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# HIGHER-DIMENSIONAL NORMALISATION STRATEGIES FOR ACYCLICITY

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YVES GUIRAUD – PHILIPPE MALBOS

**Abstract** – We introduce *acyclic track polygraphs*, a notion of complete categorical cellular models for small categories: they are polygraphs containing generators, with additional invertible cells for relations and higher-dimensional *globular syzygies*. We give a rewriting method to realise such a model by proving that a *convergent* presentation canonically extends to an acyclic track polygraph. For that, we introduce *normalising strategies*, defined as homotopically coherent ways to relate each cell of a track polygraph to its normal form, and we prove that acyclicity is equivalent to the existence of a normalisation strategy.

Using track polygraphs, we extend to every dimension the homotopical finiteness condition of finite derivation type, introduced by Squier in string rewriting theory, and we prove that it implies a new homological finiteness condition that we introduce here. The proof is based on normalisation strategies and relates acyclic track polygraphs to free Abelian resolutions of the small categories they present.

**Keywords** – rewriting; homology of small categories; low-dimensional topology; identities among relations.

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## INTRODUCTION

### The two dimensions of rewriting

The notion of presentation of higher-dimensional categories was introduced by Burroni, under the name of polygraph, [9], and by Street, under the terminology of computad, [40, 41]. Here we stick to the first name, as usual in rewriting theory, [33, 14, 15, 31]. An  $n$ -polygraph is a family  $(\Sigma_0, \dots, \Sigma_n)$ , where  $\Sigma_0$  is a set and, for every  $0 \leq k < n$ ,  $\Sigma_{k+1}$  is a (globular) cellular extension of the free  $k$ -category  $\Sigma_k^*$  over  $\Sigma_k$ , *i.e.*, a family of parallel  $k$ -cells of  $\Sigma_k^*$ .

An  $n$ -polygraph is a system of generators of an  $n$ -category and, also, a presentation of an  $(n - 1)$ -category by generators (cells up to the *syntactic* dimension  $n - 1$ ) and relations (cells of dimension  $n$ ). For example, monoids and small categories are generated by 1-polygraphs and presented by 2-polygraphs: they have syntactic dimension 1. Lawvere's algebraic theories,  $\text{pro}(p)$ s and, more generally, monoidal categories and 2-categories are generated by 2-polygraphs and presented by 3-polygraphs: they have syntactic dimension 2.

In Section 1, after some reminders on track  $n$ -categories, introduced in [15], we define the notions of *track*  $(n, p)$ -category and of *track*  $(n, p)$ -polygraph. A track  $(n, p)$ -category is an  $n$ -category enriched in  $p$ -groupoids, *i.e.*, an  $(n + p)$ -category whose cells of dimension  $n + 1$  or above are invertible. A track  $(n, p)$ -polygraph is an  $(n + 1)$ -polygraph  $(\Sigma_0, \dots, \Sigma_{n+1})$  equipped, for every  $1 < k < p$ , with a cellular extension  $\Sigma_{n+k+1}$  of the free track  $(n, k)$ -category  $\Sigma^{\top}$  over  $\Sigma_{n+k}$ .

We use track  $(n, p)$ -polygraphs as higher-dimensional presentations of  $n$ -categories, where the dimensions above  $n$  contain generating cells for higher-dimensional syzygies. We say that a track polygraph  $\Sigma$  is *acyclic* when, for every  $0 < k < p$ , every  $(n + k)$ -dimensional syzygy of  $\Sigma^{\top}$  is the boundary of an  $(n + k + 1)$ -cell of  $\Sigma^{\top}$ .

Given an  $n$ -category  $\mathcal{C}$ , an acyclic track  $(n, p - 1)$ -polygraph that presents  $\mathcal{C}$  is called a *track-polygraphic resolution* of  $\mathcal{C}$  of length  $p$ . This notion is linked to the one of polygraphic resolution, [33], and extends the ones of generation by an  $n$ -polygraph, of presentation by an  $(n + 1)$ -polygraph and of presentation by an  $(n + 1)$ -polygraph with a given homotopy basis, which respectively correspond to the cases  $p = 0$ ,  $p = 1$  and  $p = 2$ . We say that an  $n$ -category is *of finite  $p$ -derivation type* ( $\text{FDT}_p$ ) when it admits a finite track-polygraphic resolution of length  $p$ . This generalises to  $n$ -categories and in every dimension the homotopical finiteness property of finite derivation type for monoids, [39]. The notion of track-polygraphic resolution also permits the definition of the *polygraphic dimension* of an  $n$ -category, as the minimal length of its complete track-polygraphic resolutions.

### Normalisation strategies

Reduction strategies appear in many different contexts of rewriting theory and, in particular, in several rule-based programming languages, see 2.1. Among reduction strategies, we are particularly interested in normalisation ones, which are coherent choices of reduction paths from a term to a normal form. Following this idea, we introduce the notion of *normalisation strategy* for a track  $(n, p)$ -polygraph  $\Sigma$  as a homotopically coherent choice, for every  $k$ -cell  $f$  of  $\Sigma$ , of a  $k$ -cell  $\widehat{f}$ , the *normal form* of  $f$ , together with a  $(k + 1)$ -cell  $\sigma_f$  from  $f$  to  $\widehat{f}$ , see 2.2.2.

We particularly study the case of normalisation strategies for track  $(1, p)$ -polygraphs. In 2.3, we give an explicit description of normalisation strategies in the lower dimensions, up to  $p = 3$ . Then we exhibit two natural classes of normalisation strategies: *left* ones and *right* ones which, informally, normalise cells starting from the left or from the right, respectively.

Our first theorem relates the acyclicity of a track  $(1, p)$ -polygraph with the fact that it is normalising, meaning that it admits a normalisation strategy:

**Theorem 2.3.6.** *Let  $\Sigma$  be a track  $(1, p)$ -polygraph, then*

$$\Sigma \text{ is acyclic} \quad \text{iff} \quad \Sigma \text{ is normalising} \quad \text{iff} \quad \Sigma \text{ is left (resp. right) normalising.}$$

In particular, a small category  $\mathbf{C}$  is of finite  $p$ -derivation type if and only if there exists a normalising track  $(1, p - 1)$ -polygraph that presents  $\mathbf{C}$ .

Squier has used a kind of normalisation strategy in order to construct a contracting homotopy for the Fox Jacobian and, thus, to give a characterisation of the 2-dimensional homological syzygies of a convergent presentation of a monoid. The notion of normalisation strategy we introduce here allows a generalisation of Squier's construction in every dimension, for convergent track  $(1, p)$ -polygraphs.

### The case of convergent presentations

Convergent (*i.e.*, terminating and confluent) rewriting systems play an important role in rewriting theory. Indeed, they guarantee the existence of unique normal forms. In particular, when an  $n$ -category admits a presentation by a finite, convergent  $(n + 1)$ -polygraph, the normal form algorithm is a solution to its word problem.

In Section 3, we recall the notion of convergent 2-polygraph and refer the reader to [15] for a more comprehensive study of polygraphic rewriting, in particular for higher dimensions. Then, we give a methodology to extend a convergent 2-polygraph  $\Sigma$  into an acyclic track  $(1, \infty)$ -polygraph, denoted by  $c_\infty(\Sigma)$ . The  $p$ -cells of  $c_\infty(\Sigma)$ , for  $p \geq 3$ , are computed from the  $(p - 1)$ -fold *critical branchings* of  $\Sigma$ , a notion we introduce here as a higher-dimensional generalisation of critical branchings, themselves being the potential obstructions to the confluence of the 2-polygraph  $\Sigma$ . The whole construction is based on the inductive definition of a higher-dimensional normalisation strategy. The main theorem of Section 3 follows:

**Theorem 3.4.4.** *A category with a finite convergent presentation is of finite  $\infty$ -derivation type.*

In particular, if a small category admits a convergent presentation with no critical  $p$ -fold branching, for some  $p \geq 2$ , then its polygraphic dimension is at most  $p$ .

### Homological finiteness conditions

In the eighties, an important question drew the attention of the rewriting community: does a finitely presented monoid with a decidable word problem have a finite convergent presentation, [17, 4, 18, 6]? If that was true, one could always decide the word problem in a finitely presented monoid by the normal form algorithm. Squier answered this question by the negative by showing that, if a monoid admits a finite convergent presentation, then it is of homological type left-FP<sub>3</sub> and, then, by exhibiting decidable and finitely presentable monoids that are not of homological type left-FP<sub>3</sub>, [38].

Squier also introduced the homotopical condition of finite derivation type for monoids. This property characterises the existence of a finite homotopy basis making the derivation graph aspherical, [39]. He showed that, if a monoid admits a finite convergent presentation, then it is of finite derivation type. The homotopical property implies the homological one, [11, 36], and they are equivalent in the case of groups, [12], the latter result being based on the Brown-Huebschmann isomorphism between homotopical and homological syzygies.

In Section 4, we relate the homotopical property  $FDT_p$  to a new homological property  $FP_p$  we introduce here, based on natural systems. Proposition 4.2.4 relates the property  $FP_p$  to other homological finiteness conditions, the following implications being equivalences for groupoids:

$$FP_p \Rightarrow \text{bi-}FP_p \Rightarrow \text{left-}FP_p \text{ or } \text{right-}FP_p \Rightarrow \text{top-}FP_p.$$

Then we present an extension to small categories of the Fox differential calculus, originally introduced for presentations of groups, [13]. To every track  $(1, p)$ -polygraph  $\Sigma$ , we associate a chain complex of natural systems, denoted by  $F_C[\Sigma]$  and called the Reidemeister-Fox-Squier complex, see 4.3.3.

We prove that, if a track  $(1, p)$ -polygraph  $\Sigma$  is acyclic, then the complex  $F_C[\Sigma]$  is acyclic, see Theorem 4.4.3. The proof is based on using a normalisation strategy to explicitly define contracting homotopies. From this result, we deduce that, if a small category admits a finite convergent presentation, then it is of homological type  $FP_\infty$ . We also obtain the following sufficient condition for the homological property  $FP_p$ :

**Theorem 4.5.2.** *For small categories and for every  $p \geq 1$ , the property  $FDT_p$  implies the property  $FP_p$ .*

We close this section by giving an interpretation relating the properties  $FDT_3$  and  $FP_3$  of a finitely generated small category, on the one hand, to a finiteness condition on the natural system of identities among relations of any of its presentations. The notion of identities among relations was introduced for presentations of groups, [35, 37] and, by the Brown-Huebschmann isomorphism, [8], the modules of identities among relations and of 2-homological syzygies of a presentation of a group are isomorphic. Here we extend this result to small categories:

**Theorem 4.6.5.** *For every 2-polygraph  $\Sigma$ , the natural systems of homological 2-syzygies and of identities among relations of  $\Sigma$  are isomorphic.*

We also introduce the property of Abelian finite derivation type ( $FDT_{\text{ab}}$ ): an  $n$ -polygraph  $\Sigma$  is  $FDT_{\text{ab}}$  if the free Abelian track  $n$ -category it generates is of finite derivation type, see 4.7. We prove that  $\Sigma$  is  $FDT_{\text{ab}}$  if and only if the natural system of identities among relations of  $\Sigma$  is finitely generated. Thus, we obtain an equivalence between the homological property  $FP_3$  and the Abelianised version  $FDT_{\text{ab}}$  of the homotopical property  $FDT_3$ :

**Theorem 4.7.3.** *For a category  $\mathbf{C}$  with a finite presentation  $\Sigma$ , the following conditions are equivalent:*

- i) *the category  $\mathbf{C}$  is of homological type  $FP_3$ ,*
- ii) *the natural system of homological 2-syzygies of  $\Sigma$  is finitely generated,*
- iii) *the natural system of identities among relations of  $\Sigma$  is finitely generated,*
- iv) *the category  $\mathbf{C}$  is  $FDT_{\text{ab}}$ .*

Section 5 concludes this paper with examples of track-polygraphic resolutions of small categories.

## 1. RESOLUTIONS BY TRACK POLYGRAPHS

### 1.1. Higher-dimensional track categories

We recall some notions from [15]. Let  $n$  be a natural number and let  $\mathcal{C}$  be an  $n$ -category (we always consider strict, globular  $n$ -categories). We denote by  $\mathcal{C}_k$  the set (and the  $k$ -category) of  $k$ -cells of  $\mathcal{C}$ . If  $f$  is in  $\mathcal{C}_k$ , then  $s_i(f)$  and  $t_i(f)$  respectively denote the  $i$ -source and  $i$ -target of  $f$ ; we drop the suffix  $i$  when  $i = k - 1$ . The source and target maps satisfy the *globular relations*:

$$s_i \circ s_{i+1} = s_i \circ t_{i+1} \quad \text{and} \quad t_i \circ s_{i+1} = t_i \circ t_{i+1}.$$

We respectively denote by  $f : u \rightarrow v$ ,  $f : u \rightrightarrows v$ ,  $f : u \Rrightarrow v$  a 1-cell, a 2-cell, a 3-cell  $f$  with source  $u$  and target  $v$ .

If  $f$  and  $g$  are  $i$ -composable  $k$ -cells, that is when  $t_i(f) = s_i(g)$ , we denote by  $f \star_i g$  their  $i$ -composite; we simply use  $fg$  when  $i = 0$ . The compositions satisfy the *exchange relations* given, for every  $i \neq j$  and every possible cells  $f, g, h$  and  $k$ , by:

$$(f \star_i g) \star_j (h \star_i k) = (f \star_j h) \star_i (g \star_k k).$$

If  $f$  is a  $k$ -cell, we denote by  $1_f$  its identity  $(k + 1)$ -cell. When  $1_f$  is composed with cells of dimension  $k + 1$  or higher, we simply denote it by  $f$ . A cell is *degenerate* when it is an identity cell.

**1.1.1. Track  $(n, p)$ -categories.** In an  $n$ -category  $\mathcal{C}$ , a  $k$ -cell  $f$ , with  $i$ -source  $x$  and  $i$ -target  $y$ , is  *$i$ -invertible* when there exists a (necessarily unique)  $k$ -cell  $g$  in  $\mathcal{C}$ , with  $i$ -source  $y$  and  $i$ -target  $x$  in  $\mathcal{C}$ , called the  *$i$ -inverse of  $f$* , that satisfies

$$f \star_i g = 1_x \quad \text{and} \quad g \star_i f = 1_y.$$

When  $i = k - 1$ , we just say that  $f$  is *invertible* and we denote by  $f^-$  its *inverse*. As in higher-dimensional groupoids, if a  $k$ -cell  $f$  is invertible and if its  $i$ -source  $x$  and  $i$ -target  $y$  are invertible, then  $f$  is  *$i$ -invertible*, with  $i$ -inverse given by

$$x^- \star_{i-1} f^- \star_{i-1} y^-.$$

For natural numbers  $n$  and  $p$ , a *track  $(n, p)$ -category* is an  $n$ -category enriched in  $p$ -groupoids, *i.e.*, an  $(n + p)$ -category whose last  $p$ -dimensions are made of invertible cells. In particular, a track  $(n, 0)$ -category is an  $n$ -category, a track  $(n - 1, 1)$ -category is a track  $n$ -category, as defined in [15], and a track  $(0, n)$ -category is an  $n$ -groupoid. By extension, we define a *track  $(n, \infty)$ -category* as an  $n$ -category enriched in  $\infty$ -groupoids. Let us note that track  $(n, p)$ -categories are also known as  $(n + p, n)$ -categories.

**1.1.2. Acyclicity and asphericity.** Let  $\mathcal{C}$  be a track  $(n, p)$ -category and let  $k$  be a natural number, with  $0 \leq k \leq n + p$ . A  *$k$ -sphere of  $\mathcal{C}$*  is a pair  $\gamma = (f, g)$  of parallel  $k$ -cells of  $\mathcal{C}$ , *i.e.*, with  $s(f) = s(g)$  and  $t(f) = t(g)$ ; we call  $f$  the *source* of  $\gamma$  and  $g$  its *target*. A  $k$ -sphere is *degenerate* when its source and target coincide.

The track  $(n, p)$ -category  $\mathcal{C}$  is *aspherical* when each  $(n + p)$ -sphere of  $\mathcal{C}$  is degenerate. One says that  $\mathcal{C}$  is *acyclic* when, for every  $k$ -sphere  $(f, g)$  of  $\mathcal{C}$ , with  $n < k < n + p$ , there exists a  $(k + 1)$ -cell with source  $f$  and target  $g$  in  $\mathcal{C}$ . In other terms,  $\mathcal{C}$  is acyclic if and only if, for every  $0 < k < p$ , the quotient track  $(n, k)$ -category  $\mathcal{C}_{n+k}/\mathcal{C}_{n+k+1}$  is aspherical.

## 1. Resolutions by track polygraphs

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### 1.2. Track polygraphs

**1.2.1. Cellular extensions.** Let  $\mathcal{C}$  be an  $n$ -category. A *cellular extension* of  $\mathcal{C}$  is a set  $\Gamma$  equipped with a map  $\partial$  from  $\Gamma$  to the set of  $n$ -spheres of  $\mathcal{C}$ . By considering all the formal compositions of elements of  $\Gamma$ , seen as  $(n+1)$ -cells with source and target in  $\mathcal{C}$ , one builds the *free  $(n+1)$ -category generated by  $\Gamma$  over  $\mathcal{C}$* , denoted by  $\mathcal{C}[\Gamma]$ . The *size* of an  $(n+1)$ -cell  $f$  of  $\mathcal{C}[\Gamma]$  is the number of  $(n+1)$ -cells of  $\Gamma$  it contains.

The *quotient of  $\mathcal{C}$  by  $\Gamma$* , denoted by  $\mathcal{C}/\Gamma$ , is the  $n$ -category one gets from  $\mathcal{C}$  by identification of the  $n$ -cells  $s(\gamma)$  and  $t(\gamma)$ , for every  $n$ -sphere  $\gamma$  of  $\Gamma$ . We write  $f \equiv_{\Gamma} g$  when parallel  $n$ -cells  $f, g$  of  $\mathcal{C}$  are identified in  $\mathcal{C}/\Gamma$ .

If  $\mathcal{C}$  is a track  $(n, p)$ -category and  $\Gamma$  is a cellular extension of  $\mathcal{C}$ , then the *free track  $(n, p+1)$ -category generated by  $\Gamma$  over  $\mathcal{C}$*  is denoted by  $\mathcal{C}(\Gamma)$  and defined as follows:

$$\mathcal{C}(\Gamma) = \mathcal{C}[\Gamma, \Gamma^{-}] / \text{Inv}(\Gamma)$$

where  $\Gamma^{-}$  contains the same  $(n+p+1)$ -cells as  $\Gamma$ , with source and target reversed, and  $\text{Inv}(\Gamma)$  is made of two  $(n+p+2)$ -cells  $\gamma \star_{n+p+1} \gamma^{-} \rightarrow 1_{s_{\gamma}}$  and  $\gamma^{-} \star_{n+p+1} \gamma \rightarrow 1_{t_{\gamma}}$  for each  $(n+p+1)$ -cell  $\gamma$  in  $\Gamma$ .

**1.2.2. Homotopy bases.** Let  $\mathcal{C}$  be a track  $(n, p)$ -category. A cellular extension  $\Gamma$  of  $\mathcal{C}$  is a *homotopy basis* when any one of the following three equivalent conditions holds:

- The track  $(n, p)$ -category  $\mathcal{C}/\Gamma$  is aspherical.
- For every  $(n+p)$ -sphere  $\gamma$  of  $\mathcal{C}$ , we have  $s(\gamma) \equiv_{\Gamma} t(\gamma)$ .
- For every  $(n+p)$ -sphere  $\gamma$  of  $\mathcal{C}$ , there exists an  $(n+p+1)$ -cell with source  $s(\gamma)$  and target  $t(\gamma)$  in the track  $(n, p+1)$ -category  $\mathcal{C}(\Gamma)$ .

In particular, the track  $(n, p)$ -category  $\mathcal{C}$  is acyclic if and only if, for every  $k$  in  $\{0, \dots, p-1\}$ , the cellular extension  $\mathcal{C}_{n+k+1}$  of  $(n+k+1)$ -cells of the track  $(n, k)$ -category  $\mathcal{C}_{n+k}$  is a homotopy basis.

**1.2.3. Track  $(n, p)$ -polygraphs.** Let  $n$  be a natural number. A *track  $(n, 0)$ -polygraph* is an  $n$ -polygraph, *i.e.*, a family  $\Sigma = (\Sigma_0, \dots, \Sigma_n)$  made of a set  $\Sigma_0$  and, for every  $0 \leq k \leq n-1$ , a cellular extension  $\Sigma_{k+1}$  of the free  $k$ -category  $\Sigma_k^* = \Sigma_0[\Sigma_1] \cdots [\Sigma_k]$ . For  $p$  a non-zero natural number, a *track  $(n, p)$ -polygraph* is a family  $\Sigma = (\Sigma_0, \dots, \Sigma_{n+p})$  made of:

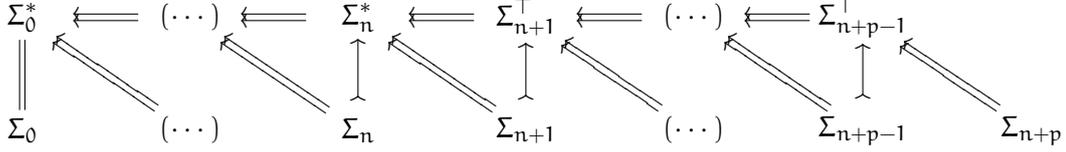
- an  $(n+1)$ -polygraph  $(\Sigma_0, \dots, \Sigma_n, \Sigma_{n+1})$ ;
- for every  $1 < k < p$ , a cellular extension  $\Sigma_{n+k+1}$  of the free track  $(n, k)$ -category

$$\Sigma_{n+k}^{\top} = \Sigma_n^*(\Sigma_{n+1}) \cdots (\Sigma_{n+k}).$$

Finally, a *track  $(n, \infty)$ -polygraph* is a family  $\Sigma = (\Sigma_k)_{k \in \mathbb{N}}$  such that, for every natural number  $p$ , the subfamily  $(\Sigma_0, \dots, \Sigma_{n+p})$  is a track  $(n, p)$ -polygraph.

