



HAL
open science

$C^{1,\alpha}$ regularity of the blow-up curve at non characteristic points for the one dimensional semilinear wave equation.

Nejla Nouaili

► To cite this version:

Nejla Nouaili. $C^{1,\alpha}$ regularity of the blow-up curve at non characteristic points for the one dimensional semilinear wave equation.. Communications in Partial Differential Equations, 2008, 33 (8), pp.1540-1548. hal-00531312

HAL Id: hal-00531312

<https://hal.science/hal-00531312>

Submitted on 2 Nov 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

$\mathcal{C}^{1,\alpha}$ regularity of the blow-up curve at non characteristic points for the one dimensional semilinear wave equation.

Nejla Nouaili

Ecole Normale Supérieure and Ecole Polytechnique de Tunisie.

February 19, 2008

Abstract

We consider $u(x, t)$ a blow-up solution of $\partial_{tt}^2 u = \partial_{xx}^2 u + |u|^{p-1}u$, that blows-up in one space dimension. We consider its blow-up curve $x \rightarrow T(x)$ and \mathcal{R} the set of non characteristic points. We prove that $T(x)$ is of class \mathcal{C}^{1,μ_0} in \mathcal{R} .

Mathematical Subject classification: 35L05, 35L67, 35A20.

Keywords: wave equation, regularity, blow-up set.

1 Introduction

We consider the one dimensional semilinear wave equation

$$\begin{cases} \partial_{tt}^2 u &= \partial_{xx}^2 u + |u|^{p-1}u, \\ u(0) &= u_0 \text{ and } u_t(0) = u_1, \end{cases} \quad (1)$$

where $u(t) : x \in \mathbb{R} \rightarrow u(x, t) \in \mathbb{R}$, $u_0 \in H_{loc,u}^1$ and $u_1 \in L_{loc,u}^2$ with

$$\|v\|_{L_{loc,u}^2}^2 = \sup_{a \in \mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx \text{ and } \|v\|_{H_{loc,u}^1}^2 = \|v\|_{L_{loc,u}^2}^2 + \|\nabla v\|_{L_{loc,u}^2}^2.$$

The Cauchy problem for equation (1) in the space $H_{loc,u}^1 \times L_{loc,u}^2$ follows from the finite speed of propagation and the wellposedness in $H^1 \times L^2$ (see Ginibre, Soffer and Velo [GSV92]). The existence of blow-up solutions follows from energy techniques (see Levine [Lev74]). More blow-up results can be found in Caffarelli and Friedman [CF85], [CF86], Alinhac [Ali95], [Ali02] and Kichenassamy and Littman [KL93b], [KL93a].

Consider u is a blow-up solution of (1). We define a continuous curve Γ as the graph of a function $x \rightarrow T(x)$ such that u cannot be extended beyond the set

$$D_u = \{(x, t) | t < T(x)\}. \quad (2)$$

The set D_u is called the maximal influence domain of u . From the finite speed of propagation, T is a 1-Lipschitz function (See Alinhac [Ali95].) Let \bar{T} be the infimum of $T(x)$

for all $x \in \mathbb{R}$. The time \bar{T} and the surface Γ are called (respectively) the blow-up time and the blow-up surface of u . A point $x_0 \in \mathbb{R}$ is called a non characteristic point if

$$\exists \delta_0 = \delta_0(x_0) \in (0, 1) \text{ and } t_0(x_0) < T(x_0) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0, T(x_0), \delta_0} \cap \{t \geq t_0\} \quad (3)$$

where

$$\mathcal{C}_{\bar{x}, \bar{t}, \bar{\delta}} = \{(x, t) | t < \bar{t} - \bar{\delta}|x - \bar{x}|\}. \quad (4)$$

We denote by \mathcal{R} , the set of non characteristic point.

