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Counting packings of generic subsets in finite groups

Roland Bacher

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*Abstract*¹: A packing of subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a group G is a sequence (g_1, \dots, g_n) such that $g_1\mathcal{S}_1, \dots, g_n\mathcal{S}_n$ are disjoint subsets of G . We give a formula for the number of packings if the group G is finite and if the subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ satisfy a genericity condition.

1 Introduction

A (left-)packing of n non-empty subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a group G is an element (g_1, \dots, g_n) of G^n such that the left-translates $g_1\mathcal{S}_1, g_2\mathcal{S}_2, \dots, g_n\mathcal{S}_n$ of the sets \mathcal{S}_i are disjoint. If G is a finite group with N elements, the number of packings of $\mathcal{S}_1, \dots, \mathcal{S}_n$ is bounded by N^n and thus finite. The sets $\mathcal{S}_1, \mathcal{S}_2, \dots$ are labelled by their indices. In particular, permuting the elements g_1, \dots, g_n of a packing $(g_1, \dots, g_n) \in G^n$ of $\mathcal{S}_1 = \mathcal{S}_2 = \dots = \mathcal{S}_n$ yields a different packing. Moreover, in the case where \mathcal{S}_1 for example is of the form $\mathcal{S}_1 = H\mathcal{S}_1$ for some subgroup H of G , a packing (g_1, \dots, g_n) gives rise to $\#(H)$ distinct packings (hg_1, g_2, \dots, g_n) , $h \in H$.

Given $(a_1, \dots, a_n) \in G^n$, remark that there is an obvious one-to-one map between packings of $\mathcal{S}_1, \dots, \mathcal{S}_n \subset G$ and packings of $a_1\mathcal{S}_1, \dots, a_n\mathcal{S}_n \subset G$.

This paper deals with enumerative properties of left-packings. Using the involutive antiautomorphism $g \mapsto g^{-1}$, its content can easily be modified in order to deal with right-packings $\mathcal{S}_1g_1, \dots, \mathcal{S}_ng_n$.

Counting packings of arbitrary subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in finite groups is probably difficult. There are however easy upper and lower bounds:

Proposition 1.1. *Let $\alpha = \alpha(G; \mathcal{S}_1, \dots, \mathcal{S}_n)$ denote the number of packings of subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G with N elements. Given an additional subset \mathcal{S}_{n+1} of G , we denote by $\tilde{\alpha} = \tilde{\alpha}(G; \mathcal{S}_1, \dots, \mathcal{S}_n, \mathcal{S}_{n+1})$ the number of packings of $\mathcal{S}_1, \dots, \mathcal{S}_n, \mathcal{S}_{n+1}$. We have*

$$\left(N - \#(\mathcal{S}_{n+1}) \sum_{i=1}^n \#(\mathcal{S}_i) \right) \alpha \leq \tilde{\alpha} \leq \left(N - \sum_{i=1}^n \#(\mathcal{S}_i) \right) \alpha .$$

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In particular, we have

$$\tilde{\alpha} = \left(N - \sum_{i=1}^n \sharp(\mathcal{S}_i) \right) \alpha \quad (1)$$

if \mathcal{S}_{n+1} is a singleton.

A family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets in a group G with identity element e is *generic* if for every sequence i_1, \dots, i_k of $k \leq n$ distinct elements in $\{1, \dots, n\}$ and for every choice of elements $g_{i_j} \in \mathcal{S}_{i_j}^{-1} \mathcal{S}_{i_j} \setminus \{e\}$, we have

$$g_{i_1} g_{i_2} \cdots g_{i_k} \neq e.$$

Otherwise stated, a subset $\mathcal{S}_1, \dots, \mathcal{S}_n$ of a group G is generic if every non-trivial relation, written as a word with letters in the alphabets $\mathcal{G}_i = \mathcal{S}_i^{-1} \mathcal{S}_i \setminus \{e\}$, in the subgroup generated by the sets $\mathcal{G}_1, \dots, \mathcal{G}_n$ involves at least two elements in one of the sets $\mathcal{G}_1, \dots, \mathcal{G}_n$.

In the case of an additive abelian group G , the genericity condition boils down to the fact that the subset $(\mathcal{S}_1 - \mathcal{S}_1) \times \cdots \times (\mathcal{S}_n - \mathcal{S}_n)$ of the group G^n intersects the subgroup $\{(x_1, \dots, x_n) \in G^n \mid \sum_{i=1}^n x_i = 0\}$ of G^n only in the identity element $(0, \dots, 0)$.

Remark 1.2. A generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets in the additive group \mathbb{Z} with prescribed cardinalities $s_i = \sharp(\mathcal{S}_i)$ can be constructed by starting with $\mathcal{S}_1 = \{0, \dots, s_1 - 1\}$ and by defining \mathcal{S}_i recursively as $\mathcal{S}_i = \{0, k_i, 2k_i, \dots, (s_i - 1)k_i\}$ where k_i is an arbitrary natural integer strictly larger than $\sum_{j=1}^{i-1} \max(\mathcal{S}_j) - \min(\mathcal{S}_j)$. A generic family is thus for example given by the sets $\mathcal{S}_1 = \{0, 1\}$, $\mathcal{S}_2 = \{0, 2\}$, \dots , $\mathcal{S}_i = \{0, 2^{i-1}\}$, \dots , $\mathcal{S}_n = \{0, 2^{n-1}\}$.

Reducing such a generic family modulo a natural integer N yields a generic family in the finite group $\mathbb{Z}/N\mathbb{Z}$ except if N is a divisor of a non-zero integer in the finite set $\{\sum_{i=1}^n \mathcal{S}_i - \mathcal{S}_i\}$.

The aim of this paper is to describe a universal formula for the number of packings for a generic family of subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G . The number of associated packings depends then only on the cardinalities of G and $\mathcal{S}_1, \dots, \mathcal{S}_n$. A trivial example is given by n subsets reduced to singletons. The associated number of packings in a finite group with N elements is then easily seen to be given by $n! \binom{N}{n} = N(N-1) \cdots (N-n+1)$.

The study of generic packings in groups is, as far as I am aware, a new addition to the already large set of classical notions of packings. Well-known and well-studied examples are Euclidean lattice-packings and sphere-packings in metric spaces. The corresponding theory has however a different flavour since one tries to pack a huge (often infinite) number of spheres as tightly as possible. Packings of generic families in finite groups are not dense at all: Typically the cardinalities of the sets $\mathcal{S}_i \subset G$ of a generic family are very small compared to the cardinality N of G and we are not interested

in density but in enumerative properties. Another source of packing-related notions is additive and elementary number theory: The existence of an infinity of twin primes for example is obviously equivalent to the question whether the set $\mathcal{P} \cap (2 + \mathcal{P})$ is infinite with $\mathcal{P} \subset \mathbb{Z}$ denoting the set of prime numbers.

