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A Liouville theorem for a heat equation and applications for quenching

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Abstract: We prove a Liouville Theorem for a semilinear heat equation with absorption term in one dimension. We then derive from this theorem uniform estimates for quenching solutions of that equation.

Mathematical Subject classification: 35K05, 35K55, 74H35.

Keywords: Quenching, Liouville theorem, heat equation.

1 Introduction

This paper is concerned with quenching solutions of the following nonlinear heat equation

$$\begin{cases} \partial_t u &= \partial_x^2 u - \frac{1}{u^\beta} \text{ in } \mathbb{R} \times [0, T), \\ u(x, 0) &= u_0(x) > 0 \text{ for } x \in \mathbb{R}, \end{cases} \quad (1)$$

where $\beta \geq 3$ and

$$u_0, \quad \frac{1}{u_0} \in L^\infty(\mathbb{R}). \quad (2)$$

We say that $u(t)$ quenches in finite time T if u exists for $t \in [0, T)$ and

$$\lim_{t \rightarrow T} \inf_{x \in \mathbb{R}} u(x, t) = 0. \quad (3)$$

Hereafter we consider a solution u of (1) which quenches at finite time T . A point a is said to be a quenching point if there is a sequence $\{(a_n, t_n)\}$ such that $a_n \rightarrow a$, $t_n \rightarrow T$ and $u(a_n, t_n) \rightarrow 0$ as $n \rightarrow \infty$.

Quenching phenomena play an important role and have many applications in physics, chemistry and biology (see Salin [Sal04]). It may arise for instance in the theory of combustion and population genetics (see Galaktionov and Vázquez [GV02]), in population dynamics (see Gaucel and Langlais [GL07]), in connection with the diffusion equation generated by a polarization phenomena in ionic conductors (see Kawarada [Kaw75]), in connection with phase transition, when we study the motion of the borderline between liquids and solids (see Fila and Kawohl [FK92a]) and also in vortex reconnection with the

boundary in type **II** superconductors (see Chapman and all [CHO98] and Merle and Zaag [MZ97]).

The study of the quenching problem (1) was initiated by Kawarada [Kaw75]. There is a considerable literature on parabolic problems with strong absorption that deal with the precise description of the qualitative behavior of solutions near quenching. Typical questions in this respect concern the existence of quenching solutions as well as the quenching rates and profiles (see Deng and Levine [DL89], Fila and Kawohl [FK92a], Fila, Kawohl and Levine [FKBL92], Merle and Zaag [MZ97], Levine [Lev93] and Dávila and Montenegro [DM05]). Recently, equation (1) was numerically studied by Liang, Lin and Tan [LLT07].

The quenching rate as $t \rightarrow T$ of the solution u of (1) near a quenching point is among the important issues. It was obtained by Guo on bounded domains [Guo90], [Guo91a], Fila and Kawohl [FK92a]. For the higher-dimensional radial problem in [Guo91b] and Fila, Kawohl and Quittner [FHQ92], under some restrictions on initial data.

In [Guo90], Guo showed that if

$$u_0'' - u_0^{-\beta} \leq 0,$$

and a is an arbitrary quenching point, then

$$\lim_{t \rightarrow T} u(x, t) = U_T(t), \text{ uniformly for } |x - a| \leq C(T - t)^{\frac{1}{2}},$$

where $C > 0$ and $U_T(t) = (\beta + 1)^{\frac{1}{\beta+1}} (T - t)^{\frac{1}{\beta+1}}$ is the solution of $U_T' = -\frac{1}{U_T^\beta}$ and $U_T(T) = 0$.

In [FG93], Filippas and Guo were able to obtain a precise description of the profile of the solution in a neighborhood of the quenching point. They proved the following:

Let u be a solution of (1) defined for $x \in (-1, 1)$, with the boundary conditions $u(\pm 1, t) = 1$, which quenches at the point a at time T . Moreover, we assume that $u_0'' - \frac{1}{u_0^\beta} \leq 0$ and that u_0 has a single minimum. Then, we have that

$$\begin{cases} u(a, t) & \rightarrow 0, \\ u(x, t) & \rightarrow u^*(x) \text{ as } t \rightarrow T \text{ if } x \neq a, \\ u^*(x) & = \left[\frac{(\beta + 1)^2}{8\beta} \right]^{\frac{1}{(\beta+1)}} \left(\frac{|x - a|^2}{|\log |x - a||} \right)^{\frac{1}{(\beta+1)}} (1 + o(1)), \end{cases} \quad (4)$$

as $|x - a| \rightarrow 0$.

However, the result of Filippas and Guo is not uniform with respect to the quenching point or initial data. We aim in this paper at obtaining uniform estimates on $u(t)$ (defined on $\mathbb{R}$) at or near the singularity, that is as $t \rightarrow T$. In order to do so, we introduce for each $a \in \mathbb{R}$ the following similarity variables:

$$y = \frac{x - a}{\sqrt{T - t}}, \quad s = -\log(T - t), \quad w_a(y, s) = (T - t)^{-\frac{1}{\beta+1}} u(x, t). \quad (5)$$

The function $w_a (= w)$ satisfies for all $s \geq -\log T$, and $y \in \mathbb{R}$:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \cdot \partial_y w + \frac{w}{\beta + 1} - w^{-\beta}, \quad (6)$$

The study of $u(t)$ near (a, T) , where a is a quenching point, is equivalent to the study of the long time behavior of w_a .

If we introduce v defined by

$$v = \frac{1}{u}, \quad (7)$$

then, we can see easily from (1) and (3), that v satisfies the following equation:

$$\partial_t v = \partial_x^2 v - 2 \frac{(\partial_x v)^2}{v} + v^{2+\beta} \text{ in } \mathbb{R} \times [0, T), \quad (8)$$

We note that u quenches at time T if and only if v blows up at time T , and that a is a quenching point for u if and only if a is a blow-up point for v .

As we did for u , we introduce for v the following function:

$$z_a(y, s) = (T - t)^{\frac{1}{\beta+1}} v(x, t), \text{ where } y \text{ and } s \text{ are defined as in (5)}. \quad (9)$$

Note from (5) and (7) that $z_a = \frac{1}{w_a}$. We get by (8) and (9), that $z_a (= z)$ satisfies for all $s \geq -\log T$, and $y \in \mathbb{R}$,

$$\partial_s z = \partial_y^2 z - \frac{1}{2} y \cdot \partial_y z - 2 \frac{(\partial_y z)^2}{z} - \frac{z}{\beta+1} + z^{2+\beta}. \quad (10)$$

We note by L_ρ^2 the Hilbert space defined by;

$$L_\rho^2 = \{g \in L_{loc}^2(\mathbb{R}^N, \mathbb{C}), \int_{\mathbb{R}^N} |g|^2 e^{-\frac{|y|^2}{4}} dy < +\infty\} \text{ where } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}.$$

If g depends only on the variable $y \in \mathbb{R}^N$, we use the notation

$$\|g\|_{L_\rho^2}^2 = \int_{\mathbb{R}^N} |g(y)|^2 e^{-\frac{|y|^2}{4}} dy.$$

If g depends only on $(y, s) \in \mathbb{R}^N \times \mathbb{R}$, we use the notation

$$\|g(\cdot, s)\|_{L_\rho^2}^2 = \int_{\mathbb{R}^N} |g(y, s)|^2 e^{-\frac{|y|^2}{4}} dy.$$

1.1 A Liouville Theorem

Our aim in this paper is to prove estimates for quenching solution of (6)), uniform with respect to the space variable x . The following Liouville theorem is crucial for that purpose.

Theorem 1 (*A Liouville Theorem for equation (6)*) Assume that $\beta \geq 3$. We consider w , a global nonnegative continuous solution of (6), satisfying

$$|\partial_y w(y, s)| + \frac{1}{w(y, s)} \leq M, \text{ for all } (y, s) \in \mathbb{R} \times \mathbb{R}, \text{ where } M > 0. \quad (11)$$

Then either

(i) $w \equiv \kappa$, or

(ii) there exists $s_0 \in \mathbb{R}$ such that for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, $w(y, s) = \varphi(s - s_0)$, where

$$\varphi(s) = \kappa(1 + e^s)^{\frac{1}{\beta+1}} \quad (12)$$

is a solution of (6) independent of y and satisfies

$$\varphi' = \frac{\varphi}{1 + \beta} - \varphi^{-\beta}, \quad \varphi(-\infty) = \kappa, \quad \varphi(+\infty) = +\infty. \quad (13)$$

Remark: If $0 < \beta < 3$, we don't have an explicit result for the global stationary solution of (6) as for the case $\beta > 3$, the only result that we know is that there exist slow orbit, for more details see Guo [Guo91a], [Guo91b] and Fila, Kawohl and Quittner [FHQ92].

Theorem 1 has an equivalent formulation for solutions of equation (1):

Corollary 2 (*A Liouville Theorem for equation (1)*) Assume that u is a nonnegative continuous solution of (1) defined for $(x, t) \in \mathbb{R} \times (-\infty, T)$. Assume in addition that $u(x, t) \geq B(T - t)^{\frac{1}{(\beta+1)}}$ for some $B > 0$ and $\|\partial_x u(\cdot, t)\|_{L^\infty} \leq M(T - t)^{\frac{1}{\beta+1} - \frac{1}{2}}$, for $M > 0$. Then, there exists $T_0 \geq T$ such that for all $(x, t) \in \mathbb{R} \times (-\infty, T)$, $u(x, T) = \kappa(T_0 - t)^{\frac{1}{(\beta+1)}}$.

1.2 Application to quenching

We note that in the blow-up recent literature ([MZ00], [MM00] and [MZ08]), Liouville Theorems have important applications to blow-up. In the quenching problem we can get similar results. In the following, we will say that a solution u is of type **I**, if it satisfy

$$\left\| \frac{1}{u(\cdot, t)} \right\|_{L^\infty(\mathbb{R})} (T - t)^{\frac{1}{\beta+1}} + \|\partial_x u(\cdot, t)\|_{L^\infty(\mathbb{R})} (T - t)^{-\frac{1}{\beta+1} + \frac{1}{2}} \leq M, \text{ (type I)} \quad (14)$$

where M is a positive constant.

Note that several authors proved the existence of type **I** solution (see for example Guo [Guo90] and [Guo91a]). We wonder whether the bound on $\|\partial_x u\|_{L^\infty}$ in (14) follows from the bound on $\|\frac{1}{u}\|_{L^\infty}$. In other words, we wonder whether we can characterize the set of type **I** solution only by the fact that

$$\left\| \frac{1}{u} \right\|_{L^\infty} (T - t)^{1/(\beta+1)} \leq M \text{ for some } M > 0.$$

In the following, we derive from the Liouville Theorem some uniform estimates of type **I** quenching solutions of equation (6).

Theorem 3 *i) (Uniform bounds on $u(t)$ on quenching time) Assume that $u(t)$ is a non-negative solution of equation (1) that quenches in finite time $T > 0$ with initial data u_0 satisfying (2). Then, we have*

$$\inf_{x \in \mathbb{R}} u(x, t) \sim U_T(t) = (\beta + 1)^{\frac{1}{\beta+1}} (T - t)^{\frac{1}{\beta+1}} \text{ as } t \rightarrow T,$$

and

$$(T-t)^{\frac{1}{\beta+1}+\frac{1}{2}}\|\partial_x v(.,t)\|_{L^\infty(\mathbb{R})} + (T-t)^{\frac{1}{\beta+1}+1}\|\partial_x^2 v(.,t)\|_{L^\infty(\mathbb{R})} \rightarrow 0 \text{ as } t \rightarrow T.$$

Equivalently, for any $a \in \mathbb{R}$,

$$\inf_{y \in \mathbb{R}} w_a(y, s) \rightarrow \kappa \text{ as } s \rightarrow +\infty \text{ and } \|\partial_y z_a(., s)\|_{L^\infty} + \|\partial_y^2 z_a(., s)\|_{L^\infty} \rightarrow 0 \text{ as } s \rightarrow +\infty,$$

where $\kappa = (1 + \beta)^{\frac{1}{\beta+1}}$.

ii) (ODE-type behavior) For all $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that, for all $t \in [\frac{T}{2}, T)$ and $x \in \mathbb{R}$,

$$\left| \frac{\partial u}{\partial t} - u^{-\beta} \right| \leq \varepsilon |u|^{-\beta} + C_\varepsilon. \quad (15)$$

1.3 Strategy of the proof of the Liouville theorem

Our method is inspired by the one of Merle and Zaag [MZ98a] and [MZ00] developed for the semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1}u \quad (16)$$

where

$$u : (x, t) \in \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R} \text{ and } p > 1, \quad p < \frac{N+2}{N-2} \text{ if } N \geq 3.$$

However, our contribution is not a simple adaptation of the proof of [MZ98a] and [MZ00]. In fact, in these papers the authors strongly rely on two blow-up criteria for the selfsimilar version of (16)

$$w_s = \Delta w - \frac{1}{2}y\partial_y w - \frac{w}{p-1} + |w|^{p-1}w. \quad (17)$$

- Criterion 1: For nonnegative solution of (17) such that

$$\int_{\mathbb{R}^N} w(y, s_0) \rho(y) dy > (p-1)^{\frac{1}{p-1}},$$

for some $s_0 \in \mathbb{R}$, then w blows up in some finite time $s > s_0$.

- Criterion 2: (with no sign condition) Assume that w is a solution of (17) such that

$$\mathcal{E}(w(s_0)) \leq \frac{(p-1)}{2(p+1)} \left(\int_{\mathbb{R}^N} |w(y, s_0)|^2 \rho(y) dy \right)^{\frac{p+1}{2}},$$

for some $s_0 \in \mathbb{R}$, where $\mathcal{E}$ is the following Lyapunov functional defined by

$$\mathcal{E}(w) = \int_{\mathbb{R}} \left(\frac{1}{2} |\partial_y w|^2 + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1} \right) \rho(y) dy \text{ with } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}.$$

Then, w blows up in finite time $s > s_0$.

Note that in [Zaa01], Zaag obtained a Liouville theorem for the non-gradient structure system

$$\begin{cases} \partial_t u = \Delta u + v^p, & \partial_t v = \Delta v + u^q \\ u(., 0) = u_0, & v(., 0) = v_0, \end{cases} \quad (18)$$

where $p - q$ is small. There, he adapts the proof of [MZ98a] and uses an infinite time blow-up criterion in similarity variables.

In our quenching problem, there is no way to find any equivalent blow-up criterion. We need new ideas which make the originality of our paper. From this point of view the difficulty level is similar to the case of the complex valued equation with no gradient structure

$$\partial_t u = \Delta u + (1 + i\delta)|u|^{p-1}u \quad (19)$$

where $u : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{C}$.

treated in [NZ10]. We proceed in five parts.

- Part 1, we show that $z (= 1/w)$ has a limit $z_{\pm\infty}$ as $s \rightarrow \pm\infty$, where $z_{\pm\infty}$ is a critical point of the stationary version of (10) or 0. That is, $z_{\pm\infty} \equiv \kappa^{-1}$ or $z_{\pm\infty} \equiv 0$, thanks to Guo. Then, we rule out the case where $z_{-\infty} \equiv 0$ with sophisticated energy argument. The following parts are dedicated to the non trivial case where $z_{-\infty} = \kappa^{-1}$.
- in Part 2, we investigate the linear problem (in w) around κ as $s \rightarrow -\infty$ and show that w behaves at most in three ways:

- (i) $w(y, s) = \kappa + C_0 e^s + o(e^s)$ as $s \rightarrow -\infty$, for some constant $C_0 \in \mathbb{R}$.
- (ii) $w(y, s) = \kappa + C_1 e^{s/2} + o(e^{s/2})$ as $s \rightarrow -\infty$, for some constant $C_1 \in \mathbb{R} \setminus \{0\}$.
- (iii) $w(y, s) = \kappa + \frac{\kappa}{2\beta s} \left(\frac{1}{2} y^2 - 1 \right) + o\left(\frac{1}{s}\right)$ as $s \rightarrow -\infty$.

In these cases, convergence take place in $L^\infty([-R, R])$ for any $R > 0$ and in L_ρ^2 .

- In Part 3, we show that (i) corresponds to $w(y, s) = \varphi(s - s_0)$ or $w(y, s) = \kappa$, for some $s_0 \in \mathbb{R}$, where φ is defined by (13).
- In Parts 4 and 5, we rule out cases (ii) and (iii). In [MZ98a] and [MZ00], the authors shows that for some $a_0 \in \mathbb{R}$ and $s_0 \in \mathbb{R}$, $w_{a_0}(y, s) = w(y + a_0 e^{s_0}, s_0)$ satisfies one of the blow-up criteria stated in page (5), which contradicts the fact that w exists for all $s \in \mathbb{R}$. In our case we don't have any blow-up criterion. It turns out that this is the major difficulty in our paper, as in [NZ10] for equation (19). Following [NZ10], we will use a geometrical method where the key idea is to extend the convergence stated in (ii) and (iii) from compact sets to larger zones, so that we find *the singular profile* for w . It appears that in both cases, for larger $|y|$, this profile becomes strictly inferior to $\frac{1}{M}$, where M is defined in (11), which is a contradiction. The originality of our paper is based on Velázquez's work in [Vel92], where he extends the convergence from compact sets to larger sets to find the profile for solutions of (17). Note that the fact that w is not in L^∞ makes it delicate to use the estimate of [Vel92]. More precisely, we obtain the following singular profiles

– If case (ii) holds, then

$$\lim_{s \rightarrow -\infty} \sup_{|y| \leq R e^{-s/2}} \left| w(y, s) - ((\beta + 1) - C_1 \kappa^{-\beta} y e^{s/2})^{\frac{1}{\beta+1}} \right| = 0$$

with $0 < R < \frac{(\beta+1)}{C_1 \kappa^{-\beta}}$ and $C_1 \neq 0$.