A track  $(n, p)$ -polygraph is *finite* when it has finitely many cells in every dimension. A track  $(n, p)$ -polygraph  $\Sigma$  is *acyclic* or *aspherical* when the free track  $(n, p)$ -category  $\Sigma^{\top}$  is.

*Remark.* A track  $(n, p)$ -polygraph yields a diagram which is similar to the one given in the original definition of  $n$ -polygraphs, [9]:



This diagram contains the source and target attachment maps of generating  $(k + 1)$ -cells on composite  $k$ -cells, their extension to composite  $(k + 1)$ -cells and the inclusion of generating  $k$ -cells into composite  $k$ -cells.

### 1.3. Resolutions by track polygraphs

**1.3.1. Track-polygraphic resolutions.** Let  $\Sigma$  be a track  $(n, p)$ -polygraph. We denote by  $\bar{\Sigma}$  the  $n$ -category presented by its underlying track  $(n, 1)$ -polygraph, that is the quotient  $n$ -category

$$\bar{\Sigma} = \Sigma_n^* / \Sigma_{n+1}.$$

We usually write  $\pi : \Sigma_n^* \twoheadrightarrow \bar{\Sigma}$  the canonical projection and, when no confusion may occur, we use  $\bar{f}$  instead of  $\pi(f)$ . If  $f$  is a  $k$ -cell of  $\Sigma^\top$ , with  $n + 1 \leq k \leq n + p$ , we also denote by  $\bar{f}$  the  $n$ -cells  $s_n(\bar{f})$  and  $t_n(\bar{f})$ , which are equal by definition of the  $n$ -category  $\bar{\Sigma}$ . A track  $(n, p)$ -polygraph  $\Sigma$  and a track  $(n, q)$ -polygraph  $\Upsilon$  are *Tietze-equivalent* when the  $n$ -categories  $\bar{\Sigma}$  and  $\bar{\Upsilon}$  are isomorphic.

Let  $\mathcal{C}$  be an  $n$ -category. For  $p$  a non-zero natural number, a *track-polygraphic resolution of  $\mathcal{C}$  of length  $p$*  is an acyclic track  $(n, p - 1)$ -polygraph  $\Sigma$  such that  $\bar{\Sigma}$  is isomorphic to  $\mathcal{C}$ . In particular, for low values of  $p$ , we have:

- A track-polygraphic resolution of length 1 of  $\mathcal{C}$  is a generating  $n$ -polygraph for  $\mathcal{C}$ , *i.e.*, an  $n$ -polygraph  $\Sigma$  such that  $\mathcal{C}$  is isomorphic to a quotient of the free  $n$ -category  $\Sigma^*$  by a set of  $n$ -spheres of  $\Sigma^*$ .
- A track-polygraphic resolution of length 2 of  $\mathcal{C}$  is a presentation by a track  $(n, 1)$ -polygraph, *i.e.*, a track  $(n, 1)$ -polygraph  $\Sigma$  such that  $\mathcal{C}$  is isomorphic to  $\bar{\Sigma}$ .
- A track-polygraphic resolution of length 3 of  $\mathcal{C}$  is a presentation by a track  $(n, 1)$ -polygraph  $\Sigma$ , extended with a homotopy basis  $\Gamma$  of the track  $(n, 1)$ -category  $\Sigma^\top$ , so that  $\mathcal{C}$  is isomorphic to  $\bar{\Sigma}$  and  $\Sigma^\top / \Gamma$  is aspherical.

*Remark.* The definition of track-polygraphic resolution is linked to the notion of polygraphic resolution, introduced in [33]. A polygraphic resolution with length  $k$  of an  $n$ -category  $\mathcal{C}$  is a  $k$ -polygraph  $\Sigma$  equipped with a surjective  $k$ -functor  $\Phi : \Sigma^* \rightarrow \mathcal{C}$  such that, for every  $0 \leq i \leq k$  and every  $i$ -sphere  $(x, y)$  of  $\Sigma^*$  with  $\Phi(x) = \Phi(y)$ , there exists an  $(i + 1)$ -cell  $u : x \rightarrow y$  in  $\Sigma^*$ . Thus, if  $\mathcal{C}$  is free up to dimension  $n - 1$  and if  $\Phi$  is the identity on dimensions up to  $n - 1$ , then a polygraphic resolution with length  $n + p$  yields a track-polygraphic resolution of  $\mathcal{C}$  with length  $p + 1$ .

## 2. Normalisation strategies for track polygraphs

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**1.3.2. Higher-dimensional finite derivation type.** Let  $\mathcal{C}$  be an  $n$ -category. For  $p$  a non-zero natural number, one says that  $\mathcal{C}$  is of *finite  $p$ -derivation type* ( $\text{FDT}_p$ ) when  $\mathcal{C}$  admits a finite track-polygraphic resolution of length  $p$ , *i.e.*, a resolution by a finite, acyclic track  $(n, p - 1)$ -polygraph. Similarly,  $\mathcal{C}$  is of *finite  $\infty$ -derivation type* ( $\text{FDT}_\infty$ ) when  $\mathcal{C}$  admits a resolution by a finite, acyclic track  $(n, \infty)$ -polygraph, *i.e.*, when  $\mathcal{C}$  is  $\text{FDT}_p$  for every non-zero natural number.

In particular,  $\mathcal{C}$  is  $\text{FDT}_1$  when it is finitely generated and  $\text{FDT}_2$  when it is finitely presented. The property  $\text{FDT}_3$  corresponds to the finite derivation type homotopical finiteness property originally introduced by Squier for monoids, [38], and extended by the authors to  $n$ -categories, [15]. The property  $\text{FDT}_4$  was introduced in [28].

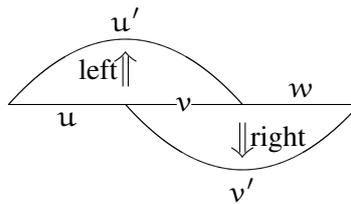
**1.3.3. Polygraphic dimension.** Let  $\mathcal{C}$  be an  $n$ -category. The *polygraphic dimension* of  $\mathcal{C}$  is defined, when it exists, as the lowest  $0 \leq p \leq \infty$  such that  $\mathcal{C}$  admits a resolution by an aspherical, acyclic track  $(n, p)$ -polygraph. In that case, we denote by  $d_{\text{pol}}(\mathcal{C})$  the polygraphic dimension of  $\mathcal{C}$ .

## 2. NORMALISATION STRATEGIES FOR TRACK POLYGRAPHS

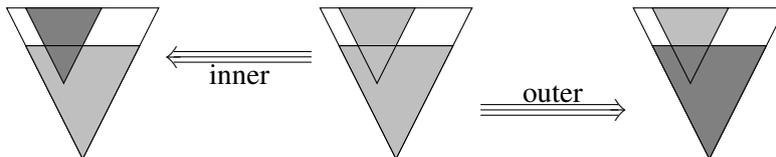
### 2.1. Strategies in rewriting theory

In a rewriting system, one specifies a set of rules that describe valid replacements of subformulas by other ones, [43, 34]. For good references on word rewriting, see [7], and, on term rewriting, see [2, 42]. On some formulas of a rewriting system, the rewriting rules may produce conflicts, when two or more rules can be applied. In order to transform a rewriting system into a computation algorithm, one needs to specify a way to apply the rules in a deterministic way. To do this, one specifies what is called a reduction strategy.

For example, in a word rewriting system, formulas are elements of a free monoid and we have two canonical reduction strategies: the leftmost one and the rightmost one, where one always uses the rewriting rule that can be applied on the leftmost or the rightmost subformula:



In term rewriting, formulas are morphisms of a Lawvere algebraic theory, [22]. There exist many possible rewriting strategies for term rewriting systems. Among them, one finds outermost and innermost reduction strategies, where one first uses the rules that apply closer to the root or closer to the leaves of the term:



In modern programming languages that are based on rewriting mechanisms, such as Caml, [23], and Haskell, [27], reduction strategies are implicitly used by the compiler to transform rewriting systems into deterministic algorithms. In that setting, the innermost strategies include the call-by-value evaluation, while the outermost strategies contain the call-by-need evaluation. Some programming languages, like Tom, [3], include a dedicated grammar to explicitly construct user-defined reduction strategies.

In order to study the computational properties of reduction strategies, several models have been introduced. In abstract rewriting, a reduction strategy is defined as a subgraph of the ambient abstract rewriting system. This definition allows the introduction of some properties: for example, a normalisation strategy is a reduction strategy that reaches normal forms, [42]. Strategies in programming languages are usually classified by corresponding notions of strategies in the  $\lambda$ -calculus, [25]. This has led to an axiomatic treatment of a general setting of standardisation in rewriting theory, where strategies are seen as standardisation systems of rewriting paths, [29].

Here, we introduce a notion of normalisation strategy for higher-dimensional rewriting systems that, in turn, induces a notion of normal forms in every dimension, together with a homotopically coherent reduction of every cell to its normal form.

## 2.2. Normalisation strategies

**2.2.1. Sections of track polygraphs.** Let  $n, p$  be natural numbers (or  $p = \infty$ ) and let  $\Sigma$  be a track  $(n, p)$ -polygraph. A *section* of  $\Sigma$  is a choice, for every  $n$ -cell  $f : u \rightarrow v$  of the presented  $n$ -category  $\bar{\Sigma}$ , of an  $n$ -cell  $\hat{f} : u \rightarrow v$  in  $\Sigma^*$  that satisfies the relation  $\pi(\hat{f}) = f$  and the functorial conditions

$$\hat{1}_x = 1_x \quad \text{and} \quad \widehat{f \star_k g} = \hat{f} \star_k \hat{g}$$

for every  $(n-1)$ -cell  $x$  of  $\bar{\Sigma}$  and every pair  $(f, g)$  of  $k$ -composable  $n$ -cells of  $\bar{\Sigma}$ , with  $0 \leq k < n-1$ .

Let us note that such an assignment  $f \mapsto \hat{f}$  is not assumed to be functorial with respect to the highest composition  $\star_{n-1}$  of  $\bar{\Sigma}$ . Indeed, such a property could only be required for a track  $(n, 0)$ -polygraph, *i.e.*, when  $\bar{\Sigma}$  is a free  $n$ -category.

Since, by hypothesis, the assignment  $f \mapsto \hat{f}$  is compatible with the quotient map  $\pi$ , it extends to a mapping of each  $n$ -cell  $f$  in  $\Sigma^*$  to a parallel  $n$ -cell in  $\Sigma^*$ , still denoted by  $\hat{f}$ , in such a way that  $\bar{f} = \bar{g}$  is equivalent to  $\hat{f} = \hat{g}$ . Thereafter, we assume that, with every track  $(n, p)$ -polygraph we consider, comes an implicitly chosen section.

**2.2.2. Normalisation strategies.** Let  $\Sigma$  be a track  $(n, p)$ -polygraph. A *normalisation strategy* for  $\Sigma$  is a mapping  $\sigma$  of every  $k$ -cell  $f$  of  $\Sigma^\top$ , with  $n \leq k < n+p$ , to a  $(k+1)$ -cell

$$f \xrightarrow{\sigma_f} \hat{f}$$

where, for  $k \geq n+1$ , the notation  $\hat{f}$  stands for the  $k$ -cell  $\hat{f} = \sigma_{s(f)} \star_{k-1} \sigma_{t(f)}^-$ , such that the relation  $\sigma_{\hat{f}} = 1_{\hat{f}}$  is satisfied, together with the functorial condition

$$\sigma_{f \star_i g} = \sigma_f \star_i \sigma_g$$

for every pair  $(f, g)$  of  $i$ -composable  $k$ -cells of  $\Sigma^\top$ , with  $n < k \leq n+p$ ,  $0 \leq i < k$  and  $i \neq n-1$ . Let us note that  $\hat{1}_x = 1_x$  implies  $\sigma_{1_x} = 1_{1_x}$  for every  $(n-1)$ -cell  $x$  in  $\Sigma^\top$ .

## 2. Normalisation strategies for track polygraphs

A track  $(n, p)$ -polygraph is *normalising* if it admits a normalisation strategy. This property is independent of the chosen section. Indeed, let us consider a track  $(n, p)$ -polygraph  $\Sigma$ , let us fix two sections  $f \mapsto \hat{f}$  and  $f \mapsto \tilde{f}$  of  $\Sigma$  and let us assume that  $\sigma$  is a normalisation strategy for  $\Sigma$ , equipped with the section  $f \mapsto \hat{f}$ . Then, one checks that we get a normalisation strategy  $\tau$  for the other section by defining  $\tau_f$  as the following composite:

$$f \xrightarrow{\sigma_f} \hat{f} \xrightarrow{(\sigma_{\hat{f}})^{-}} \tilde{f}$$

**2.2.3. Lemma.** *Let  $\Sigma$  be a track  $(n, p)$ -polygraph. A normalisation strategy  $\sigma$  for  $\Sigma$  satisfies the following properties, for every possible  $k$ -cell  $f$  of  $\Sigma^\top$ , with  $k \geq n$ :*

$$\sigma_{1_f} = 1_{1_f} \quad \text{and} \quad \sigma_{f^-} = f^- \star_{k-1} \sigma_f^- \star_{k-1} \hat{f}^-.$$

*Proof.* We have  $\sigma_{1_f} = \sigma_{1_f \star_k 1_f} = \sigma_{1_f} \star_k \sigma_{1_f}$ . Since  $\sigma_{1_f}$  is invertible for  $\star_k$ , the first relation holds. For the second relation, we have:

$$\sigma_f \star_{k-1} \sigma_{f^-} = \sigma_{f \star_{k-1} f^-} = \sigma_{1_{s(f)}} = 1_{1_{s(f)}}.$$

Thus,  $\sigma_{f^-}$  is the inverse for  $\star_{k-1}$  of  $\sigma_f$ , yielding:

$$\sigma_{f^-} = s(\sigma_f)^- \star_{k-1} \sigma_f^- \star_{k-1} t(\sigma_f)^- = f^- \star_{k-1} \sigma_f^- \star_{k-1} \hat{f}^-. \quad \square$$

### 2.3. The case of track $(1, p)$ -polygraphs

Let  $\Sigma$  be a track  $(1, p)$ -polygraph. In the lower dimensions, a normalisation strategy  $\sigma$  for  $\Sigma$  specifies the following assignments:

- For every 1-cell  $u$  of  $\Sigma^\top$ , a 2-cell

$$u \xrightarrow{\sigma_u} \hat{u}$$

of  $\Sigma^\top$  that satisfies  $\sigma_{\hat{u}} = 1_{\hat{u}}$  and thus, in particular,  $\sigma_{1_\xi} = 1_{1_\xi}$  for every 0-cell  $\xi$  of  $\Sigma$ .

- For every 2-cell  $f : u \Rightarrow v$  of  $\Sigma^\top$ , a 3-cell

$$\begin{array}{ccc} u & \xrightarrow{f} & v \\ \sigma_u \searrow & \Downarrow \sigma_f & \nearrow \sigma_v^- \\ & \hat{u} & \end{array}$$

of  $\Sigma^\top$  that satisfies  $\sigma_{\hat{f}} = 1_{\hat{f}}$  and the following relations:

- If  $u$  is a 1-cell of  $\Sigma^\top$ , then  $\sigma_{1_u} = 1_{1_u}$ :

$$\begin{array}{ccc} u & \xrightarrow{1_u} & u \\ \sigma_u \searrow & \Downarrow \sigma_{1_u} & \nearrow \sigma_u^- \\ & \hat{u} & \end{array} = \begin{array}{ccc} u & \xrightarrow{1_u} & u \\ & \Downarrow 1_{1_u} & \\ & \hat{u} & \end{array}$$

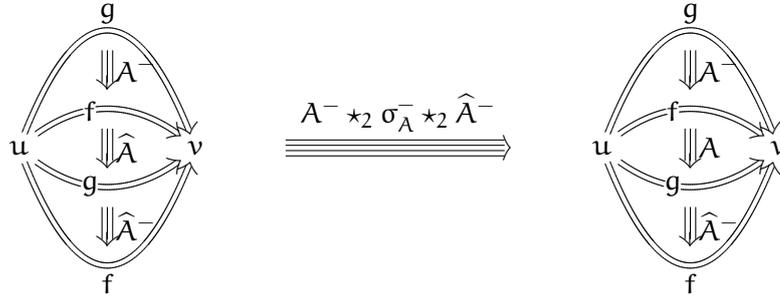


## 2. Normalisation strategies for track polygraphs

- If  $A : f \Rightarrow g : u \Rightarrow v$  and  $B : g \Rightarrow h : u \Rightarrow v$  are 3-cells of  $\Sigma^\top$ , then  $\sigma_{A \star_2 B} = \sigma_A \star_2 \sigma_B$ :



- If  $A : f \Rightarrow g : u \Rightarrow v$  is a 3-cell of  $\Sigma^\top$ , then  $\widehat{A} = \sigma_f \star_2 \sigma_g^-$  and  $\sigma_{A^-} = A^- \star_2 \sigma_A^- \star_2 \widehat{A}^-$ :



**2.3.1. Lemma.** Let  $\Sigma$  be a track  $(1, p)$ -polygraph. Normalisation strategies for  $\Sigma$  are in bijective correspondence with families of  $(k+1)$ -cells

$$\sigma_{u\varphi v} : u\varphi v \rightarrow \widehat{u\varphi v}$$

for every  $k$  in  $\{1, \dots, p-1\}$ , every  $k$ -cell  $\varphi$  of  $\Sigma$  and every pair  $(u, v)$  of 1-cells of  $\Sigma^*$  such that the composite  $k$ -cell  $u\varphi v$  is defined.

*Proof.* We proceed by induction on the size of cells of  $\Sigma^\top$ . We already know that a normalisation strategy  $\sigma$  has fixed values on identities, inverses and  $\star_i$ -composites for  $i \geq 1$ . As a consequence, using the exchange relations, we get that the values of  $\sigma$  are entirely and uniquely determined by its values on  $k$ -cells with shape  $u\varphi v$ , where  $\varphi$  is a  $k$ -cell of  $\Sigma$  and  $u, v$  are 1-cells of  $\Sigma^\top$ .  $\square$

**2.3.2. An alternative form of normalisation strategies.** Let  $\sigma$  be a normalisation strategy for a track  $(1, p)$ -polygraph  $\Sigma$ . For every  $k$ -cell  $f$  in  $\Sigma^\top$ , with  $1 \leq k \leq p$ , we denote by  $f^*$  the following  $k$ -cell of  $\Sigma^\top$ :

$$f^* = ((f \star_{k-1} \sigma_{t_{k-1}(f)}) \star_{k-2} \dots) \star_1 \sigma_{t_1(f)}.$$

This  $k$ -cell has source  $s(f)$  and target  $t(f)^*$ . If  $1 \leq k < p$ , then we have

$$\sigma_{f^*} = \sigma_f^*,$$

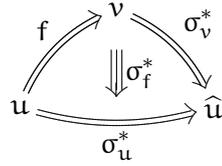
which is a  $(k+1)$ -cell of  $\Sigma^\top$  with source  $f^*$  and target  $\widehat{f^*}$ . Since every  $k$ -cell of  $\Sigma^\top$  is invertible for  $k \geq 2$ , one can recover  $\sigma$  from  $\sigma^*$ , so that the normalisation strategy  $\sigma$  is uniquely and entirely determined by the values

$$\sigma_{u\varphi v}^* : (u\varphi v)^* \rightarrow \widehat{u\varphi v}^*$$

### 2.3. The case of track $(1, p)$ -polygraphs

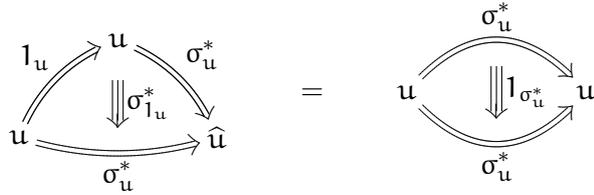
for every  $k$  in  $\{1, \dots, p-1\}$ , every  $k$ -cell  $\varphi$  of  $\Sigma$  and every pair  $(u, v)$  of 1-cells of  $\Sigma^\top$  such that  $u\varphi v$  is defined. In the lowest dimensions, the alternative form  $\sigma^*$  of the strategy  $\sigma$  consists of the following data:

- For every 1-cell  $u$  of  $\Sigma^\top$ , we have  $\sigma_u^* = \sigma_u$ .
- For every 2-cell  $f : u \Rightarrow v$  of  $\Sigma^\top$ , a 3-cell

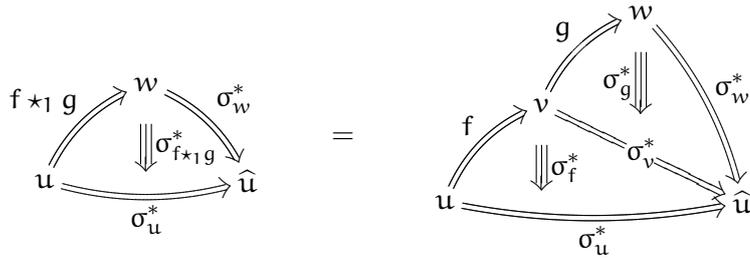


of  $\Sigma^\top$  such that the following relations hold:

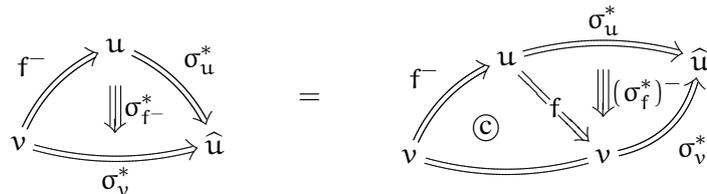
- if  $u$  is a 1-cell of  $\Sigma^\top$ , then  $\sigma_{1_u}^* = 1_{\sigma_u^*}$ ,



- if  $f : u \Rightarrow v$  and  $g : v \Rightarrow w$  are 2-cells in  $\Sigma^\top$ , then  $\sigma_{f \star_1 g}^* = (f \star_1 \sigma_g^*) \star_2 \sigma_f^*$ :

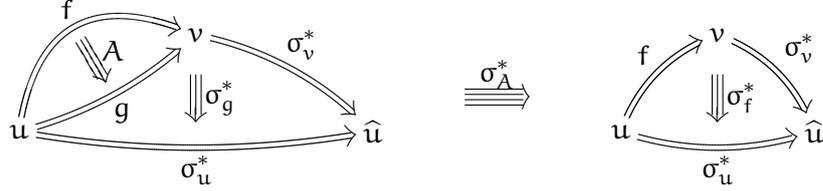


- If  $f : u \Rightarrow v$  is a 2-cell in  $\Sigma^\top$ , then  $\sigma_{f^-}^* = f^- \star_1 (\sigma_f^*)^-$ :



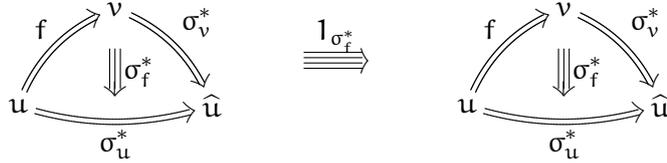
## 2. Normalisation strategies for track polygraphs

- For every 3-cell  $A : f \Rightarrow g : u \Rightarrow v$  of  $\Sigma^\top$ , a 4-cell

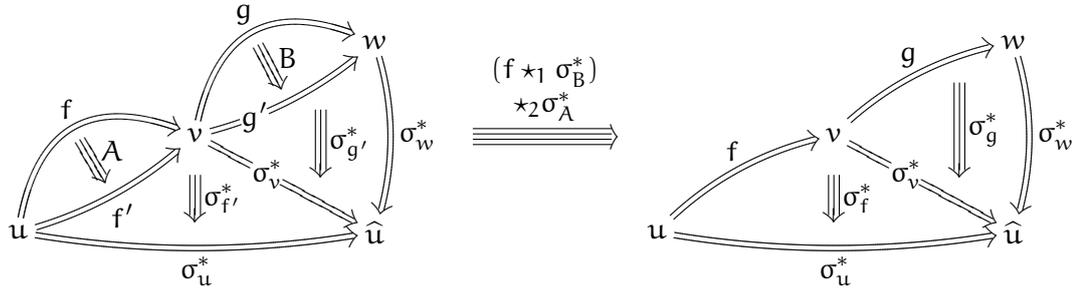


of  $\Sigma^\top$  such that the following relations hold:

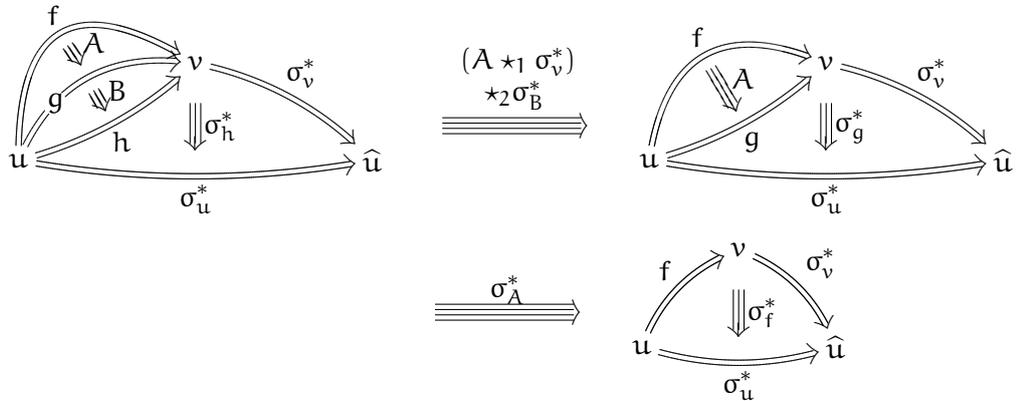
- if  $f$  is a 2-cell of  $\Sigma^\top$ , then  $\sigma_{1_f}^* = 1_{\sigma_f^*}$ :



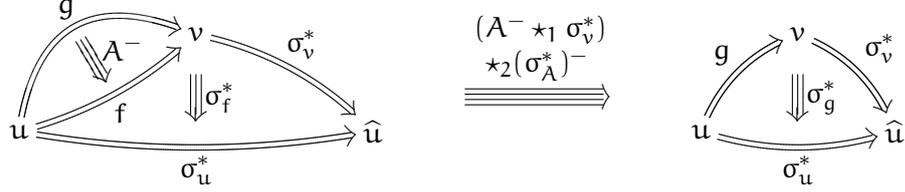
- if  $A : f \Rightarrow f' : u \Rightarrow v$  and  $B : g \Rightarrow g' : v \Rightarrow w$  are 3-cells of  $\Sigma^\top$ , then  $\sigma_{A \star_1 B}^* = (f \star_1 \sigma_B^*) \star_2 \sigma_A^*$ :



- if  $A : f \Rightarrow g : u \Rightarrow v$  and  $B : g \Rightarrow h : u \Rightarrow v$  are 3-cells of  $\Sigma^\top$ , then  $\sigma_{A \star_2 B}^* = ((A \star_1 \sigma_v^*) \star_2 \sigma_B^*) \star_3 \sigma_A^*$ :

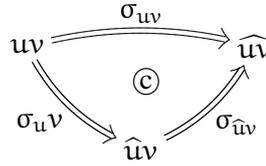


– if  $A : f \Rightarrow g : u \Rightarrow v$  is a 3-cell of  $\Sigma^\top$ , then  $\sigma_{A^-}^* = (A^- \star_1 \sigma_v^*) \star_2 (\sigma_u^*)^-$ :



**2.3.3. Left and right normalisation strategies.** Let  $\Sigma$  be a  $(1, p)$ -polygraph. A normalisation strategy  $\sigma$  for  $\Sigma$  is a *left* one when it satisfies the following properties:

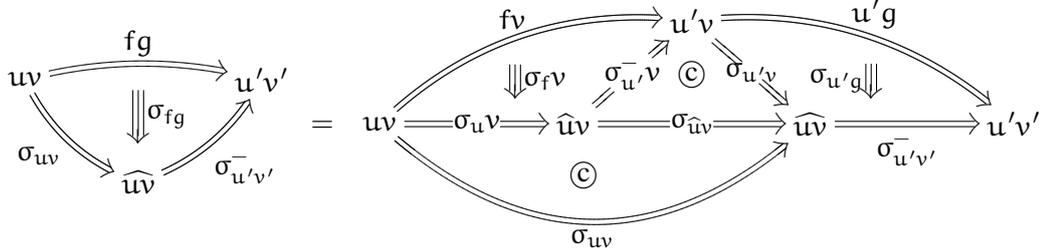
- For every pair  $(u, v)$  of 0-composable 1-cells of  $\Sigma^\top$ , we have  $\sigma_{uv} = \sigma_u v \star_1 \sigma_{\hat{u}v}$ :



- For every pair  $(f, g)$  of 0-composable  $k$ -cells of  $\Sigma^\top$ , with  $2 \leq k \leq p$ ,  $t_1(f) = u'$  and  $s_1(g) = v$ , we have

$$\sigma_{fg} = \sigma_f v \star_1 \sigma_{u'g}.$$

In particular, when  $f : u \Rightarrow u'$  and  $g : v \Rightarrow v'$  are 0-composable 2-cells of  $\Sigma^\top$ :



In a symmetric way, a normalisation strategy  $\sigma$  is a *right* one when it satisfies:

$$\sigma_{uv} = u \sigma_v \star_1 \sigma_{u\hat{v}} \quad \text{and} \quad \sigma_{fg} = u \sigma_g \star_1 \sigma_{fv'}.$$

A track  $(1, p)$ -polygraph is *left* (resp. *right*) *normalising* when it admits a left (resp. right) normalisation strategy.

**2.3.4. Lemma.** Let  $\Sigma$  be a track  $(1, p)$ -polygraph. Let  $k$  be in  $\{2, \dots, p+1\}$ , let  $f$  be  $k$ -cell  $f$  of  $\Sigma^\top$  with 1-source  $u$  and 1-target  $v$  and let  $w, w'$  be 1-cells of  $\Sigma^\top$  such that  $wfw'$  is defined. Then, if  $\sigma$  is a left normalisation strategy for  $\Sigma$ , we have:

$$\sigma_{wfw'} = \sigma_w u w' \star_1 \sigma_{\hat{w}f} w' \star_1 \sigma_w^- v w' \quad \text{and} \quad \sigma_{wfw'}^* = \sigma_w^* u w' \star_1 \sigma_{\hat{w}f}^* w' \star_1 \sigma_{w\hat{u}w'}^*.$$

Symmetrically, if  $\sigma$  is a right normalisation strategy, then we have:

$$\sigma_{wfw'} = w u \sigma_{w'} \star_1 w \sigma_{f\hat{w}'} \star_1 w v \sigma_{w'}^-, \quad \text{and} \quad \sigma_{wfw'}^* = w u \sigma_{w'}^* \star_1 w \sigma_{f\hat{w}'}^* \star_1 \sigma_{w\hat{u}w'}^*.$$

## 2. Normalisation strategies for track polygraphs

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*Proof.* In the case of a left normalisation strategy, the proof for right normalisation strategies being symmetric, we have:

$$\sigma_{fw'} = \sigma_f w' \star_1 \sigma_{1_{vw'}} = \sigma_f w' \star_1 1_{1_{vw'}} = \sigma_f w'.$$

Then, using the exchange relation, we get:

$$\sigma_{\sigma_w f} = \sigma_{wf \star_1 \sigma_w v} = \sigma_{wf} \star_1 \sigma_{\sigma_w v} = \sigma_{wf} \star_1 1_{\sigma_w v} = \sigma_{wf} \star_1 \sigma_w v.$$

Moreover, the definition of left normalisation strategy implies:

$$\sigma_{\sigma_w f} = \sigma_{\sigma_w u} \star_1 \sigma_{\widehat{w}f} = \sigma_w u \star_1 \sigma_{\widehat{w}f}.$$

From the last two computations, we deduce:

$$\sigma_{wf} = \sigma_w u \star_1 \sigma_{\widehat{w}f} \star_1 \sigma_w^{-1} v.$$

Finally, we have:

$$\begin{aligned} \sigma_{wf w'}^* &= \sigma_{(wf w')^*} \\ &= \sigma_{wf^* w'} \star_1 \sigma_{w w'} \\ &= \sigma_w u w' \star_1 \sigma_{\widehat{w}f^* w'} \star_1 \sigma_w^{-1} v w' \star_1 \sigma_{w w'} \\ &= \sigma_w u w' \star_1 \sigma_{\widehat{w}f^* w'} \star_1 \sigma_{\widehat{w}v} w' \star_1 \sigma_{\widehat{w}u w'} \\ &= \sigma_w^* u w' \star_1 \sigma_{\widehat{w}f w'}^* \star_1 \sigma_{\widehat{w}u w'}^*. \end{aligned} \quad \square$$

**2.3.5. Corollary.** *Let  $\Sigma$  be a track  $(1, p)$ -polygraph. Left (resp. right) normalisation strategies on  $\Sigma$  are in bijective correspondence with families*

$$\sigma_{\widehat{u}\varphi} : \widehat{u}\varphi \rightarrow \widehat{u}\widehat{\varphi} \quad (\text{resp. } \sigma_{\varphi\widehat{u}} : \varphi\widehat{u} \rightarrow \widehat{\varphi}\widehat{u})$$

of  $(k+1)$ -cells, indexed by  $1 \leq k \leq p+1$ , by  $k$ -cells  $\varphi$  of  $\Sigma$  and by 1-cells  $u$  of  $\mathbf{C}$  such that the composite  $\widehat{u}\varphi$  (resp.  $\varphi\widehat{u}$ ) exists.

*Proof.* Let us assume that  $\sigma$  is a left normalisation strategy. The property satisfied by  $\sigma$  on 1-cells of  $\Sigma^\top$  gives, by induction on the size of 1-cells, that the values of  $\sigma$  on 1-cells of  $\Sigma^\top$  are determined by the 2-cells  $\sigma_{\widehat{u}x}$ , for  $x$  a 1-cell of  $\Sigma$  and  $u$  a 1-cell of  $\mathbf{C}$  such that  $\widehat{u}x$  is defined. Then, Lemma 2.3.1 tells us that the values of  $\sigma$  on higher-dimensional cells of  $\Sigma^\top$  are determined by the values of  $\sigma$  on  $k$ -cells  $u\widehat{\varphi}v$  of  $\Sigma^\top$ , where  $\varphi$  is a  $k$ -cell of  $\Sigma$  and  $u, v$  are 1-cells of  $\Sigma^\top$ . We use Lemma 2.3.4 to conclude.  $\square$

**2.3.6. Theorem.** *Let  $\Sigma$  be a track  $(1, p)$ -polygraph. The following assertions are equivalent:*

- i)  $\Sigma$  is acyclic,
- ii)  $\Sigma$  is normalising,
- iii)  $\Sigma$  is left normalising,
- iv)  $\Sigma$  is right normalising.

### 3. Track-polygraphic resolutions generated by convergent 2-polygraphs

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*Proof.* Let us assume that there exists a normalisation strategy  $\sigma$  for  $\Sigma$ . We consider a  $k$ -cell  $f$  in  $\Sigma^\top$ , for some  $1 \leq k \leq p$ . By definition of a normalisation strategy, the  $(k+1)$ -cell  $\sigma_f$  has source  $f$  and target  $\widehat{f}$ . Thus, if  $g$  is a  $k$ -cell which is parallel to  $f$ , the  $(k+1)$ -cell  $\sigma_f \star_k \sigma_g^-$  of  $\Sigma^\top$  has source  $f$  and target  $g$ , proving that  $\Sigma_{k+1}$  forms a homotopy basis of  $\Sigma_k^\top$ . Hence  $\Sigma$  is acyclic.

Conversely, let us assume that  $\Sigma$  is acyclic and let us define a right normalisation strategy  $\sigma$  (the case of a left one is symmetric). We can choose a 2-cell

$$\sigma_{x\widehat{u}} : x\widehat{u} \Rightarrow \widehat{xu}$$

for every 1-cell  $x$  in  $\overline{\Sigma}$  and every 1-cell  $u$  in  $\mathbf{C}$  such that  $x\widehat{u}$  is defined. Then, let us consider  $k \in \{1, \dots, p-2\}$ . Using the fact that  $\Sigma_{k+2}$  is a homotopy basis of  $\Sigma_{k+1}^\top$ , we choose an arbitrary  $(k+2)$ -cell

$$\sigma_{\varphi\widehat{u}} : \varphi\widehat{u} \longrightarrow \widehat{\varphi u}$$

for every  $(k+1)$ -cell  $\varphi$  in  $\Sigma$  and every 1-cell  $u$  in  $\Sigma^\top$ . We use Corollary 2.3.5 to conclude.  $\square$

**2.3.7. Corollary.** *Let  $\mathbf{C}$  be a small category and let  $p$  be a non-zero natural number. Then  $\mathbf{C}$  is  $\text{FDT}_p$  if and only if there exists a finite, (left, right) normalising track  $(1, p-1)$ -polygraph presenting  $\mathbf{C}$ .*

## 3. TRACK-POLYGRAPHIC RESOLUTIONS GENERATED BY CONVERGENT 2-POLYGRAPHS

### 3.1. Convergent 2-polygraphs

Let us recall notions and results from rewriting theory for 2-polygraphs [14, 15]. We fix a 2-polygraph  $\Sigma$ .

**3.1.1. Normal forms and termination.** We say that a 1-cell  $u$  of  $\Sigma_1^*$  *reduces* to a 1-cell  $v$  when  $\Sigma^*$  contains a non-degenerate 2-cell with source  $u$  and target  $v$ . We say that  $u$  is a *normal form* when it does not reduce to a 1-cell. A *normal form of  $u$*  is a 1-cell  $v$  which is a normal form and such that  $u$  reduces to  $v$ . A *reduction sequence* is a countable family  $(u_i)_{i \in \mathbb{I}}$  of 1-cells such that each  $u_i$  reduces to the following  $u_{i+1}$ .

We say that  $\Sigma$  *terminates* when it has no infinite reduction sequence. In that case, every 1-cell has at least one normal form. Moreover, *Noetherian induction* allows definitions and proofs of properties of 1-cells by induction on the maximum size of the 2-cells leading to normal forms.

**3.1.2. Branchings and confluence.** A *branching* is a non-ordered pair  $(f, g)$  of 2-cells of  $\Sigma^*$  with the same source. For  $f : u \Rightarrow v$  and  $g : u \Rightarrow w$ , the *source* of  $(f, g)$  is  $u$  and its *target* is  $(v, w)$ , which we write  $(f, g) : u \Rightarrow (v, w)$ . A branching  $(f, g)$  is *local* when  $f$  and  $g$  have size 1 and, in that case, it is:

- *aspherical* when  $f = g$  ;
- *Peiffer* when there exist 2-cells  $f', g'$  and an  $i \in \{0, 1\}$  such that

$$f = f' \star_i s(g') \quad \text{and} \quad g = s(f') \star_i g' ;$$

- *overlapping* otherwise.

### 3. Track-polygraphic resolutions generated by convergent 2-polygraphs

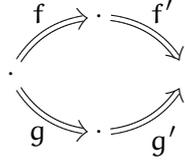
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Branchings are compared by the order  $\subseteq$  generated by the relations

$$(f, g) \subseteq (h \star_i f \star_i k, h \star_i g \star_i k)$$

given for any branching  $(f, g)$ , any 2-cells  $h, k$  and any  $i \in \{0, 1\}$ . A branching is *minimal* when it is a minimal element for  $\subseteq$ . A *critical branching* is a minimal overlapping branching. The terminology "aspherical" and "Peiffer" comes from the corresponding notions for spherical diagrams in Cayley complexes associated to presentations of groups, see [26]. The term "critical" comes from rewriting theory, [7, 2].

A branching  $(f, g)$  is *confluent* when there exists a pair  $(f', g')$  of 2-cells of  $\Sigma^*$  with the following shape:



We say that  $\Sigma$  is *confluent* (resp. *locally confluent*) when all of its branchings (resp. local branching) are confluent.

In a confluent 2-polygraph, every 1-cell has at most one normal form. Local confluence is equivalent to confluence of critical branchings. For terminating 2-polygraphs, Newman's Lemma ensures that local confluence and confluence are equivalent properties, [34].

**3.1.3. Convergence.** We say that  $\Sigma$  is *convergent* when it terminates and it is confluent. In that case, every 1-cell  $u$  has a unique normal form. Such a  $\Sigma$  is a *convergent presentation of  $\bar{\Sigma}$*  and has a canonical section  $\iota$  sending  $u$  to the corresponding normal form  $\hat{u}$ . Moreover, we have  $u \equiv_{\Sigma_2} v$  if and only if  $\hat{u} = \hat{v}$ . As a consequence, a finite and convergent 2-polygraph  $\Sigma$  yields a representation of the 1-cells of the category  $\bar{\Sigma}$ , together with a decision procedure for the corresponding word problem.

**3.1.4. Reduced 2-polygraphs.** A 2-polygraph  $\Sigma$  is *reduced* when, for every 2-cell  $\varphi : u \Rightarrow v$  in  $\Sigma$ , then  $u$  is a normal form for  $\Sigma_2 \setminus \{\varphi\}$  and  $v$  is a normal form for  $\Sigma_2$ . Let us note that, in that case, for every 1-cell  $u$  of  $\Sigma^*$ , there exists finitely many 2-cells with size 1 and source  $u$  in  $\Sigma^*$ : indeed, we have exactly one such 2-cell for every decomposition  $u = v w v'$  such that  $w$  is the source of a 2-cell of  $\Sigma$  and the number of decompositions  $u = v w v'$  is finite in a free category.

**3.1.5. Lemma.** *For every (finite) convergent 2-polygraph, there exists a (finite) Tietze-equivalent, reduced and convergent 2-polygraph.*

*Proof.* Let  $\Sigma$  be a (finite) convergent 2-polygraph  $\Sigma$ . We successively transform  $\Sigma$  as follows. First, we replace every 2-cell  $\varphi : u \Rightarrow v$  in  $\Sigma$  with  $\varphi' : u \Rightarrow \hat{u}$ . Then, if there exist several 2-cells in  $\Sigma$  with the same source, we drop all of them but one. Finally, we drop all the remaining 2-cells whose source is reducible by another 2-cell. After each step, we check that the (finite) 2-polygraph we get is convergent and that it is Tietze-equivalent to the former one. Moreover, the result is a reduced 2-polygraph.  $\square$

*Remark.* This result was proved by Métivier for term rewriting systems, [30], and by Squier for word rewriting systems, [38]. The proof works for any type of rewriting systems, including  $n$ -polygraphs for any  $n$ .

