathcal{I}(\mathbf{g})$. Since all cycles of a hyperforest are contained in hyperedges, all possible choices of the maps μ_e associated to hyperedges of F correspond to equivalence classes of \mathcal{E}_F . Different choices yield inequivalent classes. The set \mathcal{E}_F of all equivalence classes is thus in one-to-one correspondence with the set $\prod_{j=1}^n \mathcal{S}_j^{\deg_F(j)}$. \square

Proof of Proposition 2.2 Setting $s_i = \#\mathcal{S}_i$, Proposition 4.4 can be rewritten as the identity

$$\#(\mathcal{E}_F) = \prod_{j=1}^n s_j^{\deg_F(j)}$$

for every hyperforest F with vertices $\{1, \dots, n\}$. Denoting by μ the Möbius function of the poset $\mathcal{HF}(n)$ (ordered by inclusion) of hyperforests with vertices $\{1, \dots, n\}$, the number $\alpha = \alpha(G; \mathcal{S}_1, \dots, \mathcal{S}_n)$ of packings of a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a group G of order N is given by

$$\alpha = \sum_{F \in \mathcal{HF}(n)} \mu(F) N^{c(F)} \prod_{j=1}^n s_j^{\deg_F(j)} \quad (8)$$

(with $c(F)$ denoting the number of connected components of a hyperforest F). Since the Möbius function of $\mathcal{HF}(n+1)$ restricts to the Möbius function of $\mathcal{HF}(n)$, the summation over $\mathcal{HF}(n)$ in Formula (8) can be extended (after setting $s_i = 0$ for $i > n$ and using the convention $0^0 = 1$) over the poset \mathcal{HF} of all hyperforests with vertices $\mathbb{N} \setminus \{0\}$ and having only finitely many vertices of strictly positive degree.

Summing over all possible labellings of an unlabelled hyperforest and remarking that the Möbius function is invariant under permutations of labels shows that α is a symmetric function of s_1, \dots, s_n . This expresses α as a polynomial in $\sigma_1 = \sum x_i, \sigma_2 = \sum_{i < j} s_i s_j, \dots$ with contributions coming from finite unlabelled hyperforests.

More precisely, contributions to the coefficient N^{n-m} in Formula (8) come from unlabelled hyperforests consisting of $c \geq 0$ non-trivial hypertrees containing $m + c$ vertices of strictly positive degrees. Remark that the condition $m + c \leq n$ can be dropped: The contribution of a labelled hyperforest containing a non-isolated vertex $j > n$ is divisible by $s_j = 0$. We have $c \leq m$ since every non-trivial hypertree contains at least two vertices. For a fixed value of m , there are thus only finitely many unlabelled hyperforests involved in the coefficient of N^{n-m} and the set of such unlabelled hyperforests stabilizes for $n \geq 2m$. Every contribution associated to such a hyperforest is thus divisible by s_{c+m} for some c in $\{1, \dots, m\}$ if $m > 0$ and the degree in s_1, s_2, \dots (with respect to the grading $\deg(s_i) = i$) of such a contribution is maximal and equals $2m$ for the unlabelled hyperforest consisting of c isolated ordinary edges. This implies Proposition 2.2. \square

5 Proof of Proposition 2.4

For U defined by Formula (4), both sides of (6) involve only monomials of the form $\sigma_i \sigma_1^j x^n$. It is thus enough to check that coefficients of both sides of (6) agree for such monomials. This holds obviously for the constant term. For a general monomial of this form, equation (6) amounts to the identity

$$\begin{aligned} & -((-1)^j t_{i,j}(n) - (-1)^{j-1} t_{i,j-1}(n-1)) \\ = & - \left(\sum_{k=j}^{2n-i} t_{i,k}(n) (-1)^k \binom{k}{j} + \sum_{k=j}^{2n-i-1} t_{i+1,k}(n) (-1)^k \binom{k}{j} \right) \end{aligned}$$

or equivalently to

$$t_{i,j}(n) + t_{i,j-1}(n-1) = \sum_k (-1)^{k+j} \binom{k}{j} (t_{i,k}(n) + t_{i+1,k}(n)) \quad (9)$$

where $\sum_k f(k) = \sum_{k \in \mathbb{Z}} f(k)$ since $\binom{k}{j} (t_{i,k}(n) + t_{i+1,k}(n)) = 0$ for $k < j$ or $k \geq n$. We prove (9) by induction on n . Setting $t_{i,j}(0) = 0$, it holds for