– If case (iii) holds, then

$$\lim_{s \rightarrow -\infty} \sup_{|y| \leq \sqrt{-s} R} \left| w(y, s) - \left((\beta + 1) + \frac{(\beta + 1)^2 y^2}{4\beta s} \right)^{\frac{1}{\beta+1}} \right|$$

with $0 < R < \sqrt{\frac{4\beta}{(\beta+1)}}$.

Then using condition (11), case (ii) and (iii) are ruled out, which ends the proof of our Liouville Theorem.

Our paper is organised as follows:

In section 2, we prove the Liouville theorem.

In section 3, we prove the uniform estimates for quenching solutions of (6) in Theorem 3.

2 Proof of the Liouville Theorem for equation (6)

We assume that $\beta \geq 3$ and consider $w(y, s)$ a nonnegative, global solution of (6) satisfying (11), defined for all $(y, s) \in \mathbb{R} \times \mathbb{R}$. Introducing $z = \frac{1}{w}$, we know that z satisfies (10) on $\mathbb{R}^2$.

Our goal is to show that w depends only on the variable s .

2.1 Part 1: Behavior of w as $s \rightarrow \pm\infty$

The main results of this part are consequences of the parabolic estimates and the gradient structure of equation (6). Let us recall them.

Lemma 2.1 (*Parabolic estimates*) *There exists $M_0 > 0$ such that for all $(y, s) \in \mathbb{R} \times \mathbb{R}$:*

- (i) $|\partial_y w(y, s)| + \frac{1}{w(y, s)} \leq M_0$ and $w(y, s) \leq w(0, s) + M_0 |y|$,
- (ii) $|z(y, s)| + |\partial_y z(y, s)| + |\partial_y^2 z(y, s)| \leq M_0$ and $|\partial_s z| \leq M_0(1 + |y|)$.
- (iii) For all $R > 0$, z , $\partial_y z(y, s)$, $\partial_y^2 z$ and $\partial_s z$ are bounded in $C_\alpha^0([-R, R]^2)$ for some $\alpha \in (0, 1)$, where

$$C_\alpha^0([-R, R]^2) = \left\{ \psi \in L^\infty([-R, R]^2) \mid \sup_{(\xi, \tau), (\xi', \tau') \in [-R, R]^2} \frac{|\psi(\xi, \tau) - \psi(\xi', \tau')|}{(|\xi - \xi'| + |\tau - \tau'|^{1/2})^\alpha} < \infty \right\}$$

- (iv) $|\partial_s w(y, s)| \leq M_0(1 + |y|)(w(0, s) + |y|)^2$.

Proof:

(i) See (11).

(ii) and (iii) See the proof of Lemma 3.2 from [Guo90].

(iv) If we write $\partial_s w = -\partial_s z/z^2 = -\partial_s z w^2$, then we get the result by (i) and (ii).

Lemma 2.2 (*Gradient Structure*)

(i) (*Gradient Structure for equation (6)*). If we introduce

$$\mathbf{E}(w(s)) = \int_{\mathbb{R}} \partial_y w^2 \rho dy - \frac{1}{2(\beta+1)} \int_{\mathbb{R}} w^2 \rho dy - \frac{1}{\beta-1} \int_{\mathbb{R}} w^{-(\beta-1)} \rho dy \text{ with } \rho(y) = \frac{e^{-\frac{y^2}{4}}}{(4\pi)^{\frac{1}{2}}}, \quad (20)$$

then, for all $s_1 < s_2 \in \mathbb{R}$, such that

$$0 < w(y, s) \leq c_1(s)(1 + |y|) \text{ for all } s_1 \leq s \leq s_2 \text{ (with } c_1(s) > 0),$$

we have

$$\int_{s_1}^{s_2} \int_{\mathbb{R}} (\partial_s w)^2 \rho dy = \mathbf{E}(w(s_1)) - \mathbf{E}(w(s_2)) \quad (21)$$

(ii) (*Gradient Structure for equation (10)*). If we define for each z solution of (10)

$$E(z(s)) = \int_{\mathbb{R}} \frac{\partial_y z^2}{z^4} \rho dy - \frac{1}{2(\beta+1)} \int_{\mathbb{R}} \frac{1}{z^2} \rho dy - \frac{1}{\beta-1} \int_{\mathbb{R}} z^{\beta-1} \rho dy, \quad (22)$$

under the same conditions in (i), we obtain:

$$\int_a^b \int_{\mathbb{R}} \left(\frac{\partial_s z}{z^2} \right)^2 \rho dy = E(z(a)) - E(z(b)), \text{ for any real } a < b. \quad (23)$$

Proof:

(i) One may multiply equation (10) by $\partial_s w \rho$ and integrate over the ball $B_R = B(O, R)$ with $R > 0$. Then using Lemma 2.1 and the Lebesgue's theorem yields the result. (see Proposition 3 from Giga and Kohn [GK85] for more details).

(ii) Since $E(z) = \mathbf{E}(\frac{1}{z})$, this immediately follows from (i). ■

We now give the limits of z as $s \rightarrow \pm\infty$ in the following:

Proposition 2.3 (*Limit of z as $s \rightarrow \pm\infty$*) The limit $z_{+\infty}(y) = \lim_{s \rightarrow +\infty} z(y, s)$ exists and equals 0 or κ^{-1} . The convergence is uniform on every compact subset of $\mathbb{R}$. The corresponding statements also hold for the limit $z_{-\infty}(y) = \lim_{s \rightarrow -\infty} z(y, s)$.

Proof: It follows from the following:

Lemma 2.4 Consider any increasing (respectively decreasing) sequence such that $s_j \rightarrow \pm\infty$ as $j \rightarrow \infty$. Then:

(i) there is a subsequence (still denoted by (s_j)) such that

$$z(y, s + s_j) \rightarrow l \text{ in } C_{\alpha}^2([-R, R]) \text{ for all } R > 0, \text{ where } l = 0 \text{ or } \kappa^{-1}.$$

(ii) If $l = 0$, then $E(z(s_j)) \rightarrow -\infty$, if $l = \kappa^{-1}$, then $E(z(s_j)) \rightarrow E(\kappa^{-1})$.

Indeed, from Lemma 2.4, in order to get the conclusion of Proposition 2.3, it is enough to show that the limit in Lemma 2.4 is independent of the choice of the sequence. We consider the case $s \rightarrow \infty$, the other case being similar. Suppose that (s_j) and $(\bar{s}_j)$ both tend to infinity. Up to extracting subsequence, we assume that for all $j \in \mathbb{N}$, $s_j > \bar{s}_j$ and proceeding by contradiction, we assume that

$$z_j(y, s) = z(y, s + s_j) \rightarrow \kappa^{-1} \text{ and } \bar{z}_j(y, s) = z(y, s + \bar{s}_j) \rightarrow 0 \text{ as } j \rightarrow \infty.$$

By (ii) of Lemma 2.4, we have

$$E(z_j(s)) \rightarrow E(\kappa^{-1}) \text{ and } E(\bar{z}_j(s)) \rightarrow -\infty,$$

hence, for j large enough, we have

$$s_j > \bar{s}_j, \quad E(z(s_j)) > E(\kappa^{-1}) - 1 > E(z(\bar{s}_j)),$$

which contradicts the monotonicity of E . Thus the limit in Lemma 2.4 is independent of the choice of the sequence and the whole function $z(y, s)$ converges as $s \rightarrow \infty$. It remains to prove Lemma 2.4 to finish the proof.

Proof of Lemma 2.4:

(i) We only present the case $s_j \rightarrow +\infty$, the analysis for $s_j \rightarrow -\infty$ is the same. Let us recall that for all (y, s) , $z(y, s)$ satisfies (10), which is:

$$\partial_s z = \partial_y^2 z - \frac{1}{2} y \partial_y z - 2 \frac{\partial_y z^2}{z} - \frac{z}{\beta + 1} + z^{2+\beta}. \quad (24)$$

From condition (11), we get $0 \leq \frac{(\partial_y z)^2}{z} = \frac{(\partial_y w)^2}{w^2} z \leq Cz$ and $0 \leq z^{2+\beta} \leq Cz$, hence we obtain:

$$\mathcal{L}_0 z - Cz \leq \partial_s z \leq \mathcal{L}_0 z + Cz \text{ where } \mathcal{L}_0 z = \partial_y^2 z - \frac{1}{2} y \partial_y z. \quad (25)$$

Using the semigroup $S_0(\tau)$ associated to $\mathcal{L}_0$:

$$S_0(\tau)\phi(y) = \frac{1}{(4\pi(1 - e^{-\tau}))^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-\frac{\tau}{2}} - \lambda)}{4(1 - e^{-\tau})}\right) \phi(\lambda) d\lambda, \quad (26)$$

we see that for all $s \geq s'$,

$$e^{-C(s-s')} S_0(s-s') z(\cdot, s') \leq \|z(\cdot, s)\|_{L^\infty} \leq e^{C(s-s')} \|z(\cdot, s')\|_{L^\infty}. \quad (27)$$

Let (s_j) be a sequence tending to $+\infty$, and let $z_j(y, s) = z(y, s + s_j)$. From Lemma 2.1, up to extracting a sub-sequence (still denoted by (s_j)), z_j converges to $\tilde{z}_{+\infty}$ in $C^{2,1}((-R, R)^2)$ for any $R > 0$ and we obtain that $\tilde{z}_{+\infty}$ satisfy (25).

In the following, we will prove that the limit $\tilde{z}_{+\infty}$ is either 0 or κ^{-1} . We have to consider two cases. .

Case 1: There exists $(y_0, s_0) \in \mathbb{R}^2$ such that $\tilde{z}_{+\infty}(y_0, s_0) = 0$

We claim first that

$$\forall y \in \mathbb{R}, \quad \tilde{z}_{+\infty}(y, s_0) = 0. \quad (28)$$

Indeed, if for some $y_1 \in \mathbb{R}$, we have $\tilde{z}_{+\infty}(y_1, s_0) > 0$, then since we have from (11)

$$\frac{1}{z_j(y_0, s_0)} \leq \frac{1}{z_j(y_1, s_0)} + M|y_1 - y_0|,$$

letting $j \rightarrow \infty$, we get a contradiction. Thus (28) holds. Using (27), we conclude that $z \equiv 0$ on $\mathbb{R}^2$ (indeed, by (27) for $s \geq s_0$, $z(\cdot, s) \equiv 0$, and for $s \leq s_0$, $S_0(s_0 - s)z(\cdot, s) \equiv 0$, hence $z(\cdot, s) \equiv 0$).

Case 2: For all $(y, s) \in \mathbb{R}^2$, $\tilde{z}_{+\infty}(y, s) > 0$

Let us introduce $w_j(y, s) = w(y, s + s_j)$. In this case,

$$w_j(y, s) = 1/z_j(y, s) \rightarrow \tilde{w}_{+\infty} = 1/\tilde{z}_{+\infty}(y, s) \text{ in } C^{2,1}((-R, R)^2), \text{ for any } R > 0. \quad (29)$$

Since z_j and w_j are solutions of (24) and (6) respectively, the same holds for $\tilde{z}_{+\infty}$ and $\tilde{w}_{+\infty}$ respectively. Our goal in this step is to prove that $\tilde{z}_{+\infty}(y, s) = \kappa^{-1}$. First, we prove that $\tilde{z}_{+\infty}$ is independent of s , then we conclude using the result of Guo concerning the stationary global solution of (6). We will proceed as in Proposition 4, page 308 in [GK85]. We claim the following:

Claim 2.5 (i) For all $(y, s) \in \mathbb{R}^2$

$$\begin{aligned} |\partial_y w_j(y, s)| + |1/w_j(\cdot, s)| &\leq C_1, \quad |w_j(y, s)| \leq C_1(|y| + \tilde{w}_{+\infty}(0, s)) \\ \text{and } |\partial_s w_j(y, s)| &\leq C_1(1 + |y|)(|y| + \tilde{w}_{+\infty}(0, s))^2, \end{aligned}$$

where $C_1 > 0$ independent of j and $\tilde{w}_{+\infty}$ is a solution of (6).

(ii) For all $(y, s) \in \mathbb{R}^2$,

$$\begin{aligned} |\partial_y \tilde{w}_{+\infty}(y, s)| + |1/\tilde{w}_{+\infty}(y, s)| &\leq C_1, \quad |\tilde{w}_{+\infty}(y, s)| \leq C_1(|y| + \tilde{w}_{+\infty}(0, s)) \\ \text{and } |\partial_s \tilde{w}_{+\infty}(y, s)| &\leq C_1(1 + |y|)(|y| + \tilde{w}_{+\infty}(0, s))^2. \end{aligned} \quad (30)$$

Remark: To make the notation legible, we will note the partial derivative in time and space of the sequence w_j (or z_j) and the limit $\tilde{w}_{+\infty}$ (or $\tilde{z}_{+\infty}$) respectively $\partial_s w_j$, $\partial_y w_j$ and $\partial_s \tilde{w}_{+\infty}$, $\partial_y \tilde{w}_{+\infty}$.

Proof of Claim 2.5: (i) Integrating the inequality $|\partial_y w_j(y, s)| \leq M$ (by (i) of Lemma 2.1) in space between 0 and y , we obtain $|w_j(y, s)| \leq M|y| + w_j(0, s)$. Then using the fact that

$$w_j(0, s) \rightarrow \tilde{w}_{+\infty}(0, s) (= 1/\tilde{z}_{+\infty}(0, s)) < +\infty, \text{ as } j \rightarrow \infty,$$

we get the second inequality of (i) in Claim (2.5). Using Lemma 2.1 and the identity

$$\partial_s w_j(y, s) = -\partial_s z_j(y, s)/z_j(y, s)^2 = -\partial_s z_j(y, s)w_j(y, s)^2,$$

we obtain the third inequality in (i).

(ii) Using (i) and the convergence (29), the result comes immediately. ■

Up to extracting a subsequence, we can assume that $s_{j+1} - s_j \rightarrow \infty$. Therefore, using (21) with $w = w_j$, $a = m$, $b = m + s_{j+1} - s_j$, where $m \in \mathbb{Z}$, we get:

$$\begin{aligned}
\int_m^{s_{i+1}-s_j+m} \int_{\mathbb{R}} (\partial_s w_j)^2 \rho dy &= \mathbf{E}(w_j(m)) - \mathbf{E}(w_j(m + s_{j+1} - s_j)) \\
&= \mathbf{E}(w_j(m)) - \mathbf{E}(w_{j+1}(m)).
\end{aligned} \tag{31}$$

Using (i), (ii) and (iii) of Claim (2.5) together with Lesbegue's Theorem we have

$$\begin{aligned}
\int_{\mathbb{R}} \partial_y w_j^2 \rho dy &\rightarrow \int_{\mathbb{R}} \partial_y \tilde{w}_{+\infty}^2 \rho dy, & \int_{\mathbb{R}} w_j^2 \rho dy &\rightarrow \int_{\mathbb{R}} \tilde{w}_{+\infty}^2 \rho dy, \\
\text{and } \int_{\mathbb{R}} w_j^{-(\beta-1)} \rho dy &\rightarrow \int_{\mathbb{R}} \tilde{w}_{+\infty}^{-(\beta-1)} \rho dy & \text{as } j \rightarrow \infty.
\end{aligned}$$

Therefore, $\mathbf{E}(w_j(m)) \rightarrow \mathbf{E}(\tilde{w}_{+\infty}(m))$ as $j \rightarrow \infty$ (well defined by (iii) of Claim 2.5). In particular, the right-hand side of (31) tends to zero. Since $s_{j+1} - s_j \rightarrow +\infty$, it follows that for every integers $m < M$

$$\text{on the one hand } \lim_{j \rightarrow \infty} \int_m^M \int_{\mathbb{R}} |\partial_s w_j|^2 \rho dy ds = 0, \tag{32}$$

On the other hand, by the third inequality in (i) of Claim 2.5 and the continuity of $\tilde{w}_{+\infty}(0, s)$, we obtain

$$\forall (y, s) \in \mathbb{R} \times (m, M) \quad |\partial_s w_j(y, s)| \leq C(m, M)(1 + |y|)^3.$$

Since $\partial_s w_j$ converges simply to $\partial_s \tilde{w}_{+\infty}$ by (iii) of Claim 2.5, we conclude that:

$$\int_m^M \int_{\mathbb{R}} |\partial_s w_j|^2 \rho dy ds \rightarrow \int_m^M \int_{\mathbb{R}} |\partial_s \tilde{w}_{+\infty}|^2 \rho dy ds. \tag{33}$$

Using (32) and (33), we get

$$\int_m^M \int_{\mathbb{R}} |\partial_s \tilde{w}_{+\infty}|^2 \rho dy ds = 0.$$

Using the fact that m and M are arbitrary, we conclude that $\partial_s \tilde{w}_{+\infty} = 0$ on $\mathbb{R}^2$ and $\tilde{w}_{+\infty}(y, s) = \tilde{w}_{+\infty}(y)$ is independent of s . Since $\tilde{w}_{+\infty}$ is a solution of (6), it follows that $\tilde{w}_{+\infty}$ solves the following stationary equation

$$\forall y \in \mathbb{R}, \quad 0 = w'' - \frac{1}{2} y w' + \frac{1}{\beta + 1} w - w^{-\beta}. \tag{34}$$

We recall that by (ii) of Claim 2.5 we have $\tilde{w}_{+\infty}(s) > 1/C$ and $\tilde{w}_{+\infty}(s) \leq C(1 + |y|)$. Now we recall the result of Guo given by Theorem 2.1 in [Guo90]):

The only global solution of

$$w'' - \frac{1}{2} y w' + \frac{w}{\beta + 1} - \varepsilon w^{-\beta} = 0, \quad y \in \mathbb{R},$$

which is greater than or equal to some positive constant c and which grows at most polynomially as $|y| \rightarrow \infty$, is $w \equiv (\varepsilon(\beta + 1))^{\frac{1}{\beta+1}}$.