**3.1.6. The order relation on branchings.** Let  $\Sigma$  be a reduced 2-polygraph and let  $u$  be a 1-cell in  $\Sigma^*$ . We define the relation  $\preceq$  on 2-cells of  $\Sigma^*$  with size 1 and source  $u$  as follows. If  $\varphi$  and  $\psi$  are 2-cells of  $\Sigma$  and if  $f = v\varphi v'$  and  $g = w\psi w'$  have source  $u$ , then we write  $f \preceq g$  when  $v$  is smaller than  $w$ , *i.e.*, informally, when the part of  $u$  on which  $f$  acts is more at the left than the part on which  $g$  acts. By convention, we denote branchings of  $\Sigma$  in increasing order, *i.e.*,  $(f, g)$  when  $f \preceq g$ , which is always possible thanks to the following result.

**3.1.7. Lemma.** *Let  $\Sigma$  be a reduced 2-polygraph. The relation  $\preceq$  is an order, whose restriction to 2-cells with size 1 and source  $u$  is total, for any 1-cell  $u$ .*

*Proof.* From its definition, we already know that the relation  $\preceq$  is reflexive, transitive and total. For antisymmetry, we assume that  $f = v\varphi v'$  and  $g = w\psi w'$  are 2-cells with size 1 and source  $u$ , such that  $f \preceq g$  and  $g \preceq f$ , *i.e.*, such that  $v$  and  $w$  have the same size. Then, using the fact that  $\Sigma_1^*$  is free, we have  $v = w$  and either  $s(\varphi) = s(\psi)$  or  $s(\varphi) = s(\psi)\alpha$  or  $s(\varphi)\alpha = s(\psi)$ : the latter two cases cannot occur, because  $\Sigma$  is reduced and, from that same hypothesis we get, in the first case, that  $\varphi = \psi$ , hence  $f = g$ .  $\square$

**3.1.8. The leftmost and rightmost normalisation strategies.** Let  $\Sigma$  be a reduced 2-polygraph. If  $u$  is a 1-cell of  $\Sigma^*$  that is not in normal form, we denote by  $\lambda_u$  and  $\rho_u$  the minimum and maximum elements for  $\preceq$  of the (finite, non-empty) set of 2-cells with size 1 and source  $u$  in  $\Sigma^*$ . We sometimes use  $\lambda(u)$  and  $\rho(u)$  to denote the respective targets of  $\lambda_u$  and  $\rho_u$ . We note that, if  $(u, v)$  is a pair of composable 1-cells of  $\Sigma^*$ , we have  $\lambda_{uv} = \lambda_u v$  when  $u$  is reducible and  $\rho_{uv} = u \rho_v$  when  $v$  is reducible.

If  $\Sigma$  is also terminating, the *leftmost* and the *rightmost* normalisation strategies are respectively denoted by  $\lambda^*$  and  $\rho^*$  and defined by Noetherian induction on 1-cells of  $\Sigma^*$  as follows:

$$\begin{aligned} \lambda_{\hat{u}}^* &= 1_{\hat{u}} & \lambda_u^* &= \lambda_u \star 1_{\lambda_{\lambda(u)}}^* \\ \rho_{\hat{u}}^* &= 1_{\hat{u}} & \rho_u^* &= \rho_u \star 1_{\rho_{\rho(u)}}^* \end{aligned}$$

**3.1.9. Lemma.** *The normalisation strategies  $\lambda^*$  and  $\rho^*$  are respectively left and right  $\lambda$  normalisation strategies for  $\Sigma$  such that, for every 1-cell  $u$  in  $\Sigma^*$ , the 2-cells  $\lambda_u^*$  and  $\rho_u^*$  are in  $\Sigma^*$ .*

*Proof.* Let us check that  $\lambda^*$  is a leftmost normalisation strategy, the proof for  $\rho^*$  being symmetric. We must prove that, for every pair  $(u, v)$  of composable 1-cells of  $\Sigma^*$ , the following relation holds:

$$\lambda_{uv}^* = \lambda_u^* v \star 1_{\lambda_{\lambda(u)v}^*}.$$

We proceed by Noetherian induction on the 1-cell  $u$ . If  $u$  is a normal form, then  $\lambda_u^* = 1_u$  and  $\lambda_{\lambda(u)v}^* = \lambda_{uv}^*$ , so that the relation is satisfied. Otherwise, we have, using the definition of  $\lambda^*$ :

$$\lambda_{uv}^* = \lambda_{uv} \star 1_{\lambda_{\lambda(u)v}^*} = \lambda_u v \star 1_{\lambda_{\lambda(u)v}^*}.$$

We apply the induction hypothesis to  $\lambda(u)v$  to get:

$$\lambda_{uv}^* = \lambda_u v \star 1_{\lambda_{\lambda(u)v}^*} \star 1_{\lambda_{\lambda(u)v}^*} = \lambda_u^* v \star 1_{\lambda_{\lambda(u)v}^*}.$$

The fact that  $\lambda_u^*$  is in  $\Sigma^*$  is also proved by Noetherian induction on  $u$ , using the definition of  $\lambda^*$  and the facts that both  $1_{\hat{u}}$  and  $\lambda_u$  are 2-cells of  $\Sigma^*$ .  $\square$

### 3. Track-polygraphic resolutions generated by convergent 2-polygraphs

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*Remark.* A reduced and terminating 2-polygraph can have several left or right strategies, beside the leftmost and the rightmost ones. Indeed, let us consider the reduced and terminating 2-polygraph  $\Sigma$  with one 0-cell, three 1-cells  $a, b, c$  and the following three 2-cells:

$$aac \xrightarrow{\alpha} a \quad bb \xrightarrow{\beta} cc \quad acc \xrightarrow{\gamma} c$$

Let us prove that  $\Sigma$  admits at least two different left normalisation strategies. For that, we examine the 1-cell  $aabb$  and all the 2-cells of  $\Sigma^*$  from  $aabb$  to its normal form  $ac$ :

$$aabb \xrightarrow{aa\beta} aacc \begin{array}{c} \xrightarrow{\alpha c} \\ \xleftarrow{\alpha\gamma} \end{array} ac$$

Thus, if  $\sigma$  is a normalisation strategy, the 2-cell  $\sigma_{aabb}$  must be either  $aa\beta \star_1 \alpha c$  or  $aa\beta \star_1 \alpha\gamma$ . Since the 1-cells  $a, aa$  and  $aab$  are normal forms, assuming that  $\sigma$  is a left strategy still leaves us with the same choice. Hence, we can define a left normalisation strategy  $\sigma$  for  $\Sigma$  as follows:

$$\sigma_u = \begin{cases} aa\beta \star_1 \alpha\gamma & \text{if } u = aabb \\ \lambda_u^* & \text{otherwise.} \end{cases}$$

Thus, we have a left normalisation strategy for  $\Sigma$ , distinct from  $\lambda^*$ , as proved thereafter:

$$\sigma_{aabb} = aa\beta \star_1 \alpha\gamma \neq aa\beta \star_1 \alpha c = \lambda_{aabb} \star_1 \lambda_{aacc} = \lambda_{aabb}^*$$

Let us note that this phenomenon does not come from the fact that  $\Sigma$  is not confluent, since we can add the 2-cell  $\delta : bcc \Rightarrow cc b$  to  $\Sigma$  to get a reduced, convergent 2-polygraph which still has at least two left normalisation strategies. From  $\Sigma$ , we build a symmetric (for  $\star_0$ ) 2-polygraph that admits at least two right normalisation strategies.

However, we can ensure that, if  $\sigma$  is a left (resp. right) normalisation strategy for a reduced and terminating 2-polygraph  $\Sigma$  such that, for every 1-cell  $u$  of  $\Sigma^*$ , the 2-cell  $\sigma_u$  is in  $\Sigma^*$ , then this same 2-cell admits a decomposition

$$\sigma_u = \lambda_u \star_1 g_u \quad (\text{resp. } \sigma_u = \rho_u \star_1 g_u)$$

with  $g_u$  a 2-cell of  $\Sigma^*$ . Indeed, if  $\sigma$  is a left strategy, we consider the decomposition  $\lambda_u = v\varphi w$ . By definition of  $\lambda_u$ , the 1-cell  $vs(\varphi)$  is the source of only one 2-cell of  $\Sigma^*$  with size 1, namely  $v\varphi$ . Hence, since  $\sigma_{vs(\varphi)}$  is a 2-cell of  $\Sigma^*$  with source  $vs(\varphi)$ , it admits a decomposition

$$\sigma_{vs(\varphi)} = \lambda_{vs(\varphi)} \star_1 h$$

with  $h_u$  a 2-cell of  $\Sigma^*$ . We define the 2-cell  $g_u$  of  $\Sigma^*$  as

$$g_u = hw \star_1 \widehat{\sigma_{vs(\varphi)} w}$$

and use the hypothesis on  $\sigma$  to get:

$$\sigma_u = \sigma_{vs(\varphi)} w \star_1 \widehat{\sigma_{vs(\varphi)} w} = \lambda_{vs(\varphi)} w \star_1 g_u = \lambda_u \star_1 g_u.$$

The case of a right normalisation strategy is symmetric.

### 3.2. The acyclic track (1, 2)-polygraph of generating confluences

We fix a reduced convergent 2-polygraph  $\Sigma$  and its rightmost normalisation strategy, thereafter denoted by  $\sigma$ .

**3.2.1. Critical branchings of 2-polygraphs.** By case analysis on the source of critical branchings of  $\Sigma$ , we can conclude that they must have one of the following two shapes



where  $\varphi, \psi$  are 2-cells of  $\Sigma$ . The 2-polygraph  $\Sigma$  being reduced, the first case cannot occur since, otherwise, the source of  $\varphi$  would be reducible by  $\psi$ . Thus, every critical branching of  $\Sigma$  must have shape  $(\varphi v, u_1 \psi)$ . We write the branching in that order since, by definition of  $\preceq$ , we have  $\varphi v \preceq u_1 \psi$ .

We also note that the 1-cells  $u_1, u_2$  and  $v$  are normal forms and cannot be identities. Indeed, they are normal forms since, otherwise, at least one of the sources of  $\varphi$  and of  $\psi$  would be reducible by another 2-cell, preventing  $\Sigma$  from being reduced. If  $w$  was an identity, then the branching would be Peiffer. Thus, if  $u_1$  (resp.  $v$ ) was an identity, then the source of  $\psi$  (resp.  $\varphi$ ) would be reducible by  $\varphi$  (resp.  $\psi$ ).

Finally, if we write  $u = u_1 u_2$ , the definitions of  $\lambda_{uv}$  and of  $\rho_{uv}$  imply that we have:

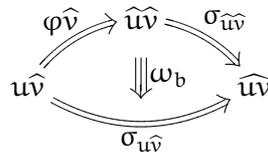
$$\lambda_{uv} = \varphi v \quad \text{and} \quad \rho_{uv} = u_1 \psi.$$

From all those observations, we conclude that every critical branching  $b$  of  $\Sigma$  must have shape

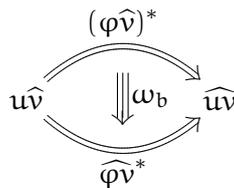
$$b = (\varphi \widehat{v}, \rho_{u\widehat{v}})$$

where  $u$  and  $v$  are 1-cells of  $\Sigma^*$  such that  $uv$  is defined and where  $\varphi$  is a 2-cell of  $\Sigma$  with source  $u$ . As a consequence, a finite, convergent and reduced 2-polygraph has finitely many critical branchings.

**3.2.2. The basis of generating confluences.** The *basis of generating confluences* of  $\Sigma$  is the cellular extension  $c_2(\Sigma)$  of  $\Sigma^\top$  made of one 3-cell



for every critical branching  $b = (\varphi \widehat{v}, \rho_{u\widehat{v}})$  of  $\Sigma$ . Alternatively,  $\omega_b$  can be pictured as follows:

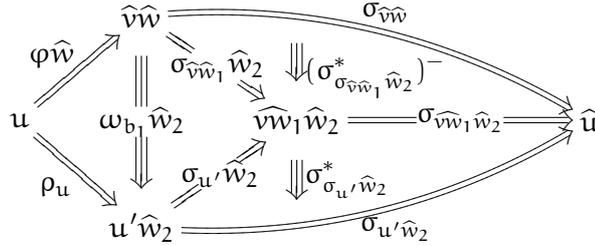


### 3. Track-polygraphic resolutions generated by convergent 2-polygraphs

**3.2.3. Lemma.** *The rightmost normalisation strategy of  $\Sigma$  extends to a right normalisation strategy of  $c_2(\Sigma)$ .*

*Proof.* First, we define a 3-cell  $\sigma_f^* : f^* \Rightarrow \widehat{f}^*$  in  $c_2(\Sigma)^\top$ , for  $f$  a 2-cell of  $\Sigma^*$ , by Noetherian induction on the source of  $f$ . If this source is a normal form  $u$ , then  $f = f^* = 1_u$  and  $\sigma_f^*$  must be  $1_{1_u}$ . Now, let us fix a 1-cell  $u$ , that is not a normal form. Let us assume that, for every 1-cell  $v$  in which  $u$  reduces and every 2-cell  $f$  in  $\Sigma^*$  with source  $v$ , we have defined  $\sigma_f^*$ . Let us start by defining a 3-cell  $\sigma_{\varphi\widehat{w}}^*$  in  $c_2(\Sigma)^\top$  for every 2-cell  $\varphi : v \Rightarrow \widehat{v}$  in  $\Sigma$  and every 1-cell  $w$  in  $\widehat{\Sigma}$  such that  $u = v\widehat{w}$ . We proceed by case analysis on the type of local branching of  $b = (\varphi\widehat{w}, \rho_u)$ .

- If  $b$  is aspherical, then  $\rho_u = \varphi\widehat{w}$ . In that case, we define  $\sigma_{\varphi\widehat{w}}^* = 1_{(\varphi\widehat{w})^*}$ .
- The branching  $b$  cannot be Peiffer, by hypothesis.
- Otherwise, we have  $\widehat{w} = \widehat{w}_1\widehat{w}_2$  and  $b_1 = (\varphi\widehat{w}_1, \rho_{v\widehat{w}_1})$  is a critical branching of  $\Sigma$ . We define the 3-cell  $\sigma_{\varphi\widehat{w}}^*$  of  $c_2(\Sigma)^\top$  as the following composite:



Hence, we have defined a 3-cell  $\sigma_{\varphi\widehat{w}}^* : (\varphi\widehat{w})^* \Rightarrow \widehat{\varphi\widehat{w}}^*$  in  $c_2(\Sigma)^\top$ . We extend this definition to every 2-cell  $f$  in  $\Sigma^*$  with source  $u$  by using the commutation properties of a right normalisation strategy with the compositions.

Then, we extend  $\sigma$  to every 2-cell of  $c_2(\Sigma)^\top$  by using the commutation properties of a normalisation strategy with the inverse.  $\square$

**3.2.4. Proposition.** *The track  $(1, 2)$ -polygraph  $c_2(\Sigma)$  is acyclic.*

*Remark.* This result is already contained in [15], with a different proof. Indeed, there it was shown that the generating confluences of a convergent  $n$ -polygraph  $\Sigma$  form a homotopy basis of the track  $n$ -category  $\Sigma^\top$ .

**3.2.5. Corollary.** *A category with a finite convergent presentation is  $FDT_3$ .*

**3.2.6. Corollary (Squier, [39]).** *A monoid with a finite convergent presentation has finite derivation type.*

### 3.3. The acyclic track $(1, 3)$ -polygraph of generating triple confluences

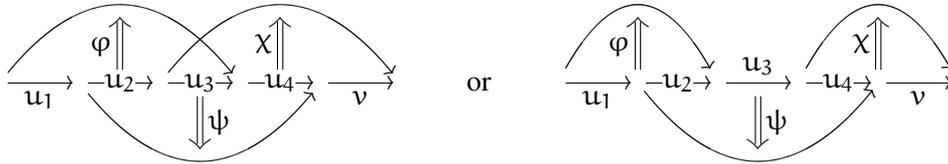
Let  $\Sigma$  be a reduced and convergent 2-polygraph.

### 3.3. The acyclic track (1, 3)-polygraph of generating triple confluences

**3.3.1. Triple branchings of 2-polygraphs.** A *triple branching* of  $\Sigma$  is a triple  $(f, g, h)$  of 2-cells of  $\Sigma^*$  with the same source and such that  $f \preceq g \preceq h$ . The triple branching  $(f, g, h)$  is *local* when  $f, g$  and  $h$  have size 1. A local triple branching  $(f, g, h)$  is:

- *aspherical* when either  $(f, g)$  or  $(g, h)$  is aspherical;
- *Peiffer* when either  $(f, g)$  or  $(g, h)$  is Peiffer;
- *overlapping*, otherwise.

Triple branchings are ordered by inclusion, similarly to branchings. A *critical* triple branching is a minimal overlapping triple branching. Such a triple branching can have two different shapes, where  $\varphi, \psi$  and  $\chi$  are generating 2-cells (those two shapes of critical triple branchings are sufficient for a reduced 2-polygraph but, in a general situation, the other possible shape of critical branchings, with an inclusion of one source into the other one, generates several other possibilities):

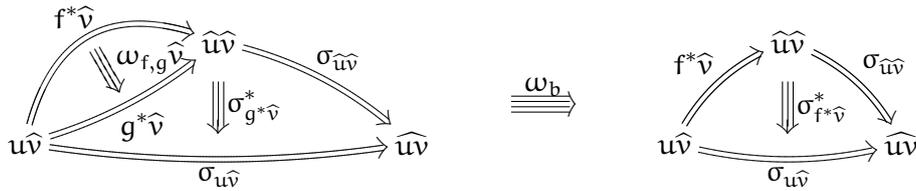


We note that, in either case, the corresponding critical triple branching  $b$  has shape

$$b = ((f, g)\widehat{v}, \rho_{u\widehat{v}}) = (f\widehat{v}, g\widehat{v}, \rho_{u\widehat{v}})$$

where  $(f, g)$  is a critical branching of  $\Sigma$  with source  $u$  and  $v$  is a 1-cell of  $\Sigma^*$ . Indeed, for the first (resp. second) case, we note that  $v$  must be a normal form for  $\Sigma$  to be reduced, we write  $u = u_1u_2u_3u_4$ ,  $f = \varphi u_4$  (resp.  $f = \varphi u_3u_4$ ),  $g = u_1\psi$  and we use the definition of  $\rho_{uv}$  to conclude that  $\rho_{uv} = u_1u_2\chi$  (resp.  $\rho_{uv} = u_1u_2u_3\chi$ ). As a consequence of this classification, a finite, reduced and convergent 2-polygraph has a finite number of critical triple branchings.

**3.3.2. The basis of generating triple confluences.** The *basis of generating triple confluences* of  $\Sigma$  is the cellular extension  $c_3(\Sigma)$  of  $c_2(\Sigma)^\top$  made of one 4-cell



for every critical triple branching  $b = (f\widehat{v}, g\widehat{v}, \rho_{u\widehat{v}})$  of  $\Sigma$ . By definition of the notations  $A^*$  and  $\widehat{A}$  for a 2-cell or 3-cell  $A$ , the 4-cell  $\omega_b$  can also be written

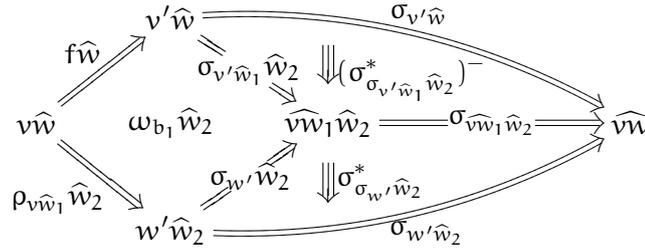


### 3. Track-polygraphic resolutions generated by convergent 2-polygraphs

**3.3.3. Lemma.** *The right normalisation strategy of  $c_2(\Sigma)$  extends to a right normalisation strategy of  $c_3(\Sigma)$ .*

*Proof.* First, we define a 4-cell  $\sigma_A^* : A^* \Rightarrow \widehat{A}^*$ , for  $A$  a 3-cell of  $c_2(\Sigma)^*$  by Noetherian induction on the 1-source of  $A$ . If this source is a normal form, then we define  $\sigma_A^* = 1_{1_{1_u}}$ . Now, let us fix a 1-cell  $u$  that is not a normal form. We assume that, for every 1-cell  $v$  in which  $u$  reduces and every 3-cell  $A$  in  $c_2(\Sigma)^*$  with 1-source  $v$ , we have defined the 4-cell  $\sigma_A^*$ . We start by defining a 4-cell  $\sigma_{\omega_{f,g}\widehat{w}}$ , for every critical branching  $(f, g)$  of  $\Sigma$  with source  $v$  and every 1-cell  $w$  in  $\overline{\Sigma}$  such that  $u = v\widehat{w}$ . We proceed by case analysis on the type of local triple branching of  $b = (f\widehat{w}, g\widehat{w}, \rho_u)$ .

- If  $b$  is aspherical, then  $\rho_u = g\widehat{w}$ . In that case, we define  $\sigma_{\omega_{f,g}\widehat{w}}^* = 1_{(\omega_{f,g}\widehat{w})^*}$ .
- The triple branching  $b$  cannot be Peiffer, by hypothesis.
- Otherwise, we have  $\widehat{w} = \widehat{w}_1\widehat{w}_2$  and  $b_1 = (f\widehat{w}_1, g\widehat{w}_1, \rho_{v\widehat{w}_1})$  is a critical triple branching of  $\Sigma$ . We define the 4-cell  $\sigma_{\omega_{f,g}\widehat{w}}^*$  as the following composite:



Hence, we have defined a 4-cell  $\sigma_{\omega_{f,g}\widehat{w}}^* : (\omega_{f,g}\widehat{w})^* \Rightarrow \widehat{\omega_{f,g}\widehat{w}}^*$  in  $c_3(\Sigma)^\top$ . We extend this definition to every 3-cell  $A$  in  $c_2(\Sigma)^*$  with 1-source  $u$  by using the commutation properties of a right normalisation strategy with the compositions.