$n = 1$ and $n = 2$. Applying the recursion relation (2) which holds for all $i, j \in \mathbb{Z}$ if $n \geq 2$ to the right side

$$R = \sum_k (-1)^{k+j} \binom{k}{j} (t_{i,k}(n) + t_{i+1,k}(n))$$

of (9) we get

$$\begin{aligned} R &= \sum_{k=j}^n (-1)^{k+j} \binom{k}{j} \left((i-2)t_{i-1,k}(n-1) + t_{i-1,k-1}(n-1) + (i-3)t_{i-2,k}(n-1) \right. \\ &\quad \left. + (i-1)t_{i,k}(n-1) + t_{i,k-1}(n-1) + (i-2)t_{i-1,k}(n-1) \right) \\ &= L + C \end{aligned}$$

where

$$\begin{aligned} L &= (i-2) \sum_k (-1)^{k+j} \binom{k}{j} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\ &\quad + \sum_k (-1)^{k+j-1} \binom{k}{j-1} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\ &\quad + (i-3) \sum_k (-1)^{k+j} \binom{k}{j} (t_{i-2,k}(n-1) + t_{i-1,k}(n-1)) \end{aligned}$$

and

$$\begin{aligned} C &= - \sum_k (-1)^{k+j-1} \binom{k}{j-1} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\ &\quad + \sum_k (-1)^{k+j} \binom{k}{j} (t_{i-1,k-1}(n-1) + t_{i,k-1}(n-1)) \\ &\quad + \sum_k (-1)^{k+j} \binom{k}{j} (t_{i,k}(n-1) + t_{i-1,k}(n-1)) \\ &= \sum_k (-1)^{k+j} \binom{k}{j-1} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\ &\quad + \sum_k (-1)^{k+j} \binom{k}{j} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\ &\quad - \sum_k (-1)^{k+j} \binom{k+1}{j} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\ &= \sum_k (-1)^{k+j} \left(\binom{k}{j-1} + \binom{k}{j} - \binom{k+1}{j} \right) (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\ &= 0 \end{aligned}$$

Using induction on n and applying (9) we get

$$\begin{aligned} L &= (i-2)(t_{i-1,j}(n-1) + t_{i-1,j-1}(n-2)) \\ &\quad + (t_{i-1,j-1}(n-1) + t_{i-1,j-2}(n-2)) \\ &\quad + (i-3)(t_{i-2,j}(n-1) + t_{i-2,j-1}(n-2)) \end{aligned}$$

We have thus

$$\begin{aligned} L &= (i-2)t_{i-1,j}(n-1) + t_{i-1,j-1}(n-1) + (i-3)t_{i-2,j}(n-1) \\ &\quad + (i-2)t_{i-1,j-1}(n-2) + t_{i-1,j-2}(n-2) + (i-3)t_{i-2,j-1}(n-2) \end{aligned}$$