Given $x_0 \in \mathbb{R}$, we introduce the following self-similar change of variables:

$$w_{x_0}(y, s) = (T(x_0) - t)^{\frac{2}{p-1}} u(x, t), \quad y = \frac{x - x_0}{T(x_0) - t}, \quad s = -\log(T(x_0) - t). \quad (5)$$

This change of variables transforms the backward light cone with vertex $(x_0, T(x_0))$ into the infinite cylinder $(y, s) \in B \times [-\log(T(x_0)), +\infty)$, where $B = B(0, 1)$. The function w_{x_0} (we write w for simplicity) satisfies the following equation for all $y \in B$ and $s \geq -\log(T(x_0))$:

$$\partial_{ss}^2 w = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial_{y,s}^2 w \quad (6)$$

$$\text{where } \mathcal{L}w = \frac{1}{\rho} \partial_y (\rho(1-y^2) \partial_y w) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \quad (7)$$

The Lyapunov functional for equation (6):

$$E(w(s)) = \int_{-1}^1 \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy \quad (8)$$

is defined in

$$\mathcal{H} = \left\{ q \in H_{loc}^1 \times L_{loc}^2(-1, 1) \mid \|q\|_{\mathcal{H}}^2 \equiv \int_{-1}^1 \left(q_1^2 + (q_1')^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}. \quad (9)$$

In [MZ07], Merle and Zaag find the behavior of $w_{x_0}(y, s)$ defined in (5) as $s \rightarrow \infty$ where x_0 is a non characteristic point. In [MZ08], they prove the \mathcal{C}^1 regularity of the blow-up set and the continuity of blow-up profile on \mathcal{R} . More precisely they prove this results (for i) see Theorem 1 from [MZ08], for ii) see Corollary 4 and Theorem 2 in [MZ07]):

Blow-up profile near a non characteristic point.

(i) \mathcal{R} is open and T is of class \mathcal{C}^1 on \mathcal{R} .

(ii) There exist positive $\mu_0 = \mu_0(p)$ and $C_0 = C_0(p)$ such that if $x_0 \in \mathcal{R}$, then there exists $d(x_0) \in (-1, 1)$, $|\theta(x_0)| = 1$, $s_0(x_0) \geq -\log(T(x_0))$ such that for all $s \geq s_0(x_0)$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta \begin{pmatrix} \kappa(d(x_0), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_0(x_0))}, \quad (10)$$

with $d(x_0) = T'(x_0)$ and for all $|d| < 1$ and $|y| \leq 1$

$$\kappa(d, y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ with } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}}. \quad (11)$$

Remark: The technique of [MZ07] and [MZ08] directly yield the fact that the convergence in (10) is locally uniform with respect to x_0 . This fact is crucial in our argument. We clearly state it in Lemma 2.2 below.

Remark: Caffarelli and Friedman proved that it is a \mathcal{C}^1 function for $N \leq 3$ under restrictive conditions on initial data that ensure that for all $x \in \mathbb{R}^N$ and $t \geq 0$, $u \geq 0$ and $\partial_t u \geq (1 + \delta_0)|\nabla u|$ for some $\delta_0 > 0$. In [CF85], they derived the same result in one dimension for $p \geq 3$ and initial data in $\mathcal{C}^4 \times \mathcal{C}^3(\mathbb{R})$.

From [MZ08], we felt that the particular value of μ_0 was not used in the proof, and that only the \mathcal{C}^1 regularity of $T(x)$ on \mathcal{R} was derived there.

In this paper, we aim at improving the regularity of $T(x)$ on \mathcal{R} . Our idea is related to the work of Zaag [Zaa02a], [Zaa02b] and [Zaa06] on the regularity of the blow-up set for the following semilinear heat equation:

$$\partial_t u = \Delta u + |u|^{p-1}u, \tag{12}$$

where $u(x, t) \in \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R}$, $p > 1$ and $(N - 2)p < N + 2$.