Then, we use the commutation properties of a normalisation strategy with the inverse to extend  $\sigma$  to any 3-cell of  $c_2(\Sigma)^\top$ .  $\square$

**3.3.4. Proposition.** *The track  $(1, 3)$ -polygraph  $c_3(\Sigma)$  is acyclic.*

**3.3.5. Corollary.** *A category with a finite convergent presentation is  $FDT_4$ .*

### 3.4. The acyclic track $(1, \infty)$ -polygraph generated by a convergent 2-polygraph

Let  $\Sigma$  be a reduced and convergent 2-polygraph and let us extend it into an acyclic track  $(1, \infty)$ -polygraph denoted by  $c_\infty(\Sigma)$  and whose generating  $p$ -cells, for  $p \geq 3$ , are (indexed by) the  $(p - 1)$ -fold critical branchings of  $\Sigma$ . We proceed by induction on  $p$ , having already seen the base cases, for  $p = 2$  and  $p = 3$ . The induction case follows the construction of  $c_3(\Sigma)$ , so we go faster here.

### 3.4. The acyclic track $(1, \infty)$ -polygraph generated by a convergent 2-polygraph

**3.4.1. Higher branchings of 2-polygraphs.** A  $p$ -fold branching of  $\Sigma$  is a family  $(f_1, \dots, f_p)$  of 2-cells of  $\Sigma^*$  with size 1, with the same source and such that  $f_1 \preceq \dots \preceq f_p$ . We define local, aspherical, Peiffer, overlapping, minimal and critical branchings in a similar way to the cases  $p = 2$  and  $p = 3$ . As before, we study the possible shapes of a  $p$ -fold critical branching  $b$  of  $\Sigma$  and we conclude that it must have shape

$$b = (c\widehat{v}, \rho_{u\widehat{v}})$$

where  $c$  is a critical  $(p - 1)$ -fold branching of  $\Sigma$  with source  $u$ . Hence, if  $\Sigma$  is finite, it has finitely many critical  $p$ -fold branchings.

**3.4.2. The basis of generating  $p$ -fold confluences.** The *basis of generating  $p$ -fold confluences* of  $\Sigma$  is the cellular extension  $c_p(\Sigma)$  of  $c_{p-1}(\Sigma)^\top$  made of one  $(p + 1)$ -cell

$$\omega_b : (\omega_c\widehat{v})^* \longrightarrow \widehat{\omega_c v}^*$$

for every critical  $p$ -fold branching  $b = (c\widehat{v}, \rho_{u\widehat{v}})$  of  $\Sigma$ .

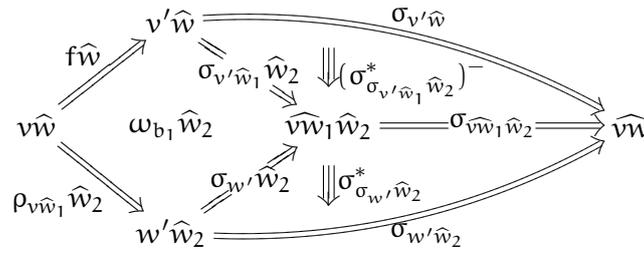
The extension of the right normalisation strategy to  $c_p(\Sigma)$  is made in the same way as in the case  $p = 3$ . It relies on a Noetherian induction and a case analysis, whose main point is to define a  $(p + 1)$ -cell

$$\sigma_{\omega_c\widehat{w}}^* : (\omega_c\widehat{w})^* \longrightarrow \widehat{\omega_c w}^*$$

in  $c_p(\Sigma)^\top$  for every local  $p$ -fold branching

$$b = (c\widehat{w}, \rho_{v\widehat{w}})$$

of  $\Sigma$  such that  $\widehat{w} = \widehat{w}_1\widehat{w}_2$  and such that  $b_1 = (c\widehat{w}_1, \rho_{v\widehat{w}_1})$  is a critical  $p$ -fold branching of  $\Sigma$ . As in the case  $p = 3$ , we define the  $(p + 1)$ -cell  $\sigma_{\omega_c\widehat{w}}^*$  as the following composite, where  $f$  is the first 2-cell of the critical  $p$ -fold branching  $c$ :



As a conclusion of this construction, we get that the track  $(1, p)$ -polygraph  $c_p(\Sigma)$  is acyclic.

**3.4.3. Theorem.** *Every convergent 2-polygraph  $\Sigma$  extends to a Tietze-equivalent, acyclic track  $(1, \infty)$ -polygraph  $c_\infty(\Sigma)$ , whose generating  $p$ -cells, for every  $p \geq 3$ , are indexed by the critical  $(p - 1)$ -fold branchings of  $\Sigma$ .*

As a consequence, we have proved:

**3.4.4. Theorem.** *A category with a finite convergent presentation is  $\text{FDT}_\infty$ .*

**3.4.5. Corollary.** *If  $\mathbf{C}$  is a category with a convergent presentation with no critical  $p$ -fold branching, for some  $p \geq 2$ , then  $d_{\text{pol}}(\mathbf{C}) \leq p$ .*

## 4. ABELIANISATION OF TRACK-POLYGRAPHIC RESOLUTIONS

### 4.1. Resolutions of finite type

**4.1.1. Modules over a category, [32].** Let  $\mathbf{C}$  be a small category. A  $\mathbf{C}$ -module is a functor from  $\mathbf{C}$  to the category of Abelian groups  $\mathbf{Ab}$ . The  $\mathbf{C}$ -modules and natural transformations between them form an Abelian category with enough projectives, denoted by  $\mathbf{Mod}(\mathbf{C})$ . Equivalently,  $\mathbf{Mod}(\mathbf{C})$  can be described as the category of additive functors from  $\mathbb{Z}\mathbf{C}$  to  $\mathbf{Ab}$ , where  $\mathbb{Z}\mathbf{C}$  is the free  $\mathbb{Z}$ -category over  $\mathbf{C}$ : its objects are the ones of  $\mathbf{C}$  and each hom-set  $\mathbb{Z}\mathbf{C}(x, y)$  is the free Abelian group generated by  $\mathbf{C}(x, y)$ .

A free  $\mathbf{C}$ -module is a coproduct of representable functors  $\mathbb{Z}\mathbf{C}(p, -)$ , denoted by  $\mathbf{C}_p$ . A  $\mathbf{C}$ -module  $M$  is *finitely generated* if there exists an epimorphism of  $\mathbf{C}$ -modules  $F \rightarrow M$ , with  $F$  free.

The tensor product over  $\mathbf{C}$  of a  $\mathbf{C}^0$ -module  $M$  and a  $\mathbf{C}$ -module  $N$  is the Abelian group  $M \otimes_{\mathbf{C}} N$  defined by:

$$M \otimes_{\mathbf{C}} N = \left( \bigoplus_{x \in \mathbf{C}_0} M(x) \otimes_{\mathbb{Z}} N(x) \right) / Q$$

where  $Q$  is the subgroup of  $\bigoplus_{x \in \mathbf{C}_0} M(x) \otimes_{\mathbb{Z}} N(y)$  generated by the elements

$$M(u)(a) \otimes b - a \otimes N(u)(b), \quad u \in \mathbf{C}(x, y), \quad a \in M(y), \quad b \in N(x).$$

**4.1.2. Modules of type  $\mathbf{FP}_p$ .** Let  $\mathbf{C}$  be a small category. A  $\mathbf{C}$ -module  $M$  is of *homological type  $\mathbf{FP}_p$* , for  $0 \leq p \leq \infty$ , when there exists a projective, finitely generated resolution of  $M$  in the category of  $\mathbf{C}$ -modules:

$$P_p \longrightarrow P_{p-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

As a generalisation of Schanuel's lemma, we have, given two exact sequences

$$0 \longrightarrow P_p \longrightarrow P_{p-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow P'_p \longrightarrow P'_{p-1} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow M \longrightarrow 0$$

with  $P_i$  and  $P'_i$  projective and finitely generated for every  $0 \leq i \leq p-1$ , then  $P_p$  is finitely generated if and only if  $P'_p$  is finitely generated. This yields the following characterisation of the property  $\mathbf{FP}_p$ :

**4.1.3. Lemma.** *Let  $\mathbf{C}$  be a small category, let  $M$  be a  $\mathbf{C}$ -module and let  $p$  be a natural number. The following assertions are equivalent:*

- i) *The  $\mathbf{C}$ -module  $M$  is of homological type  $\mathbf{FP}_p$ .*
- ii) *There exists a free, finitely generated resolution of  $M$*

$$F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

iii) The  $\mathbf{C}$ -module  $M$  is finitely generated and, for every  $0 \leq k < p$  and every finitely generated and projective resolution of  $M$

$$P_k \xrightarrow{d_k} P_{k-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

the  $\mathbf{C}$ -module  $\text{Ker } d_k$  is finitely generated.

**4.1.4. Lemma.** Let  $\text{Lan}_F : \text{Mod}(\mathbf{C}) \rightarrow \text{Mod}(\mathbf{D})$  be the additive left Kan extension along a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$ . If  $M$  is a  $\mathbf{C}$ -module of homological type  $\text{FP}_p$  then  $\text{Lan}_F(M)$  is a  $\mathbf{D}$ -module of homological type  $\text{FP}_p$ .

*Proof.* Let us assume that  $M$  is a  $\mathbf{C}$ -module of type  $\text{FP}_p$ . Then there exists a finitely generated, projective resolution  $P_* \rightarrow M$ . If  $\xi$  is a 0-cell in  $\mathbf{D}$ , then we have:

$$\text{Lan}_F(M)(\xi) = \mathbb{Z}\mathbf{D}(F, \xi) \otimes_{\mathbf{C}} M.$$

Since each  $\mathbf{C}$ -module  $P_i$  is finitely generated and projective, then so is the  $\mathbf{D}$ -module  $\text{Lan}_F(P_i)$ . Moreover, the functor  $\text{Lan}_F$  is right exact: it follows that  $\text{Lan}_F(P_*) \rightarrow \text{Lan}_F(M)$  is a finitely generated, projective resolution. This proves that  $\text{Lan}_F(M)$  is of type  $\text{FP}_p$ .  $\square$

## 4.2. Categories of finite homological type

**4.2.1. Natural systems of Abelian groups.** Let  $\mathbf{C}$  be a category. The *category of factorisations of  $\mathbf{C}$*  is the category, denoted by  $\text{FC}$ , whose objects are the morphisms of  $\mathbf{C}$  and whose morphisms from  $w$  to  $w'$  are pairs  $(u, v)$  of morphisms of  $\mathbf{C}$  such that the following diagram commutes in  $\mathbf{C}$ :

$$\begin{array}{ccc} & w & \\ u \uparrow & \xrightarrow{\quad} & \downarrow v \\ & \text{\textcircled{c}} & \\ & \xrightarrow{\quad} & w' \end{array}$$

In such a situation, the triple  $(u, w, v)$  is called a *factorisation of  $w'$* . Composition in  $\text{FC}$  is defined by pasting: if  $(u, v) : w \rightarrow w'$  and  $(u', v') : w' \rightarrow w''$  are morphisms in  $\text{FC}$ , then  $(u, v)(u', v')$  is  $(u'u, vv')$ . The identity of  $w$  is  $(1_{s(w)}, 1_{t(w)})$ .

A *natural system (of Abelian groups) on  $\mathbf{C}$*  is an  $\text{FC}$ -module  $D$ , i.e., a functor  $D : \text{FC} \rightarrow \mathbf{Ab}$ . As in [5], we denote by  $D_w$  the Abelian group which is the image of  $w$  by  $D$ . If there is no confusion, we denote by  $uav$  the image of  $a \in D_w$  through the morphism of groups  $D(u, v) : D_w \rightarrow D_{w'}$ . The category of natural systems on  $\mathbf{C}$  is denoted by  $\text{Nat}(\mathbf{C})$ .

**4.2.2. Free natural systems.** Given a subset  $X$  of the set of 1-cells of  $\mathbf{C}$ , we denote by  $\text{F}_C[X]$  the *free natural system on  $\mathbf{C}$  generated by  $X$* , which is defined by

$$\text{F}_C[X] = \bigoplus_{x \in X} \text{FC}_x.$$

## 4. Abelianisation of track-polygraphic resolutions

In particular, if  $\Sigma$  is a  $(1, p)$ -polygraph such that  $\bar{\Sigma} \simeq \mathbf{C}$ , we consider:

- The free natural system  $F_{\mathbf{C}}[\Sigma_0]$  generated by the 1-cells  $1_x$ , for  $x \in \Sigma_0$ . If  $w$  is a 1-cell in  $\mathbf{C}$ , then  $F_{\mathbf{C}}[\Sigma_0]_w$  is the free Abelian group generated by the pairs  $(u, v)$  of 1-cells of  $\mathbf{C}$  such that  $uv = w$ .
- For every  $1 \leq k \leq p$ , the free natural system  $F_{\mathbf{C}}[\Sigma_k]$  generated by the 1-cells  $\bar{\varphi}$ , for  $\varphi \in \Sigma_k$ . If  $w$  is a 1-cell in  $\mathbf{C}$ , then  $F_{\mathbf{C}}[\Sigma_k]_w$  is the free Abelian group generated by the triples  $(u, \varphi, v)$ , thereafter denoted by  $u[\varphi]v$ , made of a  $k$ -cell  $\varphi$  of  $\Sigma_k$  and 1-cells  $u, v$  of  $\mathbf{C}$  such that  $u\bar{\varphi}v = w$ .

**4.2.3. Categories of finite homological type.** The property for a small category  $\mathbf{C}$  to be of homological type  $\text{FP}_p$  is defined according to a category of modules over one of the categories in following diagram

$$\begin{array}{ccccc} & & \mathbf{C}^0 & & \\ & & \nearrow & & \searrow \\ \mathbf{FC} & \xrightarrow{\pi} & \mathbf{C}^0 \times \mathbf{C} & & \mathbf{C}^\top \\ & & \searrow & & \nearrow \\ & & \mathbf{C} & & \end{array}$$

where  $\mathbf{C}^\top$  is the groupoid generated by  $\mathbf{C}$ ,  $\pi$  is the projection  $u \mapsto (s(u), t(u))$ ,  $p_1$  and  $p_2$  are the projections of the cartesian product,  $q_1$  and  $q_2$  are the injections  $u^o \mapsto u^-$  and  $u \mapsto u$ . Let us denote by  $\mathbb{Z}$  the constant natural system on  $\mathbf{C}$  given, for any 1-cell  $u$  of  $\mathbf{C}$ , by

$$\mathbb{Z}_u = \mathbb{Z} \quad \text{and} \quad \mathbb{Z}(u, 1) = \mathbb{Z}(1, u) = 1_{\mathbb{Z}},$$

The functor  $\text{Lan}_\pi(\mathbb{Z})$  is the  $\mathbf{C}^0 \times \mathbf{C}$ -module  $\mathbb{Z}\mathbf{C}$  and the functors  $\text{Lan}_{p_i\pi}(\mathbb{Z})$ ,  $\text{Lan}_{q_i p_i\pi}(\mathbb{Z})$  are the constant modules equal to  $\mathbb{Z}$ .

A small category  $\mathbf{C}$  is of *homological type*

- i)  $\text{FP}_p$  when the constant natural system  $\mathbb{Z}$  is of type  $\text{FP}_p$ ,
- ii) *bi*- $\text{FP}_p$  when the  $\mathbf{C}^0 \times \mathbf{C}$ -module  $\mathbb{Z}\mathbf{C}$  is of type  $\text{FP}_p$ ,
- iii) *left*- $\text{FP}_p$  when the constant  $\mathbf{C}$ -module  $\mathbb{Z}$  is of type  $\text{FP}_p$ ,
- iv) *right*- $\text{FP}_p$  when the constant  $\mathbf{C}^0$ -module  $\mathbb{Z}$  is of type  $\text{FP}_p$ ,
- v) *top*- $\text{FP}_p$  when the constant  $\mathbf{C}^\top$ -module  $\mathbb{Z}$  is of type  $\text{FP}_p$ .

**4.2.4. Proposition.** i) For small categories, we have the following implications:

$$\text{FP}_p \Rightarrow \text{bi-FP}_p \Rightarrow \text{left-FP}_p \text{ or } \text{right-FP}_p \Rightarrow \text{top-FP}_p.$$

ii) For small groupoids, the conditions  $\text{FP}_p$ , *bi*- $\text{FP}_p$ , *left*- $\text{FP}_p$ , *right*- $\text{FP}_p$  and *top*- $\text{FP}_p$  are equivalent.

*Proof.* Let us prove i). We have  $\text{Lan}_\pi(\mathbb{Z}) = \mathbb{Z}\mathbf{C}$  and  $\text{Lan}_{q_i}(\mathbb{Z}) = \mathbb{Z}$ . Hence the first and last implications are consequences of Lemma 4.1.4. If  $P_* \rightarrow \mathbb{Z}\mathbf{C}$  is a finitely generated resolution of  $\mathbf{C}^0 \times \mathbf{C}$ -modules then  $P_* \otimes_{\mathbf{C}} \mathbb{Z}$  (resp.  $\mathbb{Z} \otimes_{\mathbf{C}} P_*$ ) is a finitely generated resolution of the  $\mathbf{C}$ -module (resp.  $\mathbf{C}^0$ -module)  $\mathbb{Z}$ , yielding the middle implication.

Let us prove **ii**). For a groupoid  $\mathbf{G}$ , the  $\mathbf{G}$ -modules,  $\mathbf{G}^{\circ}$ -modules,  $\mathbf{G}^{\circ} \times \mathbf{G}$ -modules and  $\mathbf{G}^{\top}$ -modules coincide. Hence the conditions  $\text{bi-FP}_p$ ,  $\text{left-FP}_p$ ,  $\text{right-FP}_p$  and  $\text{top-FP}_p$  are equivalent. There remains to prove that  $\text{left-FP}_p$  implies  $\text{FP}_p$ . First, we define, for every  $\mathbf{G}$ -module  $M$ , a natural system  $\widetilde{M}$  on  $\mathbf{G}$  by  $\widetilde{M}_g = M(\text{t}(g))$ , for any 1-cell  $g$  in  $\mathbf{G}$ , and  $\widetilde{M}(h, k) = M(k) : M(\text{t}(g)) \rightarrow M(\text{t}(g'))$ , for any factorisation  $g' = hkg$  in  $\mathbf{G}$ . As a direct consequence of this construction, if  $M$  is a finitely generated projective  $\mathbf{G}$ -module then  $\widetilde{M}$  is a finitely generated projective natural system on  $\mathbf{G}$ . Thus, if  $P_* \rightarrow \mathbb{Z}$  is a projective resolution of finitely generated  $\mathbf{G}$ -modules, then  $\widetilde{P}_* \rightarrow \mathbb{Z}$  is a projective resolution of finitely generated natural systems.  $\square$

*Remark.* The converse of the second and third implications in **i**) of Proposition 4.2.4 do not hold in general. Indeed, Cohen constructed a  $\text{right-FP}_{\infty}$  monoid which is not  $\text{left-FP}_1$ : thus, the properties  $\text{top-FP}_p$ ,  $\text{left-FP}_p$  and  $\text{right-FP}_p$  are not equivalent in general, [10]. Moreover, monoids with a finite convergent presentation are of types  $\text{left-FP}_{\infty}$  and  $\text{right-FP}_{\infty}$ , [38, 1, 19], but there exists a finitely presented monoid, of types  $\text{left-FP}_{\infty}$  and  $\text{right-FP}_{\infty}$ , which does not satisfy the homological finiteness condition FHT, introduced by Pride and Wang, [20]; since the property FHT and  $\text{bi-FP}_3$  are equivalent, [21], it follows that the properties  $\text{left-FP}_p$  and  $\text{right-FP}_p$  do not imply the property  $\text{bi-FP}_p$  in general. We conjecture that the converse of the first implication is not true either, but this is still an open problem.

**4.2.5. Finite homological type and homology.** The cohomology of categories with values in natural system was defined in [44] and [5]. Let us define the homology of a category  $\mathbf{C}$  with values in a *contravariant natural system*  $D$  on  $\mathbf{C}$ , that is an  $(\text{FC})^{\circ}$ -module.

