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Aurélien Deya

► **To cite this version:**

| Aurélien Deya. A discrete approach to Rough Elliptic Equations. 2010. hal-00530850v1

**HAL Id: hal-00530850**

**<https://hal.science/hal-00530850v1>**

Preprint submitted on 30 Oct 2010 (v1), last revised 3 Nov 2013 (v3)

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# A DISCRETE APPROACH TO ROUGH ELLIPTIC EQUATIONS

A. DEYA

ABSTRACT. By combining the formalism of [8] with a discrete approach close to the considerations of [6], we interpret and solve the rough partial differential equation  $dy_t = Ay_t dt + \sum_{i=1}^m f_i(y_t) dx_t^i$  ( $t \in [0, T]$ ) on a compact domain  $\mathcal{O}$  of  $\mathbb{R}^n$ , where  $A$  is a rather general elliptic operator of  $L^p(\mathcal{O})$  ( $p > 1$ ),  $f_i(\varphi)(\xi) := f_i(\varphi(\xi))$  and  $x$  is the generator of a 2-rough path. The (global) existence, uniqueness and continuity of a solution is established under classical regularity assumptions for  $f_i$ . Some identification procedures are also provided in order to justify our interpretation of the problem.

## 1. INTRODUCTION

The rough paths theory introduced by Lyons in [15] and then refined by several authors (see the recent monograph [11] and the references therein) has led to a very deep understanding of the standard rough systems

$$dy_t^i = \sum_{j=1}^m \sigma_{ij}(y_t) dx_t^j \quad , \quad y_0 = a \in \mathbb{R}^d, \quad (1)$$

where  $\sigma_{ij} : \mathbb{R} \rightarrow \mathbb{R}$  is a regular vector field and  $x$  is a so-called rough path, that is to say a process allowing the construction of iterated integrals (see Assumption (X) $_\gamma$  for the definition of a 2-rough path and [16] for a rough path of any order). The theory provides for instance a new pathwise interpretation of stochastic systems driven by very general Gaussian processes, as well as fruitful and highly non-trivial continuity results for the Itô solution of (1), ie when  $x$  is a standard Brownian motion.

One of the new challenges of rough paths theory now consists in adapting the machinery to infinite-dimensional (rough) equations involving a non-bounded operator, with, as a final objective, the possibility of new pathwise interpretations for stochastic PDEs. Some progresses have recently been made towards this goal, with on the one hand the viscosity-solution approach due to Friz et al (see [2, 3, 10, 9]) and on the other the development by Gubinelli and Tindel of a specific formalism designed for rough evolution equations (see [13, 8]).

The present paper is a contribution to this global project. It aims at providing, in a concise and self-contained formulation, the analysis of the following rough evolution equation:

$$y_0 = \psi \in L^p(\mathcal{O}) \quad , \quad dy_t = Ay_t dt + \sum_{i=1}^m f_i(y_t) dx_t^i \quad , \quad t \in [0, T], \quad (2)$$

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*Date:* October 30, 2010.

*2000 Mathematics Subject Classification.* 60H05, 60H07, 60G15.

*Key words and phrases.* Rough paths theory; Stochastic PDEs; Fractional Brownian motion.

where  $A$  is a rather general elliptic operator on a bounded domain  $\mathcal{O}$  of  $\mathbb{R}^n$  (see Assumptions (A1)-(A2)),  $f_i(\varphi)(\xi) := f_i(\varphi(\xi))$  and  $x$  generates a  $m$ -dimensional 2-rough path (see Assumption (X) $_\gamma$ ). Although the global form of (2) is quite similar to the equation treated in [8], several differences and notable improvements easily justify the interest of the study we suggest here, as the three main theorems 2.10-2.12 will point it out:

- (i) The equation is here analysed on a compact domain  $\mathcal{O}$  of  $\mathbb{R}^n$ . This allows to simplify the conditions relative to the vector field  $f_i$ , which reduce to the classical assumptions of rough paths theory, ie  $k$ -times differentiable ( $k \in \mathbb{N}^*$ ) with bounded derivatives (see Assumption (F) $_k$ ).
- (ii) The conditions on  $p$  are less stringent than in [8], where  $p$  has to be taken very large. It will here be possible to show the existence and uniqueness of a solution in  $L^p(\mathcal{O})$  (for a regular enough initial condition  $\psi$ ) as soon as  $p > n$ . In particular, we can go back to the Hilbert framework of [13] for the one-dimensional equation ( $n = 1, p = 2$ ).
- (iii) Last but not least, the arguments we are about to use lead to the existence of a *global* solution for (2), defined on any time interval  $[0, T]$ . We recall that only a local solution is obtained in [8], on a time interval that depends on the data of the problem, namely  $x$ ,  $f$  and  $\psi$ .

To reach those three improvements, the strategy will combine elements of the formalism used in [8] with a discrete approach of the equation, close to the machinery developed in [6] for rough standard systems. A first step consists of course in providing a reasonable interpretation of Equation (2): for sake of conciseness, we have chosen to work with an interpretation à la Davie, based on the expansion of the regular solution (see Definition 2.6), and we have left aside the sewing map at the core of the constructions in [8]. Note however that the expansion at stake leans here on the operator-valued processes  $X^{x,i}$ ,  $X^{ax,i}$ ,  $X^{xx,ij}$  already identified in the latter reference (see Subsection 2.3), and which, in some sense, plays the role of an infinite-dimensional rough path adapted to the problem. We will let the reader observe (see Subsection 2.4) that when applying the procedure to a regular driving process  $x$  (resp. a standard Brownian motion  $a$ ), the solution we retrieve coincides with the classical solution (resp. the Itô solution). Together with the continuity statement of Theorem 2.12, this identification procedure allows to justify the meaning that will be given to Equation (2) (see Corollary 2.13 and Remark 2.14).

Once endowed with an interpretation of the equation, the solving method is based on a discrete approach of the problem: as in [6], the solution is obtained as the limit of a discrete scheme the mesh of which tends to 0. Nevertheless, some fundamental differences arise when trying to mimic the strategy of [6]. To begin with, the middle-point argument at the root of the reasoning in the diffusion case (see the proof of [6, Lemma 2.4]) cannot take into account the regularization procedures that make up for the classical interplays between time and space regularities in PDEs. Therefore, the argument must be replaced with a little bit more complex algorithm described in Appendix A, and which will be used to all through the paper. Let us also mention the fact that the expansion of the vector field  $f_i(\varphi)(\xi) := f_i(\varphi(\xi))$  is not as easy to control as in the standard finite-dimensional case, even if one assumes that the functions  $f_i : \mathbb{R} \rightarrow \mathbb{R}$  are very regular. Observe for instance that if  $W^{\alpha,p}$  ( $\alpha \in (0, 1)$ ) stands for the fractional Sobolev space

likely to accommodate the solution process, and if  $f_i$  is assumed to be smooth, bounded with bounded derivative, then one can only rely on the non-uniform estimate (see [21])

$$\|f_i(\varphi)\|_{W^{\alpha,p}} \leq \|f\|_{L^\infty(\mathbb{R})} + \|f'_i\|_{L^\infty(\mathbb{R})} \|\varphi\|_{W^{\alpha,p}} \quad \text{for any } \varphi \in W^{\alpha,p}.$$

Consequently, more subtle patching arguments must be put forward so as to exhibit a global solution. This requires in particular a careful examination of the dependence of each estimate with respect to the initial condition (see for instance the controls (32) and (33)). It is worth noticing here that the high regularity of the initial condition (see the assumption in Theorem 2.10:  $\psi \in \mathcal{B}_{\gamma',p}$  with  $\gamma + \gamma' > 1$ ) will play a fundamental role in our approach: it will be resorted to in order to get rid of the quadratic terms that popped out of the estimates of [8], and which turned out to be very problematic when trying to extend the solution on any interval.

The paper is structured as follows: In Section 2, we gather all the elements that allow to introduce and understand our interpretation of Equation (2), and we state the three main results of the paper, namely Theorems 2.10-2.12. The three sections that follow are dedicated to the proof of each of these results, with the existence theorem first (Section 3) and then the uniqueness (Section 4) and continuity (Section 5) results. Finally, Appendix A contains the description and analysis of the algorithm at the root of our machinery, while Appendix B is meant to provide the details relative to the identification procedure in the Brownian case (see Proposition 2.9).

For sake of clarity, we shall only consider Equation (2) on the generic interval  $[0, 1]$ . It is however easy to realize that the whole reasoning remains valid on any (fixed) finite interval  $[0, T]$ .

All through the paper, we will denote by  $\mathcal{C}^{k,\mathbf{b}}(\mathbb{R}; \mathbb{R}^l)$  ( $k, l \in \mathbb{N}^*$ ) the set of  $\mathbb{R}^l$ -valued functions which are  $k$ -times differentiable with bounded derivatives. Moreover, following the rough paths terminology, we will call regular process any piecewise differentiable function.

## 2. INTERPRETATION OF THE EQUATION

We first give some precisions about the setting of our study, as far as the operator  $A$ , the driving process  $x$  and the vector field  $f_i$  are concerned (Subsection 2.1). Then we introduce a few notations and tools designed for the analysis of the equation (Subsections 2.2 and 2.3), and with the help of which we can define and justify our notion of solution (Subsection 2.4). We finally state the three main results associated to this definition (Subsection 2.5), and which will be proved in the next sections.