Unlike the wave equation (1), the blow-up time for (12) is unique and the set of blow-up points S_u is a subset of \mathbb{R}^N . In [Zaa02a], [Zaa02b] and [Zaa06], the author uses the idea that a better asymptotic description of the solution near blow-up points, yields geometric constraints on S_u resulting in more regularity for S_u .

In this paper, we adapt this idea to the context of the wave equation to improve the regularity of $T(x)$. More precisely, we claim the following:

Theorem 1 *Consider u a solution of (1) and $x \rightarrow T(x)$ its blow-up curve of class \mathcal{C}^1 . Then $T(x)$ is of class \mathcal{C}^{1, μ_0} in \mathcal{R} , the set of non characteristic points.*

Remark: We reasonably think that $\mu_0 \leq 1$. This comes from two different facts:

- the largest nonpositive eigenvalue of the linearized operator of the first order version of (6) around $(\kappa(d, 0), 0)$ is -1 (see page 80 in [MZ07]).
- having $\mu_0 > 1$ would imply that the blow-up set is always a flat line on the open set \mathcal{R} . This cannot occur because we know from Kichenassamy and Littman, [KL93b] and [KL93a] that any analytic, space like hypersurface can be a blow-up set of the equation

$$\partial_{tt}^2 u = \Delta u + e^u,$$

and we think that the same result should be true for equation (1), at least for some exponents.

The paper has only one section devoted to the proof of Theorem 1.

2 Refined regularity derived from asymptotic blow-up behavior

We divide this section in two parts. In Part 1, we write a crucial argument of [MZ08] which shows that the convergence in (10) is uniform with respect to x_0 in a neighborhood of a given non characteristic point. In Part 2 we give the proof of Theorem 1.

Part 1: Uniform character of the convergence in (10).

Given some $X \in \mathcal{R}$, we show here that for some $\delta > 0$, $\sup_{|x_0 - X| < \delta} s_0(x_0)$ is bounded, which gives the local uniform convergence in (10). This uniform character comes from two fundamental facts proved in [MZ07] and [MZ08] for equation (6):

- The following Liouville theorem for equation (6) proved in [MZ08]:

Proposition 2.1 (*Theorem 2' in [MZ08], A Liouville theorem for equation (6)*) Consider $w(y, s)$ a solution to equation (6) defined for all $(y, s) \in (-\frac{1}{\delta^*}, \frac{1}{\delta^*}) \times \mathbb{R}$ such that for all $s \in \mathbb{R}$,

$$\|w(s)\|_{H^1(-\frac{1}{\delta^*}, \frac{1}{\delta^*})} + \|\partial_s w(s)\|_{L^2(-\frac{1}{\delta^*}, \frac{1}{\delta^*})} \leq C^*,$$

for some $\delta^* \in (0, 1)$ and $C^* > 0$. Then, either $w \equiv 0$ or there exists $T_0 \geq 0$, $d_0 \in [-\delta^*, \delta^*]$ and $\theta_0 = \pm 1$ such that w can be extended to a function (still denoted by w) defined for all $(y, s) \in \{(y, s) \mid -1 - T_0 e^s < d_0 y\} \supset (-\frac{1}{\delta^*}, \frac{1}{\delta^*}) \times \mathbb{R}$ by

$$w(y, s) = \theta_0 \kappa_0 \frac{(1 - d_0^2)^{\frac{1}{p-1}}}{(1 + T_0 e^s + d_0 y)^{\frac{2}{p-1}}},$$

where κ_0 defined in (11).

The Liouville theorem allowed the authors in [MZ08] to show that the convergence of $(w_X, \partial_s w_X)$ to the profile (described by (10)) holds in a larger set, namely in $H^1 \times L^2(-\frac{1}{\delta_0}, \frac{1}{\delta_0})$ for some $\delta_0 > 0$. More precisely, we recall Lemma 2.2 from [MZ08].