Using this result, it follows that $\tilde{w}_{+\infty} = \kappa$ and we conclude that $\tilde{z}_{+\infty} \equiv \kappa^{-1}$. This concludes the proof of (i) of Lemma 2.4.

(ii) We recall from Lemma 2.2 that

$$E(z(s)) = \int_{\mathbb{R}} \frac{\partial_y z^2}{z^4} \rho dy - \frac{1}{2(\beta+1)} \int_{\mathbb{R}} \frac{1}{z^2} \rho dy - \frac{1}{\beta-1} \int_{\mathbb{R}} z^{\beta-1} \rho dy. \quad (35)$$

If z_j converges to κ^{-1} , using Lesbegue's Theorem and Claim 2.5, we obtain

$$E(z_j(s)) = \mathbf{E}(w_j(s)) \rightarrow E(\kappa^{-1}) = \mathbf{E}(\kappa).$$

In the case, where z_j converges to 0, we write from Lemma 2.2 and Claim 2.5

$$E(z_j(s)) = \mathbf{E}(w_j(s)) \leq C - \frac{1}{\beta-1} \int w^{-(\beta-1)} \rho dy \quad (36)$$

Since the integral in (36) tends to infinity as $j \rightarrow +\infty$ we have:

$$E(z_j(s)) \rightarrow -\infty.$$

This concludes the proof of Lemma 2.4 and Proposition 2.3. ■

To end this part, we recall the result obtained by Proposition 2.3, which says that $z_{+\infty}(y) = \lim_{s \rightarrow +\infty} z(y, s)$ and $z_{-\infty}(y) = \lim_{s \rightarrow -\infty} z(y, s)$ exist and equal 0 or κ^{-1} . Now, we will prove that the case $z_{-\infty} = 0$ can't occur.

Indeed, suppose that $z(y, s) \rightarrow 0$ as $s \rightarrow -\infty$, by (ii) of Lemma 2.4, we have:

$$E(z(s)) \rightarrow -\infty \text{ as } s \rightarrow -\infty.$$

Therefore, there exists $s_1 < 0$, such that for all $s \leq s_1$, we have

$$s < 0 \text{ and } E(z(s)) < E(z(0)).$$

which contradicts the monotonicity of E . Thus, the case $z_{-\infty} = 0$ is ruled out. In the following, we will study the case where $w_{-\infty} = \kappa$.

2.2 Part 2: Linear behavior of w near κ as $s \rightarrow -\infty$

In this part, we assume that

$$w \rightarrow \kappa \text{ as } s \rightarrow -\infty \quad (37)$$

in L^2_ρ and uniformly on every compact sets, and we classify the L^2_ρ behavior of $w - \kappa$ as $s \rightarrow -\infty$. Let us introduce $v = w - \kappa$. From (6), v satisfies the following equation:

$$\forall (y, s) \in \mathbb{R}^2, \partial_s v = \mathcal{L}v - f(v), \quad (38)$$

where

$$\begin{aligned} \mathcal{L}v &= \partial_y^2 v - \frac{1}{2} y \partial_y v + v \text{ and} \\ f(v) &= (v + \kappa)^{-\beta} - \kappa^{-\beta} + \beta \kappa^{-(\beta+1)} v. \end{aligned} \quad (39)$$

Concerning the non-linear term of equation (39), we have:

Lemma 2.6 *There exists $s_1 \in \mathbb{R}$ such that for all $s \leq s_1$:*

$$0 \leq f(v) \leq M_1 v^2 \text{ and } f(v) \leq M_2 |v|. \quad (40)$$

Proof: Using Taylor's formula, we obtain

$$(v + \kappa)^{-\beta} = \kappa^{-\beta} - \frac{\beta}{(\beta + 1)} v + \frac{1}{2} \beta(\beta + 1)(\theta + \kappa)^{-\beta-2} v^2, \quad (41)$$

for some θ between 0 and v . Therefore

$$f(v) = c(\theta, \beta) v^2, \quad c(\theta, \beta) = \frac{1}{2} \beta(\beta + 1)(\theta + \kappa)^{-\beta-2}.$$

If $v > 0$ then $0 \leq c(\theta, \beta) \leq (\beta/2)(\beta + 1)\kappa^{-\beta-2} = \beta/(2\kappa)$.

If $v < 0$, recalling that $B \leq v + \kappa \leq \theta + \kappa$ we obtain $0 \leq c(\theta, \beta) \leq (\beta/2)(\beta + 1)B^{-\beta-2}$. Thus (40) was established. To show the second inequality in (40) we observe that if $v < \kappa$, then it follows from (40). If $v > \kappa$ then from (39) and the lower bound of v , we have

$$f(v) \leq \frac{\beta}{\beta + 1} v - \frac{1}{\beta + 1} \kappa + B^{-\beta} \leq M_2 v,$$

for some constant C_2 . ■

In the following, we will discuss general properties of the operator $\mathcal{L}$. At first we note that it is self-adjoint on L_ρ^2 . Its spectrum is $\text{spec}(\mathcal{L}) = \{1 - \frac{m}{2} | m \in \mathbb{N}\}$; it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are simple and derived from Hermite polynomials. For $1 - \frac{m}{2}$ corresponds the eigenfunction

$$h_m(y) = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{n!(m-2n)!} (-1)^n y^{m-2n}.$$

The polynomial h_m satisfies $\int h_n h_m \rho dy = 2^n n! \delta_{nm}$. Let us introduce $k_m = \frac{h_m}{\|h_m\|_{L_\rho^2}^2}$. Since the eigenfunctions of $\mathcal{L}$ span all the space L_ρ^2 , we expand v as follows:

$$v(y, s) = \sum_{m=0}^2 v_m(s) \cdot h_m(y) + v_-(y, s) \quad (42)$$

where v_m is the projection of v on h_m and $v_-(y, s) = P_-(v)$, with P_- is the orthogonal projector on the negative subspace of $\mathcal{L}$.

Now we show that as $s \rightarrow -\infty$, either $v_0(s)$, $v_1(s)$ or $v_2(s)$ is predominant with respect to the expansion (42) of v in L_ρ^2 . We have the following:

Proposition 2.7 *(Classification of the behavior of $v(y, s)$ as $s \rightarrow -\infty$) As $s \rightarrow -\infty$, one of the following situations occurs:*

(i) $|v_1(s)| + |v_2(s)| + \|v_-(\cdot, s)\|_{L_\rho^2} = o(v_0(s))$, $\|v(\cdot, s) - C_0 e^s\|_{L_\rho^2} = O(e^{\frac{3}{2}s})$ for some $C_0 \in \mathbb{R}$.

(ii) $|v_0(s)| + |v_2(s)| + \|v_-(\cdot, s)\|_{L_\rho^2} = o(v_1(s))$, $\|v(\cdot, s) - C_1 y e^{s/2}\|_{L_\rho^2} = O(e^{(1-\varepsilon)s})$ for some $C_1 \in \mathbb{R} \setminus 0$ and any $\varepsilon > 0$.

(iii) $|v_0(s)| + |v_1(s)| + \|v_-(\cdot, s)\|_{L_\rho^2} = o(\|v_2(\cdot, s)\|_{L_\rho^2})$, $\|v(\cdot, s) - \frac{\kappa}{4\beta} (y^2 - 2)\|_{L_\rho^2} = O(\frac{\log(|s|)}{s^2})$.

2.3 Part 3: Case (i) of Proposition 2.7: $\exists s_0 \in \mathbb{R}$ such that $w(y, s) = \varphi(s - s_0)$

In this part, we prove the following:

Proposition 2.8 *(The relevant case (i) of Proposition 2.7) Assume that case (i) of Proposition 2.7 holds. Then there exists $s_0 \in \mathbb{R}$ such that for all $(y, s) \in \mathbb{R}^2$, $w(y, s) = \varphi(s - s_0)$, where $\varphi(s)$ is introduced in (12).*

Proof: First we recall from (i) of Proposition 2.7 and the definition of v that

$$\|w(\cdot, s) - \{\kappa + C_0 e^s\}\|_{L_p^2} \leq C e^{3/2s} \text{ as } s \rightarrow -\infty. \quad (43)$$

Let us remark that we already have a solution $\hat{\varphi}$ of (6) defined in $\mathbb{R} \times (-\infty, s_1]$ for some $s_1 \in \mathbb{R}$ and which satisfies the same expansion as $w(y, s)$ when $s \rightarrow -\infty$:

$$\begin{aligned} \text{(a) if } C_0 = 0, & \text{ just take } \hat{\varphi} \equiv \kappa, \\ \text{(b) if } C_0 > 0, & \text{ take } \hat{\varphi} \equiv \varphi(s - s_0) \text{ where } s_0 = -\log\left(\frac{(\beta+1)}{\kappa}C_0\right), \\ \text{(c) if } C_0 < 0, & \text{ take } \hat{\varphi} \equiv \varphi_0(s - s_0) \text{ where } s_0 = -\log\left(-\frac{(\beta+1)}{\kappa}C_0\right) \end{aligned} \quad (44)$$

and $\varphi_0(s) = \kappa(1 - e^s)^{\frac{1}{\beta+1}}$

is a solution of (6) that quenches at $s = 0$ but there exist $C > 0$ such that $\varphi_0 \geq C$ for all $s \leq -1$. If we introduce $V = w - \hat{\varphi}$, then we see from (6) that V is defined for all $(y, s) \in \mathbb{R} \times (-\infty, s_1]$ and satisfies

$$\frac{\partial V}{\partial s} = (\mathcal{L} + l(s))V - F(V), \quad (45)$$

where $\mathcal{L}$ is given in (39)

$$F(V) = |\hat{\varphi} + V|^{-\beta} - \hat{\varphi}^{-\beta} + \beta \hat{\varphi}^{-(\beta+1)}V$$

and

$$\begin{aligned} l(s) &= 0 & \text{if } \hat{\varphi} = \kappa, \\ l(s) &= -\frac{\beta e^s}{(\beta+1)(1+e^s)} & \text{if } \hat{\varphi} = \varphi(s - s_0), \\ l(s) &= \frac{\beta e^s}{(\beta+1)(1-e^s)} & \text{if } \hat{\varphi} = \varphi_0(s - s_0). \end{aligned} \quad (46)$$

We note that $|l(s)| \leq C e^s$ for all $s \leq s_1$ and some $C > 0$ and there exist $M_{i=1,2}$ such that

$$0 \leq F(V) \leq M_1 V^2 \text{ and } 0 \leq F(V) \leq M_2 |V|. \quad (47)$$

We omit the proof of (47) since it is quite similar to the proof of (40). Let us introduce $I(s) = \|V(\cdot, s)\|_{L_p^2}$, multiply (45) by $V\rho$ and integrate over $\mathbb{R}$. Using the fact that 1 is the greatest eigenvalue of $\mathcal{L}$ and (47), we obtain

$$I'(s) \leq (1 + l(s))I(s) + C \int V^4(y, s) \rho dy.$$

Now, we recall the following from [Vel93]:

Lemma 2.9 (*Regularizing effect of the operator $\mathcal{L}$*) Assume that $\psi(y, s)$ satisfies

$$\forall s \in [a, b], \quad \forall y \in \mathbb{R}, \quad \partial_s \psi \leq (\mathcal{L} + \sigma)\psi, \quad 0 \leq \psi(y, s),$$

for some $a \leq b$ and $\sigma \in \mathbb{R}$, where

$$\mathcal{L}\psi = \partial_y^2 \psi - \frac{1}{2}y \cdot \partial_y \psi + \psi = \frac{1}{\rho} \operatorname{div}(\rho \partial_y \psi) + \psi. \quad (48)$$

Then for any $r > 1$, there exists $C^* = C^*(r, \sigma) > 0$ and $s^*(r)$, such that

$$\forall s \in [a + s^*, b], \quad \left(\int_{\mathbb{R}} |\psi(y, s)|^r \rho(y) dy \right)^{1/r} \leq C^* \|\psi(\cdot, s - s^*)\|_{L_\rho^2}. \quad (49)$$

Proof of Lemma: See Lemma 2.3 in [HV93] ■

Using the lemma above, we obtain the existence of $C^* > 0$ and $s^* > 0$, such that $\|V(\cdot, s)^2\|_{L_\rho^2} \leq C^* \|V(\cdot, s - s^*)\|_{L_\rho^2}^2$. Then we obtain for some $s_2 \leq s_1$:

$$\forall s \leq s_2 \quad I'(s) \leq \frac{5}{4}I(s) + CI(s - s^*)^2. \quad (50)$$

Since $I(s) \leq Ce^{3/2s}$ from (43), the following lemma from [NZ10] allows us to conclude.

Lemma 2.10 (*Lemma 3.6 of [NZ10]*) Consider $I(s)$ a positive C^1 function such that (50) is satisfied and $0 \leq I(s) \leq Ce^{3/2s}$ for all $s \leq s_2$, for some s_2 . Then, for some $s_3 \leq s_2$, we have $I(s) = 0$ for all $s \leq s_3$.

Using Lemma 2.10, we obtain $V \equiv 0$ on $\mathbb{R} \times (-\infty, s_3]$. Consequently, we have

$$\forall (y, s) \in \mathbb{R} \times (-\infty, s_3], \quad w(y, s) = \hat{\varphi}(s). \quad (51)$$

From the uniqueness of the Cauchy problem for equation (6) and since w is defined for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, $\hat{\varphi}$ is defined for all $(y, s) \in \mathbb{R} \times \mathbb{R}$ and (51) holds for all $(y, s) \in \mathbb{R} \times \mathbb{R}$. Therefore, case (c) in (46) cannot hold and for all $(y, s) \in \mathbb{R}^2$, $w(y, s) = \kappa$ or $w(y, s) = \varphi(s - s_0)$. This concludes the proof of Proposition 2.8 and finishes Part 3.

2.4 Part 4: Irrelevance of the case (iii) of Proposition 2.7

We consider case (iii) of Proposition 2.7. In this part, we will proceed like in Step 4 in [NZ10]. The following proposition allows us to reach a contradiction.

Proposition 2.11 Assume that case (iii) of Proposition 2.7 holds. Then, there exists $\varepsilon_0 > 0$ such that

$$\lim_{s \rightarrow -\infty} \sup_{|y| \leq \varepsilon_0 \sqrt{-s}} \left| w(y, s) - G\left(\frac{y}{\sqrt{-s}}\right) \right| = 0, \quad (52)$$

where $G(\xi) = \left((\beta + 1) - \frac{(\beta + 1)^2}{4\beta} \xi^2 \right)^{\frac{1}{\beta + 1}}$.

Indeed, let us first use Proposition 2.11 to find a contradiction ruling out case (iii) of Proposition 2.7, and then prove Proposition 2.11.

We fundamentally rely on the following Lemma:

Lemma 2.12 (*Lemma 2.11 from [MZ98b]*) Assume that $\psi(\xi, \tau)$ satisfies for all $|\xi| \leq 4B_1$ and $\tau \in [0, \tau_*]$:

$$\begin{cases} \partial_\tau \psi & \leq \partial_{yy}^2 \psi + \lambda \psi + \mu, \\ \psi(\xi, 0) & \leq \psi_0, \psi(\xi, \tau) \leq B_2, \end{cases}$$

where $\tau_* \leq 1$. Then, for all $|\xi| \leq B_1$ and $\tau \in [0, \tau_*]$,

$$\psi(\xi, \tau) \leq e^{\lambda\tau}(\psi_0 + \mu + CB_2e^{-B_1^2/4}).$$

Let us define u_{s_0} by

$$u_{s_0}(\xi, \tau) = (1 - \tau)^{\frac{1}{\beta+1}} w(y, s) \text{ where } y = \frac{\xi + \frac{\varepsilon_0}{2}\sqrt{-s_0}}{\sqrt{1 - \tau}} \text{ and } s = s_0 - \log(1 - \tau). \quad (53)$$

We note that u_{s_0} is defined for all $\tau \in [0, 1)$ and $\xi \in \mathbb{R}$, and that u_{s_0} satisfies equation (1). From Lemma 2.1, we have

$$\forall \tau \in [0, 1), |u_{s_0}(\cdot, \tau)| \geq M(1 - \tau)^{\frac{1}{\beta+1}}. \quad (54)$$

The initial condition at time $\tau = 0$ is $u_{s_0}(\xi, 0) = w(\xi + \frac{\varepsilon_0}{2}\sqrt{-s_0}, s_0)$. Using Proposition 2.11, we get:

$$\sup_{|\xi| < 4|s_0|^{1/4}} \left| u_{s_0}(\xi, 0) - G\left(\frac{\varepsilon_0}{2}\right) \right| \equiv g(s_0) \rightarrow 0 \text{ as } s_0 \rightarrow -\infty. \quad (55)$$

If we define v , the solution of:

$$v' = -v^{-\beta} \text{ and } v(0) = G\left(\frac{\varepsilon_0}{2}\right),$$

then

$$v(\tau) = \kappa \left(1 - \frac{(\beta+1)\varepsilon_0^2}{16\beta} - \tau \right)^{\frac{1}{\beta+1}}, \quad (56)$$

which quenches at time $1 - \frac{(\beta+1)\varepsilon_0^2}{16\beta} < 1$. Therefore, there exists $\tau_0 = \tau_0(\varepsilon_0) < 1$, such that

$$v(\tau_0) = \frac{M}{3}(1 - \tau_0)^{\frac{1}{\beta+1}}. \quad (57)$$

Now, if we consider the function

$$\psi = |u_{s_0} - v|, \quad (58)$$

then the following claim allows us to conclude:

Claim 2.13 For $|s_0|$ large enough, $\tau \in [0, \tau_0]$ and $|\xi| \leq 4|s_0|^{1/4}$, we have:

- (i) $\partial_\tau \psi \leq \partial_\xi^2 \psi + C(\varepsilon_0)\psi$,
- (ii) $\psi(\xi, 0) \leq g(s_0)$,
- (iii) $\psi(\xi, \tau) \leq C(\varepsilon_0)|s_0|^{1/2}$.