**2.1. Assumptions.** As it was announced in the introduction, we mean to tackle the equation  $dy_t = Ay_t dt + f_i(y_t) dx_t^i$ ,  $t \in [0, 1]$ , in  $L^p(\mathcal{O})$ , where  $\mathcal{O}$  is a bounded domain of  $\mathbb{R}^n$ ,  $A$  is an elliptic operator,  $f_i(\varphi)(\xi) := f_i(\varphi(\xi))$  and  $x$  is a Hölder process. More precisely, to be in position to interpret and solve this equation, we will be led to assume that the following conditions are satisfied:

**Assumption (A1):**  $A$  generates an analytic semigroup of contraction  $S$  on any  $L^p(\mathcal{O})$ . Under this hypothesis, we will denote  $S_{ts} := S_{t-s}$  ( $s \leq t$ ),  $\mathcal{B}_p := L^p(\mathcal{O})$ ,  $\mathcal{B}_{\alpha,p} :=$

$\text{Dom}(A_p^\alpha)$ , and we endow the latter space with the graph norm  $\|\varphi\|_{\mathcal{B}_{\alpha,p}} := \|A_p^\alpha \varphi\|_{L^p(\mathcal{O})}$ . We also assume that for any function  $g \in \mathcal{C}^{2,b}(\mathbb{R}; \mathbb{R})$ , there exists a constant  $c_g$  such that

$$\|g(\varphi)\|_{\mathcal{B}_{1/2,p}} \leq c_g \{1 + \|\varphi\|_{\mathcal{B}_{1/2,p}}\} \quad , \quad \|g(\varphi)\|_{\mathcal{B}_{\alpha,p}} \leq c_g \{1 + \|\varphi\|_{\mathcal{B}_{\alpha,p}}^2\} \quad \text{if } \alpha \in (1/2, 1), \quad (3)$$

where  $g(\varphi)$  is just understood in the composition sense, ie  $g(\varphi)(\xi) := g(\varphi(\xi))$ .

**Assumption (A2):** If  $2\alpha p > n$ , then  $\mathcal{B}_{\alpha,p}$  is a Banach algebra continuously included in the space  $\mathcal{B}_\infty$  of continuous functions on  $\overline{\mathcal{O}}$ .

**Assumption (X) $_\gamma$ :**  $x$  allows the construction of a 2-rough path

$$(x, \mathbf{x}^2) \in \mathcal{C}_1^\gamma([0, 1]; \mathbb{R}^m) \times \mathcal{C}_2^{2\gamma}([0, 1]; \mathbb{R}^{m,m})$$

for some (fixed) coefficient  $\gamma \in (1/3, 1/2)$ . In other words, we assume that  $x$  is a  $\gamma$ -Hölder process and that there exists a 2-variable process  $\mathbf{x}^2$  (also called a Lévy area) such that for any  $0 \leq s \leq u \leq t \leq 1$ ,

$$|\mathbf{x}_{ts}^2| \leq c |t - s|^{2\gamma} \quad \text{and} \quad \mathbf{x}_{ts}^{2,ij} - \mathbf{x}_{tu}^{2,ij} - \mathbf{x}_{us}^{2,ij} = (x_t^i - x_u^i)(x_u^j - x_s^j).$$

We will then denote

$$\|\mathbf{x}\|_\gamma := \mathcal{N}[x; \mathcal{C}_1^\gamma([0, 1]; \mathbb{R}^m)] + \mathcal{N}[\mathbf{x}^2; \mathcal{C}_2^{2\gamma}([0, 1]; \mathbb{R}^{m,m})],$$

where

$$\mathcal{N}[x; \mathcal{C}_1^\gamma([0, 1]; \mathbb{R}^m)] := \sup_{0 \leq s < t \leq 1} \frac{|x_t - x_s|}{|t - s|^\gamma}, \quad \mathcal{N}[\mathbf{x}^2; \mathcal{C}_2^{2\gamma}([0, 1]; \mathbb{R}^{m,m})] := \sup_{0 \leq s < t \leq 1} \frac{|\mathbf{x}_{ts}^2|}{|t - s|^{2\gamma}}.$$

**Assumption (F) $_k$ :**  $f$  belongs to  $\mathcal{C}^{k,b}(\mathbb{R}; \mathbb{R}^m)$ .

Before pondering over the plausibility of those conditions, let us precise that we henceforth focus on the mild formulation of Equation (2), ie

$$y_t = S_t \psi + \int_0^t S_{tu} dx_u^i f_i(y_u) \quad , \quad t \in [0, 1]. \quad (4)$$

This is a classical change of viewpoint for the study of (stochastic) PDEs (see [5]), which allows to resort to the regularizing properties of the semigroup. Note however that owing to the regularity assumptions on  $f$ , it will be retrospectively possible to make a link between the mild and strong interpretations of the equation (see Remark 2.14).

**Application:** Properties (A1)-(A2) are satisfied by any elliptic operator on  $L^p((0, 1)^n)$  that can be written as

$$A = - \sum_{i,j=1}^n \partial_{\xi_i} (a_{ij} \cdot \partial_{\xi_j}) + c \quad , \quad \mathcal{D}(A_p) := W^{2,p}((0, 1)^n) \cap W_0^{1,p}((0, 1)^n), \quad (5)$$

where  $c \geq 0$  and the functional coefficients  $a_{ij}$  are bounded, differentiable with bounded derivatives on  $[0, 1]^n$ . Indeed, under those assumptions, it is proven in [7] that  $A$  generates an analytic semigroup of contraction. Then, thanks to [18], one can identify the domain  $\mathcal{D}(A_p^\alpha)$  with the complex interpolation  $[L_p, \mathcal{D}(A_p)]_\alpha$  and use the result of [20] to assert that  $\|\cdot\|_{\mathcal{D}(A_p^\alpha)} \sim \|\cdot\|_{F_{p,2}^{2\alpha}}$ , where  $F_{p,2}^{2\alpha}$  is the classical Triebel-Lizorkin space described

(for instance) in [19]. The results of [19] (resp. [21]) finally enables to check Condition (A2) (resp. the controls contained in (3)).

As far as Condition  $(X)_\gamma$  is concerned, the process we have in mind in this paper is the fractional Brownian motion  $B^H$  with Hurst index  $H > 1/3$ , for which the (a.s) existence of a Lévy area has been established in [4]. Note however that Condition  $(X)_\gamma$  is in fact satisfied by a larger class of Gaussian processes, as reported in [11].

In brief, under the above-stated regularity assumptions, the results we are about to state and prove can be applied to the equation

$$dy_t = \left[ - \sum_{i,j=1}^n \partial_{\xi_i} (a_{ij} \cdot \partial_{\xi_j} y_t) + cy_t \right] dt + \sum_{i=1}^m f_i(Y_t) dB_t^{H,i} \quad , \quad t \in [0, 1] \quad , \quad \xi \in (0, 1)^n.$$

**2.2. Hölder spaces.** We suppose in this subsection that Assumption (A1) is satisfied. In order to introduce the framework well-suited for the analysis of (4), let us only point out the following consideration: we know that the most appropriate background for the study of rough standard systems is the classical Hölder space  $\{y : [0, 1] \rightarrow \mathbb{R}^d : |y_t - y_s| \leq c|t - s|^\gamma\}$  (see [12]), and this is (among others) due to the convenient expression for the variations of the solution  $y$  of (1), namely  $y_t - y_s = \int_s^t \sigma_{ij}(y_u) dx_u^j$ . Here, if we denote by  $y$  the solution of (4) (assume for the moment that  $x$  is regular), it is readily checked that for any  $s < t$ ,

$$y_t - y_s = \int_s^t S_{tu} dx_u^i f_i(y_u) + a_{ts} y_s, \quad \text{where } a_{ts} := S_{ts} - \text{Id}.$$

With this observation in mind, the introduction of the following notations becomes quite natural:

**Notations.** For all processes  $y : [0, 1] \rightarrow \mathcal{B}_p$  and  $z : \mathcal{S}_2 \rightarrow \mathcal{B}_p$ , where  $\mathcal{S}_2 := \{(s, t) \in [0, 1]^2 : s \leq t\}$ , we set, for  $s \leq u \leq t \in [0, 1]$ ,

$$(\delta y)_{ts} := y_t - y_s \quad , \quad (\hat{\delta} y)_{ts} := (\delta y)_{ts} - a_{ts} y_s = y_t - S_{ts} y_s, \quad (6)$$

$$(\hat{\delta} z)_{tus} := z_{ts} - z_{tu} - S_{tu} z_{us}. \quad (7)$$

With these notations, the (regular) system (4) can be equivalently written as

$$y_0 = \psi \quad , \quad (\hat{\delta} y)_{ts} = \int_s^t S_{tu} dx_u^i f_i(y_u). \quad (8)$$

To make the introduction of Notations (6)-(7) even more legitimate in this convolutional context, we let the reader observe the following elementary properties:

**Proposition 2.1.** *Let  $y : [0, 1] \rightarrow \mathcal{B}_p$  and  $x : [0, 1] \rightarrow \mathbb{R}$  a regular process. Then it holds:*

- *Telescopic sum:*  $\hat{\delta}(\hat{\delta} y)_{tus} = 0$  and  $(\hat{\delta} y)_{ts} = \sum_{i=0}^{n-1} S_{t_{i+1}t} (\hat{\delta} y)_{t_{i+1}t_i}$  for any partition  $\{s = t_0 < t_1 < \dots < t_n = t\}$  of an interval  $[s, t]$  of  $[0, 1]$ .
- *Chasles relation:* if  $\mathcal{J}_{ts} := \int_s^t S_{tu} dx_u y_u$ , then  $\hat{\delta} \mathcal{J} = 0$ .