Concerning the dual notion of packings, the following question is natural: Is there an interesting notion for generic packings in arbitrary groups?

The rest of the paper is organized as follows: Section 2 contains the main result, Theorem 2.1. It expresses the number of packings of a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group in terms of a formal power series $U = U(x, \sigma_1, \sigma_2, \dots) \in A[[x]]$ with coefficients in the ring $A = \mathbb{Z}[\sigma_1, \sigma_2, \dots]$ of polynomials in elementary symmetric functions $\sigma_1, \sigma_2, \dots$ defined by $\sum_{k=0}^{\infty} \sigma_k t^k = \prod_{j=1}^n (1 + \#(\mathcal{S}_j)t)$. The series U is given explicitly by formula (4) and involves combinatorial integers $t_{i,j}(n)$ (defined recursively by formula (3)) which extend Stirling numbers of the first kind. The first few coefficients of U are given by

$$\begin{aligned} & 1 - \sigma_2 x - ((1 - \sigma_1)\sigma_3 + \sigma_4)x^2 \\ & - ((2 - 3\sigma_1 + \sigma_1^2)\sigma_4 + (5 - 3\sigma_1)\sigma_5 + 3\sigma_6)x^3 \\ & - ((6 - 11\sigma_1 + 6\sigma_1^2 - \sigma_1^3)\sigma_5 + (26 - 26\sigma_1 + 6\sigma_2^2)\sigma_6 \\ & + (35 - 15\sigma_1)\sigma_7 + 15\sigma_8)x^4 + \dots \end{aligned}$$

with omitted terms divisible by x^5 . Theorem 2.1 and Proposition 1.1 imply easily that U satisfies the functional equation

$$(1 - \sigma_1 x)U(x, \sigma_1, \sigma_2, \sigma_3, \dots) = U(x, 1 + \sigma_1, \sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \dots). \quad (2)$$

Section 3 discusses the combinatorics of packings associated to arbitrary (not necessarily generic) families $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets in a group.

In Section 4, we refine the results of section 3 by applying them to generic packings. The underlying combinatorics are then simpler and imply the existence of a formal power series $\tilde{U}(x, \sigma_1, \sigma_2, \dots)$ such that the formula

$$N^n \tilde{U}(N^{-1}, \sigma_1, \sigma_2, \dots)$$

(with $\sigma_1, \sigma_2, \dots$ defined by $\sum_{k=0}^{\infty} \sigma_k t^k = \prod_{j=1}^n (1 + \#(\mathcal{S}_j)t)$) gives the number of packings for a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of n non-empty subsets in a finite group with N elements. Although this approach does not yield an explicit formula for \tilde{U} it gives some useful information on the coefficients of \tilde{U} . Moreover, such a series \tilde{U} is unique and satisfies the functional equation (2) as an easy consequence (equivalent to identity (1) of Proposition 1.1) of its very definition.

Section 5 starts with establishing the uniqueness of a solution to the functional equation (2) under certain conditions satisfied by the series \tilde{U} considered in Section 4. Since our formulae for the series U satisfy these

We consider the formal power series $U \in A[[x]]$ with coefficients in the ring $A = \mathbb{Z}[\sigma_1, \sigma_2, \sigma_3, \dots]$ of integral polynomials in $\sigma_1, \sigma_2, \dots$ defined by

$$U(x, \sigma_1, \sigma_2, \dots) = 1 - \sum_{n=1}^{\infty} x^n \sum_{i=n+1}^{2n} \sigma_i \sum_{j=0}^{2n-i} t_{i,j}(n) (-\sigma_1)^j. \quad (4)$$

Theorem 2.1. *The number of packings of a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of n non-empty subsets in a finite group G with N elements is given by the formula*

$$N^n U(N^{-1}, \sigma_1, \sigma_2, \dots) \quad (5)$$

where

$$\sum_{i=0}^n \sigma_i t^i = \prod_{j=1}^n (1 + \#\mathcal{S}_j t)$$

and where the series $U(x, \sigma_1, \sigma_2, \dots)$ is given by formula (4).

Remark that formula (5) of Theorem 2.1 is polynomial in N for fixed complex numbers $\sigma_1, \sigma_2, \dots$ such that $\sigma_{n+1} = \sigma_{n+2} = \dots = 0$. Indeed, the coefficient of x^n in $U(x, \sigma_1, \sigma_2, \dots)$ belongs to the ideal generated by $\sigma_{n+1}, \sigma_{n+2}, \dots, \sigma_{2n} \in \mathbb{Z}[\sigma_1, \sigma_2, \dots]$.

The proof of Theorem 2.1 is based on combinatorial properties of generic packings and on a functional equation for U described by the following result which is an almost obvious consequence of Theorem 2.1 and equality (1) in Proposition 1.1.

Proposition 2.2. *Suppose that $\tilde{U} \in \mathbb{C}[[x, \sigma_1, \sigma_2, \dots]]$ gives the number of packings $N^n \tilde{U}(N^{-1}, \sigma_1, \sigma_2, \dots)$ with $\sum_{i=0}^n \sigma_i t^i = \prod_{j=1}^n (1 + \#\mathcal{S}_j t)$ for every generic family of n non-empty subsets $\mathcal{S}_1, \dots, \mathcal{S}_n \subset G$ in a finite group G with $N = \#\mathcal{S}_j$ elements.*

We have then

$$(1 - \sigma_1 x) \tilde{U}(x, \sigma_1, \sigma_2, \sigma_3, \dots) = \tilde{U}(x, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \dots) \quad (6)$$

where $\tilde{\sigma}_i = \sigma_{i-1} + \sigma_i$, using the convention $\sigma_0 = 1$.

Proof Equation (6) corresponds to equation (1) if $\sigma_1, \sigma_2, \dots$ are symmetric elementary functions of a finite set of natural integers. The general case follows by remarking that the algebra of symmetric polynomials is a free polynomial algebra on the set of elementary symmetric polynomials. \square

Remark 2.3. *Iterating the identity (6) n times with $\tilde{U} = U$ given by formula (4) we have*

$$U(x, \sigma_1, \sigma_2, \dots) \prod_{j=0}^{n-1} (1 - (\sigma_1 + j)x) = U(x, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \dots)$$

where

$$\tilde{\sigma}_k = \sum_{j=0}^{\min(k,n)} \binom{n}{j} \sigma_{k-j} .$$

A particular case is the specialization

$$U \left(x, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \dots \right) = \prod_{j=1}^{n-1} (1 - jx)$$

associated to generic families $\mathcal{S}_1, \dots, \mathcal{S}_n$ given by n singletons.