Indeed, using Lemma 2.12 with $B_1 = |s_0|^{1/4}$, $B_2 = C(\varepsilon_0)|s_0|^{1/2}$, $\tau_* = \tau_0$, $\psi_0 = g(s_0)$, $\lambda = C(\varepsilon_0)$ and $\mu = 0$, we get for all $\tau \in [0, \tau_0]$,

$$\sup_{|\xi| \leq |s_0|^{1/4}} \psi(\xi, \tau) \leq C(\varepsilon_0) \left(g(s_0) + |s_0|^{1/2} e^{-\frac{|s_0|^{1/2}}{4}} \right) \rightarrow 0 \text{ as } s_0 \rightarrow -\infty.$$

For $|s_0|$ large enough and $\xi = 0$, we get: $\psi(0, \tau_0) \leq \frac{M}{3}(1 - \tau_0)^{1/(\beta+1)}$ and by (57)

$$|u_{s_0}(0, \tau_0)| \leq v(\tau_0) + \psi(0, \tau_0) \leq \frac{2}{3}M(1 - \tau_0)^{1/(\beta+1)},$$

which is in contradiction with (54). To conclude, it remains to prove Claim 2.13.

Proof of Claim 2.13: (i) If we note by $\Psi = u_{s_0} - v$, then we get

$$\begin{aligned} \partial_s \Psi &= \partial_y^2 \Psi - \left(u_{s_0}^{-\beta} - v^{-\beta} \right), \\ &= \partial_y^2 \Psi + \beta \Psi \theta^{-\beta+1}, \text{ for some } \theta \text{ between } u_{s_0} \text{ and } v. \end{aligned}$$

Using (54), (56) and (57), we have for all $\tau \in [0, \tau_0]$ and $\xi \in \mathbb{R}$

$$\frac{1}{\theta} \leq \max \left(\frac{1}{u_0(\xi, \tau)}, \frac{1}{v(\tau)} \right) \leq C(\varepsilon_0), \text{ for some positive } \theta.$$

Since $\psi = |\Psi|$, using Kato's inequality, we conclude the proof of (i).

(ii) It is directly obtained from (55).

(iii) Using the definition (58) of ψ , (53) and (56), we write for all $\tau \in [0, \tau_0]$ and $|\xi| \leq 4|s_0|^{1/4}$,

$$\psi(\xi, \tau) \leq u_{s_0}(\xi, \tau) + v(\tau) \leq w(y, s) + \kappa, \text{ where } (y, s) \text{ are defined in (53).} \quad (59)$$

Since we have from Lemma 2.1, (37) and (53) for $|s_0|$ large enough

$$|w(y, s)| \leq |w(0, s)| + M_0|y| \leq C(1 + |y|)$$

and

$$|y| \leq \frac{|\xi + \frac{\varepsilon_0}{2}\sqrt{-s_0}|}{\sqrt{1 - \tau}} \leq \frac{4|s_0|^{1/4} + \frac{\varepsilon_0}{2}\sqrt{-s_0}}{\sqrt{1 - \tau_0}} \leq C(\varepsilon_0)|s_0|^{1/2},$$

the bound on ψ follows from (59). This concludes the proof of claim 2.13. ■

It remains to prove Proposition 2.11 to conclude Part 4.

Proof of Proposition 2.11: Consider some arbitrary $\varepsilon_0 \in (0, R^*)$, where

$$R^* = \sqrt{\frac{4\beta}{\beta + 1}}. \quad (60)$$

The parameter ε_0 will be fixed later in the proof small enough. If we note

$$f(y, s) = G \left(\frac{y}{\sqrt{-s}} \right), \quad (61)$$

and

$$F(y, s) = f(y, s) - \frac{\kappa}{2\beta s}, \quad (62)$$

then f satisfies

$$-\frac{y}{2} \cdot \partial_y f + \frac{1}{(\beta + 1)} f - |f|^{-\beta} = 0.$$

and we see from (iii) of Proposition 2.7 that

$$\| (F(., s) - w(., s)) (1 - \chi_{\varepsilon_0}) \|_{L^2_\rho} = O\left(\frac{\log |s|}{s^2}\right) \text{ as } s \rightarrow -\infty, \quad (63)$$

where

$$\chi_{\varepsilon_0}(y, s) = 1 \text{ if } \frac{|y|}{\sqrt{|s|}} \geq 3\varepsilon_0 \text{ and zero otherwise.} \quad (64)$$

The formal idea of this proof is that F solves in an approximate way the same equation as w for $s \rightarrow -\infty$. By (63), w and F are very close in the region $|y| \sim 1$. Our task is to prove that they remain close in the larger region $|y| \leq \varepsilon_0 \sqrt{-s}$, for some ε_0 chosen later. Let us consider a cut-off function

$$\gamma(y, s) = \gamma_0\left(\frac{y}{\sqrt{-s}}\right), \quad (65)$$

where $\gamma_0 \in \mathcal{C}^\infty(\mathbb{R})$ is such that $\gamma_0(\xi) = 1$ if $|\xi| \leq 3\varepsilon_0$ and $\gamma_0(\xi) = 0$ if $|\xi| \geq 4\varepsilon_0$. We introduce

$$\nu = w - F \text{ and } Z = \gamma|\nu|. \quad (66)$$

Our proof is the same as Velázquez [Vel92] and Nouaili and Zaag [NZ10]. As in [NZ10], we need to multiply by the cut-off, since our profile $F(y, s)$ defined by (62) is singular on the parabola $y = R^* \sqrt{-s}$. The cut-off function will generate an extra term, difficult to handle. Let us present the major steps of the proof in the following. The proof of the presented Lemmas will be given at the end of this step.

Lemma 2.14 (Estimates in modified L^2_ρ spaces) *There exists $\varepsilon_0 > 0$ such that the function Z satisfies for all $s \leq s_*$ and $y \in \mathbb{R}$:*

$$\partial_s Z - \partial_y^2 Z + \frac{1}{2} y \cdot \partial_y Z - (1 + \sigma) Z \leq C \left(Z^2 + \frac{(y^2 + 1)}{s^2} + (1 + \sqrt{-s}) \chi_{\varepsilon_0} \right) - 2 \operatorname{div}(|\nu| \cdot \partial_y \gamma), \quad (67)$$

where $s_* \in \mathbb{R}$, $\sigma = 1/100$ and χ_{ε_0} is defined in (64). Moreover,

$$N_{2\varepsilon_0 \sqrt{|s|}}^2(Z(s)) = o(1) \text{ as } s \rightarrow -\infty, \quad (68)$$

where the norm $N_r^q(\psi)$ is defined, for all $r > 0$ and $1 \leq q < \infty$, by

$$N_r^q(\psi) = \sup_{|\xi| \leq r} \left(\int |\psi(y)|^q \exp\left(-\frac{(y - \xi)^2}{4}\right) dy \right)^{1/q}. \quad (69)$$

Using the regularizing effect of the operator $\mathcal{L}$, we derive the following pointwise estimate, which allows us to conclude the proof of Proposition 2.11:

Lemma 2.15 (An upper bound for $Z(y, s)$ in $\{|y| \leq \varepsilon_0 \sqrt{-s}\}$) *We have:*

$$\sup_{|y| \leq \varepsilon_0 \sqrt{-s}} Z(y, s) = o(1) \text{ as } s \rightarrow -\infty.$$

Thus, Proposition 2.11 follows from Lemma 2.15 by (66), (62) and (61). It remains to prove Lemmas 2.14 and 2.15.

Proof of Lemma 2.14: The proof of (67) is straightforward and a bit technical. We leave it to Appendix B. Let us then prove (68). We take $s_0 < s^*$ and $s_0 \leq s < s^*$ such that $e^{\frac{s-s_0}{2}} \leq \sqrt{-s}$. We use the variation of constant formula in (67) to write

$$\begin{aligned} Z(y, s) &\leq S_\sigma(s - s_0)Z(., s_0) \\ &\quad + \int_{s_0}^s S_\sigma(s - \tau) \left(C \left\{ Z^2 + \frac{(y^2 + 1)}{\tau^2} + (1 + \sqrt{-\tau})\chi_{\varepsilon_0} \right\} - 2\operatorname{div}(|\nu|\partial_y \gamma) \right) d\tau, \end{aligned}$$

where S_σ is the semigroup associated to the operator $\mathcal{L}_\sigma \phi = \partial_y^2 \phi - \frac{1}{2}y\partial_y \phi + (1 + \sigma)\phi$, defined on $L_\rho^2(\mathbb{R})$. The kernel of the semigroup $S_\sigma(\tau)$ is

$$S_\sigma(\tau, y, z) = \frac{e^{(1+\sigma)\tau}}{(4\pi(1 - e^{-\tau}))^{1/2}} \exp \left[-\frac{|ye^{-\tau/2} - z|^2}{4(1 - e^{-\tau})} \right]. \quad (70)$$

Setting

$$r \equiv r(s, s_0) = 2\varepsilon_0 e^{\frac{s-s_0}{2}} = R_1 e^{\frac{s-s_0}{2}} \quad (71)$$

and taking the N_r^2 -norm we obtain

$$\begin{aligned} N_r^2(Z(., s)) &\leq N_r^2(S_\sigma(s - s_0)Z(., s_0)) + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)Z(., \tau)^2) d\tau \\ &\quad + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau) \left(\frac{(y^2 + 1)}{\tau^2} \right)) d\tau \\ &\quad + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)(1 + \sqrt{-\tau})\chi_{\varepsilon_0}(y, \tau)) d\tau + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)(\operatorname{div}(|\nu|\partial_y \gamma)) d\tau \\ &\quad \equiv J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

In comparison with [Vel92], we have a new term J_5 coming from the cut-off terms. Therefore, we just recall in the following claim the estimates on $J_1 \dots J_4$ from [Vel92], and treat J_2 and J_5 , which are new ingredients in our proof:

Claim 2.16 *We obtain as $s \rightarrow -\infty$*

$$\begin{aligned}
|J_1| &\leq C e^{(1+\sigma)(s-s_0)} \frac{\log |s_0|}{|s_0^2|}, \\
|J_2| &\leq C \int_{s_0}^{s_0+((s-R_0)-s_0)_+} \frac{e^{(1+\sigma)(s-\tau-R_0)}}{(1-e^{s-\tau-R_0})^{1/20}} (L_r^2(Z(\cdot, \tau)^2)) d\tau + C \frac{e^{(s-s_0)(1+\sigma)}}{s_0^2}, \\
&\quad \text{with } R_0 = 4\varepsilon_0, \\
|J_3| &\leq C \frac{e^{(s-s_0)(1+\sigma)}}{s_0^2} (1 + (s-s_0)) \sqrt{-s_0}, \\
|J_4| &\leq C e^{(s-s_0)(1+\sigma)} e^{\alpha s}, \text{ where } \alpha > 0, \\
|J_5| &\leq C e^{(s-s_0)(1+\sigma)} e^{\beta s}, \text{ where } \beta > 0.
\end{aligned}$$

Proof: See page 1578 in [Vel92] for J_1 , J_3 and J_4 .

To obtain the bound on J_2 , we need the following inequality, which will be proved in Appendix B

$$\partial_s Z - \partial_y^2 Z + \frac{1}{2} y \cdot Z \leq C \left(Z + \frac{(y^2 + 1)}{s^2} + (1 + \sqrt{-s}) \chi_{\varepsilon_0} \right) - 2 \operatorname{div} (|\nu| \cdot \partial_y \gamma), \quad (72)$$

then, we proceed exactly as in page 1579-1580 in [Vel92] and we obtain the wanted estimation on J_2 .

Now, we treat J_5 . We have from (70):

$$\begin{aligned}
&S_\sigma(s-\tau) (-\operatorname{div} (|\nu| \partial_y \gamma)), \\
&= -\frac{C e^{(s-\tau)(1+\sigma)}}{(1-e^{s-\tau})^{1/2}} \int_{\mathbb{R}} \exp \left(-\frac{(y e^{-(s-\tau)/2} - \lambda)^2}{4(1-e^{-(s-\tau)})} \right) \operatorname{div} (|\nu| \partial_y \gamma) d\lambda, \\
&= \frac{C e^{(s-\tau)(1+\sigma)}}{(1-e^{s-\tau})^{1/2}} \int_{\mathbb{R}} -\frac{(y e^{-(s-\tau)/2} - \lambda)}{2(1-e^{-(s-\tau)})} \exp \left(-\frac{(y e^{-(s-\tau)/2} - \lambda)^2}{4(1-e^{-(s-\tau)})} \right) |\nu| \partial_y \gamma d\lambda.
\end{aligned} \quad (73)$$

We have from Lemma 2.1 and (37)

$$|w(y, s)| \leq |w(0, s)| + M_0 |y| \leq C(1 + |y|).$$

Since F is bounded for $\frac{|y|}{\sqrt{-\tau}} \leq R^*/2$, where R^* is defined by (60) and $\operatorname{supp}(\partial_y \gamma) \subset (-4\varepsilon_0 \sqrt{-\tau}, -3\varepsilon_0 \sqrt{-\tau}) \cup (3\varepsilon_0 \sqrt{-\tau}, 4\varepsilon_0 \sqrt{-\tau})$, we have

$$\begin{aligned}
||\nu| \partial_y \gamma| &\leq C |\nu| \mathbb{I}_{\{3\varepsilon_0 \leq \frac{|y|}{\sqrt{-\tau}} \leq 4\varepsilon_0\}}, \\
&\leq C(1 + \sqrt{-\tau}) (\mathbb{I}_{3\varepsilon_0 \leq \frac{|y|}{\sqrt{-\tau}} \leq 4\varepsilon_0}) \leq C(1 + \sqrt{-\tau}) \chi_{\varepsilon_0}.
\end{aligned}$$

Using Cauchy-Schwartz inequality, we obtain:

$$|S_\sigma(s-\tau) (-\operatorname{div} (|\nu| \partial_y \gamma))| \leq \frac{C e^{(s-\tau)(1+\sigma)} (1 + \sqrt{-\tau})}{(1-e^{s-\tau})^{3/2}} \mathcal{I}_1 \mathcal{I}_2,$$

where,

$$\begin{aligned}\mathcal{I}_1 &= \left(\int_{\mathbb{R}} (ye^{-(s-\tau)/2} - \lambda)^2 \exp \left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1 - e^{-(s-\tau)})} \right) d\lambda \right)^{1/2}, \\ \mathcal{I}_2 &= \left(\int_{\mathbb{R}} \exp \left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1 - e^{-(s-\tau)})} \right) \chi_{\varepsilon_0} d\lambda \right)^{1/2}.\end{aligned}$$

Doing a change of variables, we obtain $\mathcal{I}_1 = C(1 - e^{-(s-\tau)})^{3/4}$. Furthermore, we have:

$$\mathcal{I}_2^2 \leq \mathcal{I}_3 \left(\int_{\mathbb{R}} \chi_{\varepsilon_0} e^{-\frac{\lambda^2}{4}} d\lambda \right)^{1/2},$$

where,

$$\mathcal{I}_3 = \left(\int \exp \left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{2(1 - e^{-(s-\tau)})} + \frac{\lambda^2}{4} \right) d\lambda \right)^{1/2}.$$

We introduce $\theta = ye^{-(s-\tau)/2}$, by completing squares, we readily check that:

$$\frac{\lambda^2}{4} - \frac{(\theta - \lambda)^2}{2(1 - e^{-(s-\tau)})} = -\frac{(1 + e^{-(s-\tau)})}{4(1 - e^{-(s-\tau)})} \left(\lambda - \frac{2\theta}{(1 + e^{-(s-\tau)})} \right)^2 + \frac{\theta^2}{2(1 + e^{-(s-\tau)})},$$

then we obtain:

$$\mathcal{I}_3^2 = C \left(\frac{(1 - e^{-(s-\tau)})}{(1 + e^{-(s-\tau)})} \right)^{1/2} \exp \left(\frac{\theta^2}{2(1 - e^{-(s-\tau)})} \right).$$

Therefore,

$$\begin{aligned}|N_r^2(S_\sigma(s - \tau) \operatorname{div}(|\nu| \partial_y \gamma))| &\leq C \frac{e^{(s-\tau)(1+\sigma)}(1 + \sqrt{-\tau})}{(1 + e^{-(s-\tau)})^{1/8}(1 - e^{-(s-\tau)})^{5/8}} \|\chi_{\varepsilon_0}\|_{L_\rho^2}^{1/2} \mathcal{I}_4, \\ \text{where } \mathcal{I}_4 &= N_r^2 \left(\exp \left(\frac{y^2 e^{-(s-\tau)}}{8((1 - e^{-(s-\tau)})})} \right) \right).\end{aligned}$$

Let us compute $\mathcal{I}_4$. Using the fact that

$$\begin{aligned}-\frac{(y - \mu)^2}{4} + \frac{y^2 e^{-(s-\tau)}}{4(1 - e^{-(s-\tau)})} &= \\ \frac{1}{4} \left(-\left(y(1 + e^{-(s-\tau)})^{-1/2} - \mu(1 + e^{-(s-\tau)})^{1/2} \right)^2 + \mu^2 e^{-(s-\tau)} \right),\end{aligned}$$

and doing a change of variables, we obtain:

$$\begin{aligned}&\int_{\mathbb{R}} \exp \left(-\frac{(y - \mu)^2}{4} + \frac{y^2 e^{-(s-\tau)}}{4(1 - e^{-(s-\tau)})} \right) dy \\ &\leq C \exp \left(\frac{\mu^2 e^{-(s-\tau)}}{4} \right) \int_{\mathbb{R}} \exp \left(-\frac{1}{4} \left(y(1 + e^{-(s-\tau)})^{-1/2} - \mu(1 + e^{-(s-\tau)})^{1/2} \right)^2 \right) dy.\end{aligned}$$

Hence $\mathcal{I}_4 \leq C(1 + e^{-(s-\tau)})^{1/8}$ and

$$N_r^2(S_\sigma(s-\tau))\operatorname{div}(|\nu|\partial_y\gamma) \leq C \frac{e^{(s-\tau)(1+\sigma)}(1+\sqrt{-\tau})}{(1-e^{-(s-\tau)})^{5/8}} \left(\int_{|\lambda| \geq R_1\sqrt{-\tau}} e^{-\frac{\lambda^2}{4}} d\lambda \right).$$

This gives

$$|J_5| = \int_{s_0}^s N_r^2(S_\sigma(s-\tau))\operatorname{div}(|\nu|\partial_y\gamma) d\tau \leq C(\eta)e^{(s-s_0)(1+\sigma)}e^{\alpha s_0},$$

where $\alpha > 0$. This concludes the proof of the claim 2.16.