The treatment of Equation (8) via the rough paths method will lean on the controlled expansion of the convolutional integral  $\int_s^t S_{tu} dx_u^i f_i(y_u)$ . In order to express this control with the highest accuracy, we must provide ourselves with suitable semi-norms, that

can be seen as generalizations of the classical Hölder norms: thus, if  $y : [0, 1] \rightarrow V$ ,  $z : \mathcal{S}_2 \rightarrow V$  and  $h : \mathcal{S}_3 \rightarrow V$ , where  $V$  is any Banach space and  $\mathcal{S}_3 := \{(s, u, t) \in [0, 1]^3 : s \leq u \leq t\}$ , we will denote, for any  $\lambda > 0$ ,

$$\mathcal{N}[y; \hat{\mathcal{C}}_1^\lambda([a, b]; V)] := \sup_{a \leq s < t \leq b} \frac{\|(\hat{\delta}y)_{ts}\|_V}{|t - s|^\lambda}, \quad \mathcal{N}[y; \mathcal{C}_1^0([a, b]; V)] := \sup_{t \in [a, b]} \|y_t\|_V, \quad (9)$$

$$\mathcal{N}[z; \mathcal{C}_2^\lambda([a, b]; V)] := \sup_{a \leq s < t \leq b} \frac{\|z_{ts}\|_V}{|t - s|^\lambda}, \quad \mathcal{N}[h; \mathcal{C}_3^\lambda([a, b]; V)] := \sup_{a \leq s < u < t \leq b} \frac{\|h_{tus}\|_V}{|t - s|^\lambda}. \quad (10)$$

Then, in a natural way,  $\hat{\mathcal{C}}_1^\lambda([a, b]; V)$  stands for the set of processes  $y : [0, 1] \rightarrow V$  such that  $\mathcal{N}[y; \hat{\mathcal{C}}_1^\lambda([a, b]; V)] < \infty$ , and we define  $\mathcal{C}_2^\lambda([a, b]; V)$  and  $\mathcal{C}_3^\lambda([a, b]; V)$  along the same line. With these notations, observe for instance that if  $y \in \mathcal{C}_2^\lambda([a, b]; \mathcal{L}(V, W))$  and  $z \in \mathcal{C}_2^\beta([a, b]; V)$ , the process  $h$  defined as  $h_{tus} = y_{tu}z_{us}$  ( $s \leq u \leq t$ ) belongs to  $\mathcal{C}_3^{\lambda+\beta}([a, b]; W)$ .

When  $[a, b] = [0, 1]$ , we will more simply write  $\mathcal{C}_k^\lambda(V) := \mathcal{C}_k^\lambda([a, b]; V)$ .

**2.3. Infinite-dimensional rough path.** When  $x$  is a regular process, the expansion of the convolutional integral  $\int_s^t S_{tu} dx_u^i f_i(y_u)$  puts forward the role played by the three following operator-valued processes constructed from  $x$  (see the proof of Proposition 2.8):

$$\int_s^t S_{tu} dx_u^i, \quad \int_s^t a_{tu} dx_u^i, \quad \int_s^t S_{tu} dx_u^i (\delta x^j)_{us}.$$

Of course, when  $x$  becomes irregular, these expressions don't make sense anymore. An integration by parts argument, retrospectively justified through Lemmas 2.3 and 2.4, leads in this situation to the following abstract definitions:

**Definition 2.2.** *Under Assumptions (A1) and  $(X)_\gamma$ , we define the three processes  $X^{x,i}$ ,  $X^{ax,i}$  and  $X^{xx,ij}$  by the formulas*

$$X_{ts}^{x,i} := S_{ts}(\delta x^i)_{ts} - \int_s^t AS_{tu}(\delta x^i)_{tu} du, \quad (11)$$

$$X_{ts}^{ax,i} := a_{ts}(\delta x^i)_{ts} - \int_s^t AS_{tu}(\delta x^i)_{tu} du, \quad (12)$$

$$X_{ts}^{xx,ij} := S_{ts} \mathbf{x}_{ts}^{2,ij} - \int_s^t AS_{tu} [\mathbf{x}_{tu}^{2,ij} + (\delta x^i)_{tu} (\delta x^j)_{us}] du. \quad (13)$$

If in addition Assumption  $(F)_1$  is satisfied, we set  $F_{ij}(\varphi) := f'_i(\varphi) \cdot f_j(\varphi)$  and associate to any process  $y : [0, 1] \rightarrow \mathcal{B}_p$  the two quantities

$$J_{ts}^y := (\hat{\delta}y)_{ts} - X_{ts}^{x,i} f_i(y_s) - X_{ts}^{xx,ij} F_{ij}(y_s), \quad (14)$$

$$K_{ts}^y := (\hat{\delta}y)_{ts} - X_{ts}^{x,i} f_i(y_s). \quad (15)$$

**Lemma 2.3.** *Let  $x$  a  $m$ -dimensional regular process and  $\mathbf{x}^2$  its Lévy area, understood in the classical Lebesgue sense as the iterated integral  $\mathbf{x}_{ts}^{2,ij} := \int_s^t dx_u^i (\delta x^j)_{us}$ . Then, under Assumption (A1),*

$$X_{ts}^{x,i} = \int_s^t S_{tu} dx_u^i, \quad X_{ts}^{ax,i} = \int_s^t a_{tu} dx_u^i, \quad X_{ts}^{xx,ij} = \int_s^t S_{tu} dx_u^i (\delta x^j)_{us}. \quad (16)$$

*Proof.* As aforementioned, this is just a matter of integration by parts. For instance, one has

$$\begin{aligned}
\int_s^t S_{tu} dx_u^i (\delta x^j)_{us} &= \int_s^t S_{tu} d_u (\mathbf{x}_{us}^{2,ij}) \\
&= \int_s^t S_{tu} d_u \left( -(\delta \mathbf{x}^{2,ij})_{tus} + \mathbf{x}_{ts}^{2,ij} - \mathbf{x}_{tu}^{2,ij} \right) \\
&= \int_s^t S_{tu} d_u \left( -(\delta x^i)_{tu} (\delta x^j)_{us} - \mathbf{x}_{tu}^{2,ij} \right) \\
&= S_{ts} \mathbf{x}_{ts}^{2,ij} - \int_s^t AS_{tu} \left[ \mathbf{x}_{tu}^{2,ij} + (\delta x^i)_{tu} (\delta x^j)_{us} \right] du.
\end{aligned}$$

□

Observe now that the three expressions contained in (16) can also be directly interpreted as Itô integrals when  $x$  is a standard Brownian motion. This interpretation remains consistent with Definition 2.2:

**Lemma 2.4.** *Let  $x$  a  $m$ -dimensional Brownian motion and  $\mathbf{x}^2$  its Lévy area, understood in the Itô sense as the first iterated integral of  $x$ . Then, under Assumption (A1), the three identifications of the previous lemma remain valid in this context.*

*Proof.* The integration by parts argument is just replaced with Itô's formula, after noticing that we only deal with Wiener integrals. For  $X^{xx}$ , we know indeed that for any fixed  $s$ , the process  $u \mapsto \mathbf{x}_{us}^{2,ij} = \int_s^u dx_v^i (\delta x^j)_{vs}$  is a semimartingale and  $\int_s^t S_{tu} dx_u^i (\delta x^j)_{us} = \int_s^t S_{tu} d_u (\mathbf{x}_{us}^{2,ij})$ . □

To end up with this subsection, let us underline the regularity properties that will be at our disposal in the sequel:

**Proposition 2.5.** *Under Assumptions (A1) and  $(X)_\gamma$ , one has, for all  $\alpha \in (0, 1), \kappa \in [0, \gamma)$ ,*

$$X^{x,i} \in \mathcal{C}_2^\gamma(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p})) \cap \mathcal{C}_2^{\gamma-\kappa}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha+\kappa,p})), \quad (17)$$

$$X^{ax,i} \in \mathcal{C}_2^{\gamma+\alpha}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_p)), \quad (18)$$

$$X^{xx,ij} \in \mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p})) \cap \mathcal{C}_2^{2\gamma-\kappa}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha+\kappa,p})). \quad (19)$$

We will denote by  $\|X\|_{\gamma,\alpha,\kappa}$  the norm attached to  $X := (X^x, X^{ax}, X^{xx})$  through Properties (17)-(19), that is to say

$$\|X\|_{\gamma,\alpha,\kappa} := \sum_{i,j=1}^m \left\{ \mathcal{N}[X^{x,i}; \mathcal{C}_2^\gamma(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p}))] + \dots + \mathcal{N}[X^{xx,ij}; \mathcal{C}_2^{2\gamma-\kappa}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha+\kappa,p}))] \right\}.$$

With this notation, one has  $\|X\|_{\gamma,\alpha,\kappa} \leq c_{\gamma,\alpha,\kappa} \|\mathbf{x}\|_\gamma$ . Moreover, if  $\tilde{X}$  stands for the path associated to another process  $\tilde{x}$  satisfying  $(X)_\gamma$ , then

$$\|X - \tilde{X}\|_{\gamma,\alpha,\kappa} \leq c_{\gamma,\alpha,\kappa} \{1 + \|\mathbf{x}\|_\gamma + \|\tilde{\mathbf{x}}\|_\gamma\} \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma. \quad (20)$$

*Proof.* Properties (17)-(19) are straightforward consequences of the well-known estimates (see [17])

$$\|S_{ts}\varphi\|_{\mathcal{B}_{\alpha+\kappa,p}} \leq c_\kappa |t-s|^{-\kappa} \|\varphi\|_{\mathcal{B}_{\alpha,p}}, \quad \|AS_{ts}\varphi\|_{\mathcal{B}_{\alpha+\kappa,p}} \leq c_\kappa |t-s|^{-1-\kappa} \|\varphi\|_{\mathcal{B}_{\alpha,p}},$$



$$\|a_{ts}\varphi\|_{\mathcal{B}_p} \leq c_\alpha |t-s|^\alpha \|\varphi\|_{\mathcal{B}_{\alpha,p}}.$$

