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# $\mathcal{C}^{1,\alpha}$ regularity of the blow-up curve at non characteristic points for the one dimensional semilinear wave equation.

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## Abstract

We consider  $u(x, t)$  a blow-up solution of  $\partial_{tt}^2 u = \partial_{xx}^2 u + |u|^{p-1}u$ , that blows-up in one space dimension. We consider its blow-up curve  $x \rightarrow T(x)$  and  $\mathcal{R}$  the set of non characteristic points. We prove that  $T(x)$  is of class  $\mathcal{C}^{1,\mu_0}$  in  $\mathcal{R}$ .

**Mathematical Subject classification:** 35L05, 35L67, 35A20.

**Keywords:** wave equation, regularity, blow-up set.

## 1 Introduction

We consider the one dimensional semilinear wave equation

$$\begin{cases} \partial_{tt}^2 u &= \partial_{xx}^2 u + |u|^{p-1}u, \\ u(0) &= u_0 \text{ and } u_t(0) = u_1, \end{cases} \quad (1)$$

where  $u(t) : x \in \mathbb{R} \rightarrow u(x, t) \in \mathbb{R}$ ,  $u_0 \in H_{loc,u}^1$  and  $u_1 \in L_{loc,u}^2$  with

$$\|v\|_{L_{loc,u}^2}^2 = \sup_{a \in \mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx \text{ and } \|v\|_{H_{loc,u}^1}^2 = \|v\|_{L_{loc,u}^2}^2 + \|\nabla v\|_{L_{loc,u}^2}^2.$$

The Cauchy problem for equation (1) in the space  $H_{loc,u}^1 \times L_{loc,u}^2$  follows from the finite speed of propagation and the wellposedness in  $H^1 \times L^2$  (see Ginibre, Soffer and Velo [GSV92]). The existence of blow-up solutions follows from energy techniques (see Levine [Lev74]). More blow-up results can be found in Caffarelli and Friedman [CF85], [CF86], Alinhac [Ali95], [Ali02] and Kichenassamy and Littman [KL93b], [KL93a].

Consider  $u$  is a blow-up solution of (1). We define a continuous curve  $\Gamma$  as the graph of a function  $x \rightarrow T(x)$  such that  $u$  cannot be extended beyond the set

$$D_u = \{(x, t) | t < T(x)\}. \quad (2)$$

The set  $D_u$  is called the maximal influence domain of  $u$ . From the finite speed of propagation,  $T$  is a 1-Lipschitz function (See Alinhac [Ali95].) Let  $\bar{T}$  be the infimum of  $T(x)$

for all  $x \in \mathbb{R}$ . The time  $\bar{T}$  and the surface  $\Gamma$  are called (respectively) the blow-up time and the blow-up surface of  $u$ . A point  $x_0 \in \mathbb{R}$  is called a non characteristic point if

$$\exists \delta_0 = \delta_0(x_0) \in (0, 1) \text{ and } t_0(x_0) < T(x_0) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0, T(x_0), \delta_0} \cap \{t \geq t_0\} \quad (3)$$

where

$$\mathcal{C}_{\bar{x}, \bar{t}, \bar{\delta}} = \{(x, t) | t < \bar{t} - \bar{\delta}|x - \bar{x}|\}. \quad (4)$$

We denote by  $\mathcal{R}$ , the set of non characteristic point.

Given  $x_0 \in \mathbb{R}$ , we introduce the following self-similar change of variables:

$$w_{x_0}(y, s) = (T(x_0) - t)^{\frac{2}{p-1}} u(x, t), \quad y = \frac{x - x_0}{T(x_0) - t}, \quad s = -\log(T(x_0) - t). \quad (5)$$

This change of variables transforms the backward light cone with vertex  $(x_0, T(x_0))$  into the infinite cylinder  $(y, s) \in B \times [-\log(T(x_0)), +\infty)$ , where  $B = B(0, 1)$ . The function  $w_{x_0}$  (we write  $w$  for simplicity) satisfies the following equation for all  $y \in B$  and  $s \geq -\log(T(x_0))$ :

$$\partial_{ss}^2 w = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial_{y,s}^2 w \quad (6)$$

$$\text{where } \mathcal{L}w = \frac{1}{\rho} \partial_y (\rho(1-y^2) \partial_y w) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \quad (7)$$