and applying (2) we get

$$L = t_{i,j}(n) + t_{i,j-1}(n-1)$$

which is the left side of (9). \square

6 Proof of Proposition 2.5

Assuming the existence of two distinct series U_1, U_2 fulfilling the requirements of Proposition 2.5, the difference $D = U_1 - U_2 = \sum_{n=1}^{\infty} D_n x^n$ satisfies all hypotheses except for the value of its constant term. Since U_1 and U_2 are different, there exists a minimal natural integer $n \geq 1$ such that $D_n \neq 0$. Let $m \geq n+1$ be the smallest integer such that $D_n = \sum_{k=m}^{2n} \sigma_k C_{n,k}$ with $C_{n,k} \in \mathbb{C}[\sigma_1, \sigma_2, \dots]$ and $C_{n,m} \neq 0$. Since D_n is of degree $\leq 2n$ with respect to the grading given by $\deg(\sigma_i) = i$, we have $C_{n,m} \in \mathbb{C}[\sigma_1, \dots, \sigma_{2n-m}] \subset \mathbb{C}[\sigma_1, \dots, \sigma_{n-1}]$.

Equation (6) and minimality of n imply

$$D_n(1 + \sigma_1, \sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \dots) = D_n(\sigma_1, \sigma_2, \sigma_3, \dots)$$

or equivalently

$$\sum_{k=m}^{2n} (\sigma_{k-1} + \sigma_k) C_{n,k}(1 + \sigma_1, \sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \dots) = \sum_{k=m}^{2n} \sigma_k C_{n,k}(\sigma_1, \sigma_2, \sigma_3, \dots).$$

Comparison of both sides modulo the ideal I generated by $\sigma_m, \sigma_{m+1}, \sigma_{m+2}, \dots$ gives

$$C_{n,m}(1 + \sigma_1, \sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \dots) = 0.$$

Algebraic independency of the symmetric functions $\sigma_1, \sigma_2, \dots$ shows thus $C_{n,m} = 0$ in contradiction with our assumption. \square

7 The Möbius function for the poset of finite labelled hyperforests

Let \mathcal{P} be a poset (partially ordered set) such that \mathcal{P} has a unique minimal element m and $\{y \in \mathcal{P} \mid y < x\}$ is finite for all $x \in \mathcal{P}$. This allows the recursive definition of a Möbius function μ by setting $\mu(m) = 1$ and $\mu(x) = -\sum_{y < x} \mu(y)$ for all $x > m$. Given a function $f : \mathcal{P} \rightarrow \mathbb{C}$ with finite support, the value $f(m)$ can then be recovered from the function $g(x) = \sum_{y \geq x} f(y)$ using Möbius inversion

$$f(m) = \sum_{x \in \mathcal{P}} \mu(x)g(x) ,$$

see Proposition 3.7.2 of [4] (we use only the values $\mu(m, x)$ of the Möbius function and write $\mu(x) = \mu(m, x)$ in analogy with the usual, well-known number-theoretic Möbius function of natural integers). Möbius inversion was the main ingredient in the proof of Proposition 2.2. The poset \mathcal{HF} of hyperforests consisting of all hyperforests (ordered by inclusion) with finitely many hyperedges and vertices $1, 2, 3, 4, \dots$ has a minimal element given by the trivial graph having only isolated vertices. The number

$$\#\{F' \in \mathcal{HF} \mid F' \subset F\}$$

of all hyperforests contained in a given hyperforest $F \in \mathcal{HF}$ with n hyperedges of degrees d_1, \dots, d_n is bounded by the number $2^{\binom{d_1}{2} + \dots + \binom{d_n}{2}}$ of (ordinary) subgraphs of the primal graph underlying F . The poset \mathcal{HF} has thus a Möbius function.

Proposition 7.1. *The Möbius function $\mu(F)$ of a hyperforest F in the poset \mathcal{HF} of all vertex-labelled hyperforests with finitely many hyperedges is given by*

$$\mu(F) = \prod_{j \geq 2} (-(j-2)!)^{\kappa_j}$$

where κ_j denotes the number of hyperedges involving exactly j vertices of F .