Lemma 2.2 (*Convergence extension to a larger set*) For all $X \in \mathbb{R}$, there exists $\delta_0 > 0$ such that

$$\left\| \begin{pmatrix} w_X(s) \\ \partial_s w_X(s) \end{pmatrix} - \theta(X) \begin{pmatrix} \kappa(d(X), \cdot) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-\frac{1}{\delta_0}, \frac{1}{\delta_0})} \rightarrow 0 \text{ as } s \rightarrow \infty.$$

- The following trapping result for solutions of equation (6):

Proposition 2.3 (*Theorem 3 in [MZ07], trapping near the set of non-zero stationary solutions of (6)*) There exists positive ε_0 , μ_0 and C_0 such that if $w \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution of equation (6) such that

$$\forall s \geq s^*, E(w(s)) \geq E(\kappa_0),$$

and

$$\left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \theta \begin{pmatrix} \kappa(d^*(x), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \varepsilon^*,$$

for some $d^* \in (-1, 1)$, $\theta = \pm 1$ and $\varepsilon^* \in (0, \varepsilon_0]$, where \mathcal{H} and its norm are defined in (9) and $\kappa(d, y)$ in (11), then there exists $d_\infty \in (-1, 1)$ such that

$$|d_\infty - d^*| \leq C_0 \varepsilon^* (1 - d^{*2})$$

and for all $s \geq s^*$:

$$\left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \theta \begin{pmatrix} \kappa(d_\infty, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 \varepsilon^* e^{-\mu_0(s-s^*)}.$$

In the following, we give a new version of Lemma 2.6 from [MZ08] which allows that the convergence in (10) is locally uniform with respect to x_0 .

Lemma 2.4 (Locally uniform convergence to the blow-up profile) *There exist positive $\mu_0 = \mu_0(p)$ and $C_0 = C_0(p)$ such that for all $x_0 \in \mathcal{R}$, there exist $\delta > 0$, $s^* \in \mathbb{R}$, such that for all $X \in (x_0 - \delta, x_0 + \delta)$ and $s \geq s^*$,*

$$\left\| \begin{pmatrix} w_X(s) \\ \partial_s w_X(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(T'(X), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}, \quad (13)$$

Proof: Since x_0 is non characteristic, we have from Lemma 2.2,

$$\forall s \geq s_1, \|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(-\frac{1}{\delta'_0}, \frac{1}{\delta'_0})} \leq K \quad (14)$$

for some constant K , $s_1 \in \mathbb{R}$ and $\delta'_0 \in (\delta_0, 1)$ is fixed. Again from the fact that x_0 is a non characteristic point, we note that (10) holds, hence $(w_{x_0}(s), \partial_s w_{x_0}(s))$ converges to $\theta(x_0)(\kappa(d(x_0), \cdot), 0)$ as $s \rightarrow \infty$ in the norm of \mathcal{H} .

Since for a fixed $s \geq s_1$, we have $(w_X(y, s), \partial_s w_X(y, s)) \rightarrow (w_{x_0}(y, s), \partial_s w_{x_0}(y, s))$ in \mathcal{H} as $X \rightarrow x_0$, from (14) and the continuity of solutions to equation (6) with respect to initial data, we know that for all $\varepsilon > 0$, there exists $s_2(\varepsilon) \geq s_1$ and $\delta(\varepsilon)$ such that for all $X \in (x_0 - \delta, x_0 + \delta)$,

$$\left\| \begin{pmatrix} w_X(s_2) \\ \partial_s w_X(s_2) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(d(x_0), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \varepsilon.$$

Using the convergence (10) at the point X , the continuity and the monotonicity of the Lyapunov function (8), we see that

$$\forall s \geq s_2(\varepsilon), E(w_X(s)) \geq E(\kappa(T'(X), \cdot)).$$

Since for all $d \in (-1, 1)$, $E(\kappa(d, \cdot)) = E(\kappa_0)$ (see Proposition 1 from [MZ07]), we apply Proposition 2.3 to get for all $X \in (x_0 - \delta, x_0 + \delta)$, for all $s \geq s_2$,

$$\left\| \begin{pmatrix} w_X(s) \\ \partial_s w_X(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(T'(X), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_2)}.$$