The Lyapunov functional for equation (6):

$$E(w(s)) = \int_{-1}^1 \left( \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy \quad (8)$$

is defined in

$$\mathcal{H} = \left\{ q \in H_{loc}^1 \times L_{loc}^2(-1, 1) \mid \|q\|_{\mathcal{H}}^2 \equiv \int_{-1}^1 \left( q_1^2 + (q_1')^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}. \quad (9)$$

In [MZ07], Merle and Zaag find the behavior of  $w_{x_0}(y, s)$  defined in (5) as  $s \rightarrow \infty$  where  $x_0$  is a non characteristic point. In [MZ08], they prove the  $\mathcal{C}^1$  regularity of the blow-up set and the continuity of blow-up profile on  $\mathcal{R}$ . More precisely they prove this results (for i) see Theorem 1 from [MZ08], for ii) see Corollary 4 and Theorem 2 in [MZ07]):

**Blow-up profile near a non characteristic point.**

(i)  $\mathcal{R}$  is open and  $T$  is of class  $\mathcal{C}^1$  on  $\mathcal{R}$ .

(ii) There exist positive  $\mu_0 = \mu_0(p)$  and  $C_0 = C_0(p)$  such that if  $x_0 \in \mathcal{R}$ , then there exists  $d(x_0) \in (-1, 1)$ ,  $|\theta(x_0)| = 1$ ,  $s_0(x_0) \geq -\log(T(x_0))$  such that for all  $s \geq s_0(x_0)$ ,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta \begin{pmatrix} \kappa(d(x_0), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_0(x_0))}, \quad (10)$$

with  $d(x_0) = T'(x_0)$  and for all  $|d| < 1$  and  $|y| \leq 1$

$$\kappa(d, y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ with } \kappa_0 = \left( \frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}}. \quad (11)$$

**Remark:** The technique of [MZ07] and [MZ08] directly yield the fact that the convergence in (10) is locally uniform with respect to  $x_0$ . This fact is crucial in our argument. We clearly state it in Lemma 2.2 below.

**Remark:** Caffarelli and Friedman proved that it is a  $\mathcal{C}^1$  function for  $N \leq 3$  under restrictive conditions on initial data that ensure that for all  $x \in \mathbb{R}^N$  and  $t \geq 0$ ,  $u \geq 0$  and  $\partial_t u \geq (1 + \delta_0)|\nabla u|$  for some  $\delta_0 > 0$ . In [CF85], they derived the same result in one dimension for  $p \geq 3$  and initial data in  $\mathcal{C}^4 \times \mathcal{C}^3(\mathbb{R})$ .

From [MZ08], we felt that the particular value of  $\mu_0$  was not used in the proof, and that only the  $\mathcal{C}^1$  regularity of  $T(x)$  on  $\mathcal{R}$  was derived there.

In this paper, we aim at improving the regularity of  $T(x)$  on  $\mathcal{R}$ . Our idea is related to the work of Zaag [Zaa02a], [Zaa02b] and [Zaa06] on the regularity of the blow-up set for the following semilinear heat equation:

$$\partial_t u = \Delta u + |u|^{p-1}u, \tag{12}$$

where  $u(x, t) \in \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R}$ ,  $p > 1$  and  $(N - 2)p < N + 2$ .

Unlike the wave equation (1), the blow-up time for (12) is unique and the set of blow-up points  $S_u$  is a subset of  $\mathbb{R}^N$ . In [Zaa02a], [Zaa02b] and [Zaa06], the author uses the idea that a better asymptotic description of the solution near blow-up points, yields geometric constraints on  $S_u$  resulting in more regularity for  $S_u$ .