For example, for any  $\varphi \in \mathcal{B}_{\alpha,p}$ ,

$$\begin{aligned} \|X_{ts}^{x,i}\varphi\|_{\mathcal{B}_{\alpha+\kappa,p}} &\leq \|x\|_\gamma \left\{ |t-s|^\gamma \|S_{ts}\varphi\|_{\mathcal{B}_{\alpha+\kappa,p}} + \int_s^t |t-u|^\gamma \|AS_{tu}\varphi\|_{\mathcal{B}_{\alpha+\kappa,p}} du \right\} \\ &\leq c_\kappa \|x\|_\gamma \|\varphi\|_{\mathcal{B}_{\alpha,p}} \left\{ |t-s|^{\gamma-\kappa} + \int_s^t |t-u|^{-1+\gamma-\kappa} du \right\} \\ &\leq c_{\gamma,\kappa} \|x\|_\gamma \|\varphi\|_{\mathcal{B}_{\alpha,p}} |t-s|^{\gamma-\kappa}. \end{aligned}$$

The controls of  $\|X\|_{\gamma,\alpha,\kappa}$  and  $\|X-\tilde{X}\|_{\gamma,\alpha,\kappa}$  can be readily checked from the very definitions (11)-(13). Observe for instance that

$$\begin{aligned} &\left\| \int_s^t AS_{tu}(\delta x^i)_{tu}(\delta x^j)_{us} du - \int_s^t AS_{tu}(\delta \tilde{x}^i)_{tu}(\delta \tilde{x}^j)_{us} du \right\|_{\mathcal{L}(\mathcal{B}_p,\mathcal{B}_p)} \\ &\leq \int_s^t \|AS_{tu}\|_{\mathcal{L}(\mathcal{B}_p,\mathcal{B}_p)} \left\{ |\delta(x^i - \tilde{x}^i)_{tu}| |(\delta x^j)_{us}| + |(\delta \tilde{x}^i)_{tu}| |\delta(x^j - \tilde{x}^j)_{us}| \right\} du \\ &\leq c \{1 + \|\mathbf{x}\|_\gamma + \|\tilde{\mathbf{x}}\|_\gamma\} \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \left( \int_s^t |t-u|^{-1+\gamma} |u-s|^\gamma du \right) \\ &\leq c |t-s|^{2\gamma} \{1 + \|\mathbf{x}\|_\gamma + \|\tilde{\mathbf{x}}\|_\gamma\} \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma. \end{aligned}$$

□

**2.4. Solution of the equation.** The notion of solution we will use in this paper is inspired by the considerations of [6] for rough standard systems. We first give its (abstract) definition and then provide two results meant to make the apprehension of this concept easier (Propositions 2.8 and 2.9). Remember that the notation  $J^y$  has been introduced in Definition 2.2.

**Definition 2.6.** Under Assumptions (A1),  $(X)_\gamma$  and  $(F)_1$ , for all  $\lambda \geq 0$  and  $\psi \in \mathcal{B}_{\lambda,p}$ , we will call solution in  $\mathcal{B}_{\lambda,p}$  of the equation

$$y_t = S_t\psi + \int_0^t S_{t-u} f_i(y_u) dx_u^i, \quad t \in [0, 1], \quad (21)$$

any process  $y : [0, 1] \rightarrow \mathcal{B}_{\lambda,p}$  such that  $y_0 = \psi$  and that there exists two coefficients  $\mu > 1, \varepsilon > 0$  for which

$$J^y \in \mathcal{C}_2^\mu([0, 1]; \mathcal{B}_p) \quad \text{and} \quad J^y \in \mathcal{C}_2^\varepsilon([0, 1]; \mathcal{B}_{\lambda,p}).$$

*Remark 2.7.* The reader familiar with the strategy of [6] will not be surprised by the condition  $J^y \in \mathcal{C}_2^\mu([0, 1]; \mathcal{B}_p)$  for some coefficient  $\mu > 1$ . The second condition  $J^y \in \mathcal{C}_2^\varepsilon([0, 1]; \mathcal{B}_{\lambda,p})$  may be less expected: it stems from the technical arguments that will be displayed in Section 4 so as to obtain uniqueness of the solution (observe in particular the estimate (43)).

**Proposition 2.8.** Let  $x$  a  $m$ -dimensional regular process,  $\mathbf{x}^2$  its Lévy area, understood in the Lebesgue sense. We suppose that Assumptions (A1) and  $(F)_1$  are satisfied. Then, for all  $\eta \in (0, 1)$  and  $\psi \in \mathcal{B}_{\eta,p}$ , the (classical) solution of Equation (21) is also solution in  $\mathcal{B}_{\eta,p}$  in the sense of Definition 2.6.

*Proof.* Let  $y$  the classical solution of (21), with initial condition  $\psi \in \mathcal{B}_{\eta,p}$ . Then  $y \in \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\eta,p})$  and since  $(\hat{\delta}y)_{ts} = \int_s^t S_{tu} dx_u^i f_i(y_u)$  and  $f$  is bounded, one clearly has  $y \in \hat{\mathcal{C}}_1^1([0, 1]; \mathcal{B}_p)$ . Now notice that owing to the identification (16), we get

$$K_{ts}^y = \int_s^t S_{tu} dx_u^i f_i(y_u) - X_{ts}^{x,i} f_i(y_s) = \int_s^t S_{tu} dx_u^i \delta(f_i(y))_{us},$$

and so

$$\begin{aligned} \|K_{ts}^y\|_{\mathcal{B}_p} &\leq \|\dot{x}\|_{\infty, [0,1]} \|f'\|_{\infty} \int_s^t \|(\delta y)_{us}\|_{\mathcal{B}_p} du \\ &\leq c_{x,f} \int_s^t \left\{ \|(\hat{\delta}y)_{us}\|_{\mathcal{B}_p} + \|a_{us}\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} \|y_s\| \right\} \\ &\leq c_{x,f,y} \int_s^t \{|u-s| + |u-s|^\eta\} du \leq c_{x,f,y} |t-s|^{1+\eta}. \end{aligned} \quad (22)$$

To complete the proof, observe that by resorting to the identification (16) once again, we can write  $J_{ts}^y = \int_s^t S_{tu} dx_u^i M_{us}^i$ , with

$$\begin{aligned} M_{us}^i &:= \delta(f_i(y))_{us} - (\delta x^j)_{us} f_i'(y_s) \cdot f_j(y_s) \\ &= \int_0^1 f_i'(y_s + r(\delta y)_{us}) \cdot (\delta y)_{us} - (\delta x^j)_{us} f_i'(y_s) \cdot f_j(y_s) \\ &= \int_0^1 f_i'(y_s + r(\delta y)_{us}) \cdot a_{us} y_s \int_0^1 f_i'(y_s + r(\delta y)_{us}) \cdot (\hat{\delta}y)_{us} - (\delta x^j)_{us} f_i'(y_s) \cdot f_j(y_s), \end{aligned}$$

and thus

$$\begin{aligned} M_{us}^i &= \int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot a_{us} y_s + \int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot K_{us}^y \\ &+ \int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot X_{us}^{ax,j} f_j(y_s) + \int_0^1 dr [f_i'(y_s + r(\delta y)_{us}) - f_i'(y_s)] \cdot (\delta x^j)_{us} f_j(y_s), \end{aligned} \quad (23)$$

where we have used the trivial relation  $X_{us}^{x,i} = X_{us}^{ax,i} + (\delta x^j)_{us}$ . From this expression, it is easy to show that  $\|M_{us}^i\|_{\mathcal{B}_p} \leq c_y |u-s|^\eta$ , which leads to the result with (using the notations of Definition 2.6)  $\mu = 1 + \eta$ ,  $\varepsilon = 1$ .  $\square$

**Proposition 2.9.** *Let  $x$  a  $m$ -dimensional standard Brownian motion,  $\mathbf{x}^2$  its Lévy area, understood in the Itô sense. We suppose that Assumptions (A1) and (F)<sub>2</sub> are satisfied. Then, for all  $\eta \in (1/2, 1)$  and  $\psi \in \mathcal{B}_{\eta,p}$ , the Itô solution of Equation (21) is almost surely solution in  $\mathcal{B}_{\eta,p}$  in the sense of Definition 2.6.*

*Proof.* For sake of clarity, we have postponed the proof of this result to Appendix B.  $\square$

Together with the forthcoming uniqueness Theorem 2.11, the two above-stated results allow to identify, in the two reference situations (ie when  $x$  is regular and when  $x$  is a standard Brownian motion), the solution in the sense of Definition 2.6 with the classical solution. We will then lean on the continuity Theorem 2.12 to fully justify our interpretation of the equation (see Remark 2.14).

**2.5. Main results.** With the help of the tools and definitions we have just introduced, we are in position to state the three main results of this paper, which successively provide the existence, uniqueness and continuity of the solution to (21).

**Theorem 2.10.** *Under Assumptions (A1),  $(X)_\gamma$  and  $(F)_2$ , for all  $\gamma' \in (1 - \gamma, \gamma + 1/2)$  and  $\psi \in \mathcal{B}_{\gamma', p}$ , Equation (21) admits a solution  $y$  in  $\mathcal{B}_{\gamma', p}$  in the sense of Definition 2.6, which satisfies*

$$\mathcal{N}[y; \hat{\mathcal{C}}_1^\gamma([0, 1]; \mathcal{B}_p)] + \mathcal{N}[y; \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\gamma', p})] \leq C(\|\mathbf{x}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma', p}}),$$

for some function  $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}$  growing with its arguments.