This concludes the proof of Claim 2.16. ■

Summing up $J_{i=1..5}$, from claim 2.16 we obtain

$$N_r^2(Z(., s)) \leq e^{(s-s_0)(1+\sigma)} C \frac{\log|s_0|}{s_0^2} + C \int_{s_0}^{s_0+((s-R_0)-s_0)+} \frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1-e^{s-\tau-R_0})^{1/20}} (N_r^2(Z(., \tau)))^2 d\tau.$$

Now, we recall the following from [Vel92]:

Lemma 2.17 *Let ε , C , R , σ and α be positives constants, $0 < \alpha < 1$ and assume that $H(s)$ is a family of continuous functions satisfying:*

$$H(s) \leq \varepsilon e^{s(1+\sigma)} + C \int_0^{(s-R)_+} \frac{e^{(s-\tau)(1+\sigma)} H(\tau)^2}{(1-e^{(s-\tau-R)})^\alpha} d\tau \text{ for } s > 0.$$

Then there exists $\xi = \xi(R, C, \alpha)$ such that for any $\varepsilon \in (0, \varepsilon_1)$ and any s for which $\varepsilon e^{s(1+\sigma)} \leq \xi$, we have

$$H(s) \leq 2\varepsilon e^{s(1+\sigma)}.$$

Proof: See the proof of Lemma 2.2 from [Vel92]. Note that the proof of [Vel92] is done in the case $\sigma = 0$, but it can be adapted to some $\sigma > 0$ with no difficulty. ■

We conclude that $N_{r(\tau, s_0)}^2(Z(., s)) \leq C e^{(s-s_0)(1+\sigma)} \frac{\log|s_0|}{s_0^2}$ as $s \rightarrow -\infty$. If we fix $s = -e^{(s-s_0)}$, then we obtain $s \sim s_0$, $\log|s| \sim \log|s_0|$ and $N_{R_1\sqrt{-s}}^2(Z(., s)) \leq C s^{1+\sigma} \frac{\log|s_0|}{s_0^2} \leq C \frac{\log|s|}{s^{1-\sigma}} \rightarrow 0$ as $s \rightarrow -\infty$. Since $\sigma = \frac{1}{100}$, we get $N_{R_1\sqrt{-s}}^2(Z(., s)) = o(1)$, as $s \rightarrow -\infty$.

This concludes the proof of Lemma 2.14. ■

Now we give the proof of Lemma 2.15.

Proof of Lemma 2.15.

We aim at bounding $Z(y, s)$ for $|y| \leq R_2\sqrt{-s}$ in terms of $N_{R_1\sqrt{-s'}}(Z(s'))$, where $R_2 = \varepsilon_0$ and $R_1 = 2\varepsilon_0$, for some $s' < s$. Starting from equation (72), we do as in [Vel92]:

$$\begin{aligned} Z(., s) &\leq \{e^{CR_0} S(R_0) Z(., s-R_0)\} \\ &\quad + \left\{ C \int_{s-R_0}^s e^{C(s-\tau)} S(s-\tau) \left(\frac{(y^2+1)}{\tau^2} + (1+\sqrt{-\tau})\chi_{\varepsilon_0} \right) d\tau \right\} \\ &\quad - \left\{ 2 \int_{s-R_0}^s e^{C(s-\tau)} S(s-\tau) (\operatorname{div}(|\nu|\partial_y\gamma)) d\tau \right\} \\ &= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3, \text{ where } R_0 = 4\varepsilon_0, \end{aligned}$$

where S is the semigroup associated to the operator $\mathcal{L}$ defined in (48). The terms $\mathcal{M}_1$ and $\mathcal{M}_2$ are estimated in the following:

Claim 2.18 (Velázquez) *There exists s_0 , such that for all $s \leq s_0$*

$$\begin{aligned} \sup_{|y| \leq R_2 \sqrt{-s}} |\mathcal{M}_1| &= \sup_{|y| \leq R_2 \sqrt{-s}} |e^{C R_0} S(R_0) Z(., s - R_0)| = o(1) \text{ as } s \rightarrow -\infty, \\ \sup_{|y| \leq R_2 \sqrt{-s}} |\mathcal{M}_2| &= \sup_{|y| \leq R_2 \sqrt{-s}} \int_{s-R_0}^s \left(\frac{|y|^2+1}{s^2} + (1 + \sqrt{-\tau}) \chi_{\varepsilon_0} \right) \leq \frac{C}{|s|}. \end{aligned} \quad (74)$$

Proof: See page 1581 from [Vel92] and Lemma 6.5 in [HV93] in a similar case. ■

It remains to estimate $\mathcal{M}_3$. Proceeding as in page (73) and using the fact that $\nu(y, s) \leq C(1 + \sqrt{-s})$ for all $|y| \leq C\sqrt{-s}$ (obtained from (i) of Lemma 2.1) we write

$$\begin{aligned} &|S(s - \tau) (-\operatorname{div}(|\nu| \partial_y \gamma))| \\ &= \left| \frac{C e^{s-\tau}}{(1 - e^{s-\tau})^{1/2}} \int_{\mathbb{R}} \exp \left(-\frac{(y e^{-(s-\tau)/2} - \lambda)^2}{4(1 - e^{-(s-\tau)})} \right) \operatorname{div}(|\nu| \partial_y \gamma) d\lambda \right|, \\ &= \left| \frac{C e^{s-\tau}}{(1 - e^{s-\tau})^{1/2}} \int_{\mathbb{R}} -\frac{(y e^{-(s-\tau)/2} - \lambda)}{2(1 - e^{-(s-\tau)})} \exp \left(-\frac{(y e^{-(s-\tau)/2} - \lambda)^2}{4(1 - e^{-(s-\tau)})} \right) (|\nu| \partial_y \gamma) d\lambda \right|, \\ &\leq \frac{C e^{s-\tau}}{(1 - e^{s-\tau})^{3/2}} \int_{\mathbb{R}} |y e^{-(s-\tau)/2} - \lambda| \exp \left(-\frac{(y e^{-(s-\tau)/2} - \lambda)^2}{4(1 - e^{-(s-\tau)})} \right) \chi_{\varepsilon_0} d\lambda, \\ &\leq \frac{C e^{s-\tau} \sqrt{-\tau} (1 + \sqrt{-\tau})}{(1 - e^{s-\tau})^{3/2}} \int_{\mathbb{R}} \exp \left(-\frac{(y e^{-(s-\tau)/2} - \lambda)^2}{4(1 - e^{-(s-\tau)})} \right) \chi_{\varepsilon_0} d\lambda. \end{aligned}$$

We make the change of variables $z = (1 - e^{-(s-\tau)})^{-1/2} (\lambda - e^{-(\tau-s)/2} y)$ and we obtain

$$\int_{\mathbb{R}} \exp \left(-\frac{(y e^{-(s-\tau)/2} - \lambda)^2}{4(1 - e^{-(s-\tau)})} \right) \chi_{\varepsilon_0} d\lambda \leq (1 - e^{s-\tau})^{1/2} \int_{\Sigma} e^{-z^2/4} dz,$$

where,

$$\Sigma = \left\{ z \in \mathbb{R} : \left| z + e^{-(\tau-s)/2} (1 - e^{s-\tau})^{-1/2} y \right| \geq 3\varepsilon_0 (1 - e^{s-\tau})^{-1/2} \sqrt{-\tau} \right\}.$$

Since $|y e^{-(\tau-s)/2}| \leq \varepsilon_0 \sqrt{-s}$, we readily see that $\Sigma \subset \{z \in \mathbb{R} : |z| \geq \varepsilon_0 \sqrt{-s}\}$. Then we conclude that

$$|S(s - \tau) (-\operatorname{div}(|\nu| \partial_y \gamma))| \leq \frac{C e^{s-\tau}}{(1 - e^{s-\tau})} e^{\beta s}, \text{ where } \beta > 0,$$

and we obtain

$$\sup_{|y| \leq R_2 \sqrt{-s}} |\mathcal{M}_3| = o\left(\frac{1}{|s|}\right) \text{ as } s \rightarrow -\infty.$$

Putting together $\mathcal{M}_{i=1..3}$, the proof of lemma 2.15 is complete. This concludes also the proof of Proposition 2.11 and rules out case (iii) of Proposition 2.7. ■

2.5 Part 5: Irrelevance of the case (ii) of Proposition 2.7

To conclude the proof of Theorem 1, we consider case (ii) of Proposition 2.7. We claim that the following proposition allows us to reach a contradiction in this case.

Proposition 2.19

$$\lim_{s \rightarrow -\infty} \sup_{|y| \leq \varepsilon_0 e^{-s/2}} \left| w(y, s) - G(ye^{-s/2}) \right| = 0, \text{ where } G(\xi) = \kappa \left(1 - C_1 \kappa^\beta \xi \right)^{\frac{1}{\beta+1}}. \quad (75)$$

As in the previous part, first, we will find a contradiction ruling out case (ii) of Proposition 2.7 and then prove Proposition 2.19.

Let us define u_{s_0} by

$$u_{s_0}(\xi, \tau) = (1 - \tau)^{\frac{1}{\beta+1}} w(y, s) \text{ where } y = \frac{\xi + \frac{\varepsilon_0}{2} e^{-s_0/2}}{\sqrt{1 - \tau}} \text{ and } s = s_0 - \log(1 - \tau). \quad (76)$$

We note that u_{s_0} is defined for all $\tau \in [0, 1)$ and $\xi \in \mathbb{R}$, and that u_{s_0} satisfies equation (1). From Lemma 2.1, we have

$$\forall \tau \in [0, 1), |u_{s_0}(\cdot, \tau)| \geq M(1 - \tau)^{\frac{1}{\beta+1}}. \quad (77)$$

The initial condition at time $\tau = 0$ is $u_{s_0}(\xi, 0) = w(\xi + \frac{\varepsilon_0}{2} e^{-s_0/2}, s_0)$. Using Proposition 2.11, we get:

$$\sup_{|\xi| < 4e^{-s_0/4}} \left| u_{s_0}(\xi, 0) - G\left(\frac{\varepsilon_0}{2}\right) \right| \equiv g(s_0) \rightarrow 0 \text{ as } s_0 \rightarrow -\infty. \quad (78)$$

If we define v , the solution of:

$$v' = -v^{-\beta} \text{ and } v(0) = G\left(\frac{\varepsilon_0}{2}\right),$$

then

$$v(\tau) = \kappa \left(1 - C_1 \kappa^\beta \frac{\varepsilon_0}{2} - \tau \right)^{\frac{1}{\beta+1}}, \quad (79)$$

which quenches at time $1 - C_1 \kappa^\beta \frac{\varepsilon_0}{2} < 1$. Therefore, there exists $\tau_0 = \tau_0(\varepsilon_0) < 1$, such that

$$v(\tau_0) = \frac{M}{3} (1 - \tau_0)^{\frac{1}{\beta+1}}. \quad (80)$$

Now, if we consider the function

$$\psi = |u_{s_0} - v|, \quad (81)$$

then the following claim allows us to conclude (we omit the proof since it is the same as the proof of Lemma 2.13):

Claim 2.20 *For $|s_0|$ large enough and for all $\tau \in [0, \tau_0]$ and $|\xi| \leq 4e^{-s_0/4}$, we have:*

- (i) $\partial_\tau \psi \leq \partial_\xi^2 \psi + C(\varepsilon_0) \psi$,
- (ii) $\psi(\xi, 0) \leq g(s_0)$,
- (iii) $\psi(\xi, \tau) \leq C(\varepsilon_0) e^{-s_0/2}$.

Indeed, using Lemma 2.12 with $B_1 = e^{-s_0/4}$, $B_2 = C(\varepsilon_0)e^{-s_0/2}$, $\tau_* = \tau_0$, $\psi_0 = g(s_0)$, $\lambda = C(\varepsilon_0)$ and $\mu = 0$, we get for all $\tau \in [0, \tau_0]$,

$$\sup_{|\xi| \leq e^{-s_0/4}} \psi(\xi, \tau) \leq C(\varepsilon_0) \left(g(s_0) + e^{-s_0/2} e^{-\frac{e^{-s_0/2}}{4}} \right) \rightarrow 0 \text{ as } s_0 \rightarrow -\infty.$$

For $|s_0|$ large enough and $\xi = 0$, we get: $\psi(0, \tau_0) \leq \frac{M}{3}(1 - \tau_0)^{1/(\beta+1)}$ and by (57)

$$|u_{s_0}(0, \tau_0)| \leq v(\tau_0) + \psi(0, \tau_0) \leq \frac{2}{3}M(1 - \tau_0)^{1/(\beta+1)},$$

which is in contradiction with (54).

Proof of Proposition 2.19: The proof is very similar to that of Proposition 2.11. If we note $f(y, s) = G(ye^{s/2})$, then f satisfies

$$-\partial_s f - \frac{1}{2}y \cdot \partial_y f + \frac{f}{\beta+1} - f^{-\beta} = 0. \quad (82)$$

Consider an arbitrary $\varepsilon_0 \in (0, \frac{R^*}{10})$, where $R^* = \frac{\kappa^p}{C_1}$. ε_0 will be fixed small enough later. Let us consider a cut-off function $\gamma(y, s) = \gamma_0(ye^{s/2})$, where $\gamma_0 \in \mathcal{C}^\infty(\mathbb{R})$ such that $\gamma_0(\xi) = 1$ if $|\xi| \leq 3\varepsilon_0$ and $\gamma_0(\xi) = 0$ if $|\xi| \geq 4\varepsilon_0$. We note $\nu = w - f$ and $Z = \gamma|\nu|$. From (ii) of Proposition 2.7, we have

$$\|Z\|_{L_\rho^2} \leq Ce^{s(1-\varepsilon)} \text{ as } s \rightarrow -\infty, \text{ for some } \varepsilon > 0. \quad (83)$$

As in the previous part, we divide our proof in two parts given in the following lemmas.

Lemma 2.21 (Estimates in the modified L_ρ^2 spaces) *There exists $\varepsilon_0 > 0$ such that the function Z satisfies for all $s \leq s_*$ and $y \in \mathbb{R}$,*

$$\partial_s Z - \partial_y^2 Z + \frac{1}{2}y \cdot \partial_y Z - (1 + \sigma)Z \leq C(Z^2 + e^s + (1 + e^{-s/2})\chi_{\varepsilon_0}) - 2\operatorname{div}(|\nu|\partial_y \gamma), \quad (84)$$

where $s_* \in \mathbb{R}$, $\sigma = \frac{1}{100}$ and

$$\chi_{\varepsilon_0}(y, s) = 1 \text{ if } |y|e^{s/2} \geq 3\varepsilon_0 \text{ and zero otherwise.} \quad (85)$$

Moreover, we have

$$N_{2\varepsilon_0 e^{-s/2}}^2(Z(s)) = o(1) \text{ as } s \rightarrow -\infty. \quad (86)$$

As in Part 4, the following lemma allows us to conclude the proof of Proposition 2.19:

Lemma 2.22 (An upper bound for $Z(y, s)$ in $|y| \leq \varepsilon_0 e^{-s/2}$) *We have:*

$$\sup_{|y| \leq \varepsilon_0 e^{-s/2}} Z(y, s) = o(1) \text{ as } s \rightarrow -\infty. \quad (87)$$

Remains to prove Lemmas 2.21 and 2.22 to conclude the proof of Proposition 2.19. Here, we only sketch the proof of Lemma 2.21, since it is completely similar to Part 4. We don't give the proof of Lemma 2.22. We refer the reader to Part 4 and Proposition 2.4 from Velázquez [Vel92] for similar situations.