In this paper, we adapt this idea to the context of the wave equation to improve the regularity of  $T(x)$ . More precisely, we claim the following:

**Theorem 1** *Consider  $u$  a solution of (1) and  $x \rightarrow T(x)$  its blow-up curve of class  $\mathcal{C}^1$ . Then  $T(x)$  is of class  $\mathcal{C}^{1, \mu_0}$  in  $\mathcal{R}$ , the set of non characteristic points.*

**Remark:** We reasonably think that  $\mu_0 \leq 1$ . This comes from two different facts:

- the largest nonpositive eigenvalue of the linearized operator of the first order version of (6) around  $(\kappa(d, 0), 0)$  is  $-1$  (see page 80 in [MZ07]).
- having  $\mu_0 > 1$  would imply that the blow-up set is always a flat line on the open set  $\mathcal{R}$ . This cannot occur because we know from Kichenassamy and Littman, [KL93b] and [KL93a] that any analytic, space like hypersurface can be a blow-up set of the equation

$$\partial_{tt}^2 u = \Delta u + e^u,$$

and we think that the same result should be true for equation (1), at least for some exponents.

The paper has only one section devoted to the proof of Theorem 1.

## 2 Refined regularity derived from asymptotic blow-up behavior

We divide this section in two parts. In Part 1, we write a crucial argument of [MZ08] which shows that the convergence in (10) is uniform with respect to  $x_0$  in a neighborhood of a given non characteristic point. In Part 2 we give the proof of Theorem 1.

**Part 1: Uniform character of the convergence in (10).**

Given some  $X \in \mathcal{R}$ , we show here that for some  $\delta > 0$ ,  $\sup_{|x_0 - X| < \delta} s_0(x_0)$  is bounded, which gives the local uniform convergence in (10). This uniform character comes from two fundamental facts proved in [MZ07] and [MZ08] for equation (6):

- The following Liouville theorem for equation (6) proved in [MZ08]:

**Proposition 2.1** (*Theorem 2' in [MZ08], A Liouville theorem for equation (6)*) Consider  $w(y, s)$  a solution to equation (6) defined for all  $(y, s) \in (-\frac{1}{\delta^*}, \frac{1}{\delta^*}) \times \mathbb{R}$  such that for all  $s \in \mathbb{R}$ ,

$$\|w(s)\|_{H^1(-\frac{1}{\delta^*}, \frac{1}{\delta^*})} + \|\partial_s w(s)\|_{L^2(-\frac{1}{\delta^*}, \frac{1}{\delta^*})} \leq C^*,$$

for some  $\delta^* \in (0, 1)$  and  $C^* > 0$ . Then, either  $w \equiv 0$  or there exists  $T_0 \geq 0$ ,  $d_0 \in [-\delta^*, \delta^*]$  and  $\theta_0 = \pm 1$  such that  $w$  can be extended to a function (still denoted by  $w$ ) defined for all  $(y, s) \in \{(y, s) \mid -1 - T_0 e^s < d_0 y\} \supset (-\frac{1}{\delta^*}, \frac{1}{\delta^*}) \times \mathbb{R}$  by

$$w(y, s) = \theta_0 \kappa_0 \frac{(1 - d_0^2)^{\frac{1}{p-1}}}{(1 + T_0 e^s + d_0 y)^{\frac{2}{p-1}}},$$

where  $\kappa_0$  defined in (11).

The Liouville theorem allowed the authors in [MZ08] to show that the convergence of  $(w_X, \partial_s w_X)$  to the profile (described by (10)) holds in a larger set, namely in  $H^1 \times L^2(-\frac{1}{\delta_0}, \frac{1}{\delta_0})$  for some  $\delta_0 > 0$ . More precisely, we recall Lemma 2.2 from [MZ08].

**Lemma 2.2** (*Convergence extension to a larger set*) For all  $X \in \mathbb{R}$ , there exists  $\delta_0 > 0$  such that

$$\left\| \begin{pmatrix} w_X(s) \\ \partial_s w_X(s) \end{pmatrix} - \theta(X) \begin{pmatrix} \kappa(d(X), \cdot) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-\frac{1}{\delta_0}, \frac{1}{\delta_0})} \rightarrow 0 \text{ as } s \rightarrow \infty.$$

- The following trapping result for solutions of equation (6):