**Theorem 2.11.** *If  $p > n$  and if Assumptions (A1), (A2),  $(X)_\gamma$  and  $(F)_3$  are satisfied, then for all  $\gamma' \in (1 - \gamma, \gamma + 1/2)$  and  $\psi \in \mathcal{B}_{\gamma', p}$ , the solution  $y$  in  $\mathcal{B}_{\gamma', p}$  given by Theorem 2.10 is unique. Moreover, for any*

$$0 < \beta < \inf(3\gamma - 1, \gamma + \gamma' - 1, \gamma - (\gamma' - 1/2)),$$

there exists a constant  $c_{x, \psi, f, \beta}$  such that for all  $n$ ,

$$\max_{k=0, \dots, 2^n} \|y_{t_k^n} - y_{t_k^n}^n\|_{\mathcal{B}_{\gamma', p}} \leq \frac{c_{x, \psi, f, \beta}}{(2^n)^\beta},$$

where  $y^n$  stands for the process given by the discrete scheme (26).

**Theorem 2.12.** *Under the conditions of Theorem 2.11, the solution of the equation is continuous with respect to the initial condition and the driving rough path. More precisely, if  $y$  (resp.  $\tilde{y}$ ) is the solution in  $\mathcal{B}_{\gamma', p}$  associated to  $(x, \mathbf{x}^2)$  (resp.  $(\tilde{x}, \tilde{\mathbf{x}}^2)$ ), with initial condition  $\psi$  (resp.  $\tilde{\psi}$ ), then*

$$\begin{aligned} & \mathcal{N}[y - \tilde{y}; \hat{\mathcal{C}}_1^\gamma([0, 1]; \mathcal{B}_p)] + \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\gamma', p})] \\ & \leq C \left( \|\mathbf{x}\|_\gamma, \|\tilde{\mathbf{x}}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma', p}}, \|\tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} \right) \left\{ \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\}, \end{aligned} \quad (24)$$

for some functions  $C : (\mathbb{R}^+)^4 \rightarrow \mathbb{R}^+$  growing with its arguments.

Together with the identification result established in Proposition 2.8, these three theorems offer another viewpoint on the solution of Equation (21), which may be more in accordance with the formalism used in [11] for rough standard systems:

**Corollary 2.13.** *Under the assumptions of Theorem 2.11, let  $\psi \in \mathcal{B}_{\gamma', p}$  and  $(\tilde{x}^n)_n$  a sequence of regular processes such that  $\|x - \tilde{x}^n\|_\gamma + \|\mathbf{x}^2 - \tilde{\mathbf{x}}^{2, n}\|_{2\gamma} \rightarrow 0$  as  $n$  tends to infinity, where  $\tilde{\mathbf{x}}^{2, n}$  stands for the standard Lévy area constructed from  $\tilde{x}^n$ . Let  $\tilde{y}^n$  the (classical) solution of (21) associated to each  $\tilde{x}^n$ . If  $y$  is the solution of (21) given by Theorem 2.11, then*

$$\mathcal{N}[y - \tilde{y}^n; \hat{\mathcal{C}}_1^\gamma([0, 1]; \mathcal{B}_p)] + \mathcal{N}[y - \tilde{y}^n; \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\gamma', p})] \rightarrow 0 \quad (25)$$

as  $n$  tends to infinity.

*Remark 2.14.* Through the latter result, one can see that the solution  $y$  we exhibit is a solution *in the rough paths sense*, that is to say a limit of regular solutions with respect to some particular topology. In this context,  $y$  can legitimately be called a mild solution of (2), as a limit of classical mild solutions. Furthermore, it is worth noticing

that given the regularity assumptions on  $f_i$ , if we suppose in addition that the initial condition  $\psi$  belongs to the domain  $\mathcal{D}(A_p)$ , then each regular mild solution  $\tilde{y}^n$  is also a strong solution (see [17, Theorem 6.1.6]). Consequently, if  $\psi \in \mathcal{D}(A_p)$ ,  $y$  can in some way be also considered as a strong solution of (2), keeping in mind the topology of the underlying convergence result (25).

### 3. EXISTENCE OF A SOLUTION

This section is devoted to the proof of Theorem 2.10. Thus, we henceforth suppose that the assumptions of the theorem, namely (A1),  $(X)_\gamma$  and  $(F)_2$ , are satisfied. Besides, we fix a parameter  $\gamma' \in (1 - \gamma, \gamma + 1/2)$  and an initial condition  $\psi \in \mathcal{B}_{\gamma', p}$ .

**3.1. A few additional notations.** We consider the sequence  $(\Pi^n)_n$  of dyadic partitions of  $[0, 1]$  and define the discrete process  $y^n$  following the iteration formula:

$$y_0^n := \psi \quad , \quad y_{t_{k+1}^n}^n := S_{t_{k+1}^n t_k^n} y_{t_k^n}^n + X_{t_{k+1}^n t_k^n}^{x, i} f_i(y_{t_k^n}^n) + X_{t_{k+1}^n t_k^n}^{xx, ij} F_{ij}(y_{t_k^n}^n) \quad (26)$$

for any point  $t_k^n$  of  $\Pi^n$ .  $y^n$  is then extended on  $[0, 1]$  by linear interpolation. For sake of clarity, we will denote in this section  $J^n := J^{y^n}$  and  $K^n := K^{y^n}$ . Observe that owing to the very definition of  $y^n$ , one has  $J_{t_{k+1}^n t_k^n}^n = 0$ .

In the rest of the paper, we will also appeal to the discrete versions of the generalized Hölder norms introduced in Subsection 2.2. Thus, for any  $n \in \mathbb{N}$ , we denote  $\llbracket a, b \rrbracket_n := [a, b] \cap \Pi^n$  and

$$\mathcal{N}[h; \hat{\mathcal{C}}_1^\lambda(\llbracket t_p^n, t_q^n \rrbracket_n, \mathcal{B}_{\alpha, p})] := \sup_{\substack{t_p^n \leq s < t \leq t_q^n \\ s, t \in \Pi^n}} \frac{\|(\hat{\delta}h)_{ts}\|_{\mathcal{B}_{\alpha, p}}}{|t - s|^\lambda},$$

We define the two quantities  $\mathcal{N}[\cdot; \mathcal{C}_2^\lambda(\llbracket a, b \rrbracket_n; \mathcal{B}_{\alpha, p})]$  and  $\mathcal{N}[\cdot; \mathcal{C}_3^\lambda(\llbracket a, b \rrbracket_n; \mathcal{B}_{\alpha, p})]$  along the same line.

Now, for any (not necessarily uniform) partition  $\tilde{\Pi}$  of  $[0, 1]$  made of points of  $\Pi^n$ , we introduce the process  $J^{n, \tilde{\Pi}}$  defined for all  $s < t \in \Pi^n$  as

$$J_{ts}^{n, \tilde{\Pi}} := \begin{cases} 0 & \text{if } (s, t) \cap \tilde{\Pi} = \emptyset \\ (\hat{\delta}J^n)_{ts} & \text{if } (s, t) \cap \tilde{\Pi} = u \\ J_{ts}^n - J_{t\tilde{t}_N}^n - \sum_{k=1}^{N-1} S_{t\tilde{t}_{k+1}} J_{\tilde{t}_{k+1}\tilde{t}_k}^n - S_{t\tilde{t}_1} J_{\tilde{t}_1 s}^n & \text{if } (s, t) \cap \tilde{\Pi} = \{\tilde{t}_1, \dots, \tilde{t}_N\} \end{cases}$$

*Remark 3.1.* Since  $J_{t_{k+1}^n t_k^n}^n = 0$ , one has in particular  $J_{ts}^{n, \Pi^n} = J_{ts}^n$ . Besides, if  $\tilde{\Pi}, \hat{\Pi}$  are two partitions of  $[0, 1]$  made of points of  $\Pi^n$  and such that  $\tilde{\Pi} \cap (s, t) = \{\tilde{t}_1, \dots, \tilde{t}_N\}$  ( $N \geq 3$ ) and  $\hat{\Pi} \cap (s, t) = \{\tilde{t}_1, \dots, \tilde{t}_{k-1}, \tilde{t}_{k+1}, \dots, \tilde{t}_N\}$  for  $1 \leq k \leq N - 1$ , then  $J_{ts}^{n, \tilde{\Pi}} - J_{ts}^{n, \hat{\Pi}} = S_{t\tilde{t}_{k+1}} (\hat{\delta}J^n)_{\tilde{t}_{k+1}\tilde{t}_k\tilde{t}_{k-1}}$ . With the same notations, if  $\hat{\Pi} \cap (s, t) = \{\tilde{t}_1, \dots, \tilde{t}_{N-1}\}$ , then  $J_{ts}^{n, \tilde{\Pi}} - J_{ts}^{n, \hat{\Pi}} = (\hat{\delta}J^n)_{t\tilde{t}_N\tilde{t}_{N-1}}$ .