3 Combinatorics of packings for arbitrary families $\mathcal{S}_1, \dots, \mathcal{S}_n$ of subsets in a group G

Proof of Proposition 1.1 Let $\mathcal{S}_1, \dots, \mathcal{S}_n$ be n subsets in a finite group G with N elements and let \mathcal{S}_{n+1} be an additional subset of G . A packing of $\mathcal{S}_1, \dots, \mathcal{S}_n$ given by $(g_1, \dots, g_n) \in G^n$ extends to a packing $(g_1, \dots, g_n, g_{n+1}) \in G^{n+1}$ of $\mathcal{S}_1, \dots, \mathcal{S}_{n+1}$ if and only if $g_{n+1} \in G \setminus (\cup_{i=1}^n g_i \mathcal{S}_i (\mathcal{S}_{n+1})^{-1})$ where \mathcal{S}^{-1} denotes the set of inverses. Since $g_i \mathcal{S}_i (\mathcal{S}_{n+1})^{-1}$ contains at most $\#(\mathcal{S}_{n+1}) \#(\mathcal{S}_i)$ elements, we have the first inequality.

Consider now a fixed element $x \in \mathcal{S}_{n+1}$. We have

$$\#(\cup_{i=1}^n g_i \mathcal{S}_i (\mathcal{S}_{n+1})^{-1}) \geq \#(\cup_{i=1}^n g_i \mathcal{S}_i x^{-1}) = \#(\cup_{i=1}^n g_i \mathcal{S}_i)$$

Since (g_1, \dots, g_n) is a packing, we have

$$\#(\cup_{i=1}^n g_i \mathcal{S}_i) = \sum_{i=1}^n \#(\mathcal{S}_i)$$

showing the second inequality.

Equality (1) is obvious for $\#(\mathcal{S}_{n+1}) = 1$. □

We fix a group G and a family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of n non-empty subsets in G . Given an element $\mathbf{g} = (g_1, \dots, g_n)$ of G^n , we consider the corresponding *intersection graph* $\mathcal{I}(\mathbf{g})$ with vertices $1, \dots, n$ and edges $\{i, j\}$ between distinct vertices i, j if $g_i \mathcal{S}_i \cap g_j \mathcal{S}_j \neq \emptyset$ in G . Observe that $\mathbf{g} = (g_1, \dots, g_n)$ in G^n defines a packing if and only if $\mathcal{I}(\mathbf{g})$ is the trivial graph with n isolated vertices.

Given a finite simple graph Γ with vertices $1, \dots, n$ and edges $E(\Gamma)$, we consider the sets

$$\mathcal{R}_\Gamma = \{(g_1, \dots, g_n) \in G^n \mid g_i \mathcal{S}_i \cap g_j \mathcal{S}_j \neq \emptyset \text{ for every } \{i, j\} \in E(\Gamma)\} .$$

An element \mathbf{g} in G^n belongs thus to \mathcal{R}_Γ if and only if Γ is a subgraph of the intersection graph $\mathcal{I}(\mathbf{g})$.

We denote by \mathcal{E}_Γ the equivalence classes of \mathcal{R}_Γ defined by $(g_1, \dots, g_n) \sim (h_1, \dots, h_n)$ if $g_i h_i^{-1} = g_j h_j^{-1}$ for every edge $\{i, j\}$ of Γ . Two elements $\mathbf{g} = (g_1, \dots, g_n)$ and $\mathbf{h} = (h_1, \dots, h_n)$ of \mathcal{R}_Γ represent thus the same equivalence class of \mathcal{E}_Γ if and only if the function $i \mapsto g_i h_i^{-1}$ is constant on (vertices of) connected components.

Proposition 3.1. *Suppose that G is a finite group having N elements. We have then*

$$\#\mathcal{R}_\Gamma = \#\mathcal{E}_\Gamma N^{c(\Gamma)}$$

where $c(\Gamma)$ denotes the number of connected components of Γ .

Proof We set $c = c(\Gamma)$ and we denote the connected components of Γ by $\Gamma_1, \dots, \Gamma_c$. We get a free action of G^c on \mathcal{R}_Γ by considering

$$(a_1, \dots, a_c) \cdot (g_1, \dots, g_n) \mapsto (a_{\gamma(1)}^{-1} g_1, \dots, a_{\gamma(n)}^{-1} g_n)$$

where $\gamma(i) \in \{1, \dots, c\}$ is defined by the inclusion of the vertex i in the $\gamma(i)$ -th connected component $\Gamma_{\gamma(i)}$ of Γ . This action is transitive and elements of \mathcal{E}_Γ are thus in one-to-one correspondence with orbits of this action on the set G^n . \square

Remark 3.2. *Fixing an element (g_1, \dots, g_n) representing an equivalence class of \mathcal{E}_Γ and choosing elements $a_{i,j} \in \mathcal{S}_i, a_{j,i} \in \mathcal{S}_j$ such that $g_i a_{i,j} = g_j a_{j,i}$ for every edge $\{i, j\}$ in a spanning forest of Γ , one sees that \mathcal{E}_Γ consists of at most $(\max_i \#\mathcal{S}_i)^{2c-2}$ distinct equivalence classes.*

Proposition 3.3. *The number $\alpha = \alpha(G; \mathcal{S}_1, \dots, \mathcal{S}_n)$ of packings of a family $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G with N elements is given by*

$$\alpha = \sum_{\Gamma \in \mathcal{B}} (-1)^{e(\Gamma)} \#\mathcal{E}_\Gamma N^{c(\Gamma)}$$

where the sum is over the Boolean poset \mathcal{B} of all $2^{\binom{n}{2}}$ simple graphs with vertices $1, \dots, n$ and where $e(\Gamma) = \#\mathcal{E}(\Gamma)$, respectively $c(\Gamma)$, denotes the number of edges, respectively connected components, of a graph $\Gamma \in \mathcal{B}$.

Proof Proposition 3.1 shows that it is enough to prove the equality

$$\alpha = \sum_{\Gamma \in \mathcal{B}} (-1)^{e(\Gamma)} \#\mathcal{R}_\Gamma .$$

An element $\mathbf{g} = (g_1, \dots, g_n) \in G^n$ defines a packing if and only if its intersection graph $\mathcal{I}(\mathbf{g})$ is trivial. It provides thus a contribution of 1 to α in this case since it is only involved as an element of \mathcal{R}_Γ if Γ is the trivial graph with isolated vertices $1, \dots, n$ and no edges.