**Proposition 2.3** (*Theorem 3 in [MZ07], trapping near the set of non-zero stationary solutions of (6)*) There exists positive  $\varepsilon_0$ ,  $\mu_0$  and  $C_0$  such that if  $w \in C([s^*, \infty), \mathcal{H})$  for some  $s^* \in \mathbb{R}$  is a solution of equation (6) such that

$$\forall s \geq s^*, E(w(s)) \geq E(\kappa_0),$$

and

$$\left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \theta \begin{pmatrix} \kappa(d^*(x), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \varepsilon^*,$$

for some  $d^* \in (-1, 1)$ ,  $\theta = \pm 1$  and  $\varepsilon^* \in (0, \varepsilon_0]$ , where  $\mathcal{H}$  and its norm are defined in (9) and  $\kappa(d, y)$  in (11), then there exists  $d_\infty \in (-1, 1)$  such that

$$|d_\infty - d^*| \leq C_0 \varepsilon^* (1 - d^{*2})$$

and for all  $s \geq s^*$ :

$$\left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \theta \begin{pmatrix} \kappa(d_\infty, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 \varepsilon^* e^{-\mu_0(s-s^*)}.$$

In the following, we give a new version of Lemma 2.6 from [MZ08] which allows that the convergence in (10) is locally uniform with respect to  $x_0$ .

**Lemma 2.4 (Locally uniform convergence to the blow-up profile)** *There exist positive  $\mu_0 = \mu_0(p)$  and  $C_0 = C_0(p)$  such that for all  $x_0 \in \mathcal{R}$ , there exist  $\delta > 0$ ,  $s^* \in \mathbb{R}$ , such that for all  $X \in (x_0 - \delta, x_0 + \delta)$  and  $s \geq s^*$ ,*

$$\left\| \begin{pmatrix} w_X(s) \\ \partial_s w_X(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(T'(X), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}, \quad (13)$$

*Proof:* Since  $x_0$  is non characteristic, we have from Lemma 2.2,

$$\forall s \geq s_1, \|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(-\frac{1}{\delta'_0}, \frac{1}{\delta'_0})} \leq K \quad (14)$$

for some constant  $K$ ,  $s_1 \in \mathbb{R}$  and  $\delta'_0 \in (\delta_0, 1)$  is fixed. Again from the fact that  $x_0$  is a non characteristic point, we note that (10) holds, hence  $(w_{x_0}(s), \partial_s w_{x_0}(s))$  converges to  $\theta(x_0)(\kappa(d(x_0), \cdot), 0)$  as  $s \rightarrow \infty$  in the norm of  $\mathcal{H}$ .

Since for a fixed  $s \geq s_1$ , we have  $(w_X(y, s), \partial_s w_X(y, s)) \rightarrow (w_{x_0}(y, s), \partial_s w_{x_0}(y, s))$  in  $\mathcal{H}$  as  $X \rightarrow x_0$ , from (14) and the continuity of solutions to equation (6) with respect to initial data, we know that for all  $\varepsilon > 0$ , there exists  $s_2(\varepsilon) \geq s_1$  and  $\delta(\varepsilon)$  such that for all  $X \in (x_0 - \delta, x_0 + \delta)$ ,

$$\left\| \begin{pmatrix} w_X(s_2) \\ \partial_s w_X(s_2) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(d(x_0), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \varepsilon.$$

Using the convergence (10) at the point  $X$ , the continuity and the monotonicity of the Lyapunov function (8), we see that

$$\forall s \geq s_2(\varepsilon), E(w_X(s)) \geq E(\kappa(T'(X), \cdot)).$$

Since for all  $d \in (-1, 1)$ ,  $E(\kappa(d, \cdot)) = E(\kappa_0)$  (see Proposition 1 from [MZ07]), we apply Proposition 2.3 to get for all  $X \in (x_0 - \delta, x_0 + \delta)$ , for all  $s \geq s_2$ ,

$$\left\| \begin{pmatrix} w_X(s) \\ \partial_s w_X(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(T'(X), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_2)}.$$

This concludes the proof of Lemma 2.4. ■

## Part 2: Proof of Theorem 1

This Part is devoted to the proof of Theorem 1. The starting point is Lemma 2.4 proved in [MZ07], which we translate back in the variables  $u(x, t)$  in the following:

**Corollary 2.5** (*Corollary of Lemma 2.4*) *There exists  $\mu_0$  and  $C_0$  such that for all  $x_0 \in \mathcal{R}$ , there exist  $\delta > 0$ ,  $0 < t^*(x_0) < \inf_{|X-x_0| \leq \delta} T(X)$ , such that for all  $X \in (x_0 - \delta, x_0 + \delta)$  and  $t \in [t^*(x_0), T(X))$ ,*

$$\sup_{|\xi - X| \leq \frac{3}{4}(T(X) - t)} \left| u(\xi, t) - \theta(x_0)\kappa_0 \frac{(1 - T'(X)^2)^{\frac{1}{p-1}}}{(T(X) - t + T'(X)(\xi - X))^{\frac{2}{p-1}}} \right| \leq C(T(X) - t)^{\mu_0 - \frac{2}{p-1}}. \quad (15)$$

*Proof of Theorem 1:* We fix  $x_0 \in \mathcal{R}$  and consider an arbitrary  $\sigma \geq \frac{4}{3}$ . For any  $x \in (x_0 - \delta, x_0 + \delta)$ , where  $\delta > 0$ , we define  $t = t(x, \sigma)$  by:

$$\frac{|x_0 - x|}{T(x_0) - t} = \frac{1}{\sigma}. \quad (16)$$

Note that  $(x, t)$  is on the edge of the backward cone with vertex  $(x_0, T(x_0))$  and slope  $\sigma$ . We also note that

$$t \rightarrow T(x_0) \text{ as } x \rightarrow x_0. \quad (17)$$

This is our idea: using (15), we get 2 different main terms for  $u(x, t)$ , depending on the choice of  $X$  and  $\xi$ . These two terms have to agree up to error terms, which yields constraints on  $T(x)$  and  $T'(x)$  leading to more regularity.

Since from (17), we have  $t = t(x, \delta) \geq t^*(x_0)$  defined in Corollary 2.5 for  $x$  close enough to  $x_0$ , we are able to apply this corollary, first with  $X = \xi = x$ , and then with  $X = x_0$  and  $\xi = x$ .

Using (15) with  $X = \xi = x$ , we get on the one hand:

$$\left| u(x, t) - \theta\kappa_0 \frac{(1 - T'(x)^2)^{\frac{1}{p-1}}}{(T(x) - t)^{\frac{2}{p-1}}} \right| \leq C(T(x) - t)^{\mu_0 - \frac{2}{p-1}}. \quad (18)$$

On the other hand, using (15) with  $X = x_0$  and  $\xi = x$ , we get

$$\left| u(x, t) - \theta\kappa_0 \frac{(1 - T'(x_0)^2)^{\frac{1}{p-1}}}{(T(x_0) - t + T'(x_0)(x - x_0))^{\frac{2}{p-1}}} \right| \leq C(T(x_0) - t)^{\mu_0 - \frac{2}{p-1}}. \quad (19)$$

Since we have from (16)  $t = T(x_0) - \sigma|x_0 - x|$ , hence

$$T(x_0) - t = \sigma|x - x_0| \text{ and } (\sigma - 1)|x - x_0| \leq T(x) - t \leq (\sigma + 1)|x - x_0|,$$

we derive from (18) and (19),

$$\left| \frac{(1 - T'(x_0)^2)^{\frac{1}{p-1}}}{(T'(x_0)\text{sign}(x - x_0) + \sigma)^{\frac{2}{p-1}}} - \frac{(1 - T'(x)^2)^{\frac{1}{p-1}}}{\left(\frac{T(x) - T(x_0)}{|x - x_0|} + \sigma\right)^{\frac{2}{p-1}}} \right| \leq C|x_0 - x|^{\mu_0},$$

hence

$$\left| \frac{1 - T'(x_0)^2}{(T'(x_0)\text{sign}(x - x_0) + \sigma)^2} - \frac{1 - T'(x)^2}{\left(\frac{T(x) - T(x_0)}{|x - x_0|} + \sigma\right)^2} \right| \leq C|x_0 - x|^{\mu_0}, \quad (20)$$

where  $\text{sign}(X) = \frac{X}{|X|}$  for  $X \neq 0$ .