Remark 7.2. *The poset \mathcal{HF} is in fact a lattice with wedge $F_1 \wedge F_2$ given by the intersection and join $F_1 \vee F_2$ given by the smallest hyperforest containing F_1 and F_2 .*

Proof Remark first that the order relation induced on subforests of a given hyperforest $F \in \mathcal{HF}$ is the product order of all order-relations on hyperedges of F . We have thus

$$\mu(F) = \prod_{e \in E(F)} \mu(e)$$

where $E(F)$ denotes the set of hyperedges of F and where $\mu(e)$ is the Möbius function restricted to a hyperedge $e \in E(F)$. This can of course be rewritten as

$$\mu(F) = \prod_{j \geq 2} \mu(K_j)^{\kappa_j}$$

where K_j is an arbitrary hyperedge on j labelled vertices and where κ_j is the number of hyperedges having j vertices of F .

The proof of Proposition 2.2 shows that $\mu(K_j)$ coincides with the coefficient of $\sigma_{j+1}x^j$ in U . By Theorem 2.1 (whose proof needs only the existence but not the exact determination of the Möbius function), this coefficient equals $-t_{j+1,0}(j) = -(j-2)!$ where the last identity follows easily from Formula (2) defining the integers $t_{i,j}(n)$ recursively. \square

Remark 7.3. *It would be interesting to have a simple direct proof that $\mu(K_n) = -(n-2)!$ for a hypergraph $K_n \in \mathcal{HF}$ reduced to a unique hyperedge involving $n \geq 2$ vertices.*

Remark 7.4. *Let $\mathcal{HT}_k(n)$ be the finite set of all hypertrees with k hyperedges and n labelled vertices. Denoting by $\sharp(e)$ the number of vertices involved in a hyperedge e , Proposition 7.1 and the proof of Proposition 2.2 show the identity*

$$\sum_{T \in \mathcal{HT}_k(n)} \prod_{e \in \mathcal{E}(T)} (\sharp(e) - 2)! \prod_{j=1}^n s_j^{\deg(j)} = (-1)^{n+k+1} \sigma_n \sigma_1^{k-1} S_1(n-1, k) \quad (10)$$

where $\sigma_1 = \sum_{j=1}^n s_j$ and $\sigma_n = \prod_{j=1}^n s_j$ and where $S_1(n, k)$ denotes the Stirling number of the first kind defined by

$$\sum_{k=0}^n S_1(n, k) x^k = x(x-1)(x-2) \cdots (x-n+1) = \prod_{j=0}^{n-1} (x-j) .$$

Setting $s_1 = \cdots = s_n = 1$, Formula (10) specializes to the identity

$$\sum_{T \in \mathcal{HT}_k(n)} \prod_{e \in \mathcal{E}(T)} (\sharp(e) - 2)! = n^{k-1} S_1(n-1, k) (-1)^{n+k+1}$$

which is analogous to a Theorem of Husimi (see [2] or [1]) expressing the total number

$$n^{k-1} S_2(n-1, k)$$

of elements in $\mathcal{HT}_k(n)$ (labelled hypertrees with k hyperedges on n vertices) in terms of Stirling numbers of the second kind.

All these results are of course generalizations and variations of Cayley's theorem enumerating the n^{n-2} labelled trees on n vertices.

Observe that all these identities can also be deduced for example from Exercise 5.30 in [4] using a well-known map between hypergraphs and bipartite (ordinary) graphs.

8 Computational aspects and examples

The computation of $U(x, \sigma_1, \sigma_2, \dots)$ up to $o(x^n)$ is straightforward using the recurrence relation (2). For a given fixed numerical value of σ_1 , the following trick reduces memory requirement and speeds the computation up: Setting

$$c_n(\sigma_1) = (\gamma_{n+1}(\sigma_1, n), \gamma_{n+2}(\sigma_1, n), \dots, \gamma_{2n}(\sigma_1, n))$$

with $\gamma_i(\sigma_1, n) = \sum_{j=0}^{2n-i} t_{i,j}(n)(-\sigma_1)^j$ we have

$$U(x, \sigma_1, \sigma_2, \dots) = 1 - \sum_{n=1}^{\infty} \langle c_n(\sigma_1), (\sigma_{n+1}, \dots, \sigma_{2n}) \rangle x^n$$