An element $\mathbf{g} = (g_1, \dots, g_n) \in G^n$ with non-trivial intersection graph $\mathcal{I}(\mathbf{g})$ containing $e \geq 1$ edges yields a contribution of 0 to α since contributions coming from the 2^{e-1} subgraphs of $\mathcal{I}(\mathbf{g})$ containing an even number of edges cancel out with contributions associated to the 2^{e-1} subgraphs having an odd number of edges. \square

Remark 3.4. *Introducing*

$$\alpha_\Gamma = \{\mathbf{g} \in G^n \mid \mathcal{I}(\mathbf{g}) = \Gamma\},$$

we have $\alpha = \alpha_T$ where T denotes the trivial graph with n isolated vertices $1, \dots, n$. The above proof of Proposition 3.3 computes α by applying Moebius inversion (more precisely, its dual form, see Proposition 3.7.2 of [4])

$$\alpha = \sum_{\Gamma \in \mathcal{B}} \mu(\Gamma) \#(\mathcal{R}_\Gamma)$$

(with $\mu(\Gamma) = (-1)^{e(\Gamma)}$ denoting the Moebius function of the Boolean lattice \mathcal{B} of all simple graphs on $1, \dots, n$) to the numbers

$$\#(\mathcal{R}_\Gamma) = \sum_{\Gamma \subset \Gamma'} \alpha_{\Gamma'}$$

given by Proposition 3.1.

4 Proving the existence of U

We consider a fixed generic family of n non-empty finite subsets $\mathcal{S}_1, \dots, \mathcal{S}_n$ in a finite group G having N elements.

In this section we prove the existence of a series $\tilde{U}(x, \sigma_1, \sigma_2, \dots)$ such that the number of associated packings is given by

$$N^n \tilde{U}(N^{-1}, \sigma_1, \sigma_2, \dots)$$

(see equation 5) with $\sigma_1, \sigma_2, \dots$ defined by $\sum_{i=0}^n \sigma_i t^i = \prod_{j=1}^n (1 + \#(\mathcal{S}_j)t)$.

We recall that a simple graph Γ is a *block graph* (or a cordal and diamond-free graph) if all its cycles occur in maximal cliques (ie. in maximal complete subgraphs) of Γ . Block graphs can be identified with hyperforests, see Section 7. Examples are given by forests and disjoint unions of complete graphs.

Proposition 4.1. *All intersection graphs $\mathcal{I}(\mathbf{g})$, $\mathbf{g} \in G^n$ associated to a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n \subset G$ are block graphs.*

Proof Consider an oriented cycle formed by k cyclically consecutive vertices $i_1, i_2, \dots, i_{k-1}, i_k, i_{k+1} = i_1$ of $\mathcal{I}(\mathbf{g})$. For every $j \in \{1, \dots, k\}$ there exist thus two (not necessarily distinct) elements $a_{i_j}, b_{i_j} \in \mathcal{S}_{i_j}$ such that $g_{i_j} a_{i_j} = g_{i_{j+1}} b_{i_{j+1}}$. This implies the relation

$$g_{i_1} a_{i_1} (g_{i_2} b_{i_2})^{-1} \cdots g_{i_k} a_{i_k} (g_{i_1} b_{i_1})^{-1} = e. \quad (7)$$

Setting $c_{i_j} = b_{i_j}^{-1} a_{i_j}$, relation (7) is equivalent to the relation $c_{i_1} \cdots c_{i_k} = e$ with $c_{i_j} \in \mathcal{S}_{i_j}^{-1} \mathcal{S}_{i_j}$. Genericity of the family $\mathcal{S}_1, \dots, \mathcal{S}_n$ shows $a_{i_j} = b_{i_j}$ for all j and the sets $g_{i_j} \mathcal{S}_{i_j}$ intersect in the common element $g_{i_1} a_{i_1} = \cdots = g_{i_k} a_{i_k}$. All vertices i_1, \dots, i_k of $\mathcal{I}(\mathbf{t})$ are thus adjacent and contained in a maximal clique of $\mathcal{I}(\mathbf{t})$. \square

Lemma 4.2. *The intersection $g_i\mathcal{S}_i \cap g_j\mathcal{S}_j$ associated to an edge $\{i, j\}$ in an intersection graph $\mathcal{I}(\mathbf{g})$ is reduced to a unique element if $\mathcal{S}_1, \dots, \mathcal{S}_n$ is a generic family of G .*

Proof Otherwise there exist two distinct elements $a_i, b_i \in \mathcal{S}_i$ and two distinct elements $a_j, b_j \in \mathcal{S}_j$ such that $g_i a_i = g_j a_j$ and $g_i b_i = g_j b_j$. This implies the relation $a_i^{-1} b_i b_j^{-1} a_j = e$ in contradiction with the definition of genericity. \square

Proposition 4.3. *Let $\mathcal{S}_1, \dots, \mathcal{S}_n$ be a generic family of subsets in a group G and let Γ be a block graph with vertices $1, \dots, n$.*

We have

$$\#(\mathcal{E}_\Gamma) = \prod_{j=1}^n (\#(\mathcal{S}_j))^{d_h(j)}$$

where $d_h(j)$ denotes the number of non-trivial maximal cliques containing j .

Proof Let $\mathbf{g} \in G^n$ represent a class of \mathcal{E}_Γ . The proof of Proposition 4.1 and Lemma 4.2 show that each maximal clique of Γ with vertices i_1, \dots, i_k corresponds to a unique element $a = \bigcap_{j=1}^k g_{i_j} \mathcal{S}_{i_j}$ of G . This defines maps $\mathbf{g} \rightarrow g_{i_j}^{-1} a$ which depend only on the class of \mathbf{g} in \mathcal{E}_Γ and extend to a map from \mathcal{E}_Γ to $\prod_{j=1}^n \mathcal{S}_j^{d_h(j)}$ which is one-to-one. \square

Proposition 4.4. *There exists a series $\tilde{U} \in \mathbb{Z}[x, \sigma_1, \sigma_2, \dots]$ such that the number of packings of a generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of n non-empty subsets in a finite group G with N elements is given by*

$$N^n \tilde{U}(N^{-1}, \sigma_1, \sigma_2, \dots)$$

where $\sum_{i=0}^n \sigma_i t^i = \prod_{j=1}^n (1 + \#(\mathcal{S}_j) t)$.

Moreover, every monomial in $\sigma_1, \sigma_2, \dots$ contributing to the coefficient x^k of \tilde{U} is divisible by an element of the set $\{\sigma_{k+1}, \dots, \sigma_{2k}\}$ and is of degree at most $2k$ with respect to the grading given by $\deg(\sigma_i) = i$ for $i = 1, 2, \dots$.