Proof of Lemma 2.21: As in the previous step, we leave the proof of (84) to Appendix B. Let us now apply the variation of constants formula and take the norm $N_{r(s,s_0)}^2$, where $r(s, s_0)$ is as in (71). Assume that $s_0 < 2s_*$, then for all $s_0 \leq s \leq \frac{s_0}{2}$, we have

$$\begin{aligned} N_r^2(Z(., s)) &\leq N_r^2(S_\sigma(s - s_0)Z(., s_0)) + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)(Z(., \tau)^2)) d\tau \\ &\quad + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)(e^\tau)) d\tau \\ &\quad + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)((1 + e^{-\tau/2})\chi_{\varepsilon_0}(., \tau))) d\tau \\ &\quad - 2 \int_{s_0}^s N_r^2(S_\sigma(s - \tau)(\operatorname{div}(\nu|\partial_y \gamma))) d\tau \\ &= J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Arguing as in Part 4 and using (83), we prove:

Claim 2.23

$$\begin{aligned} |J_1| &\leq C e^{(s-s_0)(1+\sigma)} e^{s_0(1-\varepsilon)}, \\ |J_2| &\leq C \int_{s_0}^{s_0+((s-R_0)-s_0)_+} \frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1 - e^{s-\tau-R_0})^{1/20}} (L_r^2(Z(., s)^2)) d\tau \\ &\quad + C e^{(s-s_0)(1+\sigma)} e^s \text{ with } R_0 = 4\varepsilon_0, \\ |J_3| &\leq C e^{(s-s_0)(1+\sigma)} e^s (1 + e^{-s/2}), \\ |J_4| &\leq C e^{(s-s_0)(1+\sigma)} e^{-\alpha e^{-s}} \text{ where } \alpha > 0, \\ |J_5| &\leq C e^{(s-s_0)(1+\sigma)} e^{-\beta e^{-s}} \text{ where } \beta > 0. \end{aligned}$$

Proof: To estimate $J_{i=1,3,4}$, see page 1584 in [Vel92]. To treat J_2 and J_5 , we proceed as in the proof of Lemma 2.14 of the previous part. ■

Summing up $J_{i=1..5}$, we obtain:

$$\begin{aligned} N_r^2(Z(., s)) &\leq \\ &C e^{(s-s_0)(1+\sigma)} e^{(1-\varepsilon)s} + C \int_{s_0}^{s_0+((s-R_0)-s_0)_+} \frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1 - e^{s-\tau-R_0})^{1/20}} (L_r^2(Z(., s)^2)) d\tau, \end{aligned}$$

then using Proposition 2.17, we get $N_{r(s,s_0)}^2(Z(., s)) \leq C e^{(s-s_0)(1+\sigma)} e^{(1-\varepsilon)s}$ as $s \rightarrow -\infty$ for $s_0 \leq s \leq \frac{s_0}{2}$. If we fix $s = s_0/2$, then we obtain $N_{r(s,s_0)}^2(Z(., s)) \leq C e^{s(2(1-\varepsilon)-(1+\sigma))} \leq C e^{s(1-(2\varepsilon+\sigma))} \rightarrow 0$ as $s \rightarrow -\infty$, since ε is small enough and $\sigma = \frac{1}{100}$. This concludes the proof of Lemma 2.21. ■

As announced earlier, we don't give the proof of Lemma 2.22 and refer the reader to Part 4 and Section 2 from [Vel92]. This concludes the proof of Proposition 2.19 and rules out case (ii) of Proposition 2.7.

Conclusion of Part 3 and the proof of the Liouville theorem

We conclude from Part 4 and 5 that cases (ii) and (iii) of Proposition 2.7 are ruled out. By Part 3, we obtain that $w \equiv \kappa$ or $w \equiv \varphi(s - s_0)$ for some real s_0 , where φ is defined in (12), which is the desired conclusion of Theorem 1.

3 Uniform bounds on $u(t)$ on quenching time and ODE type behavior

We prove here Theorem 3. Let us first state the following convergence result which derives from [Guo90].

Proposition 4 (Guo) *Consider $u(x, t)$ a type **I** solution of (1), which quenches in finite time T . Then, for any quenching point a*

$$\lim_{t \rightarrow T} u(a, t)(T - t)^{-\frac{1}{\beta+1}} = \kappa,$$

Proof: See Theorem 3.10 page 77 in [Guo90]. Although Guo studied equation (1) on a finite interval with Dirichlet boundary conditions. One can easily check that this method holds on the whole space $\mathbb{R}$, whenever the solution is of type **I**. ■

Consider $u(t)$ a nonnegative solution of type **I** of equation (1) that quenches at time T and

$$|\partial_y w(y, s)| + \frac{1}{w(y, s)} \leq M, \text{ for all } (y, s) \in \mathbb{R} \times \mathbb{R}, \text{ where } M > 0. \quad (88)$$

In this section we proceed as in the proof of Theorem 1.1 in [MZ98a]. In the following, Part 1 is devoted to the proof of (i) of Theorem 3. In Part 2, we will prove (ii) of Theorem 3.

Part 1: Uniform quenching estimates

From the definition (6) and (10) of w_a and z_a , it is enough to prove the second statement of i) for *some* $a \in \mathbb{R}$.

Consider a a quenching point of u . For simplicity, we assume that $a = 0$ and write w (respectively z) instead of w_a (respectively z_a).

We want to prove that $\inf_{y \in \mathbb{R}} w(y, s) \rightarrow \kappa$ and $\|\partial_y z(\cdot, s)\|_{L^\infty} + \|\partial_y z(\cdot, s)\|_{L^\infty} \rightarrow 0$ as $s \rightarrow +\infty$ (where $z \equiv 1/w$).

Using Proposition 4, we know that $w(0, s) \rightarrow \kappa$ as $s \rightarrow +\infty$. Since $\inf_{y \in \mathbb{R}} w(y, s) \leq w(0, s)$, we obtain

$$\limsup_{s \rightarrow +\infty} \inf_{y \in \mathbb{R}} w(y, s) \leq \kappa. \quad (89)$$

and

$$\liminf_{s \rightarrow +\infty} \|z(\cdot, s)\|_{L^\infty} \geq \kappa^{-1}, \quad (90)$$

hence,

$$\liminf_{s \rightarrow +\infty} \|z(\cdot, s)\|_{L^\infty} + \|\partial_y z(\cdot, s)\|_{L^\infty} + \|\partial_y^2 z(\cdot, s)\|_{L^\infty} \geq \kappa^{-1}.$$

The conclusion follows if we prove that

$$\limsup_{s \rightarrow +\infty} \|z(\cdot, s)\|_{L^\infty} + \|\partial_y z(\cdot, s)\|_{L^\infty} + \|\partial_y^2 z(\cdot, s)\|_{L^\infty} \leq \kappa^{-1}.$$

We argue by contradiction and assume that there exists a sequence $(s_n)_n$ such that $s_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and

$$\lim_{n \rightarrow +\infty} \|z(\cdot, s_n)\|_{L^\infty} + \|\partial_y z(\cdot, s_n)\|_{L^\infty} + \|\partial_y^2 z(\cdot, s_n)\|_{L^\infty} = \kappa^{-1} + 3\varepsilon_0,$$

where $\varepsilon_0 > 0$. Up to extracting a subsequence, we have either

$$\lim_{n \rightarrow +\infty} \|z(\cdot, s_n)\|_{L^\infty} \geq \kappa^{-1} + \varepsilon_0, \text{ or } \lim_{n \rightarrow +\infty} \|\partial_y z(\cdot, s_n)\|_{L^\infty} \geq \varepsilon_0, \text{ or } \lim_{n \rightarrow +\infty} \|\partial_{yy} z(\cdot, s_n)\|_{L^\infty} \geq \varepsilon_0.$$

We deduce the existence of $(y_n^i)_{i=1..3}$ such that

$$\begin{aligned} & \|z(\cdot, s_n)\|_{L^\infty} - \frac{\varepsilon_0}{2} \leq z(y_n^1, s_n), \\ \text{or } & \|\partial_y z(\cdot, s_n)\|_{L^\infty} - \frac{\varepsilon_0}{2} \leq |\partial_y z(y_n^2, s_n)|, \\ \text{or } & \|\partial_{yy} z(\cdot, s_n)\|_{L^\infty} - \frac{\varepsilon_0}{2} \leq |\partial_y^2 z(y_n^3, s_n)|. \end{aligned}$$

Introducing $y_n = y_n^i$, where $i = 1, 2$ or 3 , we define for each $n \in \mathbb{N}$

$$W_n(y, s) = w(y + y_n e^{s/2}, s + s_n) \text{ and } Z_n(y, s) = 1/W_n(y, s).$$

We can see that W_n satisfies (6) and Z_n satisfies (10). More precisely, we claim the following:

Lemma 3.1 *$(Z_n)_n$ is a sequence of solutions of (10) with the following properties:*

i) $\lim_{n \rightarrow +\infty} |Z_n(0, 0)| \geq \kappa^{-1} + \varepsilon_0/2$, or $\lim_{n \rightarrow +\infty} |\partial_y Z_n(0, 0)| \geq \varepsilon_0/2$, or $\lim_{n \rightarrow +\infty} |\partial_y^2 Z_n| \geq \varepsilon_0/2$.

ii) For all $R > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$,
- Z_n is defined for all $(y, s) \in [-R, R]^2$ and $W_n = 1/Z_n$ satisfies

$$|\partial_y W_n(y, s)| + \frac{1}{W_n(y, s)} \leq M.$$

- $Z_n \geq 0$ and $\|Z_n(\cdot, s)\|_{L^\infty([-R, R]^2)} + \|\partial_y Z_n(\cdot, s)\|_{L^\infty([-R, R]^2)} \leq M$.
- There exists $m(R) > 0$ such that $\|Z_n(\cdot, s)\|_{C^{2,1}([-R, R]^2)} \leq m(R)$, where $C^{2,1}([-R, R]^2)$ stands for the functions which are continuously differentiable twice in space and once in time.

Proof of Lemma 3.1: We give the following version of Lemma 2.1,

Claim 3.2 *(Parabolic estimates) For all $(y, s) \in [-R, R]^2$:*

(i) $|Z_n(y, s)| + |\partial_y Z_n(y, s)| + |\partial_y^2 Z_n(y, s)| \leq M$ and $|\partial_s Z_n| \leq M(1 + |y|)$.

(ii) For all $R > 0$, Z_n , $\partial_y Z_n(y, s)$, $\partial_y^2 Z_n$ and $\partial_s Z_n$ are bounded in $C_\alpha^0([-R, R]^2)$ for some $\alpha \in (0, 1)$, where

$$C_\alpha^0([-R, R]^2) = \left\{ \psi \in L^\infty([-R, R]^2) \mid \sup_{(\xi, \tau), (\xi', \tau') \in [-R, R]^2} \frac{|\psi(\xi, \tau) - \psi(\xi', \tau')|}{(|\xi - \xi'| + |\tau - \tau'|^{1/2})^\alpha} < \infty \right\}$$

Using the fact that u is of type **I** and claim 3.2, we proceed as in the proof of Lemma 2.2 in [MZ98a]. ■

Now, using the compactness property of (Z_n) , shown in Lemma 3.1, we find $Z_R \in C^{2,1}([-R, R]^2)$ such that up to extracting a subsequence, $Z_n \rightarrow Z_R \in C^{2,1}([-R, R]^2)$. Using a diagonal process, we find $Z \in C^{2,1}(\mathbb{R}^2)$ such that $Z_n \rightarrow Z$ in $C_{loc}^{2,1}(\mathbb{R}^2)$. From Lemma 3.1, we have

- (a) Z satisfies (10) for $(y, s) \in \mathbb{R} \times \mathbb{R}$.
(b) $Z \geq 0$ and $\|Z\|_{L^\infty(\mathbb{R} \times \mathbb{R})} + \|\partial_y Z\|_{L^\infty(\mathbb{R} \times \mathbb{R})} \leq C$.
(c) $Z(0, 0) \geq \kappa^{-1} + \varepsilon_0/2$, or $|\partial_y Z(0, 0)| \geq \varepsilon_0/2$, or $|\partial_y^2 Z(0, 0)| \geq \varepsilon_0/2$.

Now, we have to consider two cases:

Case 1: If there exists (y_0, s_0) such that $Z(y_0, s_0) = 0$. Then, doing as in the case 1 of proof of Lemma 2.4, we obtain $Z \equiv 0$, which is impossible because of (c).

Case 2: For all (y, s) , $Z(y, s) > 0$. In this case, we have $W_n = 1/Z_n \rightarrow W = 1/V$ in $C_{loc}^{2,1}(\mathbb{R}^2)$ and $|\partial_y Z_n(0, 0)|^2/Z_n(0, 0) \rightarrow |\partial_y Z(0, 0)|^2/Z(0, 0)$.

Since W_n satisfies (1), the same holds for W . Using ii) of claim 3.2, we see that

$$\forall (y, s) \in \mathbb{R}^2 \quad |\partial_y W(y, s)| + \frac{1}{W(y, s)} \leq M.$$

Using our Liouville theorem, we get $W \equiv \kappa$ or $W(y, s) = \varphi(s - s_0)$, where $\varphi(s - s_0) = \kappa(1 + e^s)^{1/(\beta+1)}$. In all cases, this contradicts (c). This concludes the proof of i) of Theorem 3. ■

Part 2: ODE type behavior

Consider a type **I** solution $u(t)$ of (1), that quenches in finite time $T > 0$, with initial condition satisfying (2). We know from (88) that:

$$\forall t \in [0, T), x \in \mathbb{R}, |u_x(x, t)|(T - t)^{\frac{1}{2} - \frac{1}{\beta+1}} + \frac{1}{u(x, t)}(T - t)^{\frac{1}{\beta+1}} \leq M, \text{ where } M > 0. \quad (91)$$

Let us prove now the uniform pointwise control of the diffusion term by the nonlinear term, which asserts that the solution $u(t)$ behaves everywhere like the ODE $u' = -u^{-\beta}$ (up to a constant).

Since $u_0 \in L^\infty$, hence

$$\|u(t, \cdot)\|_{L^\infty} \leq \|u_0\|_{L^\infty},$$

from equation (1), it is enough to prove that

$$\forall \varepsilon, \quad \exists c_\varepsilon > 0 \text{ such that for all } (x, t) \in \mathbb{R} \times [T/2, T),$$

$$|\partial_t v - v^{2+\beta}| \leq \varepsilon v^{2+\beta} + c_\varepsilon, \text{ where } v = 1/u.$$

The plan of the proof is the same as in [MZ98a] and [MZ00]. However, the Giga-Kohn property "small local energy implies no blow-up locally" breaks down. We proceed as in [NZ10], where the authors proved that "small L_ρ^2 norm implies no blow-up locally". In our case we have the following Lemma:

Lemma 3.3 *For all $M > 0$, there exist positive ϵ_0 , C_0 and M_0 such that if U is a type **I** solution of (1) and*

$$\forall |x_0| \leq 1, \quad \|Z_{x_0}(\cdot, 0)\|_{L_\rho^2} \leq \epsilon, \quad (92)$$

for some $0 < \epsilon \leq \eta_0$, where

$$y = \frac{x - x_0}{\sqrt{1 - t}}, \quad s = -\log(1 - t), \quad Z_{x_0}(y, s) = (1 - t)^{\frac{1}{\beta+1}} \frac{1}{U(x, t)},$$

then:

(i) For all $|x_0| \leq 1$ and $s \in [0, +\infty)$,

$$\|Z_{x_0}(\cdot, s)\|_{L_\rho^2} \leq C_0 \epsilon e^{-\frac{s}{\beta+1}}. \quad (93)$$

(ii) For all $|x| \leq 1$ and $t \in [0, 1)$, we have $|U(x, t)| \geq M_0$.

Proof: The idea of the proof is the same as in the proof of Proposition 3.3 in [NZ10].

(i) For simplicity, we write Z instead of Z_{x_0} . Since Z is a solution of (10), we multiply (10) by $Z\rho$ and integrate to get

$$I'(s) \leq -\int -\frac{2}{\beta+1} I(s) + 2 \int |Z(y, s)|^{\beta+3} \rho(y) dy, \text{ where } I(s) = \int Z(y, s)^2 \rho(y) dy. \quad (94)$$

We note by $(\partial_y Z)^2/Z = (\partial_y w)^2 Z^3$, (10) and (91) that Z satisfies

$$\partial_s Z \leq \left(\partial_y^2 Z - \frac{1}{2} y \cdot \partial_y Z + Z \right) + \sigma Z, \text{ where } \sigma \text{ is a positive constant.}$$

Using Lemma 2.9 (for Z), we see that there exist $C^*(M) > 0$ and $s^* = s^*(p+1) > 0$ such that for all $s \geq s^*$

$$\int |Z(y, s)|^{\beta+3} \rho(y) dy \leq C^* \|Z(\cdot, s - s^*)^2\|_{L_\rho^2}^{\frac{\beta+3}{2}} \quad (95)$$

Now, we divide the proof in two steps:

Step 1: $0 \leq s \leq s^*$. Using (94) and the fact that Z is bounded by $M > 0$ (see (92)), we get

$$I'(s) \leq \lambda I(s) \text{ for some } \lambda = \lambda(M) > 0,$$

hence $I(s) \leq e^{\lambda s} I(0) \leq e^{\lambda s} \epsilon^2 \leq \frac{C_0^2}{2} \epsilon^2 e^{-\frac{2s}{\beta+1}}$, where we define $C_0^2 = 2e^{(\lambda + \frac{2}{\beta+1})s^*}$. This gives (93) for $0 \leq s \leq s^*$.