We consider the nerve  $N_*(\mathbf{C})$  of  $\mathbf{C}$ , with boundary maps  $d_i : N_n(\mathbf{C}) \rightarrow N_{n-1}(\mathbf{C})$ , for  $0 \leq i \leq n$ . For  $s = (u_1, \dots, u_n)$  in  $N_n(\mathbf{C})$ , we denote by  $\bar{s}$  the composite 1-cell  $u_1 \cdots u_n$  of  $\mathbf{C}$ . For every natural number  $n$ , the  $n$ -th chain group  $C_n(\mathbf{C}, D)$  is defined as the Abelian group

$$C_n(\mathbf{C}, D) = \bigoplus_{s \in N_n(\mathbf{C})} D_{\bar{s}}.$$

We denote by  $\iota_s$  the embedding of  $D_{\bar{s}}$  into  $C_n(\mathbf{C}, D)$ . The boundary map  $d : C_n(\mathbf{C}, D) \rightarrow C_{n-1}(\mathbf{C}, D)$  is defined, on the component  $D_{\bar{s}}$  of  $C_n(\mathbf{C}, D)$ , by:

$$d\iota_s = \iota_{d_0(s)}u_{1*} + \sum_{i=1}^{n-1} (-1)^i \iota_{d_i(s)} + (-1)^n \iota_{d_n(s)}u_n^*,$$

with  $s = (u_1, \dots, u_n)$  and where  $u_{1*}$  and  $u_n^*$  respectively denote  $D(u_1, 1)$  and  $D(1, u_n)$ . The homology of  $\mathbf{C}$  with coefficients in  $D$  is defined as the homology of the complex  $(C_*(\mathbf{C}, D), d_*)$ :

$$H_*(\mathbf{C}, D) = H_*(C_*(\mathbf{C}, D), d_*).$$

We denote by  $\text{Tor}_*^{\text{FC}}(D, -)$  the left derived functor from the functor  $D \otimes_{\text{FC}} -$ . One proves that there is an isomorphism which is natural in  $D$ :

$$H_*(\mathbf{C}, D) \simeq \text{Tor}_*^{\text{FC}}(D, \mathbb{Z}).$$

As a consequence, using Lemma 4.1.3, we get:

**4.2.6. Proposition.** *If a category  $\mathbf{C}$  is of homological type  $\text{FP}_p$ , for a natural number  $p$ , then the Abelian group  $H_k(\mathbf{C}, \mathbb{Z})$  is finitely generated for every  $0 \leq k \leq p$ .*

## 4. Abelianisation of track-polygraphic resolutions

### 4.3. The Reidemeister-Fox-Squier complex

**4.3.1. Derivations of a category.** Let  $\mathbf{C}$  be a small category and let  $D$  be a natural system on  $\mathbf{C}$ . We recall from [5] that a *derivation of  $\mathbf{C}$  with values into  $D$*  is a mapping  $d$  that sends every 1-cell  $u$  of  $\mathbf{C}$  to an element of  $D_u$ , such that, for every composable 1-cells  $u$  and  $v$ , the following relation holds:

$$d(uv) = ud(v) + d(u)v.$$

Thus, in the particular case of a free category  $\Sigma^*$ , a derivation of  $\Sigma^*$  into  $D$  is characterised by its values on the 1-cells of  $\Sigma$ .

**4.3.2. Lemma.** *Let  $\Sigma$  be a  $(1, p)$ -polygraph. For every  $1 \leq k \leq p$ , there exists a unique map  $[\cdot]$  from  $\Sigma_k^\top$  to  $F_{\bar{\Sigma}}[\Sigma_k]$  that extends the inclusion of  $\Sigma_k$  into  $F_{\bar{\Sigma}}[\Sigma_k]$  and such that the following relations hold:*

$$[1_x] = 0 \quad [x^-] = -[x] \quad [x \star_i y] = \begin{cases} [x]\bar{y} + \bar{x}[y] & \text{if } i = 0 \\ [x] + [y] & \text{otherwise} \end{cases}$$

*Proof.* Those relations give a way to define an element  $[x]$  of  $F_{\bar{\Sigma}}[\Sigma_k]$  for every  $k$ -cell  $x$  in  $\Sigma_k^\top$ . To prove that  $[x]$  is uniquely defined, one checks that  $[\cdot]$  is compatible with the defining relations of a track  $(1, p)$ -category. For example, we have, for every  $0 \leq i < j \leq p$ :

$$[(x \star_i y) \star_j (z \star_i t)] = [(x \star_j z) \star_i (y \star_j t)] = \begin{cases} [x]\bar{y} + \bar{x}[y] + [z]\bar{t} + \bar{z}[t] & \text{if } i = 0 \\ [x] + [y] + [z] + [t] & \text{otherwise} \end{cases} \quad \square$$

**4.3.3. The Reidemeister-Fox-Squier complex.** Let  $\Sigma$  be a track  $(1, p)$ -polygraph. For  $1 \leq k \leq p + 1$ , the  $k$ -th Reidemeister-Fox-Squier boundary map of  $\Sigma$  is the morphism of natural systems

$$\delta_k : F_{\bar{\Sigma}}[\Sigma_k] \longrightarrow F_{\bar{\Sigma}}[\Sigma_{k-1}]$$

defined, on a  $k$ -cell  $x$  in  $\Sigma$ , by:

$$\delta_k[x] = \begin{cases} (\bar{x}, 1) - (1, \bar{x}) & \text{if } k = 1 \\ [s(x)] - [t(x)] & \text{otherwise.} \end{cases}$$

The *augmentation map* of  $\Sigma$  is the morphism of natural systems  $\varepsilon : F_{\bar{\Sigma}}[\Sigma_0] \rightarrow \mathbb{Z}$  defined, for every pair  $(u, v)$  of composable 1-cells of  $\bar{\Sigma}$ , by:

$$\varepsilon(u, v) = 1.$$

By induction on the size of cells of  $\Sigma^\top$ , one proves that, for every  $k$ -cell  $f$  in  $\Sigma^\top$ , with  $k \geq 1$ , the following holds:

$$\delta_k[f] = \begin{cases} (\bar{f}, 1) - (1, \bar{f}) & \text{if } k = 1 \\ [s(f)] - [t(f)] & \text{otherwise.} \end{cases}$$

As a consequence, we have  $\varepsilon\delta_1 = 0$  and  $\delta_k\delta_{k+1} = 0$ , for every  $1 \leq k \leq p$ . Thus, we get the following chain complex of natural systems on  $\bar{\Sigma}$

$$F_{\bar{\Sigma}}[\Sigma_{p+1}] \xrightarrow{\delta_{p+1}} F_{\bar{\Sigma}}[\Sigma_p] \xrightarrow{\delta_p} \cdots \xrightarrow{\delta_1} F_{\bar{\Sigma}}[\Sigma_0] \xrightarrow{\varepsilon} \mathbb{Z},$$

which we denote by  $F_{\bar{\Sigma}}[\Sigma]$  and call the *Reidemeister-Fox-Squier complex* of the track  $(1, p)$ -polygraph  $\Sigma$ .

## 4.4. Abelianisation of track-polygraphic resolutions

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**4.3.4. Homological syzygies.** For every  $k$  in  $\{1, \dots, p+1\}$ , the kernel of  $\delta_k$  is denoted by  $h_k(\Sigma)$  and called *the natural system of homological  $k$ -syzygies of  $\Sigma$* . The kernel of  $\varepsilon$  is denoted by  $h_0(\bar{\Sigma})$  and called *the augmentation ideal of  $\bar{\Sigma}$* .

The natural system  $h_0(\bar{\Sigma})$  is finitely generated if and only if the small category  $\bar{\Sigma}$  has homological type  $FP_1$ . If  $\Sigma$  is a generating 1-polygraph for a small category  $\mathbf{C}$ , one checks that  $h_0(\mathbf{C})$  is generated by the set  $\{(x, 1) - (1, x) \mid x \in \Sigma_1\}$ . It follows that a category has homological type  $FP_1$  if and only if it is finitely generated.

### 4.4. Abelianisation of track-polygraphic resolutions

Let us fix a small category  $\mathbf{C}$  with a track-polygraphic resolution of length  $p \geq 1$ , *i.e.*, an acyclic track  $(1, p-1)$ -polygraph  $\Sigma$  such that  $\bar{\Sigma}$  is isomorphic to  $\mathbf{C}$ .

**4.4.1. Contracting homotopies.** Since  $\Sigma$  is acyclic, it admits a left normalisation strategy  $\sigma$ . We denote by  $\sigma_k$ , for  $-1 \leq k \leq p$ , the following families of morphisms of groups, indexed by the 1-cells of  $\mathbf{C}$ :

$$\begin{array}{lll} (\sigma_{-1})_w : \mathbb{Z} \longrightarrow F_{\mathbf{C}}[\Sigma_0]_w & (\sigma_0)_w : F_{\mathbf{C}}[\Sigma_0]_w \longrightarrow F_{\mathbf{C}}[\Sigma_1]_w & (\sigma_k)_w : F_{\mathbf{C}}[\Sigma_k]_w \longrightarrow F_{\mathbf{C}}[\Sigma_{k+1}]_w \\ 1 \longmapsto (1, w) & (u, v) \longmapsto [\hat{u}]v & u[x]v \longmapsto [\sigma_{\hat{u}x}]v \end{array}$$

**4.4.2. Lemma.** *For every  $k \in \{1, \dots, p-1\}$ , every  $k$ -cell  $f$  of  $\Sigma^\top$  and every 1-cells  $u, v$  of  $\mathbf{C}$  such that  $u\bar{f}v$  exists, we have*

$$\sigma_k(u[f]v) = [\sigma_{\hat{u}f}]v.$$

*Proof.* We proceed by induction on the size of  $f$ , using the relations satisfied by the derivation  $[\cdot]$  and by the normalisation strategy  $\sigma$ . If  $f = 1_w$ , for some  $(k-1)$ -cell  $w$  of  $\Sigma^\top$ , then we have:

$$\sigma_k(u[1_w]v) = \sigma_k(0) = 0 = [1_{\hat{u}w}]v = [\sigma_{1_{\hat{u}w}}]v.$$

If  $f$  has size 1, then the result holds by definition of  $\sigma_k$ . Let us assume that  $f = gh$ , where  $g$  and  $h$  are non-degenerate  $k$ -cells of  $\Sigma^\top$ . Then we use the induction hypothesis on  $g$  and  $h$  to get, on the one hand:

$$\sigma_k(u[gh]v) = \sigma_k(u[g]\bar{h}v) + \sigma_k(u\bar{g}[h]v) = [\sigma_{\hat{u}g}]\bar{h}v + [\sigma_{\hat{u}gh}]v.$$

On the other hand, since  $\sigma$  is a left normalisation strategy, we have:

$$[\sigma_{\hat{u}gh}]v = [\sigma_{\hat{u}g}s(h) \star_1 \sigma_{\hat{u}gh}]v = [\sigma_{\hat{u}g}]\bar{h}v + [\sigma_{\hat{u}gh}]v.$$

Finally, let us assume that  $f = g \star_i h$ , where  $g$  and  $h$  are non-degenerate  $k$ -cells of  $\Sigma^\top$  and  $i \geq 1$ . Then we get:

$$\sigma_k(u[g \star_i h]v) = \sigma_k(u[g]v) + \sigma_k(u[h]v) = [\sigma_{\hat{u}g}]v + [\sigma_{\hat{u}h}]v.$$

And we also have:

$$[\sigma_{\hat{u}(g \star_i h)}]v = [\sigma_{\hat{u}g \star_i \hat{u}h}]v = [\sigma_{\hat{u}g} \star_i \sigma_{\hat{u}h}]v = [\sigma_{\hat{u}g}]v + [\sigma_{\hat{u}h}]v. \quad \square$$

**4.4.3. Theorem.** *If a small category  $\mathbf{C}$  admits a track-polygraphic resolution  $\Sigma$  of length  $p$ , then the Reidemeister-Fox-Squier complex  $F_{\mathbf{C}}[\Sigma]$  is a free resolution of the constant natural system  $\mathbb{Z}$  on  $\mathbf{C}$ .*

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*Proof.* Let us prove that  $\sigma_*$  is a contracting homotopy. Each  $(\sigma_{-1})_w$  is a section of  $\varepsilon_w$ , hence  $\varepsilon$  is an epimorphism of natural systems. Then we check the relation

$$\delta_1 \sigma_0(u, v) = (u, v) - (1, uv) = (u, v) - \sigma_{-1} \varepsilon(u, v),$$

yielding the exactness at  $F_{\mathbf{C}}[\Sigma_0]$ . Then, we compute

$$\delta_2 \sigma_1(u[x]v) = \delta_2([\sigma_{\widehat{u}x}]v) = [\widehat{u}x]v - [\widehat{u}x]v = [\widehat{u}]xv + u[x]v - [\widehat{u}x]v.$$

and

$$\sigma_0 \delta_1(u[x]v) = \sigma_0(u\bar{x}, v) - \sigma_0(u, \bar{x}v) = [\widehat{u}x]v - [\widehat{u}]xv.$$

Hence  $\delta_2 \sigma_1 + \sigma_0 \delta_1 = 1_{F_{\mathbf{C}}[\Sigma_1]}$ , proving exactness at  $F_{\mathbf{C}}[\Sigma_1]$ . Finally, for  $k \in \{2, \dots, p-1\}$ , we have:

$$\begin{aligned} \delta_{k+1} \sigma_k(u[\varphi]v) &= \delta_{k+1}[\sigma_{\widehat{u}\varphi}]v \\ &= [\widehat{u}\varphi]v - [\sigma_{\widehat{u}s(\varphi)} \star_{k-1} \sigma_{\widehat{u}t(\varphi)}^-]v \\ &= u[\varphi]v - [\sigma_{\widehat{u}s(\varphi)}]v + [\sigma_{\widehat{u}t(\varphi)}]v \\ &= u[\varphi]v - \sigma_{k-1}(u[s\varphi]v) + \sigma_{k-1}(u[t\varphi]v) \\ &= u[\varphi]v - \sigma_{k-1} \delta_k(u[\varphi]v). \end{aligned}$$

Thus, we get  $\delta_{k+1} \sigma_k + \sigma_{k-1} \delta_k = 1_{F_{\mathbf{C}}[\Sigma_k]}$ , proving exactness at  $F_{\mathbf{C}}[\Sigma_k]$  and concluding the proof.  $\square$

As a consequence, we get:

**4.4.4. Theorem.** *If a small category admits a convergent presentation, then it is of type  $FP_{\infty}$ .*

### 4.5. Description of homological syzygies and cohomological dimension

From Theorem 4.4.3, we get a characterisation of the homological properties  $FP_p$  in terms of track-polygraphic resolutions:

**4.5.1. Proposition.** *If a small category  $\mathbf{C}$  admits a track-polygraphic resolution  $\Sigma$  of length  $p \geq 1$ , then  $\mathbf{C}$  is of homological type  $FP_p$ . Moreover, if the natural system  $h_p(\Sigma)$  of homological  $p$ -syzygies of  $\Sigma$  is finitely generated, then  $\mathbf{C}$  is of homological type  $FP_{p+1}$ .*

In particular, every small category is of homological type  $FP_0$ . Finitely generated (resp. presented) categories are of homological type  $FP_1$  (resp.  $FP_2$ ). More generally, we have the following result, generalising the fact that a finite derivation type monoid is of homological type  $FP_3$ , [11, 36]:

**4.5.2. Theorem.** *For small categories and for every  $p \geq 1$ , the property  $FDT_p$  implies the property  $FP_p$ .*

Theorem 4.4.3 also gives a description of homological  $p$ -syzygies in terms of critical  $p$ -fold branchings of a convergent presentation:

**4.5.3. Proposition.** *Let  $\mathbf{C}$  be a small category with a convergent presentation  $\Sigma$ . Then, for every  $p \geq 2$ , the natural system  $h_p(\Sigma)$  of homological  $p$ -syzygies of  $\Sigma$  is generated by the elements*

$$\delta_{p+1}[\omega_b] = [(\omega_c \widehat{v})^*] - [\widehat{\omega_c v}^*]$$

where  $b = (c\widehat{v}, \rho_{c\widehat{v}})$  ranges over the critical  $p$ -fold branchings of  $\Sigma$ . As a consequence, a small category with a finite, convergent presentation is of homological type  $FP_p$ .

## 4.6. Homological syzygies and identities among relations

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Finally, Theorem 4.4.3 gives the following bounds for the cohomological dimension of a small category. We recall that the *cohomological dimension* of a small category  $\mathbf{C}$ , is defined, when it exists as the lowest  $0 \leq n \leq \infty$  such that the constant natural system  $\mathbb{Z}$  on  $\mathbf{C}$  admits a projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

In that case, the cohomological dimension of  $\mathbf{C}$  is denoted by  $\text{cd}(\mathbf{C})$ . In particular, when  $\mathbf{C}$  is free, then  $\text{cd}(\mathbf{C}) \leq 1$ , see [5].

**4.5.4. Theorem.** *Let  $\mathbf{C}$  be a small category. Then:*

- i)  $\text{cd}(\mathbf{C}) \leq d_{\text{pol}}(\mathbf{C})$ .
- ii) *If  $\mathbf{C}$  admits a track-polygraphic resolution of length  $p$ , then  $\text{cd}(\mathbf{C}) \leq p$ .*
- iii) *If  $\mathbf{C}$  admits a convergent presentation with no critical  $p$ -fold branching, then  $\text{cd}(\mathbf{C}) \leq p$ .*

### 4.6. Homological syzygies and identities among relations

In [16], the authors have introduced the natural system on  $\bar{\Sigma}$  of identities among relations of an  $n$ -polygraph  $\Sigma$ . If  $\Sigma$  is a convergent 2-polygraph, this natural system is generated by the critical branchings of  $\Sigma$ . In Proposition 4.5.3, we have seen that this is also the case of the natural system of homological 2-syzygies of  $\Sigma$ . In this section, we prove that, more generally, the natural systems of homological 2-syzygies and of identities among relations of any 2-polygraph are isomorphic.

**4.6.1. Natural systems on  $n$ -categories.** We recall from [15], that a *context* of an  $n$ -category  $\mathcal{C}$  is an  $(n+1)$ -cell with size 1 in the  $(n+1)$ -category  $\mathcal{C}[x]$ , where  $x$  is a  $n$ -sphere of  $\mathcal{C}$ , seen as a cellular extension of  $\mathcal{C}$  with only one element. Such a context  $C$  admits a decomposition

$$C = f_n \star_{n-1} (f_{n-1} \star_{n-2} (\cdots \star_1 f_1 \times g_1 \star_1 \cdots) \star_{n-2} g_{n-1}) \star_{n-1} g_n,$$

where, for every  $k$  in  $\{1, \dots, n\}$ ,  $f_k$  and  $g_k$  are  $k$ -cells of  $\mathcal{C}$ . If  $f$  is an  $n$ -cell of  $\mathcal{C}$  which is parallel to  $x$ , one denotes by  $C[f]$  the  $n$ -cell of  $\mathcal{C}$  obtained by replacing  $x$  with  $f$  in  $C$ . The context  $C$  is a *whisker* of  $\mathcal{C}$  if  $f_n$  and  $g_n$  are degenerate. Every context  $C$  of  $\mathcal{C}_{n-1}$  yields a whisker of  $\mathcal{C}$  such that  $C[f \star_{n-1} g] = C[f] \star_{n-1} C[g]$  holds.

If  $\Gamma$  is a cellular extension of  $\mathcal{C}$ , then every non-degenerate  $(n+1)$ -cell  $f$  of  $\mathcal{C}[\Gamma]$  has a decomposition

$$f = C_1[\varphi_1] \star_n \cdots \star_n C_k[\varphi_k],$$

with  $k \geq 1$  and, for every  $i$  in  $\{1, \dots, k\}$ ,  $\varphi_i$  in  $\Gamma$  and  $C_i$  a context of  $\mathcal{C}$ , *i.e.*, a whisker of  $\mathcal{C}[\Gamma]$ .

The *category of contexts* of  $\mathcal{C}$  is denoted by  $\text{Ct}(\mathcal{C})$ , its objects are the  $n$ -cells of  $\mathcal{C}$  and its morphisms from  $f$  to  $g$  are the contexts  $C$  of  $\mathcal{C}$  such that  $C[f] = g$  holds. When  $n = 1$ , the category  $\text{Ct}(\mathcal{C})$  is isomorphic to the category  $\text{FC}$  of factorisations of a small category  $\mathbf{C}$ . We denote by  $\text{Wk}(\mathcal{C})$  the subcategory of  $\text{Ct}(\mathcal{C})$  with the same objects and with whiskers as morphisms. A *natural system* on  $\mathcal{C}$  is a  $\text{Ct}(\mathcal{C})$ -module. We denote by  $D_u$  and  $D_C$  the images of an  $n$ -cell  $u$  and of a context  $C$  of  $\mathcal{C}$  by the functor  $D$ .

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**4.6.2. Identities among relations.** Let  $\Sigma$  be an  $n$ -polygraph. An  $n$ -cell  $f$  in  $\Sigma^\top$  is *closed* when  $s(f) = t(f)$ . The natural system  $\Pi(\Sigma)$  on  $\bar{\Sigma}$  of *identities among relations* of  $\Sigma$  is defined as follows:

- If  $u$  is an  $(n-1)$ -cell of  $\bar{\Sigma}$ , the Abelian group  $\Pi(\Sigma)_u$  is generated by one element  $[f]$ , for each  $n$ -cell  $f : v \Rightarrow v$  of  $\Sigma^\top$  such that  $\bar{v} = u$ , submitted to the following relations:

- if  $f : v \rightarrow v$  and  $g : v \rightarrow v$  are  $n$ -cells of  $\Sigma^\top$ , with  $\bar{v} = u$ , then

$$[f \star_{n-1} g] = [f] + [g] ;$$

- if  $f : v \rightarrow w$  and  $g : w \rightarrow v$  are  $n$ -cells of  $\Sigma^\top$ , with  $\bar{v} = \bar{w} = u$ , then

$$[f \star_{n-1} g] = [g \star_{n-1} f] .$$

- If  $g = C[f]$  is a factorisation in  $\bar{\Sigma}$ , then the morphism  $\Pi(\Sigma)_C : \Pi(\Sigma)_f \rightarrow \Pi(\Sigma)_g$  of groups is defined by

$$\Pi(\Sigma)_C([f]) = [\widehat{C}[f]] ,$$

where  $\widehat{C}$  is any representative context for  $C$  in  $\Sigma^*$ . We recall from [16] that the value of  $\Pi(\Sigma)$  does not depend on the choice of  $\widehat{C}$ , proving that  $\Pi(\Sigma)$  is a natural system on  $\bar{\Sigma}$  and allowing one to denote this element of  $\Pi(\Sigma)_g$  by  $C[f]$ .

The identities among relations satisfy the relations

$$[f^-] = -[f] \quad \text{and} \quad [g \star_{n-1} f \star_{n-1} g^-] = [f]$$

for every  $n$ -cells  $f : u \rightarrow u$  and  $g : v \rightarrow u$  in  $\Sigma^\top$ .