Proof Using Moebius inversion in the poset of block graphs (ordered by inclusion) with n labelled vertices $1, \dots, n$, one gets an expression for the number of packings which involves only symmetric functions of $\#(\mathcal{S}_1), \dots, \#(\mathcal{S}_n)$ by Proposition 4.3. The contribution to N^{n-k} for $k \geq 1$ comes only from block graphs involving $k + c$ non-isolated vertices in exactly c non-trivial connected components. Such a block graph contributes a monomial divisible by σ_{k+c} . Since $c \leq \frac{k+c}{2}$, the degree of such a monomial contribution is maximal and equals $2k$ if and only if Γ is a forest consisting of k disjoint edges involving $2k$ vertices. \square

Proposition 4.5. *Let $\tilde{U} \in \mathbb{Z}[[x, \sigma_1, \sigma_2, \dots]]$ be a series such that $N^n \tilde{U}(N^{-1}, \sigma_1, \sigma_2, \dots)$ is the number of packings for every generic family $\mathcal{S}_1, \dots, \mathcal{S}_n$ of n non-empty subsets in a finite group G with N elements. Then \tilde{U} is unique and satisfies the functional equation (6) of Proposition 2.2.*

Proof If $\tilde{U}_1 \neq \tilde{U}_2$ are two such series, the difference

$$D = \sum_{n=0}^{\infty} D_n x^n = \tilde{U}_1(x, \sigma_1, \sigma_2, \sigma_3, \dots) - \tilde{U}_2(x, \sigma_1, \sigma_2, \sigma_3, \dots)$$

is not identically zero. Thus there exists an integer n such that the polynomial $D_n \in \mathbb{C}[\sigma_1, \sigma_2, \dots, \sigma_{2n}]$ is non-zero. The definition of \tilde{U}_i shows that the symmetric polynomial $P_n(s_1, \dots, s_{2n})$ obtained by the polynomial substitution given by $\sum_{k=0}^{2n} \sigma_k t^k = \prod_{j=1}^{2n} (1 + s_j t)$ in D_n is identically zero for all $s_1, \dots, s_{2n} \in \mathbb{N}$. This implies $P_n(s_1, \dots, s_{2n}) = 0$ and $D_n = 0$ since P determines $D_n \in \mathbb{C}[\sigma_1, \dots, \sigma_{2n}]$ uniquely.

Proposition 2.2 states that \tilde{U} satisfies the functional equation (6). \square

5 Proof of Theorem 2.1

Proposition 5.1. *There exists at most a unique series $\tilde{U} \in \mathbb{Z}[[x, \sigma_1, \sigma_2, \dots]]$ with the following properties:*

- (i) \tilde{U} satisfies the identity (6) of Proposition 2.2,
- (ii) \tilde{U} is of the form $\tilde{U} = 1 + \sum_{n=1}^{\infty} a_n x^n$ with $a_n \in \mathbb{Z}[\sigma_1, \sigma_2, \dots]$ a polynomial of degree $\leq 2n$ with respect to the grading $\deg(\sigma_i) = i$ and a_n is an element of the ideal generated by $\sigma_{n+1}, \sigma_{n+2}, \sigma_{n+3}, \dots$.

Proof Consider $D = \tilde{U}_1 - \tilde{U}_2 = \sum_{n=1}^{\infty} D_n x^n$ for two different series \tilde{U}_1, \tilde{U}_2 satisfying the conditions of Proposition 5.1. Let $n \geq 1$ be the smallest integer such that $D_n \neq 0$. Let $m \geq n + 1$ be the smallest integer such that $D_n = \sum_{k=m}^{2n} \sigma_k C_{n,k}$ with $C_{n,k} \in \mathbb{C}[\sigma_1, \sigma_2, \dots]$ and $C_{n,m} \neq 0$. Since D_n is of degree $\leq 2n$ with respect to the grading given by $\deg(\sigma_i) = i$, we have $C_{n,m} \in \mathbb{C}[\sigma_1, \dots, \sigma_{2n-m}] \subset \mathbb{C}[\sigma_1, \dots, \sigma_{n-1}]$.

Equation (6) and minimality of n imply

$$D_n(1 + \sigma_1, \sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \dots) = D_n(\sigma_1, \sigma_2, \sigma_3, \dots)$$

or equivalently

$$\sum_{k=m}^{2n} (\sigma_{k-1} + \sigma_k) C_{n,k} (1 + \sigma_1, \sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \dots) = \sum_{k=m}^{2n} \sigma_k C_{n,k} (\sigma_1, \sigma_2, \sigma_3, \dots).$$

Comparison of both sides modulo the ideal I generated by $\sigma_m, \sigma_{m+1}, \sigma_{m+2}, \dots$ gives

$$C_{n,m}(1 + \sigma_1, \sigma_1 + \sigma_2, \sigma_2 + \sigma_3, \dots) = 0$$

contradicting $C_{n,m} \neq 0$. \square

Proposition 5.2. *The series U defined by formula (4) satisfies equation (6) of Proposition 2.2.*

Proof Remark first that both series $(1 - \sigma_1 x)U(x, \sigma_1, \sigma_2, \dots) - 1$ and $U(x, 1 + \sigma_1, \sigma_1 + \sigma_2, \dots) - 1$ are linear in $\sigma_2, \sigma_3, \dots$. Considering the coefficient of $\sigma_i \sigma_1^j x^n$ of both series, equation (6) amounts to the identity

$$t_{i,j}(n) + t_{i,j-1}(n-1) = \sum_{k=j}^n (-1)^{k+j} \binom{k}{j} (t_{i,k}(n) + t_{i+1,k}(n))$$

or equivalently to

$$t_{i,j}(n) + t_{i,j-1}(n-1) = \sum_k (-1)^{k+j} \binom{k}{j} (t_{i,k}(n) + t_{i+1,k}(n)) \quad (8)$$

where $\sum_k f(k) = \sum_{k \in \mathbb{Z}} f(k)$ since $\binom{k}{j} (t_{i,k}(n) + t_{i+1,k}(n)) = 0$ for $k < j$ or $k \geq n$. We prove (8) by induction on n . Setting $t_{i,j}(0) = 0$ it holds for $n = 1$ and $n = 2$. Applying the recursion relation (3) which holds for all $i, j \in \mathbb{Z}$ if $n \geq 2$ to the right-hand side