where $\langle a, b \rangle = \sum_{i \in I} a_i b_i$ for two finite-dimensional vectors a, b with coefficients indexed by a common finite set I . The coefficients $\gamma_i(\sigma_1, n)$ of $c_n(\sigma_1)$ can be computed from the coefficients of $c_{n-1}(\sigma_1)$ by the formula

$$\gamma_i(\sigma_1, n) = (i - 2 - \sigma_1)\gamma_{i-1}(\sigma_1, n - 1) + (i - 3)\gamma_{i-2}(\sigma_1, n - 1)$$

with missing coefficients omitted in the case of $i = n + 1$ or $i = 2n$.

The coefficients of the first vectors $c_1(0), c_2(0), c_3(0), \dots$ are given by the rows of

$$\begin{array}{cccccc} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 5 & 3 & & & \\ 6 & 26 & 35 & 15 & & \\ 24 & 154 & 340 & 315 & 105 & , \end{array}$$

see A112486 of [3].

8.1 The examples $U(x, -1, -1, -1, \dots)$ and $U(x, 0, -1, -1, -1, \dots)$

The series

$$U(x, -1, -1, -1, -1, \dots) - 1$$

is the generating series of the sequence

$$S(n) = \sum_{i,j} t_{i,j}(n)$$

enumerating the sums of the triangles $T(n)$ defined by the integers $t_{i,j}(n)$.

We have

$$\begin{aligned} & (1+x)U(x, -1, -1, -1, -1, \dots) \\ &= U(x, 0, -2, -2, -2, -2, \dots) \\ &= 2U(x, 0, -1, -1, -1, -1, \dots) - 1 \end{aligned}$$

where $U(x, 0, -1, -1, -1, \dots) - 1$ corresponds to the generating series of the sequence

$$s(n) = \sum_{i=n+1}^{2n} t_{i,0}(n)$$

starting as

$$1, 2, 10, 82, 938, 13778, 247210, 5240338, 128149802, 3551246162, \dots,$$

cf. A112487 of [3], and obtained by summing the integers of the first column of the triangles $T(1), T(2), \dots$. In particular, we have $2s(n) = S(n-1) + S(n)$ or equivalently

$$2 \sum_{i=n+1}^{2n} t_{i,0}(n) = \sum_{i=n+1}^{2n} \sum_{j=0}^{2n-i} t_{i,j}(n) + \sum_{i=n}^{2n-2} \sum_{j=0}^{2n-2-i} t_{i,j}(n-1)$$

for all $n \geq 2$.

8.2 A family of rational examples

Proposition 8.1. *Let $\sigma_1, \sigma_2, \dots$ be a sequence of complex numbers of the form $\sigma_n = (-1)^n P(n)$ for all $n \geq A$ where A is some natural integer and where $P(s) \in \mathbb{C}[s]$ is a polynomial. Then $U(x, \sigma_1, \sigma_2, \dots)$ is a rational series.*

Proof Let d denote the degree of P . Applying identity (6) of Proposition 2.3 iteratively $d+1$ times we get a series of the form $U(x, \tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{A+d+2}, 0, 0, 0, \dots)$ which is a polynomial. \square

As an illustration we consider the series $U(x, y, 1, -1, 1, \dots)$. Proposition 2.3 shows

$$(1 - xy)U(x, y, 1, -1, 1, -1, \dots) = (U(1 + y, 1 + y, 0, 0, \dots)) = 1 - (1 + y)x.$$

$$\text{We have thus } U(x, y, 1, -1, 1, \dots) = 1 - \frac{x}{1-xy}.$$

8.3 Coefficients of $U(x, \sigma_1, P(2), P(3), P(4), \dots)$

Proposition 8.2. *Let $P(s) \in \mathbb{C}[s]$ be a polynomial of degree d . There exist constants $\alpha_0, \dots, \alpha_d \in \mathbb{C}$ such that*

$$[x^n]U(x, \sigma_1, P(2), P(3), P(4), \dots) = \sum_{h=0}^d \alpha_h [x^{n+h}]U(x, \sigma_1, 1, 1, 1, 1, \dots)$$

for all $n \geq 1$ with $[x^n]U$ denoting the coefficient of x^n in the series U .