This concludes the proof of Lemma 2.4. ■

Part 2: Proof of Theorem 1

This Part is devoted to the proof of Theorem 1. The starting point is Lemma 2.4 proved in [MZ07], which we translate back in the variables $u(x, t)$ in the following:

Corollary 2.5 (Corollary of Lemma 2.4) *There exists μ_0 and C_0 such that for all $x_0 \in \mathcal{R}$, there exist $\delta > 0$, $0 < t^*(x_0) < \inf_{|X-x_0| \leq \delta} T(X)$, such that for all $X \in (x_0 - \delta, x_0 + \delta)$ and $t \in [t^*(x_0), T(X))$,*

$$\sup_{|\xi - X| \leq \frac{3}{4}(T(X) - t)} \left| u(\xi, t) - \theta(x_0)\kappa_0 \frac{(1 - T'(X)^2)^{\frac{1}{p-1}}}{(T(X) - t + T'(X)(\xi - X))^{\frac{2}{p-1}}} \right| \leq C(T(X) - t)^{\mu_0 - \frac{2}{p-1}}. \quad (15)$$

Proof of Theorem 1: We fix $x_0 \in \mathcal{R}$ and consider an arbitrary $\sigma \geq \frac{4}{3}$. For any $x \in (x_0 - \delta, x_0 + \delta)$, where $\delta > 0$, we define $t = t(x, \sigma)$ by:

$$\frac{|x_0 - x|}{T(x_0) - t} = \frac{1}{\sigma}. \quad (16)$$

Note that (x, t) is on the edge of the backward cone with vertex $(x_0, T(x_0))$ and slope σ . We also note that

$$t \rightarrow T(x_0) \text{ as } x \rightarrow x_0. \quad (17)$$

This is our idea: using (15), we get 2 different main terms for $u(x, t)$, depending on the choice of X and ξ . These two terms have to agree up to error terms, which yields constraints on $T(x)$ and $T'(x)$ leading to more regularity.

Since from (17), we have $t = t(x, \delta) \geq t^*(x_0)$ defined in Corollary 2.5 for x close enough to x_0 , we are able to apply this corollary, first with $X = \xi = x$, and then with $X = x_0$ and $\xi = x$.

Using (15) with $X = \xi = x$, we get on the one hand:

$$\left| u(x, t) - \theta\kappa_0 \frac{(1 - T'(x)^2)^{\frac{1}{p-1}}}{(T(x) - t)^{\frac{2}{p-1}}} \right| \leq C(T(x) - t)^{\mu_0 - \frac{2}{p-1}}. \quad (18)$$

On the other hand, using (15) with $X = x_0$ and $\xi = x$, we get

$$\left| u(x, t) - \theta\kappa_0 \frac{(1 - T'(x_0)^2)^{\frac{1}{p-1}}}{(T(x_0) - t + T'(x_0)(x - x_0))^{\frac{2}{p-1}}} \right| \leq C(T(x_0) - t)^{\mu_0 - \frac{2}{p-1}}. \quad (19)$$