Now, if we introduce:

$$f(\xi) = T(\xi + x_0) - T(x_0) - \xi T'(x_0) \text{ and } \lambda = T'(x_0) \quad (21)$$

then we have  $f(0) = f'(0) = 0$  and (20) becomes:

$$\left| \frac{1 - \lambda^2}{(\lambda \text{sign}(\xi) + \sigma)^2} - \frac{1 - (f'(\xi) + \lambda)^2}{\left(\frac{f(\xi)}{|\xi|} + \lambda \text{sign}(\xi) + \sigma\right)^2} \right| \leq C|\xi|^{\mu_0}. \quad (22)$$

Therefore,

$$\left| (1 - \lambda^2) \left( \frac{f(\xi)}{|\xi|} + \lambda \text{sign}(\xi) + \sigma \right)^2 - (1 - (f'(\xi) + \lambda)^2) (\lambda \text{sign}(\xi) + \sigma)^2 \right| \leq C|\xi|^{\mu_0},$$

and

$$\left| \left( \frac{f(\xi)}{\xi} \right)^2 \frac{1 - \lambda^2}{2(\lambda \text{sign}(\xi) + \sigma)^2} + \frac{f(\xi)}{\xi} \frac{1 - \lambda^2}{\lambda \text{sign}(\xi) + \sigma} + f'(\xi)^2 + f'(\xi)2\lambda \right| \leq C|\xi|^{\mu_0}. \quad (\text{E}_\sigma)$$

Now, we consider two different values of  $\sigma$ ,  $\sigma_1$  and  $\sigma_2$  and we take the difference between  $(\text{E}_\sigma, \sigma = \sigma_1)$  and  $(\text{E}_\sigma, \sigma = \sigma_2)$  to get:

$$\left| \left( \frac{f(\xi)}{\xi} \right)^2 \left[ \frac{1 - \lambda^2}{2(\lambda \text{sign}(\xi) + \sigma_1)^2} - \frac{1 - \lambda^2}{2(\lambda \text{sign}(\xi) + \sigma_2)^2} \right] + \frac{f(\xi)}{\xi} \left[ \frac{1 - \lambda^2}{\lambda \text{sign}(\xi) + \sigma_1} - \frac{1 - \lambda^2}{\lambda \text{sign}(\xi) + \sigma_2} \right] \right| \leq C|\xi|^{\mu_0}.$$

Since  $\frac{f(\xi)}{\xi} \rightarrow f'(0) = 0$  as  $\xi \rightarrow 0$ , this yields

$$\left| \frac{f(\xi)}{\xi} \right| \leq C|\xi|^{\mu_0}. \quad (23)$$

Using this last estimate in equation  $(\text{E}_\sigma)$ , this gives the following:

$$|f'(\xi)^2 + f'(\xi)2\lambda| \leq C|\xi|^{\mu_0}. \quad (24)$$

At this level of the proof we distinguish two cases:

**Case 1:**  $\lambda \neq 0$ . Since  $f'(\xi) \rightarrow f'(0) = 0$  as  $\xi \rightarrow 0$ , we get immediately from (24)  $|f'(\xi)| \leq C|\xi|^{\mu_0}$ . Using back the change of variables (21), we see that  $T$  is  $\mathcal{C}^{1, \mu_0}$  near  $x_0$ .

**Case 2:**  $\lambda = 0$ . We note first that by (24), we have:

$$|f'(\xi)| \leq C|\xi|^{\mu_0/2}. \quad (25)$$

Switching  $x$  and  $x_0$  in (20), we get:

$$\left| \frac{1 - T'(x)^2}{(T'(x)\text{sign}(x_0 - x) + \sigma)^2} - \frac{1 - T'(x_0)^2}{\left(\frac{T(x_0) - T(x)}{|x - x_0|} + \sigma\right)^2} \right| \leq C|x_0 - x|^{\mu_0}.$$



Using the change of variable (21) and the same arguments used to obtain  $(E_\sigma)$ , we have after simplifications:

$$\left| (1 - f'(\xi)^2) \left( \sigma - \frac{f(\xi)}{|\xi|} \right)^2 - (\sigma - f'(\xi) \operatorname{sign}(\xi))^2 \right| \leq C|\xi|^{\mu_0}.$$

Using (23) and (25) in the last inequality, we obtain  $|f'(\xi)| \leq C|\xi|^{\mu_0}$ . Using again (21), we see that  $T(x)$  is  $\mathcal{C}^{1,\mu_0}$  near  $x_0$ . This concludes the proof of Theorem 1. ■

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