**4.6.3. Lemma.** *Let  $\Sigma$  be a 2-polygraph and let  $f$  be a closed 2-cell of  $\Sigma^\top$ . Then we have  $[f] = 0$  in  $F_C[\Sigma_2]$  if and only if  $[f] = 0$  holds in  $\Pi(\Sigma)$ .*

*Proof.* To prove that  $[f] = 0$  implies  $[f] = 0$ , we check that the relations defining  $\Pi(\Sigma)$  are satisfied in  $F_C[\Sigma_2]$ . The first relation is given by the definition of the map  $[\cdot]$ . The second relation is given by

$$[f \star_1 g] = [f] + [g] = [g] + [f] = [g \star_1 f] .$$

Conversely, let us consider a 2-cell  $f : u \Rightarrow u$  in  $\Sigma^\top$  such that  $[f] = 0$  holds. We decompose  $f$  into:

$$f = u_1 \varphi_1^{\varepsilon_1} v_1 \star_1 \cdots \star_1 u_p \varphi_p^{\varepsilon_p} v_p$$

where  $\varphi_i$  is a 2-cell of  $\Sigma$ ,  $u_i$  and  $v_i$  are 1-cells of  $\Sigma^\top$ ,  $\varepsilon_i$  is an element of  $\{-, +\}$ . Then we get:

$$0 = [f] = \sum_{i=1}^p \varepsilon_i \bar{u}_i [\varphi_i] \bar{v}_i$$

Since  $F_C[\Sigma_2]$  is free over  $\Sigma_2$ , this implies that there exists a permutation  $\tau$  of  $\{1, \dots, p\}$  such that:

$$\varphi_i = \varphi_{\tau(i)} \quad \bar{u}_i = \bar{u}_{\tau(i)} \quad \bar{v}_i = \bar{v}_{\tau(i)} \quad \varepsilon_i = -\varepsilon_{\tau(i)} .$$

## 4.6. Homological syzygies and identities among relations

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Now, let us choose a 2-cell  $h_i : u \Rightarrow u_i s(\varphi_i) v_i$  in  $\Sigma^\top$ , in such a way that, if  $u_i s(\varphi_i) v_i = u_j s(\varphi_j) v_j$ , then  $h_i = h_j$ . Then we have, with the convention  $h_{p+1} = h_1$ :

$$[f] = [h_1 \star_1 f \star_1 h_1^-] = \sum_{i=1}^p [h_i \star_1 u_i \varphi_i^{\varepsilon_i} v_i \star_1 h_{i+1}^-].$$

From the properties of the permutation  $\tau$ , we get:

$$\begin{aligned} [h_{\tau(i)} \star_1 u_{\tau(i)} \varphi_{\tau(i)}^{\varepsilon_{\tau(i)}} v_{\tau(i)} \star_1 h_{\tau(i+1)}^-] &= [h_{\tau(i)} \star_1 u_i \varphi_i^{-\varepsilon_i} v_i \star_1 h_{\tau(i+1)}^-] \\ &= [h_{i+1} \star_1 u_i \varphi_i^{-\varepsilon_i} v_i \star_1 h_i^-] \\ &= - [h_i \star_1 u_i \varphi_i^{\varepsilon_i} v_i \star_1 h_{i+1}^-]. \end{aligned}$$

Hence we get, by induction on  $p$ , the relation  $[f] = 0$ . □

**4.6.4. Lemma.** *Let  $\Sigma$  be a 2-polygraph. For every element  $a$  in  $h_2(\Sigma)$ , there exists a closed 2-cell  $f$  in  $\Sigma^\top$  such that  $a = [f]$  holds.*

*Proof.* Let  $w$  be the 1-cell of  $\bar{\Sigma}$  such that  $a$  belongs to  $F_{\bar{\Sigma}}[\Sigma_2]_w$ . We consider a homotopy basis  $\Sigma_3$  of  $\Sigma^\top$ . Then, there exists an element  $b$  in  $F_{\bar{\Sigma}}[\Sigma_3]_w$  such that  $a = \delta_3(b)$  holds. By definition of  $F_{\bar{\Sigma}}[\Sigma_3]_w$ , we can write

$$b = \sum_{i=1}^k \varepsilon_i u_i [\alpha_i] v_i$$

with  $\alpha_i$  in  $\Sigma_3$ ,  $u_i$  and  $v_i$  in  $\bar{\Sigma}$ ,  $\varepsilon_i$  in  $\{-, +\}$ , such that, for every  $i$ ,  $u_i \bar{\alpha}_i v_i = w$ . We choose 2-cells  $g_i : \hat{w} \Rightarrow \hat{u}_i \alpha_i \hat{v}_i$  in  $\Sigma^\top$ . Then we get  $b = [A]$  with

$$A = (g_1 \star_1 u_1 \alpha_1^{\varepsilon_1} v_1 \star_1 g_1^-) \star_1 \cdots \star_1 (g_k \star_1 u_k \alpha_k^{\varepsilon_k} v_k \star_1 g_k^-).$$

Finally, we get:

$$a = \delta_3([A]) = [s(A)] - [t(A)] = [s(A) \star_1 t(A)^-].$$

Hence  $f = s(A) \star_1 t(A)^-$  is a closed 2-cell of  $\Sigma^\top$  that satisfies  $a = [f]$  □

**4.6.5. Theorem.** *For every 2-polygraph  $\Sigma$ , the natural systems of homological 2-syzygies and of identities among relations of  $\Sigma$  are isomorphic.*

*Proof.* We define an Abelianisation map as the morphism of natural systems  $\Phi : \Pi(\Sigma) \rightarrow h_2(\Sigma)$  given by

$$\Phi([f]) = [f].$$

This definition is correct, since the defining relations of  $\Pi(\Sigma)$  also hold in  $F_{\mathbb{C}}[\Sigma_2]$ , hence in  $h_2(\Sigma)$ . Let us check that  $\Phi$  is a morphism of natural systems. Indeed, we have

$$\Phi(u [f] v) = [\hat{u} f \hat{v}] = u [f] v = u \Phi([f]) v$$

for every 2-cell  $f : w \Rightarrow w$  in  $\Sigma^\top$  and 1-cells  $u, v$  in  $\bar{\Sigma}$  such that  $\hat{u} f \hat{v}$  is defined.

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Now, let us define a morphism of natural systems  $\Psi : h_2(\Sigma) \rightarrow \Pi(\Sigma)$ . Let  $\alpha$  be an element of  $h_2(\Sigma)_w$ . Then there exists a closed 2-cell  $f : u \Rightarrow u$  such that  $\alpha = [f]$  and  $w = \bar{u}$ . We define

$$\Psi(\alpha) = [f].$$

This definition does not depend on the choice of  $f$ . Indeed, let us assume that  $g : v \Rightarrow v$  is a closed 2-cell such that  $\alpha = [g]$  holds. It follows that  $\bar{v} = w = \bar{u}$ . Hence, we can choose a 2-cell  $h : u \Rightarrow v$  in  $\Sigma^\top$ . Then we have:

$$\alpha = [f] = [g] = [h \star_1 g \star_1 h^-].$$

As a consequence, we get:

$$0 = [f] - [h \star_1 g \star_1 h^-] = [f \star_1 h^- \star_1 g^- \star_1 h].$$

Thus:

$$0 = [f \star_1 h^- \star_1 g^- \star_1 h] = [f] - [h \star_1 g \star_1 h^-] = [f] - [g].$$

The relations  $\Psi\Phi = 1_{\Pi(\Sigma)}$  and  $\Phi\Psi = 1_{h_2(\Sigma)}$  are direct consequences of the definitions of  $\Phi$  and  $\Psi$ .  $\square$

#### 4.7. Abelian finite derivation type

A track  $n$ -category  $\mathcal{T}$  is *Abelian* if, for every  $(n-1)$ -cell  $u$  in  $\mathcal{T}$ , the group  $\text{Aut}_u^\mathcal{T}$  of closed  $n$ -cells of  $\mathcal{T}$  with source  $u$  is Abelian. The Abelianized track category of  $\mathcal{T}$  is the track  $n$ -category, denoted by  $\mathcal{T}_{\text{ab}}$  and defined as the quotient of  $\mathcal{T}$  by the  $n$ -spheres  $(f \star_{n-1} g, g \star_{n-1} f)$ , where  $f$  and  $g$  are closed  $n$ -cells with same source.

One says that an  $n$ -polygraph  $\Sigma$  is of *Abelian finite derivation type*,  $\text{FDT}_{\text{ab}}$  for short, is the Abelianized track  $n$ -category  $\Sigma_{\text{ab}}^\top$  is  $\text{FDT}_3$ . In this section, we prove that an  $n$ -polygraph is  $\text{FDT}_{\text{ab}}$  if and only if the natural system  $\Pi(\Sigma)$  of identities among relations of  $\Sigma$  is finitely generated.

In [16], it is proved that  $\Pi(\Sigma)$  is the only natural system on  $\bar{\Sigma}$ , up to isomorphism, such that there exists an isomorphism of natural systems on the free  $n$ -category  $\Sigma^*$

$$\widehat{\Pi(\Sigma)} \simeq \text{Aut}_{\text{ab}}^{\Sigma^\top}$$

with  $\widehat{\Pi(\Sigma)}$  defined, on every  $n$ -cell  $u$  of  $\Sigma^*$ , by

$$\widehat{\Pi(\Sigma)}_u = \Pi(\Sigma)_{\bar{u}}.$$

**4.7.1. Lemma ([16]).** *Let  $\mathcal{T}$  be a track  $n$ -category and let  $\mathcal{B}$  be a family of closed  $n$ -cells of  $\mathcal{T}$ . The following assertions are equivalent:*

1. *The cellular extension  $\tilde{\mathcal{B}} = \{\tilde{\beta} : \beta \rightarrow 1_{s\beta}, \beta \in \mathcal{B}\}$  is a homotopy basis of  $\mathcal{T}$ .*
2. *Every closed  $n$ -cell  $f$  in  $\mathcal{T}$  can be written*

$$f = (g_1 \star_{n-1} C_1 [\beta_1^{\varepsilon_1}] \star_{n-1} g_1^-) \star_{n-1} \cdots \star_{n-1} (g_k \star_{n-1} C_k [\beta_k^{\varepsilon_k}] \star_{n-1} g_k^-) \quad (1)$$

where, for every  $i \in \{1, \dots, k\}$ , we have  $\beta_i \in \mathcal{B}$ ,  $\varepsilon_i \in \{-, +\}$ ,  $C_i \in \text{Wk}(\mathcal{T})$  and  $g_i \in \mathcal{T}_n$ .

**4.7.2. Proposition.** *An  $n$ -polygraph  $\Sigma$  is  $\text{FDT}_{\text{ab}}$  if and only if the natural system  $\Pi(\Sigma)$  on  $\bar{\Sigma}$  is finitely generated.*

*Proof.* Let us assume that the  $n$ -polygraph  $\Sigma$  is  $\text{FDT}_{\text{ab}}$ . Then the Abelian track  $n$ -category  $\Sigma_{\text{ab}}^{\top}$  has a finite homotopy basis  $\mathcal{B}$ . Let  $\partial\mathcal{B}$  be the set of closed  $n$ -cells of  $\Sigma_{\text{ab}}^{\top}$  defined by:

$$\partial\mathcal{B} = \left\{ \partial\beta = s(\beta) \star_{n-1} t(\beta)^{-} \mid \beta \in \mathcal{B} \right\}.$$

By Lemma 4.7.1, any closed  $n$ -cell  $f$  in  $\Sigma_{\text{ab}}^{\top}$  can be written

$$f = (g_1 \star_{n-1} C_1[\partial\beta_1^{\varepsilon_1}] \star_{n-1} g_1^{-}) \star_{n-1} \cdots \star_{n-1} (g_k \star_{n-1} C_k[\partial\beta_k^{\varepsilon_k}] \star_{n-1} g_k^{-}),$$

where, for every  $i$  in  $\{1, \dots, k\}$ ,  $\beta_i$  in  $\mathcal{B}$ ,  $\varepsilon_i$  in  $\{-, +\}$ ,  $C_i$  a whisker of  $\Sigma^*$  and  $g_i$  an  $n$ -cell of  $\Sigma_n^*$ . As a consequence, for any identity among relations  $[f]$  in  $\Pi(\Sigma)$ , we have:

$$[f] = \sum_{i=1}^k \varepsilon_i [g_i \star_{n-1} C_i[\partial\beta_i] \star_{n-1} g_i^{-}] = \sum_{i=1}^k \varepsilon_i C_i [\partial\beta_i].$$

Thus, the elements of  $[\partial\mathcal{B}]$  form a generating set for the natural system  $\Pi(\Sigma)$ . Hence  $\Pi(\Sigma)$  is finitely generated.

Conversely, suppose that the natural system  $\Pi(\Sigma)$  is finitely generated. Then there exists a finite set  $\mathcal{B}$  of closed  $n$ -cells in  $\Sigma_{\text{ab}}^{\top}$  such that, for every  $(n-1)$ -cell  $u$  of  $\bar{\Sigma}$  and every closed  $n$ -cell  $f$  with source  $v$  of  $\Sigma_{\text{ab}}^{\top}$ , such that  $\hat{v} = u$ , one can write

$$[f] = \sum_{i=1}^p \varepsilon_i C_i [\beta_i]$$

with, for every  $i$  in  $\{1, \dots, p\}$ ,  $\beta_i$  in  $\mathcal{B}$ ,  $C_i$  a whisker of  $\bar{\Sigma}$  and  $\varepsilon_i$  an integer, such that, for every representative  $\hat{C}_i$  of  $C_i$  in  $\Sigma_{\text{ab}}^{\top}$ ,  $\hat{C}_i[\beta_i]$  is a closed  $n$ -cell of  $\Sigma_{\text{ab}}^{\top}$  whose source  $v_i$  satisfies  $\bar{v}_i = v$ . We fix, for every  $i$ , an  $n$ -cell  $g_i : v \Rightarrow v_i$  in  $\Sigma^{\top}$ . Then, by properties of  $\Pi(\Sigma)$ , we have:

$$\begin{aligned} [f] &= \sum_{i=1}^p \left[ g_i \star_{n-1} \hat{C}_i[\beta_i^{\varepsilon_i}] \star_{n-1} g_i^{-} \right] \\ &= \left[ (g_1 \star_{n-1} \hat{C}_1[\beta_1^{\varepsilon_1}] \star_{n-1} g_1^{-}) \star_{n-1} \cdots \star_{n-1} (g_p \star_{n-1} \hat{C}_p[\beta_p^{\varepsilon_p}] \star_{n-1} g_p^{-}) \right]. \end{aligned}$$

We use the isomorphism between  $\Pi(\Sigma)_u$  and  $\text{Aut}_{\Sigma_{\text{ab}}^{\top}}^{\Sigma_{\text{ab}}^{\top}}$  and Lemma 4.7.1 to deduce that the cellular extension  $\tilde{\mathcal{B}} = \left\{ \tilde{\beta} : \beta \rightarrow 1_{s\beta}, \beta \in \mathcal{B} \right\}$  of  $\Sigma_{\text{ab}}^{\top}$  is a homotopy basis. Thus  $\Sigma$  is  $\text{FDT}_{\text{ab}}$ .  $\square$

In [16], the authors have proved that the property to be finitely generated for  $\Pi(\Sigma)$  is Tietze-invariant for finite polygraphs: if  $\Sigma$  and  $\Upsilon$  are two Tietze-equivalent finite  $n$ -polygraphs, then the natural system  $\Pi(\Sigma)$  is finitely generated if and only if the natural system  $\Pi(\Upsilon)$  is finitely generated.

By Proposition 4.7.2, we deduce that the property  $\text{FDT}_{\text{ab}}$  is Tietze-invariant for finite polygraphs, so that one can say that an  $n$ -category is  $\text{FDT}_{\text{ab}}$  when it admits a presentation by a finite  $(n+1)$ -polygraph which is  $\text{FDT}_{\text{ab}}$ . In this way, the following result relates the homological property  $\text{FP}_3$  and the homotopical property  $\text{FDT}_{\text{ab}}$ .

## 5. Examples

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**4.7.3. Theorem.** *Let  $\mathbf{C}$  be a category with a finite presentation  $\Sigma$ . The following conditions are equivalent:*

- i)  $\mathbf{C}$  is of homological type  $\text{FP}_3$ ,
- ii)  $h_2(\Sigma)$  is finitely generated,
- iii)  $\Pi(\Sigma)$  is finitely generated,
- iv)  $\mathbf{C}$  is  $\text{FDT}_{\text{ab}}$ .

*Proof.* The equivalence between **i)** and **ii)** comes from the definition of the property  $\text{FP}_3$ . The equivalence between **ii)** and **iii)** is a consequence of Proposition 4.6.5. The last equivalence is given by Proposition 4.7.2.  $\square$

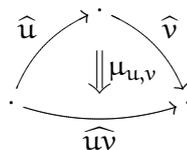
In Theorem 4.5.2, we have seen that  $\text{FDT}_3$  implies  $\text{FP}_3$ . We expect that the reverse implication is false in general, which is equivalent to proving that  $\text{FDT}_{\text{ab}}$  does not imply  $\text{FDT}$ , since  $\text{FP}_3$  is equivalent to  $\text{FDT}_{\text{ab}}$  for finitely presented categories.

## 5. EXAMPLES

### 5.1. The standard track-polygraphic resolution of a small category

We fix a small category  $\mathbf{C}$ . The *standard presentation of  $\mathbf{C}$*  is the 2-polygraph denoted by  $\text{N}\mathbf{C}$  and whose cells are the following ones:

- one 0-cell for each 0-cell of  $\mathbf{C}$ ,
- one 1-cell  $\hat{u}$  for every non-degenerate 1-cell  $u$  of  $\mathbf{C}$ ,
- one 2-cell



for every pair  $(u, v)$  of non-degenerate and composable 1-cells in  $\mathbf{C}$ .

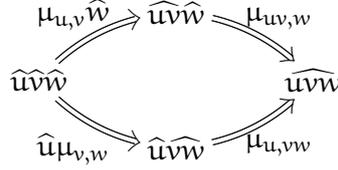
The 2-polygraph  $\text{N}\mathbf{C}$  is reduced. Let us prove that it is convergent. For termination, one checks that each 2-cell  $\mu_{u,v} : \hat{u}\hat{v} \Rightarrow \widehat{uv}$  of  $\text{N}\mathbf{C}$  has source of size 2 and target of size 1. As a consequence, for every non-degenerate 2-cell  $f : u \Rightarrow v$  of the free 2-category  $\text{N}\mathbf{C}^*$ , the size of  $u$  is strictly greater than the size of  $v$ .

For confluence, we check that  $\text{N}\mathbf{C}$  has one critical branching for every triple  $(u, v, w)$  of non-degenerate composable 1-cells in  $\mathbf{C}$

$$(\mu_{u,v}\hat{w}, \widehat{u}\mu_{v,w})$$

## 5.1. The standard track-polygraphic resolution of a small category

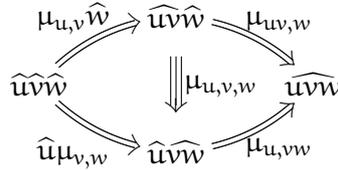
and that this critical branching is confluent:



The rightmost normalisation strategy for  $\mathbf{NC}$  is given, on 1-cells, by:

$$\sigma_{\widehat{u}_1 \dots \widehat{u}_n}^* = (\widehat{u}_1 \dots \widehat{u}_{n-2} \mu_{u_{n-1}, u_n}) \star_1 \sigma_{\widehat{u}_1 \dots \widehat{u}_{n-1} \widehat{u}_n}^* .$$

Hence we get the acyclic track  $(1, 2)$ -polygraph  $c_2(\mathbf{NC})$ , by extending  $\mathbf{NC}$  with one 3-cell

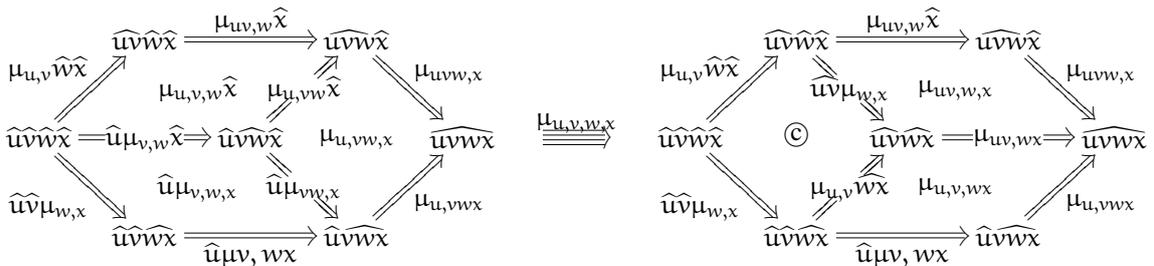


for every triple  $(u, v, w)$  of non-degenerate composable 1-cells of  $\mathbf{C}$ . The acyclic track  $(1, 2)$ -polygraph  $c_2(\mathbf{NC})$  is a track-polygraphic resolution of  $\mathbf{C}$  of length 2. One iterates this process to build a resolution of  $\mathbf{C}$  by the acyclic track  $(1, \infty)$ -polygraph  $c_\infty(\mathbf{NC})$ , called the *standard track-polygraphic resolution of  $\mathbf{C}$* .