Since we have from (16) $t = T(x_0) - \sigma|x_0 - x|$, hence

$$T(x_0) - t = \sigma|x - x_0| \text{ and } (\sigma - 1)|x - x_0| \leq T(x) - t \leq (\sigma + 1)|x - x_0|,$$

we derive from (18) and (19),

$$\left| \frac{(1 - T'(x_0)^2)^{\frac{1}{p-1}}}{(T'(x_0)\text{sign}(x - x_0) + \sigma)^{\frac{2}{p-1}}} - \frac{(1 - T'(x)^2)^{\frac{1}{p-1}}}{\left(\frac{T(x) - T(x_0)}{|x - x_0|} + \sigma\right)^{\frac{2}{p-1}}} \right| \leq C|x_0 - x|^{\mu_0},$$

hence

$$\left| \frac{1 - T'(x_0)^2}{(T'(x_0)\text{sign}(x - x_0) + \sigma)^2} - \frac{1 - T'(x)^2}{\left(\frac{T(x) - T(x_0)}{|x - x_0|} + \sigma\right)^2} \right| \leq C|x_0 - x|^{\mu_0}, \quad (20)$$

where $\text{sign}(X) = \frac{X}{|X|}$ for $X \neq 0$.

Now, if we introduce:

$$f(\xi) = T(\xi + x_0) - T(x_0) - \xi T'(x_0) \text{ and } \lambda = T'(x_0) \quad (21)$$

then we have $f(0) = f'(0) = 0$ and (20) becomes:

$$\left| \frac{1 - \lambda^2}{(\lambda \text{sign}(\xi) + \sigma)^2} - \frac{1 - (f'(\xi) + \lambda)^2}{\left(\frac{f(\xi)}{|\xi|} + \lambda \text{sign}(\xi) + \sigma\right)^2} \right| \leq C|\xi|^{\mu_0}. \quad (22)$$

Therefore,

$$\left| (1 - \lambda^2) \left(\frac{f(\xi)}{|\xi|} + \lambda \text{sign}(\xi) + \sigma \right)^2 - (1 - (f'(\xi) + \lambda)^2) (\lambda \text{sign}(\xi) + \sigma)^2 \right| \leq C|\xi|^{\mu_0},$$

and

$$\left| \left(\frac{f(\xi)}{\xi} \right)^2 \frac{1 - \lambda^2}{2(\lambda \text{sign}(\xi) + \sigma)^2} + \frac{f(\xi)}{\xi} \frac{1 - \lambda^2}{\lambda \text{sign}(\xi) + \sigma} + f'(\xi)^2 + f'(\xi)2\lambda \right| \leq C|\xi|^{\mu_0}. \quad (\text{E}_\sigma)$$

Now, we consider two different values of σ , σ_1 and σ_2 and we take the difference between $(\text{E}_\sigma, \sigma = \sigma_1)$ and $(\text{E}_\sigma, \sigma = \sigma_2)$ to get:

$$\left| \left(\frac{f(\xi)}{\xi} \right)^2 \left[\frac{1 - \lambda^2}{2(\lambda \text{sign}(\xi) + \sigma_1)^2} - \frac{1 - \lambda^2}{2(\lambda \text{sign}(\xi) + \sigma_2)^2} \right] + \frac{f(\xi)}{\xi} \left[\frac{1 - \lambda^2}{\lambda \text{sign}(\xi) + \sigma_1} - \frac{1 - \lambda^2}{\lambda \text{sign}(\xi) + \sigma_2} \right] \right| \leq C|\xi|^{\mu_0}.$$

Since $\frac{f(\xi)}{\xi} \rightarrow f'(0) = 0$ as $\xi \rightarrow 0$, this yields

$$\left| \frac{f(\xi)}{\xi} \right| \leq C|\xi|^{\mu_0}. \quad (23)$$

Using this last estimate in equation (E_σ) , this gives the following:

$$|f'(\xi)^2 + f'(\xi)2\lambda| \leq C|\xi|^{\mu_0}. \quad (24)$$

At this level of the proof we distinguish two cases:

Case 1: $\lambda \neq 0$. Since $f'(\xi) \rightarrow f'(0) = 0$ as $\xi \rightarrow 0$, we get immediately from (24) $|f'(\xi)| \leq C|\xi|^{\mu_0}$. Using back the change of variables (21), we see that T is \mathcal{C}^{1, μ_0} near x_0 .