**3.2. Preliminary results on  $J^n$ .** We fix  $t_p^n < t_q^n \in \Pi^n$  and apply the algorithm described in Appendix A to the uniform partition  $\{t_p^n, t_{p+1}^n, \dots, t_q^n\}$ . We set  $N := q - p$ , so that for any  $k \in \{p, \dots, q\}$ ,  $t_k^n = t_p^n + \frac{(k-p)(t_q^n - t_p^n)}{N}$ . We also denote by  $(\Pi^{n,m})_{m \in \{1, \dots, N-1\}}$  the (decreasing) sequence of partitions of  $[t_p^n, t_q^n]$  deduced from the algorithm, and  $\Pi^{n,0} := \{t_p^n, t_{p+1}^n, \dots, t_q^n\}$ . Finally,  $J_{t_q^n t_p^n}^{n,m} := J_{t_q^n t_p^n}^{n, \Pi^{n,m}}$ . With these notations in hand, one has

$$J_{t_q^n t_p^n}^n = \sum_{m=0}^{N-1} \left\{ J_{t_q^n t_p^n}^{n,m} - J_{t_q^n t_p^n}^{n,m+1} \right\}.$$

Once endowed with this decomposition, we are in position to show the following result, which turns out to be the starting point of our reasoning:

**Lemma 3.2.** *Let  $\mu > 1$  and  $\kappa > 0$ . There exists a constant  $c = c_{\mu, \kappa}$  such that*

$$\|J_{t_q^n t_p^n}^n\|_{\mathcal{B}_{\gamma', p}} \leq c \left\{ |t_q^n - t_p^n|^\kappa + |t_q^n - t_p^n|^{\mu - \gamma'} \right\} \left\{ \mathcal{N}[\hat{\delta} J^n; \mathcal{C}_3^\kappa(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_{\gamma', p})] + \mathcal{N}[\hat{\delta} J^n; \mathcal{C}_3^\mu(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_p)] \right\}, \quad (27)$$

and

$$\|J_{t_q^n t_p^n}^n\|_{\mathcal{B}_p} \leq c |t_q^n - t_p^n|^\mu \mathcal{N}[\hat{\delta} J^n; \mathcal{C}_3^\mu(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_p)]. \quad (28)$$

*Proof.* We use the notations of Appendix A. By referring to Remark 3.1, one easily deduces

$$\begin{aligned} & \sum_{m=0}^{N-1} \left\{ J_{t_q^n t_p^n}^{n,m} - J_{t_q^n t_p^n}^{n,m+1} \right\} \\ &= \sum_{r=1}^{M-1} \left\{ (\hat{\delta} J^n)_{t_q^n t_{p+k_{A_{r-1}+1}}^n}^{t_{p+k_{A_{r-1}+1}}^n} + \sum_{m=A_{r-1}+2}^{A_r} S_{t_q^n t_{p+k_m^+}^n}^{t_{p+k_m^+}^n} (\hat{\delta} J^n)_{t_{p+k_m^+}^n t_{p+k_m^-}^n}^{t_{p+k_m^+}^n t_{p+k_m^-}^n} \right\} \\ & \quad + (\hat{\delta} J^n)_{t_q^n t_{p+k_{A_{M-1}+1}}^n}^{t_{p+k_{A_{M-1}+1}}^n} + \mathbf{1}_{\{A_{M-1}+1 < N-1\}} (\hat{\delta} J^n)_{t_q^n t_{p+k_{N-1}}^n}^{t_{p+k_{N-1}}^n t_p^n}. \end{aligned}$$

Then, if  $C_n := \mathcal{N}[\hat{\delta} J^n; \mathcal{C}_3^\kappa(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_{\gamma', p})] + \mathcal{N}[\hat{\delta} J^n; \mathcal{C}_3^\mu(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_p)]$ ,

$$\begin{aligned} & \sum_{m=0}^{N-1} \|J_{t_q^n t_p^n}^{n,m} - J_{t_q^n t_p^n}^{n,m+1}\|_{\mathcal{B}_{\gamma', p}} \\ & \leq 2 C_n |t_q^n - t_p^n|^\kappa \\ & \quad + C_n \sum_{r=0}^{M-1} \left\{ \left| t_q^n - t_{p+k_{A_{r-1}+1}}^n \right|^\kappa + \sum_{m=A_{r-1}+2}^{A_r} \left| t_q^n - t_{p+k_m^+}^n \right|^{-\gamma'} \left| t_{p+k_m^+}^n - t_{p+k_m^-}^n \right|^\mu \right\} \\ & \leq C_n \left\{ |t_q^n - t_p^n|^\kappa + |t_q^n - t_p^n|^{\mu - \gamma'} \right\} \\ & \quad \left\{ 2 + \sum_{r=0}^{M-1} \left\{ \left| 1 - \frac{k_{A_{r-1}+1}^-}{N} \right|^\kappa + \frac{1}{N^\mu} \sum_{m=A_{r-1}+2}^{A_r} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} |k_m^+ - k_m^-|^\mu \right\} \right\} \\ & \leq c_{\kappa, \mu, \gamma'} \left\{ |t_q^n - t_p^n|^\kappa + |t_q^n - t_p^n|^{\mu - \gamma'} \right\} C_n, \end{aligned}$$

thanks to Proposition 6.2. The second control (28) can be obtained with the same arguments, once one has noticed that (51) entails in particular

$$\sum_{r=1}^{M-1} \left\{ \left| 1 - \frac{k_{A_{r-1}+1}^-}{N} \right|^\mu + \frac{1}{N^\mu} \sum_{m=A_{r-1}+2}^{A_r} |k_m^+ - k_m^-|^\mu \right\} \leq c_\mu < \infty.$$

□

**Lemma 3.3.** *For any process  $y : [0, 1] \rightarrow \mathcal{B}_p$  and all  $s < u < t \in [0, 1]$ ,*

$$(\hat{\delta}J^y)_{tus} = X_{tu}^{x,i} \delta(f_i(y))_{us} - X_{tu}^{x,i} (\delta x^j)_{us} F_{ij}(y_s) + X_{tu}^{xx,ij} \delta(F_{ij}(y))_{us} \quad (29)$$

and also

$$(\hat{\delta}J^y)_{tus} = I_{tus} + II_{tus} + III_{tus} + IV_{tus}, \quad (30)$$

with

$$\begin{aligned} I_{tus} &:= X_{tu}^{x,i} \left( \int_s^1 dr f'_i(y_s + r(\delta y)_{us}) \cdot K_{us}^y \right), \\ II_{tus} &:= X_{tu}^{x,i} \left( \int_s^1 dr f'_i(y_s + r(\delta y)_{us}) \cdot \{a_{us} y_s + X_{us}^{xx,j} f_j(y_s)\} \right), \\ III_{tus} &:= X_{tu}^{x,i} \left( \int_0^1 dr [f'_i(y_s + r(\delta y)_{us}) - f'_i(y_s)] \cdot (\delta x^j)_{us} f_j(y_s) \right), \\ IV_{tus} &:= X_{tus}^{xx,ij} \delta(F_{ij}(y))_{us}. \end{aligned}$$

*Proof.* Those are only straightforward computations. For (29), we use the fact that if  $m_{ts} := g_{ts} h_s$ , then  $(\hat{\delta}m)_{tus} = (\hat{\delta}g)_{tus} h_s - g_{tu} (\delta h)_{us}$ , together with the algebraic relations

$$(\hat{\delta}X^{x,i})_{tus} = 0 \quad , \quad (\hat{\delta}X^{xx,ij})_{tus} = X_{tu}^{x,i} (\delta x^j)_{us} \quad \text{for all } s \leq u \leq t,$$

that can be readily checked from the expressions (11) and (13). The expansion of  $\delta(f_i(y))_{us} - (\delta x^j)_{us} F_{ij}(y_s)$  which then leads to (30) has already been elaborated on in the proof of Proposition 2.8. □

**3.3. Existence of a solution.** Thanks to the above preliminary results, we are first able to control  $J^n$  on successive time intervals independent of  $n$ :

**Proposition 3.4.** *Let  $\mu, \varepsilon$  such that*

$$3\gamma > \mu > 1 \quad , \quad \gamma + \gamma' > \mu > 1 \quad , \quad \gamma - (\gamma' - \frac{1}{2}) > \varepsilon > 0. \quad (31)$$

*There exists a time  $T_0 = T_0(x, f, \gamma, \gamma', \mu, \varepsilon) > 0$ ,  $T_0 \in \Pi^n$ , such that for any  $k$ ,*

$$\mathcal{N}[J^n; \mathcal{C}_2^\varepsilon(\llbracket kT_0, (k+1)T_0 \wedge 1 \rrbracket_n; \mathcal{B}_{\gamma', p})] \leq 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma', p}} \quad (32)$$

and

$$\mathcal{N}[J^n; \mathcal{C}_2^\mu(\llbracket kT_0, (k+1)T_0 \wedge 1 \rrbracket_n; \mathcal{B}_p)] \leq 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma', p}}. \quad (33)$$

*Proof.* This is an iteration procedure over the points of the partition, for which we first focus on the case  $k = 0$  in (32) and (33). Assume that both estimates hold true on  $\llbracket 0, t_q^n \rrbracket_n$ . Then, for any  $t \in \llbracket 0, t_q^n \rrbracket_n$ , one has, thanks to (17), (19) and (3),

$$\begin{aligned} \|y_t^n\|_{\mathcal{B}_{\gamma',p}} &\leq \|J_{t0}^n\|_{\mathcal{B}_{\gamma',p}} + \|S_{t0}\psi\|_{\mathcal{B}_{\gamma',p}} + c_x^0 t^{\gamma-(\gamma'-\frac{1}{2})} \left\{ \|f_i(\psi)\|_{\mathcal{B}_{1/2,p}} + \|F_{ij}(\psi)\|_{\mathcal{B}_{1/2,p}} \right\} \\ &\leq \|J_{t0}^n\|_{\mathcal{B}_{\gamma',p}} + \|S_{t0}\psi\|_{\mathcal{B}_{\gamma',p}} + c_{x,f}^1 t^{\gamma-(\gamma'-\frac{1}{2})} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} \\ &\leq c_{x,f}^2 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}, \end{aligned} \quad (34)$$

so that  $\mathcal{N}[y^n; \mathcal{C}_1^0(\llbracket 0, t_q^n \rrbracket_n, \mathcal{B}_{\gamma',p})] \leq c_{x,f}^2 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}$ . Besides, if  $s < t \in \llbracket 0, t_q^n \rrbracket_n$ ,

$$\begin{aligned} \|(\hat{\delta}y^n)_{ts}\|_{\mathcal{B}_p} &\leq \|J_{ts}^n\|_{\mathcal{B}_p} + \|X_{ts}^{xx,ij} f_i(y_s^n)\|_{\mathcal{B}_p} + \|X_{ts}^{xx,ij} F_{ij}(y_s^n)\|_{\mathcal{B}_p} \\ &\leq |t-s|^\gamma c_{x,f}^3 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}, \end{aligned}$$

hence

$$\mathcal{N}[y^n; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, t_q^n \rrbracket_n; \mathcal{B}_p)] \leq c_{x,f}^3 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}. \quad (35)$$