Proof The proof is by induction on d and holds certainly for $d = 0$. Setting $\gamma_i(n) = \sum_{j=n+1}^{2n} t_{i,j}(n)(-\sigma_1)^j$ we have

$$\begin{aligned} 0 &= -i^d \gamma_i(n+1) + i^d(i-2-\sigma_1) \gamma_{i-1}(n) + i^d(i-3) \gamma_{i-2}(n) \\ &= -i^d \gamma_i(n+1) + (i-1)^{d+1} \gamma_{i-1}(n) + (i-2)^{d+1} \gamma_{i-2}(n) + \\ &\quad + Q_1(i-1) \gamma_{i-1}(n) + Q_2(i-2) \gamma_{i-2}(n) \end{aligned}$$

where Q_1 and Q_2 are polynomials of degree $\leq d$.

Summing over i (for a fixed integer n) and using induction on d implies the result. \square

9 Conjectural asymptotics for $s(1), s(2), \dots$

Computations with a few thousand values of $s(n)$ suggest the following asymptotic formula for the integral sequence $s(n) = \sum_{i=n+1}^{2n} t_{i,0}(n)$:

Conjecture 9.1. *There exists a real constant χ and a sequence A_0, A_1, \dots of rational polynomials $A_i(x) \in \mathbb{Q}[x]$ with A_i of degree i such that*

$$s(n) = \frac{n^{n-1}}{\chi^{n-1/2} e^n} \left(\sum_{k=0}^m \frac{A_k(\chi)}{n^k} + o(n^{-m}) \right)$$

for all $m \in \mathbb{N}$.

The real constant χ is approximatively given by

.3068528194400546905827678785418234319244998656397447458793199905

and the first few polynomials A_0, A_1, \dots are

$$\begin{aligned} A_0 &= 1 \\ A_1 &= \frac{11}{24} - \frac{x}{12} \\ A_2 &= \frac{265}{1152} - \frac{47x}{288} + \frac{x^2}{288} \\ A_3 &= \frac{48703}{414720} - \frac{3649x}{13824} + \frac{107x^2}{6912} + \frac{139x^3}{51840} \\ A_4 &= \frac{2333717}{39813120} - \frac{2019163x}{4976640} + \frac{16489x^2}{331776} + \frac{26549x^3}{1244160} - \frac{571x^4}{2488320} \\ A_5 &= \frac{38180761}{1337720832} - \frac{293093189x}{477757440} + \frac{16859263x^2}{119439360} + \\ &\quad + \frac{6752203x^3}{59719680} - \frac{170729x^4}{59719680} - \frac{163879x^5}{209018880} \end{aligned}$$

The coefficients B_k of the formal power series $\sum_{k=1}^{\infty} B_k(x)t^k = \log(\sum_{k=0}^{\infty} A_k(x)t^k)$ seem to be simpler and start as

$$\begin{aligned} B_1 &= \frac{11}{24} - \frac{x}{12} \\ B_2 &= \frac{1}{8} - \frac{x}{8} \\ B_3 &= \frac{127}{2880} - \frac{3x}{16} + \frac{x^2}{288} + \frac{x^3}{360} \\ B_4 &= \frac{1}{64} - \frac{9x}{32} + \frac{11x^2}{576} + \frac{x^3}{48} \\ B_5 &= \frac{221}{40320} - \frac{27x}{64} + \frac{41x^2}{576} + \frac{1381x^3}{12960} - \frac{x^4}{1440} - \frac{x^5}{1260} \end{aligned}$$