Case 2: $\lambda = 0$. We note first that by (24), we have:

$$|f'(\xi)| \leq C|\xi|^{\mu_0/2}. \quad (25)$$

Switching x and x_0 in (20), we get:

$$\left| \frac{1 - T'(x)^2}{(T'(x)\text{sign}(x_0 - x) + \sigma)^2} - \frac{1 - T'(x_0)^2}{\left(\frac{T(x_0) - T(x)}{|x - x_0|} + \sigma\right)^2} \right| \leq C|x_0 - x|^{\mu_0}.$$

Using the change of variable (21) and the same arguments used to obtain (E_σ) , we have after simplifications:

$$\left| (1 - f'(\xi)^2) \left(\sigma - \frac{f(\xi)}{|\xi|} \right)^2 - (\sigma - f'(\xi) \operatorname{sign}(\xi))^2 \right| \leq C|\xi|^{\mu_0}.$$

Using (23) and (25) in the last inequality, we obtain $|f'(\xi)| \leq C|\xi|^{\mu_0}$. Using again (21), we see that $T(x)$ is \mathcal{C}^{1,μ_0} near x_0 . This concludes the proof of Theorem 1. ■

References

- [Ali95] S. Alinhac. *Blowup for nonlinear hyperbolic equations*. Progress in Nonlinear Differential Equations and their Applications, 17. Birkhäuser Boston Inc., Boston, MA, 1995.
- [Ali02] S. Alinhac. A minicourse on global existence and blowup of classical solutions to multidimensional quasilinear wave equations. In *Journées “Équations aux Dérivées Partielles” (Forges-les-Eaux, 2002)*, pages Exp. No. I, 33. Univ. Nantes, Nantes, 2002.
- [CF85] L.A. Caffarelli and A. Friedman. Differentiability of the blow-up curve for one-dimensional nonlinear wave equations. *Arch. Rational Mech. Anal.*, 91(1):83–98, 1985.
- [CF86] L.A. Caffarelli and A. Friedman. The blow-up boundary for nonlinear wave equations. *Trans. Amer. Math. Soc.*, 297(1):223–241, 1986.
- [GSV92] J. Ginibre, A. Soffer, and G. Velo. The global Cauchy problem for the critical nonlinear wave equation. *J. Funct. Anal.*, 110(1):96–130, 1992.
- [KL93a] S. Kichenassamy and W. Littman. Blow-up surfaces for nonlinear wave equations. I. *Comm. Partial Differential Equations*, 18(3-4):431–452, 1993.
- [KL93b] S. Kichenassamy and W. Littman. Blow-up surfaces for nonlinear wave equations. II. *Comm. Partial Differential Equations*, 18(11):1869–1899, 1993.
- [Lev74] H.A. Levine. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. *Trans. Amer. Math. Soc.*, 192:1–21, 1974.
- [MZ07] F. Merle and H. Zaag. Existence and universality of the blow-up profile for the semilinear wave equation in one space dimension. *J. Funct. Anal.*, 253(1):43–121, 2007.
- [MZ08] F. Merle and H. Zaag. Openness of the set of non characteristic points and regularity of the blow-up curve for the 1 d semilinear wave equation. *Comm. Math. Phys.*, 2008. To appear.
- [Zaa02a] H. Zaag. On the regularity of the blow-up set for semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 19(5):505–542, 2002.

- [Zaa02b] H. Zaag. One-dimensional behavior of singular N -dimensional solutions of semilinear heat equations. *Comm. Math. Phys.*, 225(3):523–549, 2002.
- [Zaa06] H. Zaag. Determination of the curvature of the blow-up set and refined singular behavior for a semilinear heat equation. *Duke Math. J.*, 133(3):499–525, 2006.

Nejla Nouaili

Département de mathématiques et applications -École Normale supérieure
45 rue d'Ulm 75230 PARIS cedex 05
France
E-mail: Nejla.Nouaili@ens.fr