$$R = \sum_k (-1)^{k+j} \binom{k}{j} (t_{i,k}(n) + t_{i+1,k}(n))$$

of (8) we get

$$\begin{aligned} R &= \sum_{k=j}^n (-1)^{k+j} \binom{k}{j} \left(\right. \\ &\quad (i-2)t_{i-1,k}(n-1) + t_{i-1,k-1}(n-1) + (i-3)t_{i-2,k}(n-1) \\ &\quad \left. + (i-1)t_{i,k}(n-1) + t_{i,k-1}(n-1) + (i-2)t_{i-1,k}(n-1) \right) \\ &= L + C \end{aligned}$$

where

$$\begin{aligned} L &= (i-2) \sum_k (-1)^{k+j} \binom{k}{j} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\ &\quad + \sum_k (-1)^{k+j-1} \binom{k}{j-1} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\ &\quad + (i-3) \sum_k (-1)^{k+j} \binom{k}{j} (t_{i-2,k}(n-1) + t_{i-1,k}(n-1)) \end{aligned}$$

and

$$\begin{aligned}
C &= -\sum_k (-1)^{k+j-1} \binom{k}{j-1} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\
&\quad + \sum_k (-1)^{k+j} \binom{k}{j} (t_{i-1,k-1}(n-1) + t_{i,k-1}(n-1)) \\
&\quad + \sum_k (-1)^{k+j} \binom{k}{j} (t_{i,k}(n-1) + t_{i-1,k}(n-1)) \\
&= \sum_k (-1)^{k+j} \binom{k}{j-1} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\
&\quad + \sum_k (-1)^{k+j} \binom{k}{j} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\
&\quad - \sum_k (-1)^{k+j} \binom{k+1}{j} (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\
&= \sum_k (-1)^{k+j} \left(\binom{k}{j-1} + \binom{k}{j} - \binom{k+1}{j} \right) (t_{i-1,k}(n-1) + t_{i,k}(n-1)) \\
&= 0
\end{aligned}$$

Using induction on n and applying (8) we get

$$\begin{aligned}
L &= (i-2)(t_{i-1,j}(n-1) + t_{i-1,j-1}(n-2)) \\
&\quad + (t_{i-1,j-1}(n-1) + t_{i-1,j-2}(n-2)) \\
&\quad + (i-3)(t_{i-2,j}(n-1) + t_{i-2,j-1}(n-2))
\end{aligned}$$

We have thus

$$\begin{aligned}
L &= (i-2)t_{i-1,j}(n-1) + t_{i-1,j-1}(n-1) + (i-3)t_{i-2,j}(n-1) \\
&\quad + (i-2)t_{i-1,j-1}(n-2) + t_{i-1,j-2}(n-2) + (i-3)t_{i-2,j-1}(n-2)
\end{aligned}$$

and applying (3) we get

$$L = t_{i,j}(n) + t_{i,j-1}(n-1)$$

which is the left-hand-side involved in (8). \square

Proof of Theorem 2.1 By Proposition 5.2, the series U defined by formula (4) satisfies condition (i) of Proposition 5.1. It satisfies condition (ii) by construction. By Proposition 4.5 it coincides with the series \tilde{U} given by Proposition 4.4. \square

6 The Moebius function for the poset of block graphs

Let \mathcal{P} be a poset (partially ordered set) with a unique minimal element m such that $\{y \in \mathcal{P} \mid y < x\}$ is finite for all $x \in \mathcal{P}$. This allows the

definition of a Moebius function μ given recursively by $\mu(m) = 1$ and $\mu(x) = -\sum_{y < x} \mu(y)$ for all $x > m$. Given a function $f : \mathcal{P} \rightarrow \mathbb{C}$ with finite support, the value $f(m)$ can then be recovered from the function $g(x) = \sum_{y \geq x} f(y)$ using Moebius inversion

$$f(m) = \sum_{x \in \mathcal{P}} \mu(x)g(x) ,$$

see Proposition 3.7.2 of [4] (we use only the values $\mu(m, x)$ of the Moebius function and write $\mu(x) = \mu(m, x)$ in analogy with the usual, well-known number-theoretic Moebius function of natural integers). Moebius inversion was the main ingredient in the proof of Proposition 4.4. We did however not compute the Moebius function for the poset \mathcal{HF} (with \mathcal{HF} standing for HyperForest, see Remark 6.2 and Section 7) of all block graphs (ordered by inclusion) with vertices $1, 2, 3, 4, \dots$ and finitely many edges. The poset \mathcal{HF} has a minimal element given by the trivial graph having only isolated vertices. The subset

$$\{\Gamma' \in \mathcal{HF} \mid \Gamma' \subset \Gamma\}$$

of all block subgraphs of a fixed block graph $\Gamma \in \mathcal{HF}$ with n edges contains at most 2^n elements given by removing suitable subsets of edges from Γ . The poset \mathcal{HF} has thus a Moebius function which is described by the following result.

Proposition 6.1. *The Moebius function $\mu(\Gamma)$ on a vertex-labelled block graph Γ with respect to the poset \mathcal{HF} of all vertex-labelled block graphs having finitely many edges is given by*

$$\mu(\Gamma) = \prod_{j \geq 2} (-(j-2)!)^{\kappa_j}$$

where κ_j denotes the number of maximal cliques with j vertices in Γ .

Remark 6.2. *The poset \mathcal{HF} coincides with the poset of all hyperforests (ordered by inclusion) having vertices \mathbb{N} and a finite number of hyperedges. It is in fact a lattice with wedge $\Gamma_1 \wedge \Gamma_2$ given by the intersection and join $\Gamma_1 \vee \Gamma_2$ given by the smallest block graph containing Γ_1 and Γ_2 as subgraphs.*

Proof Remark first that the order relation on the poset of all (vertex-labelled) block subgraphs of Γ is the product over all order-relations on block graphs induced by maximal cliques of Γ . We have thus $\mu(\Gamma) = \prod_{j \geq 2} (\mu(K_j))^{\kappa_j}$ where K_j is a complete subgraph on j labelled vertices and where κ_j is the number of maximal cliques with j vertices in Γ .

In order to compute $\mu(K_j)$, the combinatorial description for the existence of a suitable series \tilde{U} given by the proof of Proposition 4.4 shows that $\mu(K_{j+1})$ is given by the coefficient of $\sigma_{j+1}x^j$ in U . By Theorem 2.1 whose proof does not depend on Proposition 6.1 this coefficient equals $-t_{j+1,0}(j) = -(j-2)!$ where the last identity follows easily from formula (3) defining $t_{i,j}(n)$ recursively. \square

Remark 6.3. *It would be interesting to have a simple direct proof that $\mu(K_n) = -(n-2)!$ for a complete graph K_n with $n \geq 2$ vertices in the poset \mathcal{HF} .*

7 Enumeration of weighted hypertrees

A *hypergraph* is a generalized graph with edges replaced by *hyperedges* defined as finite subsets of at least two vertices. Calling n -simplex a hyperedge with $n+1$ vertices, allowing simplices reduced to one vertex and adding a closure property we get the definition of a simplicial complex.