For each natural number  $n \geq 2$ ,  $c_\infty(\mathbf{NC})$  has one  $n$ -cell  $\mu_{u_1, \dots, u_n}$  for every family  $(u_1, \dots, u_n)$  of non-degenerate composable 1-cells of  $\mathbf{C}$ , with the shape of an  $n$ -simplex, representing all the possible ways to transform  $\widehat{u}_1 \dots \widehat{u}_n$  into  $u_1 \dots u_n$  and all the homotopies between those different ways. More precisely, the  $n$ -cell  $\mu_{u_1, \dots, u_n}$  has the shape of an  $n$ -oriental [41], the higher-categorical equivalent of an  $n$ -simplex. In particular, the source and the target of the  $n$ -cell  $\mu_{u_1, \dots, u_n}$  are  $\star_{n-2}$ -composites of the following  $(n - 1)$ -cells

$$d_i(u_1, \dots, u_n) = \begin{cases} \widehat{u}_1 \mu_{u_2, \dots, u_n} & \text{if } i = 0 \\ \mu_{u_1, \dots, u_i u_{i+1}, \dots, u_n} & \text{if } 1 \leq i \leq n - 1 \\ \mu_{u_1, \dots, u_{n-1}} \widehat{u}_n & \text{if } i = n \end{cases}$$

For example, the 4-cell  $\mu_{u,v,w,x}$  is given thereafter:



## 5. Examples

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From this track-polygraphic resolution of  $\mathbf{C}$ , we get the following Abelian resolution of the constant natural system  $\mathbb{Z}$ :

$$\cdots \xrightarrow{\delta_5} F_{\mathbf{C}}[c_3(\mathbf{N}\mathbf{C})] \xrightarrow{\delta_4} F_{\mathbf{C}}[c_2(\mathbf{N}\mathbf{C})] \xrightarrow{\delta_3} F_{\mathbf{C}}[\mathbf{N}\mathbf{C}_2] \xrightarrow{\delta_2} F_{\mathbf{C}}[\mathbf{C}_1] \xrightarrow{\delta_1} F_{\mathbf{C}}[\mathbf{C}_0] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

with differential maps given by

$$\begin{aligned} \delta_n[\mu_{u_1, \dots, u_n}] &= \sum_{i=0}^n (-1)^{n-i} [d_i(u_1, \dots, u_n)] \\ &= (-1)^n u_1 [\mu_{u_2, \dots, u_n}] + \sum_{i=1}^{n-1} (-1)^{n-i} [\mu_{u_1, \dots, u_i u_{i+1}, \dots, u_n}] + [\mu_{u_1, \dots, u_{n-1}}] u_n. \end{aligned}$$

Moreover, those elements  $\delta_n[\mu_{u_1, \dots, u_n}]$  form a generating set for the natural system  $h_{n-1}(\mathbf{N}\mathbf{C})$  of homological  $(n-1)$ -syzygies of  $\mathbf{N}\mathbf{C}$ .

### 5.2. A simple example of standard track-polygraphic resolution

Let us denote by  $A$  the monoid with one non-unit element,  $a$ , and with product given by  $aa = a$ . The standard presentation of  $A$ , seen as a category, is the reduced and convergent 2-polygraph, denoted by  $As$ , with one 0-cell (for 1), one 1-cell (for  $a$ ) and one 2-cell  $aa \Rightarrow a$ . Here we use diagrammatic notations, where  $a$  is denoted by a vertical string  $|$  and the 2-cell  $aa \Rightarrow a$  is pictured as  $\blacktriangledown$ . The 2-polygraph  $As$  has one critical branching:

$$(\blacktriangledown|, | \blacktriangledown).$$

The corresponding generating confluence is the 3-cell:

By extending  $As$  with that 3-cell, one gets a finite, acyclic track  $(1, 2)$ -polygraph, still denoted by  $As$  and which is a track-polygraphic resolution of  $A$  of length 3. We conclude that the monoid  $A$  has  $FDT_3$  and, thus, type  $FP_3$ . In particular, the natural system  $h_2(As)$  of homological 2-syzygies of  $As$  is generated by the following element:

$$\delta_3[\blacktriangledown] = [\blacktriangledown] - [\blacktriangledown] = [\blacktriangledown] + [\blacktriangledown] - [|\blacktriangledown] - [\blacktriangledown|] = [\blacktriangledown]a - a[\blacktriangledown].$$

The 2-polygraph  $As$  has exactly one critical triple branching:

$$b = (\blacktriangledown||, | \blacktriangledown|, || \blacktriangledown).$$

This triple critical branching  $b$  has shape  $(f\hat{a}, g\hat{a}, \rho_{u\hat{a}})$  with

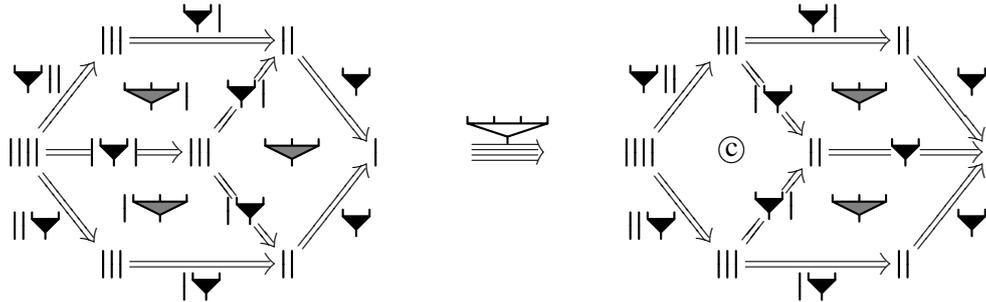
$$f = \blacktriangledown| \quad \text{and} \quad g = | \blacktriangledown$$

## 5.2. A simple example of standard track-polygraphic resolution

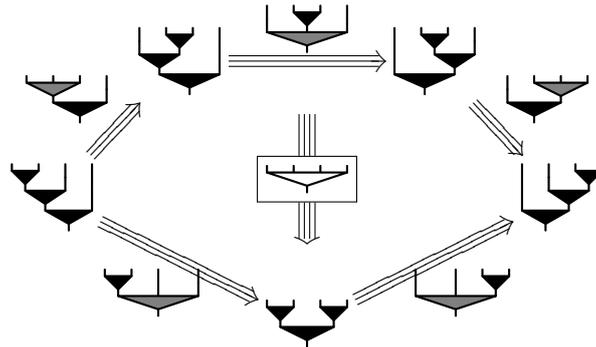
We compute the 3-cells  $\omega_{f,g}$ ,  $\sigma_{f\hat{a}}^*$  and  $\sigma_{g\hat{a}}^*$ , using their definitions and the properties of the rightmost normalisation strategy  $\sigma$ , to get:

$$\omega_{f,g\hat{a}} = \text{[Diagram: a 3-cell with a downward-pointing triangle and a vertical line]}, \quad \sigma_{f\hat{a}}^* = \text{[Diagram: two downward-pointing triangles connected by a horizontal line with a star symbol]}, \quad \sigma_{g\hat{a}}^* = \text{[Diagram: two downward-pointing triangles connected by a horizontal line with a star symbol]}$$

We fill the diagram defining  $\omega_b = \text{[Diagram: a downward-pointing triangle with a vertical line]},$  obtaining:



Contracting one dimension, we see that the 4-cell  $\text{[Diagram: a downward-pointing triangle with a vertical line]}$  is, in fact, Mac Lane's pentagon, or Stasheff's polytope  $K_4$ :



We get a finite, acyclic track  $(1, 3)$ -polygraph  $c_3(As)$  which is a track-polygraphic resolution of  $A$  of length 4, proving that  $A$  has  $FDT_4$  and, as a consequence, that it is of type  $FP_4$ . In particular, the natural system  $h_3(As)$  of homological 3-szygies of  $As$  is generated by the following element:

$$\begin{aligned} \delta_4[\text{[Diagram: a downward-pointing triangle with a vertical line]}] &= [\text{[Diagram: two downward-pointing triangles connected by a horizontal line]}] + [\text{[Diagram: two downward-pointing triangles connected by a horizontal line]}] + [\text{[Diagram: two downward-pointing triangles connected by a horizontal line]}] - [\text{[Diagram: two downward-pointing triangles connected by a horizontal line]}] - [\text{[Diagram: two downward-pointing triangles connected by a horizontal line]}] \\ &= [\text{[Diagram: a downward-pointing triangle with a vertical line]}] + [\text{[Diagram: a downward-pointing triangle with a vertical line]}] + [\text{[Diagram: a downward-pointing triangle with a vertical line]}] - [\text{[Diagram: a downward-pointing triangle with a vertical line]}] - [\text{[Diagram: a downward-pointing triangle with a vertical line]}] \\ &= a [\text{[Diagram: a downward-pointing triangle with a vertical line]}] - [\text{[Diagram: a downward-pointing triangle with a vertical line]}] + [\text{[Diagram: a downward-pointing triangle with a vertical line]}] a. \end{aligned}$$

Iterating the process, we get a resolution of  $A$  by an acyclic track  $(1, \infty)$ -polygraph  $c_\infty(As)$ . For every natural number,  $c_\infty(As)$  has exactly one  $n$ -cell, whose shape is Stasheff's polytope  $K_n$ . For example, in dimension 5, the generating 5-cell  $\omega_b$  is associated to the following critical quadruple branching:

$$b = (\text{[Diagram: a downward-pointing triangle with a vertical line}], \text{[Diagram: a downward-pointing triangle with a vertical line}], \text{[Diagram: a downward-pointing triangle with a vertical line}], \text{[Diagram: a downward-pointing triangle with a vertical line]}).$$

To compute the source and target of the corresponding 4-cell  $\omega_b$ , we use the inductive construction of the rightmost strategy  $\sigma$ . Alternatively, one can also start from the 2-dimensional source and target of  $\omega_b$ ,

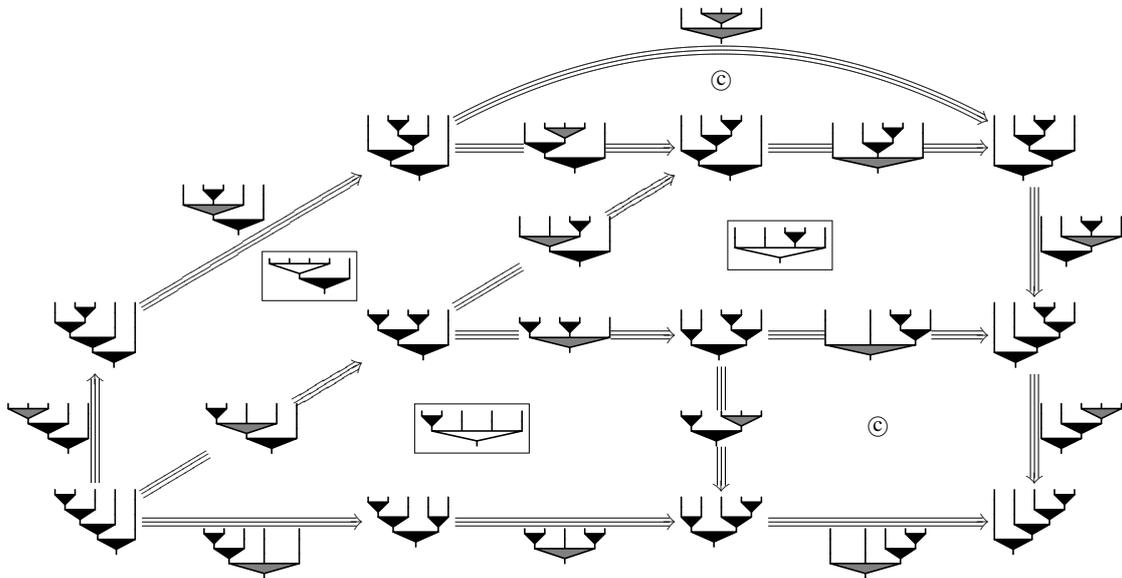
## 5. Examples

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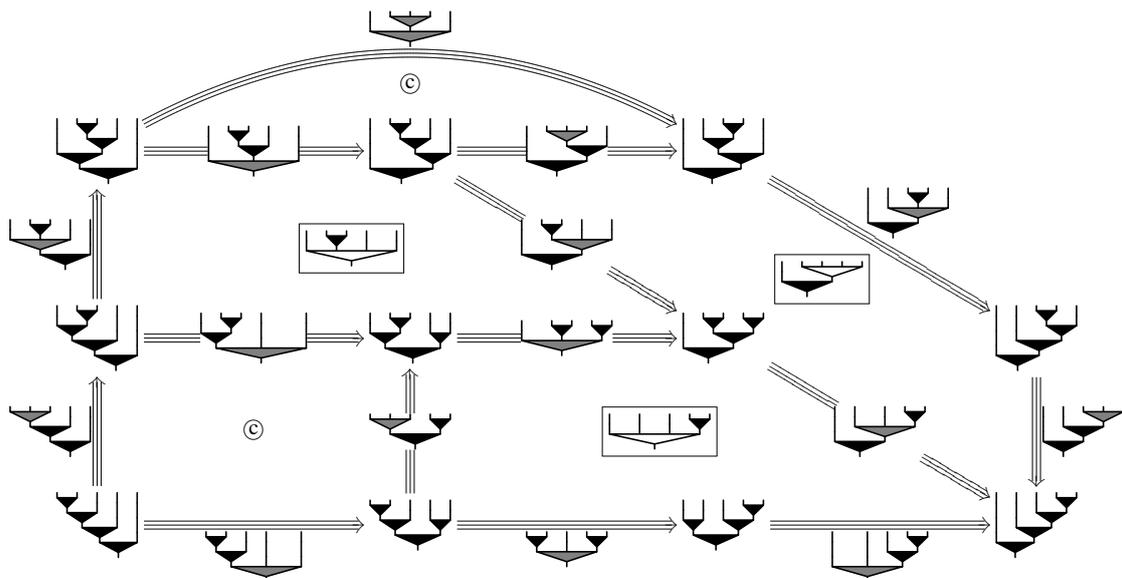
which are obtained as the 2-cells associated to the source  $\alpha\alpha\alpha\alpha$  of the critical quadruple branching  $b$  by the leftmost and the rightmost strategies, respectively:

$$s_2(\omega_b) = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \quad \text{and} \quad t_2(\omega_b) = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$$

Then one computes all the possible 3-cells from  $s_2(\omega_b)$  to  $t_2(\omega_b)$  and one fills all the 3-dimensional spheres with 4-cells built from the generating 4-cell  $\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}$ . Either way, we obtain the following composite 4-cell as the source of  $\omega_b$ :



And the following composite 4-cell is the target of  $\omega_b$ :



The corresponding generator of the natural system  $h_4(As)$  of homological 4-syzygies of  $As$  is:

$$\begin{aligned} \delta_5[\omega_b] &= \left[ \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \right] + \left[ \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \right] + \left[ \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \right] - \left[ \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \right] - \left[ \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \right] - \left[ \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \right] \\ &= \left[ \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \right] a - a \left[ \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \\ \text{---} \end{array} \right]. \end{aligned}$$

### 5.3. The category Epi

We denote by **Epi** the subcategory of the simplicial category whose 0-cells are the natural numbers and whose morphisms from  $n$  to  $p$  are the ordered-preserving surjections from  $\{0, \dots, n\}$  to  $\{0, \dots, p\}$ . This category is studied in [24], where it is denoted by  $\Delta^{\text{epi}}$ .

The category **Epi** admits a presentation by the (infinite) 2-polygraph  $\Sigma$  with the natural numbers as 0-cells, with 1-cells

$$n + 1 \xrightarrow{s_i} n \quad 0 \leq i \leq n,$$

where  $s_i$  represents the map

$$s_i(j) = \begin{cases} j & \text{if } 0 \leq j \leq i, \\ j - 1 & \text{if } i + 1 \leq j \leq n + 1, \end{cases}$$

and with 2-cells

$$\begin{array}{ccc} & s_i \rightarrow n + 1 & s_j \rightarrow n \\ n + 2 & \Downarrow s_{i,j} & n \\ & s_{j+1} \rightarrow n + 1 & s_i \rightarrow n \end{array} \quad 0 \leq i \leq j \leq n + 1.$$

Let us prove that this 2-polygraph is convergent. For termination, given a 1-cell  $u = s_{i_1} \dots s_{i_k}$  of  $\Sigma^*$ , we define the natural number  $\nu(u)$  as the number of pairs  $(i_p, i_q)$  such that  $i_p \leq i_q$ , with  $1 \leq p < q \leq k$ . In particular, we have  $\nu(s_i s_j) = 1$  and  $\nu(s_{j+1} s_i) = 0$  when  $i \leq j$ , *i.e.*, when  $s_i s_j$  is the source and  $s_{j+1} s_i$  is the target of a 2-cell of  $\Sigma$ . Moreover, we have  $\nu(wu w') > \nu(wv w')$  when  $\nu(u) > \nu(v)$  holds. Thus, for every non-degenerate 2-cell  $f : u \Rightarrow v$  in  $\Sigma^*$ , the strict inequality  $\nu(u) > \nu(v)$  is satisfied, yielding termination of  $\Sigma$ .

The 2-polygraph  $\Sigma$  has one critical branching  $(s_{i,j} s_k, s_i s_{j,k})$  for every possible  $0 \leq i \leq j \leq k \leq n + 2$  and it is confluent, so that we get a resolution of **Epi** by the acyclic track (1, 2)-polygraph  $c_2(\Sigma)$  obtained by extending  $\Sigma$  with all the 3-cells filling the confluence diagrams associated to the critical branchings:

$$\begin{array}{ccccc} & & s_{j+1} s_i s_k & \xrightarrow{s_{j+1} s_{i,k}} & s_{j+1} s_{k+1} s_i & & \\ & s_{i,j} s_k \nearrow & & & & \searrow s_{j+1, k+1} s_i & \\ s_i s_j s_k & & & & & & s_{k+2} s_{j+1} s_i \\ & s_i s_{j,k} \searrow & & & & \nearrow s_{k+2} s_{i,j} & \\ & & s_i s_{k+1} s_j & \xrightarrow{s_{i, k+1} s_j} & s_{k+2} s_i s_j & & \end{array}$$

## 5. Examples

To simplify notations, we draw each 1-cell  $s_i$  as a vertical string  $|_i$ , each 2-cell  $s_{i,j}$  as  $\bowtie_{i,j}$ , so that each 3-cell  $s_{i,j,k}$  has the same shape as the Yang-Baxter relation, or permutohedron of order 3:

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} \xrightarrow{\text{Diagram 4}} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \end{array}$$

Using those notations, the natural system  $h_2(\Sigma)$  of homological 2-szygies of  $\Sigma$  is generated by the elements

$$\delta_3 [\bowtie_{i,j,k}] = \left[ \text{Diagram 8} \right] - \left[ \text{Diagram 9} \right] = \begin{cases} ([\bowtie_{i,j}]s_k - s_{k+2}[\bowtie_{i,j}]) \\ + (s_{j+1}[\bowtie_{i,k}] - [\bowtie_{i,k+1}]s_j) \\ + ([\bowtie_{j+1,k+1}]s_i - s_i[\bowtie_{j,k}]) \end{cases}$$

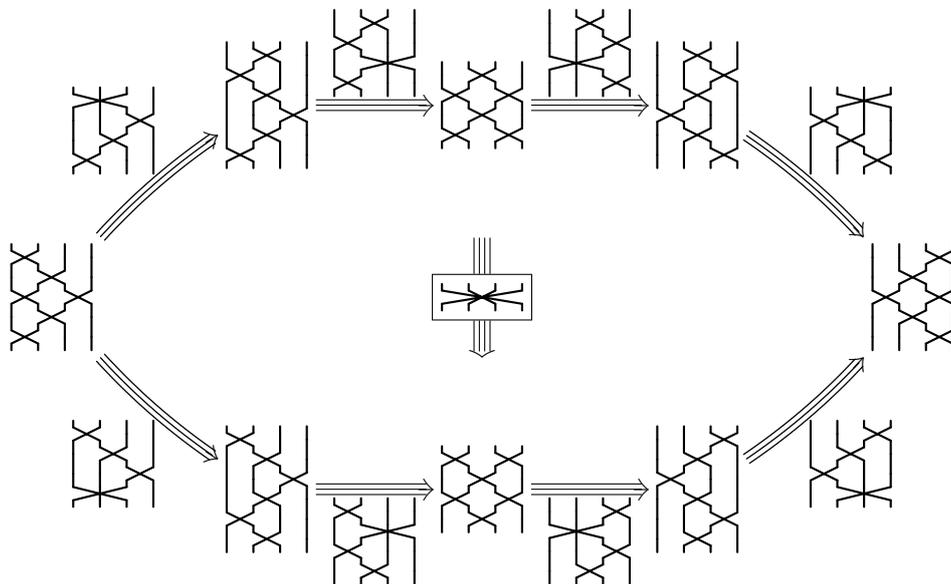
The 2-polygraph  $\Sigma$  has one critical triple branching

$$(s_{i,j}s_k s_l, s_i s_{j,k} s_l, s_i s_j s_{k,l})$$

for every possible  $0 \leq i \leq j \leq k \leq l \leq n + 3$ . This yields a resolution of **Epi** by the acyclic track  $(1, 3)$ -polygraph  $c_3(\Sigma)$  with one 4-cell  $s_{i,j,k,l}$  for every possible  $0 \leq i \leq j \leq k \leq l \leq n + 3$ . In string diagrams, omitting the subscripts, each critical triple branching is written

$$(\bowtie ||, | \bowtie |, || \bowtie)$$

With the same conventions, the corresponding 4-cell has the shape of the permutohedron of order 4:



As usual, the elements  $\delta_4 [\text{Diagram 10}]_{i,j,k,l}$ , for  $0 \leq i \leq j \leq k \leq l \leq n + 3$ , form a generating set for the natural system  $h_3(\Sigma)$  of homological 3-szygies of  $\Sigma$ .

More generally, this construction extends to a track-polygraphic resolution of **Epi** of infinite length where, for every natural number  $n$ , the generating  $n$ -cell has the shape of a permutohedron of order  $n$ .

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