Remark 9.2. *The constant χ appearing Conjecture 9.1 seems also to be related to the maximal index m_n such that $t_{m_n,0}(n) = \max_i(t_{i,0}(n))$ with m_n given asymptotically by $\frac{1}{2\chi}$. Moreover, we have seemingly $\lim_{n \rightarrow \infty} \frac{t_{m_n,0}(n) \sqrt{n}}{s(n)} \sim .87$ (and the numbers $t_{i,0}(n)$, suitably rescaled, should satisfy a central limit Theorem).*

10 Modular properties of the sequence $s(1), s(2), \dots$

Proposition 8.1 implies the following result:

Proposition 10.1. *The series $U(x, \sigma_1, \sigma_2, \dots) \in \mathbb{F}_p[[x]]$ is rational if $\sigma_1, \sigma_2, \dots$ is an ultimately periodic sequence of elements in \mathbb{F}_p .*

One can also prove Proposition 10.1 as follows: Since we work over \mathbb{F}_p , one can restrict the indices i, j of the coefficients $t_{i,j}(n)$ to finite subsets. This implies that the coefficients of $U(x, \sigma_1, \dots)$ are ultimately periodic. \square

The first non-trivial case of Proposition 10.1 is perhaps given by the generating series $U(x, 0, -1, -1, -1, \dots)$ with coefficients of $U(x, 0, -1, -1, \dots) - 1$ given by the sequence

$$s(n) = \sum_{i=n+1}^{2n} t_{i,0}(n)$$

obtained by summing all coefficients in the first column of the triangular arrays $T(1), T(2), \dots$.

Conjecture 10.2. *There exists a sequence*

$$\begin{aligned} \alpha_0 &= -1, \alpha_1 = 2, \alpha_2 = 0, \alpha_3 = \frac{1}{3}, \alpha_4 = \frac{5}{18}, \alpha_5 = \frac{149}{540}, \alpha_6 = \frac{553}{2025}, \\ \alpha_7 &= \frac{1849741}{6804000}, \alpha_8 = \frac{775167119}{2857680000}, \alpha_9 = \frac{325214957371}{1200225600000}, \dots \end{aligned}$$

of rational numbers such that

$$(1 + x^{p-1}) \sum_{n=1}^{\infty} s(n)x^n \equiv x + \sum_{n=0}^{p-2} \alpha_n x^{p-n} \pmod{p}$$

for every prime number p .

Conjecture 10.3. *The rational sequence $\alpha_0, \alpha_1, \dots$ has an asymptotic expansion given by*

$$\alpha_n \sim \sum_{k=1}^{\infty} \frac{k^{k-n}}{k!} \left(\frac{2}{e^2}\right)^k$$

and converges with limit given by $2e^{-2} = .27067056647322538378799\dots$

The error term

$$\epsilon_n = \alpha_n - \sum_{k=1}^{\infty} \frac{k^{k-n}}{k!} \left(\frac{2}{e^2}\right)^k$$

is given by

$$\epsilon_n = \frac{(-1)^{n+1}}{s(n+1)} \frac{1}{\chi} \left(\sum_{k=0}^m \frac{\gamma_{2k}(\chi)}{n^{2k}} + o(n^{-2m-1}) \right)$$

where χ is as in Conjecture 9.1 and where $\gamma_{2k}(x) \in \mathbb{Q}[x]$ is a polynomial of degree at most $2k$. The first few polynomials are given by

$$\begin{aligned} \gamma_0 &= 1 \\ \gamma_2 &= -\frac{x}{12} \\ \gamma_4 &= -\frac{x}{48} + \frac{x^2}{48} + \frac{x^3}{40} \\ \gamma_6 &= -\frac{x}{192} + \frac{5x^2}{96} + \frac{193x^3}{864} - \frac{x^4}{72} - \frac{5x^5}{252} \end{aligned}$$

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