A *path* in a hypergraph is a finite sequence v_0, v_1, \dots, v_n of vertices such that v_{i-1} and v_i belong to a common hyperedge for $i = 1, \dots, n$. A hypergraph is *connected* if every pair v, w of vertices can be joined by a path $v_0 = v, v_1, \dots, v_n = w$. A *cycle* of a hypergraph is a cyclic path $v_0, v_1, \dots, v_{n-1}, v_n = v_0$ of n distinct vertices such that the n ordinary edges $\{v_i, v_{i+1}\}, i = 0, \dots, n-1$ belong to n distinct hyperedges. A *hyperforest* is a hypergraph without cycles. A connected hyperforest is a *hypertree*. Associating to every maximal clique of a block graph the hyperedge with the same vertices we get a one-to-one map between block graphs and hyperforests.

We recall that Stirling numbers of the second kind $S_2(n, k)$, defined by the equality

$$e^{t(e^x-1)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(\sum_{k=0}^n S_2(n, k) t^k \right),$$

count the number of partitions of $\{1, \dots, n\}$ into k non-empty subsets. The following result, first proven by Husimi, see [2] or [1] generalizes Cayley's theorem (corresponding to the case $k = n-1$ of ordinary trees) to hypertrees.

Theorem 7.1. (*Husimi*) *The number of hypertrees with k hyperedges and n labelled vertices is given by*

$$n^{k-1} S_2(n-1, k).$$

Identifying hypertrees with the corresponding block graphs and introducing the resulting Moebius function $\mu(T) = \prod_{j \geq 2} (-(j-2)!)^{\kappa_j}$ of a hypertree having κ_j hyperedges involving j vertices for $j \geq 2$ we have the following result:

Theorem 7.2. *We have*

$$(-1)^n \sum_{T \in \mathcal{T}(n, k)} \mu(T) = n^{k-1} S_1(n-1, k)$$

where $\mathcal{T}(n, k)$ denotes the set of all labelled hypertrees with n vertices and k hyperedges and where $S_1(n, k)$ denotes the Stirling number of the first kind

defined by

$$\sum_{k=0}^n S_1(n, k)x^k = x(x-1)(x-2)\cdots(x-n+1) = \prod_{j=0}^{n-1} (x-j) .$$

Proof Each hypertree T with vertices $1, \dots, n$ consisting of k hyperedges yields a contribution of $\mu(T) \prod_{i=1}^n s_i^{\deg_h(i)}$ (with $\deg_h(i)$ denoting the number of hyperedges containing the vertex i) to the coefficient $(-1)^k t_{n, k-1}(n-1) = -(-1)^n S_1(n-1, k)$ of the monomial $\sigma_n \sigma_1^{k-1} x^{n-1}$ in U and all contributions are of this form. Setting $s_1 = s_2 = \dots = s_n = 1$ yields the result. \square

8 Computational aspects and examples

The computation of $U(x, \sigma_1, \sigma_2, \dots)$ up to $o(x^n)$ is straightforward using the recurrence relation (3). For a given fixed numerical value of σ_1 , the following trick reduces memory requirement and speeds the computation up: Setting

$$c_n(\sigma_1) = (\gamma_{n+1}(\sigma_1, n), \gamma_{n+2}(\sigma_1, n), \dots, \gamma_{2n}(\sigma_1, n))$$

with $\gamma_i(\sigma_1, n) = \sum_{j=0}^{n-1} t_{i,j}(n)(-\sigma_1)^j$ we have

$$U(x, \sigma_1, \sigma_2, \dots) = 1 - \sum_{n=1}^{\infty} \langle c_n(\sigma_1), (\sigma_{n+1}, \dots, \sigma_{2n}) \rangle x^n$$

where $\langle a, b \rangle = \sum_{i=1}^n a_i b_i$ for $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. The coefficients $\gamma_i(\sigma_1, n)$ of $c_n(\sigma_1)$ can be computed from the coefficients of $c_{n-1}(\sigma_1)$ by the formula

$$\gamma_i(\sigma_1, n) = (i-2-\sigma_1)\gamma_{i-1}(\sigma_1, n-1) + (i-3)\gamma_{i-2}(\sigma_1, n-1)$$

with missing coefficients omitted in the case of $i = n+1$ or $i = 2n$.

The coefficients of the first vectors $c_1(0), c_2(0), c_3(0), \dots$ are given by the rows of

$$\begin{array}{cccccc} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 5 & 3 & & & \\ 6 & 26 & 35 & 15 & & \\ 24 & 154 & 340 & 315 & 105 & , \end{array}$$

see A112486 of [3].

8.1 The examples $U(x, -1, -1, -1, \dots)$ and $U(x, 0, -1, -1, -1, \dots)$

The series

$$U(x, -1, -1, -1, -1, \dots) - 1$$

is the generating series of the sequence

$$S(n) = \sum_{i,j} t_{i,j}(n)$$

enumerating the sums of the triangles $T(n)$ defined by the integers $t_{i,j}(n)$. We have

$$\begin{aligned} & (1+x)U(x, -1, -1, -1, -1, \dots) \\ &= U(x, 0, -2, -2, -2, -2, \dots) \\ &= 2U(x, 0, -1, -1, -1, -1, \dots) - 1 \end{aligned}$$

where $U(x, 0, -1, -1, -1, \dots) - 1$ corresponds to the generating series of the sequence

$$s(n) = \sum_{i=n+1}^{2n} t_{i,0}(n)$$

starting as

$$1, 2, 10, 82, 938, 13778, 247210, 5240338, 128149802, 3551246162, \dots,$$

cf. A112487 of [3], and obtained by summing the integers of the first column of the triangles $T(1), T(2), \dots$. In particular, we have $2s(n) = S(n-1) + S(n)$ or equivalently

$$2 \sum_{i=n+1}^{2n} t_{i,0}(n) = \sum_{i=n+1}^{2n} \sum_{j=0}^{2n-i} t_{i,j}(n) + \sum_{i=n}^{2n-2} \sum_{j=0}^{2n-2-i} t_{i,j}(n-1)$$

for all $n \geq 2$.