One can also rely on the estimate

$$\|K_{ts}^n\|_{\mathcal{B}_p} \leq \|J_{ts}^n\|_{\mathcal{B}_p} + \|X_{ts}^{xx,ij} F_{ij}(y_s^n)\|_{\mathcal{B}_p} \leq c_{x,f}^4 |t-s|^{2\gamma} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}. \quad (36)$$

Now, from the decomposition (30), we easily deduce, for all  $0 \leq s < u < t \in \llbracket 0, t_{q+1}^n \rrbracket_n$ ,

$$\|(\hat{\delta}J^n)_{tus}\|_{\mathcal{B}_p} \leq c_{x,f}^5 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} \left\{ |t-s|^{3\gamma} + |t-s|^{\gamma+\gamma'} \right\}.$$

Indeed, one has for instance

$$\begin{aligned} &\| [f'_i(y_s + r(\delta y)_{us}) - f'_i(y_s)] \cdot (\delta x^j)_{us} f_j(y_s) \|_{\mathcal{B}_p} \\ &\leq c_{x,f} |u-s|^\gamma \|(\delta y)_{us}\|_{\mathcal{B}_p} \\ &\leq c_{x,f} |u-s|^\gamma \left\{ \|(\hat{\delta}y)_{us}\|_{\mathcal{B}_p} + \|a_{us} y_s\|_{\mathcal{B}_p} \right\} \\ &\leq c_{x,f} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} \left\{ |u-s|^{2\gamma} + |u-s|^{\gamma+\gamma'} \right\} \leq c_{x,f} |u-s|^{2\gamma} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}, \end{aligned}$$

where the constant  $c_{x,f}$  may of course vary from line to line. Consequently,

$$\mathcal{N}[\hat{\delta}J^n; \mathcal{C}_3^\mu(\llbracket 0, t_{q+1}^n \rrbracket_n; \mathcal{B}_p)] \leq c_{x,f}^5 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} \left\{ T_0^{3\gamma-\mu} + T_0^{\gamma+\gamma'-\mu} \right\}.$$

On the other hand, it is readily checked from (29) that

$$\|(\hat{\delta}J^n)_{tus}\|_{\mathcal{B}_{\gamma',p}} \leq c_{x,f}^6 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} |t-s|^{\gamma-(\gamma'-\frac{1}{2})},$$

and therefore

$$\mathcal{N}[\hat{\delta}J^n; \hat{\mathcal{C}}_3^{\gamma-(\gamma'-\frac{1}{2})}(\llbracket 0, t_{q+1}^n \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{x,f}^6 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}.$$

By using the estimate (27), we get

$$\mathcal{N}[J^n; \mathcal{C}_2^\varepsilon(\llbracket 0, t_{q+1}^n \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{x,f}^7 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} \left( T_0^{3\gamma-\mu} + T_0^{\gamma+\gamma'-\mu} + T_0^{\gamma-(\gamma'-\frac{1}{2})-\varepsilon} \right).$$

It only remains to pick  $T_0$  such that

$$c_{x,f}^7 \left( T_0^{3\gamma-\mu} + T_0^{\gamma+\gamma'-\mu} + T_0^{\gamma-(\gamma'-\frac{1}{2})-\varepsilon} \right) \leq 1.$$

We can follow the same lines to show (33) from the estimate (28).

It is now easy to realize that the same reasoning (with the same constants) can be applied on the interval  $[T_0, 2T_0]$  by replacing  $\psi$  with  $y_{T_0}^n$ , and then on the interval  $[2T_0, 3T_0]$ , etc.  $\square$

**Corollary 3.5.** *With the notations of Proposition 3.4, there exists a constant  $c_{x,f}$  such that for any  $k$ ,*

$$\mathcal{N}[J^n; \mathcal{C}_2^\mu(\llbracket kT_0, (k+2)T_0 \wedge 1 \rrbracket_n; \mathcal{B}_p)] \leq \left\{ 1 + \|y_{(k+1)T_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\} + c_{x,f} \left\{ 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\}, \quad (37)$$

$$\mathcal{N}[J^n; \mathcal{C}_2^\varepsilon(\llbracket kT_0, (k+2)T_0 \wedge 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq \left\{ 1 + \|y_{(k+1)T_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\} + c_{x,f} \left\{ 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\}. \quad (38)$$

*Proof.* If  $kT_0 \leq s < (k+1)T_0 \leq t < (k+2)T_0$ ,

$$J_{ts}^n = J_{t,(k+1)T_0}^n - S_{t,(k+1)T_0} J_{(k+1)T_0,s}^n - (\hat{\delta} J^n)_{t,(k+1)T_0,s}.$$

We already know that

$$\|J_{t,(k+1)T_0}^n\|_{\mathcal{B}_p} + \|J_{(k+1)T_0,s}^n\|_{\mathcal{B}_p} \leq |t-s|^\mu \left\{ 2 + \|y_{(k+1)T_0}^n\|_{\mathcal{B}_{\gamma',p}} + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\}.$$

By using the decomposition (30), together with the estimates (34), (35) and (36), we get  $\|(\hat{\delta} J^n)_{t,(k+1)T_0,s}\|_{\mathcal{B}_p} \leq c_x |t-s|^\mu \left\{ 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\}$ , which yields (37). (38) can be shown with the same arguments.  $\square$

*Proof of Theorem 2.10.* With the same estimates as in (34), we first deduce from Proposition 3.4

$$\mathcal{N}[y^n; \mathcal{C}_1^0(\llbracket kT_0, (k+1)T_0 \wedge 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{x,f}^1 \left\{ 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\},$$

where the constant  $c_{x,f}^1$  does not depend on  $k$ . As  $T_0$  is independent of  $y^n$ , this leads to

$$\mathcal{N}[y^n; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{x,f}^2 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}. \quad (39)$$

From this uniform control, we get, by repeating the argument of Corollary 3.5,

$$\mathcal{N}[J^n; \mathcal{C}_2^\mu(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] \leq c_{x,f}^4 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}, \quad (40)$$

and then

$$\mathcal{N}[y^n; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] \leq c_{x,f}^5 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}. \quad (41)$$

Now remember that  $y^n$  is extended on  $[0, 1]$  by linear interpolation, so that

$$\begin{aligned} \mathcal{N}[y^n; \mathcal{C}_1^\gamma([0, 1]; \mathcal{B}_p)] &\leq 3\mathcal{N}[y^n; \mathcal{C}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] \\ &\leq 3\mathcal{N}[y^n; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] + c_{\gamma'} \mathcal{N}[y^n; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] \\ &\leq c_{x,f}^6 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}. \end{aligned}$$

Thus, we are in position to apply Ascoli Theorem and assert the existence of a subsequence  $y^{n_k}$  of  $y^n$  that converges to an element  $y$  in  $\mathcal{C}_1^0([0, 1]; \mathcal{B}_p)$ . It remains to check that  $y$  is a solution of (21). To do so, let  $s < t \in [0, 1]$  and consider two sequences  $s_{n_k} < t_{n_k} \in \Pi^{n_k}$  such that  $s_{n_k}$  decreases to  $s$  and  $t_{n_k}$  increases to  $t$ . Then

$$\|J_{ts}^y\|_{\mathcal{B}_p} \leq \|J_{ts}^y - J_{ts}^{y^{n_k}}\|_{\mathcal{B}_p} + \|J_{ts}^{y^{n_k}} - J_{t_{n_k} s_{n_k}}^{y^{n_k}}\|_{\mathcal{B}_p} + \|J_{t_{n_k} s_{n_k}}^{y^{n_k}}\|_{\mathcal{B}_p}. \quad (42)$$



On the one hand,

$$\|J_{ts}^y - J_{ts}^{y^{n_k}}\|_{\mathcal{B}_p} \leq c_{x,f} \mathcal{N}[y - y^{n_k}; \mathcal{C}_1^0([0, 1]; \mathcal{B}_p)] \rightarrow 0,$$

while on the other hand

$$\begin{aligned} \|J_{ts}^{y^{n_k}} - J_{t_{n_k} s_{n_k}}^{y^{n_k}}\|_{\mathcal{B}_p} &\leq c_f \left\{ \|X_{ts}^{x,i} - X_{t_{n_k} s_{n_k}}^{x,i}\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} + \|X_{ts}^{xx,ij} - X_{t_{n_k} s_{n_k}}^{xx,ij}\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} \right\} \\ &\quad + c_{x,f} \left\{ \|y_t^{n_k} - y_{t_{n_k}}^{n_k}\|_{\mathcal{B}_p} + \|y_{s_{n_k}}^{n_k} - y_s^{n_k}\|_{\mathcal{B}_p} \right\}, \end{aligned}$$

from which we easily deduce, with the uniform controls (39) and (41) in mind,

$$\|J_{ts}^{y^{n_k}} - J_{t_{n_k} s_{n_k}}^{y^{n_k}}\|_{\mathcal{B}_p} \rightarrow 0.$$