Step 2: $s \geq s^*$. In this step, we argue by contradiction to prove (93) for all $s \geq s^*$. We suppose that there exists $s_1 > s^*$, such that

$$I(s) < (C_0 \epsilon)^2 e^{-\frac{2s}{\beta+1}}, \text{ for all } s^* \leq s < s_1 \quad (96)$$

$$I(s_1) = (C_0 \epsilon)^2 e^{-\frac{2s_1}{\beta+1}}. \quad (97)$$

Let $F(s) = I(s)(C_0 \epsilon)^{-2} e^{\frac{2s}{\beta+1}}$. From (94), (95), (96) and Step 1, we have for all $s^* \leq s \leq s_1$,

$$\begin{aligned} F'(s) &\leq C^*(C_0 \epsilon)^{-2} e^{\frac{2s}{\beta+1}} I(s - s^*)^{\frac{\beta+3}{2}} \\ &\leq C^*(C_0 \epsilon)^{\beta+1} e^{\frac{2s}{\beta+1}} e^{-(s-s^*)\frac{\beta+3}{\beta+1}} \leq C^*(C_0 \epsilon)^{\beta-1} e^{\frac{\beta+3}{\beta+1}s^*} e^{-s}. \end{aligned}$$

Since $F(s^*) \leq \frac{1}{2}$ from the step above, we integrate the last inequality to obtain

$$\begin{aligned} F(s_1) &\leq C^*(C_0 \epsilon)^{\beta-1} e^{s^*} \left(e^{-s^*} - e^{-s_1} \right) + F(s^*), \\ &\leq C^*(C_0 \epsilon)^{\beta-1} e^{\frac{2s^*}{\beta+1}} + \frac{1}{2} \leq \frac{3}{4}, \end{aligned}$$

for $\epsilon \leq \epsilon_0(M)$ small enough. This contradicts (97). Therefore, (93) holds.

(ii) Applying parabolic regularity to equation (6) and using estimate (92), we get for all $|x_0| \leq 1$, $R > 0$ and $|y| < R$, $|Z_{x_0}(y, s)| \leq M_0 e^{-\frac{s}{\beta+1}}$, hence for all $t \in [0, 1]$, $|U(x_0, t)| \geq M_0$, for some $M_0 = M_0(M)$. This ends the proof of Lemma 3.3. ■

We argue by contradiction and assume that for some $\varepsilon_0 > 0$, there exists $(x_n, t_n)_{n \in \mathbb{N}}$, a sequence of elements of $\mathbb{R} \times [\frac{T}{2}, T)$, such that

$$\forall n \in \mathbb{N}, |\partial_x^2 v(x_n, t_n)| \geq \varepsilon_0 |v(x_n, t_n)|^{2+\beta} + n. \quad (98)$$

From the uniform estimates of i) of Theorem 3 and the parabolic regularity, since $\|\partial_{xx} v(\cdot, t)\|_{L^\infty}$ is bounded on compact sets of $[\frac{T}{2}, T)$, we have

$$T - t_n \rightarrow 0, \text{ as } n \rightarrow \infty.$$

We note that i) of Theorem 3 implies that $v(x_n, t_n)(T - t_n)^{\frac{1}{\beta+1}}$ is uniformly bounded, therefore, we can assume that it converges as $n \rightarrow +\infty$. Let us consider two cases:

i) *Estimates in the very singular region:* $v(x_n, t_n)(T - t_n)^{\frac{1}{\beta+1}} \rightarrow \kappa_0 > 0$. From (98), it follows that

$$\|\partial_x^2 v(\cdot, t_n)\|_{L^\infty} \geq |\partial_x^2 v(x_n, t_n)| \geq \varepsilon_0 \left(\frac{\kappa_0}{2}\right)^{2+\beta} (T - t_n)^{-\frac{\beta+2}{\beta+1}},$$

with $t_n \rightarrow T$, which contradicts i) Theorem 3.

ii) *Estimates in the singular region:* $v(x_n, t_n)(T - t_n)^{\frac{1}{\beta+1}} \rightarrow 0$.

We consider n large enough, such that

$$v(x_n, t_n)(T - t_n)^{\frac{1}{\beta+1}} \leq \eta_0/3, \text{ where } \eta_0 \text{ is defined in Lemma 3.3.}$$

We take $t_n^0 \rightarrow T$ such that

$$(T - t_n^0)^{-\frac{\beta+2}{\beta+1}} = \sqrt{n}. \quad (99)$$

Using (98) and i) of Theorem 3, we obtain:

$$n \leq |\partial_x^2 v(x_n, t_n)| \leq C_0 (T - t_n)^{-\frac{\beta+2}{\beta+1}},$$

hence $t_n^0 < t_n$. Now we distinguish two cases:

Case 1. We assume that (up to extracting a subsequence) there exists $t'_n \in (t_n^0, t_n)$, such that $v(x_n, t'_n)(T - t'_n)^{\frac{1}{\beta+1}} = \frac{2}{3}\eta_0$. If we consider

$$v_n(\xi, \tau) = (T - t'_n)^{\frac{1}{\beta+1}} v(x_n + \xi \sqrt{T - t'_n}, t'_n + \tau(T - t'_n)), \quad (100)$$

then, we have from i) of Theorem 3,

$$v_n(0, 0) = \frac{2}{3}\eta_0, \|\partial_\xi v_n(\cdot, 0)\|_{L^\infty} + \|\partial_\xi^2 v_n(\cdot, 0)\|_{L^\infty} \rightarrow 0. \quad (101)$$

If we note by $u_n = 1/v_n$, then we have from (2) and (91):

$$\forall \tau < 1, \xi \in \mathbb{R}, |\partial_\xi u_n(\xi, \tau)|(1 - \tau)^{\frac{1}{2} - \frac{1}{\beta+1}} + \frac{1}{u_n(\xi, \tau)}(1 - \tau)^{\frac{1}{\beta+1}} \leq M \text{ and } \partial_\tau u_n = \partial_x^2 u_n - u_n^{-\beta}. \quad (102)$$

Then, we can see that v_n is bounded and satisfies (8). Using parabolic regularity, we can extract a subsequence (still denoted by t_n) such that, $v_n(\xi, \tau) \rightarrow \hat{v}(\xi, \tau)$ in $\mathcal{C}^{2,1}$ of every compact set of $\mathbb{R} \times (-\infty, 1)$. We note that if there exists (ξ_0, τ_0) such that $\hat{v}(\xi_0, \tau_0) = 0$, then $\hat{v} \equiv 0$ (see case 1 in the proof of Lemma 2.4). This case is impossible because $\hat{v}(0, 0) = 3/2\eta_0$. Therefore, it holds that for all (ξ, τ) , $\hat{v}(\xi, \tau) > 0$. Recalling that $u_n(\xi, \tau) = 1/v_n(\xi, \tau)$, we have

$$u_n(\xi, \tau) = 1/v_n(\xi, \tau) \rightarrow \hat{u}(\xi, \tau) = 1/\hat{v}(\xi, \tau) \text{ in } \mathcal{C}^{2,1} \text{ of every compact set of } \mathbb{R} \times (-\infty, 1)$$

and by (102), we get

$$\partial_\tau \hat{u} = \partial_\xi^2 \hat{u} - \hat{u}^{-\beta}, \quad \hat{u}(0, 0) = \frac{3}{2\eta_0} \text{ and } |\partial_\xi \hat{u}(\xi, \tau)|(1 - \tau)^{\frac{1}{2} - \frac{1}{\beta+1}} + \frac{1}{\hat{u}(\xi, \tau)}(1 - \tau)^{\frac{1}{\beta+1}} \leq M.$$

Using the Liouville Theorem (see Theorem 1), we get

$$\hat{u}(\xi, \tau) = \kappa \left(\left(\frac{3\kappa}{2\eta_0} \right)^{-(\beta+1)} - \tau \right)^{\frac{1}{\beta+1}}.$$

We claim that it is enough to prove that the convergence of $u_n \rightarrow \hat{u}$ takes place in $C^{2,1}((-1, 1) \times [0, 1))$ to conclude. Indeed, if we have this extended convergence, then we write from (98) and the definition (100) of v_n ,

$$\begin{aligned} |\partial_\xi^2 v_n(0, \tau_n)| &= (T - t'_n)^{\frac{\beta+2}{\beta+1}} |\partial_{\xi\xi} v(0, t_n)| \\ &\geq \varepsilon_0 |v(0, t_n)|^{\beta+2} (T - t'_n)^{\frac{\beta+2}{\beta+1}} \\ &= \varepsilon_0 |v_n(0, \tau_n)|^{2+\beta} \\ &\geq \varepsilon_0 \min_{\tau \in [0, 1]} |v_n(0, \tau)|^{2+\beta}, \end{aligned}$$

with $\tau_n = \frac{t_n - t'_n}{T - t'_n}$. Letting $n \rightarrow \infty$, we obtain

$$\partial_\xi^2 v_n(0, \tau_n) \rightarrow 0 \text{ and } \min_{\tau \in [0, 1]} |v_n(0, \tau)|^{2+\beta} \rightarrow \min_{\tau \in [0, 1]} |\hat{v}(\tau)|^{2+\beta}.$$

Hence

$$0 \geq \varepsilon_0 \min_{\tau \in [0, 1]} |\hat{v}(\tau)|^{2+\beta} = \varepsilon_0 |\hat{0}(\tau)|^{2+\beta} > 0 \quad (103)$$

which is a contradiction.

Let us then extend the convergence. If we consider the following similarity variables,

$$y = \frac{\xi - \xi_0}{\sqrt{1 - \tau}}, \quad s = -\log(1 - \tau), \quad Z_{n, \xi_0}(y, s) = (1 - \tau)^{-\frac{1}{\beta+1}} v_n(\xi, \tau), \quad (104)$$

then, we see from (101) that for all $|\xi_0| \leq 1$, $\|Z_{n, \xi_0}(\cdot, 0)\|_{L^2_\rho} \leq \eta_0$, for n large enough. Using Lemma 3.3, we get for all $|\xi| \leq 1$ and $\tau \in [0, 1)$, $|v_n(\xi, \tau)| \leq \frac{1}{M_0}$. Using the parabolic regularity, we can extend the convergence, and then reach the contradiction (103). This concludes Case 1.

Case 2. We assume that for some $n_0 \in \mathbb{N}$, for all $n \geq n_0$ and $t \in [t_n^0, t_n]$, we have:

$$v(x_n, t)(T - t)^{\frac{1}{\beta+1}} < \frac{2}{3}\eta_0.$$

Then, we take $t'_n = t_n^0$ and introduce v_n by (100). As in Case 1, we obtain by Lemma 3.3 and the parabolic regularity:

$$\forall |\xi| \leq 1 \text{ and } \tau \in [0, 1), |v_n(\xi, \tau)| \leq 1/M_0, \quad |\partial_\xi^2 v_n(0, \tau_n)| \leq C_0 \eta_0 \text{ where } \tau_n = \frac{t_n - t_n^0}{T - t_n^0}.$$

Therefore, we get from (98), (100) and (99):

$$n \leq |\partial_\xi^2 v_n(x_n, t_n)| = (T - t_n^0)^{-\frac{\beta+2}{\beta+1}} |\partial_\xi^2 v_n(0, \tau_n)| \leq_0 \eta_0 (T - t_n^0)^{-\frac{\beta+2}{\beta+1}} = C_0 \eta_0 \sqrt{n},$$

which is a contradiction, as $n \rightarrow \infty$. This ends Case 2 and concludes the proof of Theorem 3.

A Proof of Proposition 2.7

We proceed as in Appendix A [MZ98a], the most important difference, is that in our case we have not w bounded. Let us introduce some notations,

$$v_+(y, s) = v_0(s)h_0(y) + v_1(s)h_1(y), \quad v_{null}(y, s) = v_2(s)h_2(y).$$

We divide the proof in two parts: in part 1, we show that either v_{null} or v_+ is predominant in L_ρ^2 as $s \rightarrow -\infty$. In part 2, we show that in the case where v_+ is predominant, then either $v_0(s)$ or $v_1(s)$ predominates the other.

Step 1 of the Proof: Competition between v_+ , v_{null} and v_-

First, we recall from (38)

$$\forall (y, s) \in \mathbb{R}^2, \partial_s v = \mathcal{L}v - f(v). \quad (105)$$

Let us introduce some notations

$$z(s) = \|v_+(\cdot, s)\|_{L_\rho^2}, \quad x(s) = \|v_{null}(\cdot, s)\|_{L_\rho^2} \text{ and } y(s) = \|v_-(\cdot, s)\|_{L_\rho^2}.$$

Projection (105) onto the unstable subspace of $\mathcal{L}$ forming the L_ρ^2 -inner product with v_+ , and using standard inequalities, Lemma 2.6 and (40). We get

$$\dot{z} \geq \frac{1}{2}z - N, \text{ where } N = \|v^2\|_{L_\rho^2}.$$

Working similarly with v_{null} and v_- we arrive at the system

$$\begin{aligned} \dot{z} &\geq \frac{1}{2}z - N \\ |\dot{x}| &\leq N, \\ \dot{y} &\leq -\frac{1}{2}y + N. \end{aligned} \quad (106)$$

If we knew for $|s|$ large enough

$$N \leq \varepsilon(x + y + z), \quad (107)$$

we could use ODE techniques to conclude the step. However, we do not have this information at this stage. We thus estimate N proceeding as in Section 4 in [FK92b]. We

note that in [FK92b] (107) was proved under the additional assumption that $v(y, s)$ is uniformly bounded. One can check in [FK92b] that this assumption is only used for the derivation of the estimate $|f(v)| \leq c|v|$, which in our case is true by (40). Then, we get by (107) and (106),

$$\begin{aligned} \dot{z} &\geq \left(\frac{1}{2} - \varepsilon\right) z - \varepsilon(x + y) \\ |\dot{x}| &\leq \varepsilon(x + y + z), \\ \dot{y} &\leq -\left(\frac{1}{2} + \varepsilon\right) y + \varepsilon(x + z). \end{aligned} \tag{108}$$

Now, let us recall Lemma A.1 from [NZ10].

Lemma A.1 (Merle-Zaag) *Let $x(s)$, $y(s)$, and $z(s)$ be absolutely continuous, real-valued functions that are nonnegative and satisfy*
(i) $(x, y, z)(s) \rightarrow 0$ as $s \rightarrow \infty$, and $\forall s \leq s^$, $x(s) + y(s) + z(s) \neq 0$, and*
(ii) $\forall \varepsilon > 0$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \leq s_0$

$$\begin{cases} \dot{z} &\geq c_0 z - \varepsilon(x + y) \\ |\dot{x}| &\leq \varepsilon(x + y + z) \\ \dot{y} &\leq -c_0 y + \varepsilon(x + y) \end{cases} \tag{109}$$

Then, either $x + y = o(z)$ or $y + z = o(x)$ as $s \rightarrow -\infty$.

Applying Lemma above to (108), we get either

$$\|v_-(\cdot, s)\|_{L_\rho^2} + \|v_+(\cdot, s)\|_{L_\rho^2} = o(\|v_{null}(\cdot, s)\|_{L_\rho^2}),$$

or

$$\|v_-(s)\|_{L_\rho^2} + \|v_{null}(\cdot, s)\|_{L_\rho^2} = o(\|v_+(\cdot, s)\|_{L_\rho^2}).$$

Step 2 of the Proof: Competition between v_0 and v_1

In this step we focus on the case where $\|v_-(\cdot, s)\|_{L_\rho^2} + \|v_{null}(\cdot, s)\|_{L_\rho^2} = o(\|v_+(\cdot, s)\|_{L_\rho^2})$. We will show that it leads to either case (ii) or (iii) of Proposition 2.7. We want to derive from (105) the equations satisfied by v_0 and v_1 . For this we will estimate in the following $\int f(v)k_m(y)\rho(y)dy$, for $m = 0, 1$ where

$$k_m(y) = h_m(y)/\|h_m(y)\|_{L_\rho^2}^2.$$

Lemma A.2 *There is $\delta_0 > 0$ and an integer $k' > 4$ such that for all $\delta \in (0, \delta_0)$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \leq s_0$, $\int v^2|y|^{k'}\rho dy \leq c_0(k')\delta^{4-k'}z(s)^2$*

Poof: This lemma is analogous to Lemma A.3 p 175 from [MZ98a], it have been proved under the additional assumption that $v(y, s)$ is uniformly bounded. One can check that this assumption is only used for the derivation of the estimate $f(v) < Cv$, which is in our case true. ■

Proceeding as in Appendix A from [MZ98a] and doing the projection of equation (105) respectively on $k_0(y)$ and $k_1(y)$, we obtain

$$v'_0(s) = v_0(s) - \frac{\beta}{2\kappa}z(s)^2(1 + \alpha(s)). \tag{110}$$

Using the same type of calculations as for $\int v^3 \rho dy$ and Lemma A.2, we can prove that $\int v^2 k_1(y) \rho dy = O(z(s)^2)$. Therefore, (105) yields the following vectorial equation:

$$v_1'(s) = \frac{1}{2}v_1(s) + \gamma(s)z(s)^2, \quad (111)$$

where γ is bounded.

Proceeding as in [MZ98a], we get:

$$\forall \varepsilon > 0, v_0(s) = O(e^{(1-\varepsilon)s}) \text{ and } v_1(s) = C_1 e^{\frac{s}{2}} + O(e^{(1-\varepsilon)s})$$

and

$$\forall \varepsilon > 0, v_0(s) = -\frac{\beta}{\kappa}|C_1|^2 s e^s (1 + o(1)) + C_0 e^s + O(e^{2(1-\varepsilon)s}) \text{ as } s \rightarrow -\infty \quad (112)$$

Two cases then arise:

- If $C_1 \neq 0$, then $v_1(s) \equiv C_1 e^{\frac{s}{2}} \gg v_0(s) O(se^s)$, from (112). We note that applying Lemma 2.9 to $|v|$, we have for s large enough (and $s < 0$),

$$N^2 = \int |v(y, s)|^4 \rho(y) dy \leq C^* \|v(\cdot, s - s^*)\|_{L_\rho^2}^2, \quad (113)$$

for some positive s^* and C^* .

Recalling (105) and using (113), we obtain $\dot{y} \leq -\frac{1}{2}y + c^* \|v(\cdot, s - s^*)\|_{L_\rho^2}^2 \leq -\frac{1}{2}y + ce^s$. Then, we obtain $x = \|v_{null}(\cdot, s)\|_{L_\rho^2} = O(e^s)$. We conclude that $\|v(\cdot, s) - C_1 e^{s/2} y\|_{L_\rho^2} = O(e^{s(1-\varepsilon)})$ as $s \rightarrow -\infty$, for some $\varepsilon > 0$. This is case (ii) of Proposition 2.7.