8.2 A family of rational examples

Proposition 8.1. *Let $\sigma_1, \sigma_2, \dots$ be a sequence of complex numbers of the form $\sigma_n = (-1)^n P(n)$ for all $n \geq A$ where A is some natural integer and where $P(s) \in \mathbb{C}[s]$ is a polynomial. Then $U(x, \sigma_1, \sigma_2, \dots)$ is a rational series.*

Proof Let d denote the degree of P . Applying identity (6) of Proposition 2.2 iteratively $d+1$ times we get a series of the form $U(x, \tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{A+d+2}, 0, 0, 0, \dots)$ which is a polynomial. \square

As an illustration we consider the series $U(x, y, 1, -1, 1, \dots)$. Proposition 2.2 shows

$$(1-xy)U(x, y, 1, -1, 1, -1, \dots) = (U(1+y, 1+y, 0, 0, \dots)) = 1 - (1+y)x.$$

We have thus $U(x, y, 1, -1, 1, \dots) = 1 - \frac{x}{1-xy}$.

8.3 Coefficients of $U(x, \sigma_1, P(2), P(3), P(4), \dots)$

Proposition 8.2. *Let $P(s) \in \mathbb{C}[s]$ be a polynomial of degree d . There exist constants $\alpha_0, \dots, \alpha_d \in \mathbb{C}$ such that*

$$[x^n]U(x, \sigma_1, P(2), P(3), P(4), \dots) = \sum_{h=0}^d \alpha_h [x^{n+h}]U(x, \sigma_1, 1, 1, 1, 1, \dots)$$

for all $n \geq 1$ with $[x^n]U$ denoting the coefficient of x^n in the series U .

Proof The proof is by induction on d and holds certainly for $d = 0$. Setting $\gamma_i(n) = \sum_{j=n+1}^{2n} t_{i,j}(n)(-\sigma_1)^j$ we have

$$\begin{aligned} 0 &= -i^d \gamma_i(n+1) + i^d(i-2-\sigma_1)\gamma_{i-1}(n) + i^d(i-3)\gamma_{i-2}(n) \\ &= -i^d \gamma_i(n+1) + (i-1)^{d+1} \gamma_{i-1}(n) + (i-2)^{d-2} \gamma_{i-2}(n) + \\ &\quad + Q_1(i-1)\gamma_{i-1}(n) + Q_2(i-2)\gamma_{i-2}(n) \end{aligned}$$

where Q_1 and Q_2 are polynomials of degree $\leq d$.

Summing over i (for a fixed integer n) and using induction on d implies the result. \square

9 Modular properties

Proposition 9.1. *The series $U(x, \sigma_1, \sigma_2, \dots) \in \mathbb{F}_p[[x]]$ is rational if $\sigma_1, \sigma_2, \dots$ is an ultimately periodic sequence of elements in \mathbb{F}_p .*

Proof Since we work over \mathbb{F}_p , one can restrict the indices i, j of the coefficients $t_{i,j}(n)$ to finite subsets. This implies that the coefficients of $U(x, \sigma_1, \dots)$ are ultimately periodic. \square

The easiest non-trivial case of Proposition 9.1 is perhaps given by the generating series $U(x, 0, -1, -1, -1, \dots) - 1$ associated to the sequence

$$s(n) = \sum_{i=n+1}^{2n} t_{i,0}(n) = 1 - U(x, 0, -1, -1, -1, \dots)$$

obtained by summing all coefficients in the first column of the triangular arrays $T(1), T(2), \dots$.

Experimentally, there exists seemingly a sequence

$$\begin{aligned} \alpha_0 &= -1, \alpha_1 = 2, \alpha_2 = 0, \alpha_3 = \frac{1}{3}, \alpha_4 = \frac{5}{18}, \alpha_5 = \frac{149}{540}, \alpha_6 = \frac{553}{2025}, \\ \alpha_7 &= \frac{1849741}{6804000}, \alpha_8 = \frac{775167119}{2857680000}, \alpha_9 = \frac{325214957371}{1200225600000}, \dots \end{aligned}$$

of rational numbers such that

$$(1 + x^{p-1}) \sum_{n=1}^{\infty} s(n)x^n \equiv \sum_{n=0}^{p-1} \alpha_n x^{p-n} \pmod{p}$$

for every prime number p .

The rational sequence $\alpha_0, \alpha_1, \dots$ has experimentally an asymptotic expansion given by

$$\alpha_n \sim \sum_{k=1}^{\infty} \frac{k^{k-n}}{k!} \left(\frac{2}{e^2}\right)^k .$$

In particular, it seems to converge to

$$\frac{2}{e^2} = .27067056647322538378799 \dots .$$

The generating series $\sum_{n=0}^{\infty} \alpha_n x^n$ seems to have a positive real root approximately given by

$$.469988171695013162992878063240573355384683977952459810170161 .$$

The error term

$$\epsilon_n = \alpha_n - \sum_{k=1}^{\infty} \frac{k^{k-n}}{k!} \left(\frac{2}{e^2}\right)^k$$

seems to be of the form

$$\epsilon_n \sim \frac{(-1)^{n+1}}{s(n+1)} C(n)$$

with $C(n) = \gamma_0 - \frac{1}{12n^2} + \frac{\gamma_4}{n^4} + \frac{\gamma_6}{n^6} + \dots \in \mathbb{R}[\frac{1}{n^2}]$ where

$$\begin{aligned} \gamma_0 &\sim 3.25889135327093 \\ \gamma_4 &\sim -.0120865999417078 \\ \gamma_6 &\sim .0312295100177430 \end{aligned}$$

More (probably correct) digits for γ_0 are given by

$$3.25889135327092945459791735692 .$$

Remark 9.2. *The constant γ_0 appearing in the error term $C(n)$ seems also to be related to the maximal index m_n such that $t_{m_n,0}(n) = \max_i(t_{i,0}(n))$ with m_n given asymptotically by $\frac{\gamma_0}{2}$ and it seems also to be involved in the asymptotics of $s(n)$, given experimentally by*

$$s(n) \sim \left(\frac{n\gamma_0}{e}\right)^n \sum_{k \geq 1} \tilde{\gamma}_k n^{-k}$$

where $\gamma_0 \sim 3.258891 \dots$ is as above and where

$$\begin{aligned} \tilde{\gamma}_1 &\sim .553942974899091 \\ \tilde{\gamma}_2 &\sim .239725616524017 \\ \tilde{\gamma}_3 &\sim .099867601803607 . \end{aligned}$$

References

- [1] I.M. Gessel, L.H. Kalikow, *Hypergraphs and a functional equation of Bouwkamp and de Bruijn*. J. Combin. Theory Ser. A 110 (2005), no. 2, 275–289.
- [2] K. Husimi, *Note on Mayer's theory of cluster integrals*, Journal of Chemical Physics **18** (1950), 682–684.
- [3] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at <http://oeis.org>.
- [4] R. P. Stanley, *Enumerative Combinatorics, Volume I*, Cambridge University Press (1997).

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