- If $C_1 = 0$, we obtain case (iii) of Proposition 2.7. Indeed, let us first improve the estimate of v . In fact, from (112) we have $v_0 = C_0 e^s + O(e^{3/2s})$ and from (111) $v_1 = O(e^{3/2s})$.

Recalling (105) and using (113), we obtain $\dot{y} \leq -\frac{1}{2}y + c^* \|v(\cdot, s - s^*)\|_{L_\rho^2}^2 \leq -\frac{1}{2}y + ce^{2s}$.

Then, we have $y = O(e^{3/2s})$. Similarly, we obtain that $x = O(e^{3/2s})$ and we conclude that

$$\|v(\cdot, s) - v_0(s)\|_{L_\rho^2} = O(e^{3/2s}).$$

This is case (iii) of Proposition 2.7.

Step 3: Case where v_{null} dominates

In the following, we prove that (iii) of Proposition 2.7 holds. First, we prove the following Lemma:

Lemma A.3 *Assume that*

$$\|v_+(\cdot, s)\|_{L_\rho^2} + \|v_-(\cdot, s)\|_{L_\rho^2} = o(\|v_{null}(\cdot, s)\|_{L_\rho^2}) \quad (114)$$

holds, then

$$v(y, s) = \frac{\kappa}{4\beta s} (y^2 - 2) + o\left(\frac{1}{s}\right),$$

in L_ρ^2 as $s \rightarrow -\infty$.

Proof: Since $v_{null}(y, s) = v_2(s)h_2(y)$, we note that $v_2(s) = \int v k_2(y)\rho$. Using (41), we rewrite equation (105) as follows

$$v_s = \mathcal{L}v - \frac{\beta}{2\kappa}v^2 + g(v), \quad (115)$$

where $g(v) = O(v^3)$. Projecting equation (115) onto $h_2(y)$ we get

$$\begin{aligned} \partial_s v_2 &= -\frac{\beta}{2\kappa} \int v^2 k_2(y)\rho(y) + \int g(v)k_2(y)\rho(y) \\ &= -\frac{\beta}{2\kappa} \int v_{null}^2 k_2(y)\rho(y) + \frac{\beta}{2\kappa} \int (v_{null}^2 - v^2) k_2(y)\rho(y) + \int g(v)k_2(y)\rho(y) \\ &= -\frac{\beta}{2\kappa} 8v_2^2 + \frac{\beta}{2\kappa} \mathcal{E}_1 + \mathcal{E}_2, \end{aligned} \quad (116)$$

where we use the fact that $\int v_{null} k_2 \rho = v_2^2 \int h_2^2 k_2 \rho = 8v_2^2$. we next estimate $\mathcal{E}_1$ and $\mathcal{E}_2$. For this we need the following lemma

Lemma A.4 *There exists $\alpha_0 > 0$ and an integer $k' > 4$ such that for all $\alpha \in (0, \alpha_0)$, there exists $s_0 \in \mathbb{R}$ such that for all $s \leq s_0$,*

$$\int |v|^2 |y|^{k'} \rho dy \leq c_0(k') \alpha^{4-k'} \int v_{null}^2 \rho dy.$$

Proof: See Lemma C.1 in [MZ98a] (page 187). ■

Recalling that $v = v_+ + v_{null} + v_-$, we write on the one hand:

$$\begin{aligned} |\mathcal{E}_1| &\leq \int |v_+ + v_-| \times |v + v_{null}| |k_2(y)| \rho, \\ &\leq c \left(\int |v_+ + v_-|^2 \rho \right)^{1/2} \left\{ (v^2 k_2^2(y) \rho)^{1/2} + \left(\int v_{null}^2 k_2^2 \rho \right)^{1/2} \right\}. \end{aligned}$$

We know by (114) $(\int |v_+ + v_-|^2 \rho)^{1/2} = o(v_2)$ and

$$(v^2 k_2^2(y) \rho)^{1/2} + \left(\int v_{null}^2 k_2^2 \rho \right)^{1/2} \leq \underbrace{(v^2 k_2^2(y) \rho)^{1/2}}_I + c|v_2|.$$

To treat I , we have from Lemma A.4

$$\int v^2 k_2^2 \rho \leq c \int |v|^2 \rho + \int |v|^2 |y|^{k'} \rho \leq c \int v^2 \rho \leq c v_2^2,$$

we conclude that $\mathcal{E}_1 = o(v_2^2)$. It remains to estimate $\mathcal{E}_2$, we consider $\alpha \in (0, \alpha_0)$ and we proceed as in Appendix C from [MZ98a], (page 189). We write for $m = 0$ or $m = 2$:

$$\begin{aligned} \int |v|^3 |y|^m \rho dy &\leq \int_{|y| \leq \alpha^{-1}} |v|^3 |y|^m \rho dy + \int_{|y| \geq \alpha^{-1}} |v|^3 |y|^m \rho, \\ &\leq \varepsilon \alpha^{-m} \int_{|y| \leq \alpha^{-1}} |v|^2 \rho dy + C M \alpha^{k'-m-1} (1 + \alpha) \int_{|y| \geq \alpha^{-1}} |v|^2 |y|^{k'} \rho, \\ &\leq C (\varepsilon \alpha^{-m} + M c_0(k') \alpha^{4-m}) \int v_{null}^2 \rho dy, \end{aligned}$$

where, we used the fact that $|v| \rightarrow 0$ as $s \rightarrow -\infty$ in $L^\infty(B(0, \alpha^{-1}))$, $|v(y, s)| \leq M(1 + |y|)$, Lemma A.4 and $\int |v|^2 \rho dy \leq \int v_{null}^2 \rho dy$. We can then choose ε and α such that for $s \leq s_0$, $\int |v|^3 |y|^m \rho \leq \varepsilon \int v_{null}^2$ and we get $\mathcal{E}_2 = o(v_2^2)$.

So finally, we have

$$\partial_s v_2 = -\frac{\beta}{\kappa} 4v_2^2 + o(v_2^2).$$

Solving the above, we obtain

$$v_{null} = \frac{\kappa}{4\beta s} (1 + o(1))(y^2 - 2).$$

This concludes the proof of Lemma A.3. ■

In order to finish the proof of (iii) of Proposition 2.7, we need to refine the estimates of Lemma A.3 to catch the $O(\log(|s|)/s^2)$. Recalling system (106) and using (113), we obtain,

$$\dot{y} \leq -\frac{1}{2}y + c\|v(\cdot, s - s^*)\|_{L_\rho^2} \leq -\frac{1}{2}y + c\frac{1}{s^2}.$$

Then, integrating $(ye^{s/2})' \leq C\frac{e^s}{s^2}$ between $-\infty$ and s , we get $y \leq \frac{C}{s^2}$. Doing the same for $z = \|v_+(\cdot, s)\|_{L_\rho^2}$, we obtain $(ze^{-s/2})' \geq C\frac{e^s}{s^2}$, integrating between s and $s_0 \geq s$, we get $z \leq \frac{C}{s^2}$.

Proceeding as in the proof of Lemma A.3, we write

$$\begin{aligned} \partial_s v_2 &= -\frac{\beta}{2\kappa} \int v_{null}^2 k_2(y) \rho(y) + \frac{\beta}{2\kappa} \int (v_{null}^2 - v^2) k_2(y) \rho(y) + \int g(v) k_2(y) \rho(y), \\ &= -\frac{\beta}{2\kappa} 8v_2^2 + \frac{\beta}{2\kappa} \mathcal{E}_1 + \mathcal{E}_2. \end{aligned} \tag{117}$$

Then, we have

$$\begin{aligned} |\mathcal{E}_1| &\leq \int |v_+ + v_-| \times |v + v_{null}| |k_2(y)| \rho, \\ &\leq c \left(\int |v_+ + v_-|^2 \rho \right)^{1/2} \left\{ (v^2 k_2^2(y) \rho)^{1/2} + \left(\int v_{null}^2 k_2^2 \rho \right)^{1/2} \right\}, \\ &\leq \varepsilon \left(\int v_{null}^2 \rho \right)^{1/2} \left\{ c \left(\int v^4 \rho \right)^{1/4} + c \left(\int v_{null}^2 \rho \right)^{1/2} \right\}. \end{aligned}$$

Using the fact that $\|v_{null}(\cdot, s)\|_{L_\rho^2} \sim \frac{C}{s}$ and (113), we have

$$\int v^4 \rho \leq c (v^2(\cdot, s - s^*) \rho)^2 \leq \frac{c}{(s - s^*)^2} \leq \frac{c}{s^2}.$$

Thus, $\mathcal{E}_1 \leq \frac{C}{s^3}$. Similarly, we obtain $\mathcal{E}_2 \leq \frac{C}{s^3}$. Then, we have from (117):

$$\partial_s v_2 = -\frac{4\beta}{\kappa} v_2^2 + O\left(\frac{1}{s^3}\right) = -\frac{4\beta}{\kappa} v_2^2 \left(1 + O\left(\frac{1}{s}\right)\right).$$

By integrating, we obtain the desired estimation

$$v_2(s) = \frac{\kappa}{4\beta s} + O(\log(|s|)/s^2).$$

This concludes the proof of Proposition 2.7. ■

B Equations of Z in Parts 4 and 5

Equation of Z in Part 4: In this part we establish the equations (67) and (72) satisfied by Z and finishes the proof of Lemma 2.14 and Claim 2.16. If we recall from (66) that $\nu = w - F$, where F is defined by (62), then we see from (6) that ν satisfies the following equation for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, such that for $|y| < 4\varepsilon_0\sqrt{-s}$.

$$\partial_s \nu = \mathcal{L}\nu + l(y, s)\nu - B(\nu) + R(y, s), \quad (118)$$

where $\mathcal{L}$ is defined in (48),

$$\begin{aligned} l(y, s) &= \left(-\frac{\beta}{\beta+1} + \beta F^{-(\beta+1)} \right), \\ B(\nu) &= (F + \nu)^{-\beta} - F^{-\beta} + \beta F^{-(\beta+1)}\nu, \\ R(y, s) &= -\partial_s F + \partial_y^2 F - \frac{1}{2}y \cdot \partial_y F + \frac{F}{\beta+1} - F^{-\beta}. \end{aligned} \quad (119)$$

Using Taylor's formula, the fact that F is bounded for $|y| \leq 4\varepsilon_0\sqrt{-s}$ and proceeding as in the proof of Lemma 2.6, we readily obtain for all $s \leq s_0$ and $|y| < 4\varepsilon_0\sqrt{-s}$

$$|B(\nu)| \leq C|\nu|^2, \quad |R(y, s)| \leq C \left(\frac{|y|^2 + 1}{s^2} + \chi_{\varepsilon_0} \right), \quad |l(y, s)| \leq C \min \left[\frac{(1 + |y|^2)}{|s|}, 1 \right]$$

with χ_{ε_0} defined in (64). Therefore, we write for $|s|$ large enough and $|y| \leq 4\varepsilon_0\sqrt{-s}$:

$$|l(y, s)| \leq C \left\{ \frac{(1 + \varepsilon_0^2 |s|)}{|s|} + \chi_{\varepsilon_0} \right\} \leq C \{2\varepsilon_0^2 + \chi_{\varepsilon_0}\}.$$

Using Kato's inequality, we obtain for $z = |\nu|$, $|s|$ large enough and $|y| \leq 4\varepsilon_0\sqrt{-s}$:

$$\partial_s z - \partial_y^2 z + \frac{1}{2}y \cdot \partial_y z - (1 + \sigma)z \leq C \left(z^2 + \frac{(y^2 + 1)}{s^2} + \chi_{\varepsilon_0} \right),$$

where we fix ε_0 small enough so that $\sigma = C\varepsilon_0^2 \leq \frac{1}{100}$ (use the fact that $\chi_{\varepsilon_0}\gamma \leq \gamma^2 + \chi_{\varepsilon_0}$), with the cut-off function γ defined by (65).

We define $Z = z\gamma$ and we obtain for $|s|$ large enough:

$$\begin{aligned} \partial_s Z - \partial_y^2 Z + \frac{1}{2}y \cdot \partial_y Z - (1 + \sigma)Z &\leq C \left(Z^2 + \frac{(y^2 + 1)}{s^2} + \chi_{\varepsilon_0} \right) \\ &\quad + z \left(\partial_s \gamma - \partial_y^2 \gamma + \frac{y}{2} \cdot \partial_y \gamma \right) - 2\partial_y \gamma \cdot \partial_y z, \end{aligned}$$

(here, we used the fact that $\gamma z^2 = Z^2 + (\gamma - \gamma^2)z^2 \leq Z^2 + C\chi_{\varepsilon_0}$). The last terms in this equation are the cut-off terms. By (i) of Lemma 2.1 and (37), we get

$$z(y, s) \leq C(1 + \sqrt{-s}) \text{ for all } |y| \leq 4\varepsilon_0\sqrt{-s},$$

then we obtain using the fact that $z(\partial_s \gamma - \partial_y^2 \gamma + \frac{y}{2} \cdot \partial_y \gamma) - 2\partial_y \gamma \cdot \partial_y z \leq C(1 + \sqrt{-s})\chi_{\varepsilon_0} - 2\text{div}(z\partial_y \gamma)$, we obtain for $|s|$ large enough:

$$\partial_s Z - \partial_y^2 Z + \frac{1}{2}y \cdot \partial_y Z - (1 + \sigma)Z \leq C \left(Z^2 + \frac{(y^2 + 1)}{s^2} + (1 + \sqrt{-s})\chi_{\varepsilon_0} \right) - 2\text{div}(|\nu|\partial_y \gamma),$$

which is the desired equation (67) in Lemma 2.14.

In the following, we will establish inequality (72). Let us rewrite (118) as follow

$$\partial_s \nu = \left(\mathcal{L} - \frac{\beta}{\beta+1} \right) \nu - B_1(\nu) + R(y, s),$$

where

where $B_1(\nu) = (F + \nu)^{-\beta} - F^{-\beta}$ and R is defined in (119)

Proceeding as in the proof of Lemma 2.6, we get $|B(\nu)| \leq C|\nu|$. Now, we do as in the proof of (67), and we obtain the wanted result (72).

Equation of Z in Part 5: In the following, we determine the equations satisfied by Z in Part 5. We note by $\nu = w - f$. We can see from (82), that ν satisfies the following equation for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, such that for $|y| < 4\varepsilon_0 e^{-s/2}$

$$\partial_s \nu = \mathcal{L}\nu + l(\nu) - B(\nu) + R(y, s),$$

where

$$\begin{aligned} l(\nu) &= \left(-\frac{\beta}{\beta+1} + \beta f^{-(\beta+1)} \right) \nu = l(y, s)\nu, \\ B(\nu) &= \left((f + \nu)^{-\beta} - f^{-\beta} + \beta f^{-(\beta+1)} \right) \nu, \\ R(y, s) &= e^s \partial_y^2 G(y e^{s/2}). \end{aligned}$$

Using Taylor's formula, the fact that f is bounded for $|y| \leq 4\varepsilon_0 e^{s/2}$ and proceeding as in the proof of Lemma 2.6, we prove that for $|s|$ large and $|y| \leq 4\varepsilon_0 e^{-s/2}$

$$|B(\nu)| \leq C|\nu|^2, |R(y, s)| \leq C e^s + \chi_{\varepsilon_0}(y, s),$$

with χ_{ε_0} is defined by (85). We obtain for $|y| e^{s/2} \leq 4\varepsilon_0$ and s large

$$|l(y, s)| \leq C \min \left[|y| e^{s/2}, 1 \right].$$

If we consider χ_{ε_0} defined in (85), then, we write for $|s|$ large and $|y| \leq 4\varepsilon_0 e^{-s/2}$:

$$|l| \leq C \left\{ |y| e^{s/2} + \chi_{\varepsilon_0} \right\} \leq C \{ \varepsilon_0 + \chi_{\varepsilon_0} \}.$$

Using Kato's inequality, we obtain for $z = |\nu|$, $|s|$ large enough and $|y| e^s \leq 4\varepsilon_0$,

$$\partial_s z - \partial_y^2 z + \frac{1}{2} y \cdot \partial_y z - (1 + \sigma) z \leq C (z^2 + e^s + \chi_{\varepsilon_0}),$$

where $\sigma = C\varepsilon_0 = \frac{1}{100}$. Now, we consider the cut-off function γ , we define $Z = z\gamma$ and we obtain for $|s|$ large:

$$\begin{aligned} & \partial_s Z - \partial_y^2 Z + \frac{1}{2} y \cdot \partial_y Z - (1 + \sigma) Z \\ & \leq C (Z^2 + e^s + \chi_{\varepsilon_0}) - z \left(\partial_s \gamma - \partial_y^2 \gamma + \frac{y}{2} \cdot \partial_y \gamma \right) + 2 \partial_y \gamma \partial_y z. \end{aligned}$$

The last terms in this equation are the cut-off terms. By (i) of Lemma 2.1 we have $|\nu| \leq C(1 + e^{-s/2})$. Then, using $z(\partial_s \gamma - \partial_y^2 \gamma + \frac{y}{2} \cdot \partial_y \gamma) - 2 \partial_y \gamma \partial_y z \leq C\chi_{\varepsilon_0} + 2 \operatorname{div}(z \partial_y \gamma)$, we obtain for $|s|$ large:

$$\partial_s Z - \partial_y^2 Z + \frac{1}{2} y \cdot \partial_y Z - (1 + \sigma) Z \leq C \left(Z^2 + e^s + (1 + e^{-s/2}) \chi_{\varepsilon_0} \right) - 2 \operatorname{div}(|\nu| \partial_y \gamma),$$

which is the desired equation in (84).

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