Finally, owing to (40),

$$\|J_{t_{n_k} s_{n_k}}^{y^{n_k}}\|_{\mathcal{B}_p} \leq c_{x,f}^7 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} |t - s|^\mu.$$

Going back to (42), this proves that  $J^y \in \mathcal{C}_3^\mu([0, 1]; \mathcal{B}_p)$ . Then we follow the same lines starting with the estimate  $\mathcal{N}[J^n; \mathcal{C}_3^\varepsilon([0, 1]_n; \mathcal{B}_{\gamma',p})] \leq c_{x,f}^4 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}$  to get  $J^y \in \mathcal{C}_3^\varepsilon([0, 1]; \mathcal{B}_{\gamma',p})$ , so that  $y$  is indeed a solution of (21) in  $\mathcal{B}_{\gamma',p}$ .  $\square$

#### 4. UNIQUENESS OF THE SOLUTION

In this section, we mean to prove Theorem 2.11. As a consequence, we assume that  $p > n$  and that Conditions (A1), (A2),  $(X)_\gamma$  and  $(F)_3$  are checked. Let  $y$  a solution of (21) in  $\mathcal{B}_{\gamma',p}$ , for some (fixed) parameter  $\gamma' \in (1 - \gamma, 1/2 + \gamma)$ , with initial condition  $\psi \in \mathcal{B}_{\gamma',p}$ , and  $y^n$  the process described by the scheme (26), with the same initial condition  $\psi$ .

We introduce, for all  $s < t \in \Pi^n$ , the quantity

$$\begin{aligned} \mathcal{N}[y - y^n; \mathcal{Q}([s, t]_n)] &:= \\ \mathcal{N}[y - y^n; \hat{\mathcal{C}}_1^\gamma([s, t]_n; \mathcal{B}_p)] &+ \mathcal{N}[y - y^n; \mathcal{C}_1^0([s, t]_n; \mathcal{B}_{\gamma',p})] + \mathcal{N}[K^y - K^{y^n}; \mathcal{C}_2^{2\gamma}([s, t]_n; \mathcal{B}_p)]. \end{aligned}$$

Besides, we fix  $\mu > 1$ ,  $\varepsilon > 0$  such that  $\|J_{ts}^y\|_{\mathcal{B}_p} \leq c |t - s|^\mu$  and  $\|J_{ts}^y\|_{\mathcal{B}_{\gamma',p}} \leq c |t - s|^\varepsilon$ .

The proof of Theorem 2.11 is based on the two following preliminary results, which aim at controlling, as in the previous section, the residual terms  $J$ :

**Lemma 4.1.** *For all  $\tilde{\mu} > 1$  and  $\kappa > 0$ , there exists two constants  $c_y, c_{\tilde{\mu}}$  such that if  $s < t \in \Pi^n$ ,*

$$\begin{aligned} \|J_{ts}^y - J_{ts}^{y^n}\|_{\mathcal{B}_{\gamma',p}} &\leq c_y \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\} + c_{\tilde{\mu}} \left\{ |t - s|^\kappa + |t - s|^{\tilde{\mu}-\gamma'} \right\} \\ &\quad \left\{ \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^\kappa([s, t]_n; \mathcal{B}_{\gamma',p})] + \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^{\tilde{\mu}}([s, t]_n; \mathcal{B}_p)] \right\}. \end{aligned}$$

$$\|J_{ts}^y - J_{ts}^{y^n}\|_{\mathcal{B}_p} \leq \frac{c_y |t - s|}{(2^n)^{\mu-1}} + c_{\tilde{\mu}} |t - s|^{\tilde{\mu}} \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^{\tilde{\mu}}([s, t]_n; \mathcal{B}_p)].$$

*Proof.* Going back to the notations of Subsection 3.1, we decompose  $J_{ts}^y - J_{ts}^{y^n}$  as

$$J_{ts}^y - J_{ts}^{y^n} = \left[ J_{ts}^{y, \Pi^n} - J_{ts}^{y^n, \Pi^n} \right] + R_{ts}^{y, \Pi^n},$$

with, if  $s = t_k^n$  and  $t = t_l^n$ ,

$$R_{ts}^{y, \Pi^n} := J_{tt_{l-1}^n}^y + \sum_{i=k}^{l-2} S_{tt_{i+1}^n} J_{t_{i+1}^n t_i^n}^y.$$

To handle the term into brackets, we use the same arguments as in the proof of Lemma 3.2, which yield here

$$\begin{aligned} \|J_{ts}^{y, \Pi^n} - J_{ts}^{y^n, \Pi^n}\|_{\mathcal{B}_{\gamma', p}} &\leq c_{\tilde{\mu}, \gamma'} \left\{ |t - s|^\kappa + |t - s|^{\tilde{\mu} - \gamma'} \right\} \\ &\quad \left\{ \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^\kappa(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma', p})] + \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] \right\} \end{aligned}$$

and

$$\|J_{ts}^{y, \Pi^n} - J_{ts}^{y^n, \Pi^n}\|_{\mathcal{B}_p} \leq c_{\mu, \gamma'} |t - s|^{\tilde{\mu}} \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)].$$

Then it suffices to observe that

$$\|R_{ts}^{y, \Pi^n}\|_{\mathcal{B}_p} \leq \frac{c_y}{(2^n)^{\mu-1}} \left\{ |t - t_{l-1}^n| + \sum_{i=k}^{l-2} |t_{i+1}^n - t_i^n| \right\} \leq \frac{c_y |t - s|}{(2^n)^{\mu-1}},$$

$$\text{and } \|R_{ts}^{y, \Pi^n}\|_{\mathcal{B}_{\gamma', p}} \leq \frac{c_y}{(2^n)^\varepsilon} + \sum_{i=k}^{l-2} |t - t_{i+1}^n|^{-\gamma'} \frac{c_y}{(2^n)^\mu} \leq c_{y, \gamma'} \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\}. \quad (43)$$

□

**Lemma 4.2.** *Set  $\tilde{\mu} := \inf(\gamma + \gamma', 3\gamma)$ . Then for every  $s < t \in \Pi^n$ ,*

$$\mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] \leq c_{y, x, f, \psi} \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket s, t \rrbracket_n)], \quad (44)$$

$$\mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^\gamma(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma', p})] \leq c_{y, x, f, \psi} \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket s, t \rrbracket_n)]. \quad (45)$$

*Proof.* (44) is a consequence of the decomposition (30). Indeed, one has for instance, if  $\mathcal{N}_y := \mathcal{N}[y; \hat{\mathcal{C}}_1^\gamma([0, 1]; \mathcal{B}_p)] + \mathcal{N}[y; \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\gamma', p})]$ ,

$$\begin{aligned} &\|f'_i(y_s + r(\delta y)_{us}) - f'_i(y_s) - f'_i(y_s^n + r(\delta y^n)_{us}) + f'_i(y_s^n)\|_{\mathcal{B}_p} \\ &\leq \|r \int_0^1 dr' [f''_i(y_s + rr'(\delta y)_{us}) - f''_i(y_s^n + rr'(\delta y^n)_{us})] (\delta y)_{us}\|_{\mathcal{B}_p} \\ &\quad + \|r \int_0^1 dr' f''_i(y_s^n + rr'(\delta y^n)_{us}) \delta(y - y^n)_{us}\|_{\mathcal{B}_p} \\ &\leq c_f \mathcal{N}_y |u - s|^\gamma \int_0^1 dr' \|(y_s + rr'(\delta y)_{us}) - (y_s^n + rr'(\delta y^n)_{us})\|_{\mathcal{B}_\infty} \\ &\quad + c_f |u - s|^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(I)] \\ &\leq c_{f, \mathcal{N}_y} |u - s|^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(I)], \end{aligned}$$

where we have used the continuous inclusion  $\mathcal{B}_{\gamma',p} \subset \mathcal{B}_\infty$ . As for (45), it suffices to observe, with the expression (29) in mind, that one has for instance

$$\begin{aligned}
& \|X_{tu}^{x,i}(f_i(y_u) - f_i(y_u^n))\|_{\mathcal{B}_{\gamma',p}} \\
& \leq c_x |t - s|^\gamma \|f_i(y_u) - f_i(y_u^n)\|_{\mathcal{B}_{\gamma',p}} \\
& \leq c_x |t - s|^\gamma \left\| \int_0^1 dr f'_i(y_u + r(y_u^n - y_u))(y_u^n - y_u) \right\|_{\mathcal{B}_{\gamma',p}} \\
& \leq c_x |t - s|^\gamma \|y_u^n - y_u\|_{\mathcal{B}_{\gamma',p}} \left\| \int_0^1 dr f'_i(y_u + r(y_u^n - y_u)) \right\|_{\mathcal{B}_{\gamma',p}} \\
& \leq c_{x,f,\mathcal{N}_y,\mathcal{N}_{y^n}} |t - s|^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(I)] \\
& \leq c_{x,f,\mathcal{N}_y,\psi} |t - s|^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(I)], \tag{46}
\end{aligned}$$

where, to get the last estimate, we have appealed to the uniform control  $\mathcal{N}_{y^n} \leq c_{x,f,\psi}$  established in the proof of Theorem 2.10.  $\square$

*Proof of Theorem 2.11.* Let  $T_1 \leq 1 \in \Pi^n$ . Write

$$\hat{\delta}(y - y^n)_{ts} = X_{ts}^{x,i} [f_i(y_s) - f_i(y_s^n)] + X_{ts}^{xx,ij} [F_{ij}(y_s) - F_{ij}(y_s^n)] + [J_{ts}^y - J_{ts}^{y^n}],$$

and use the two previous lemmas to deduce first

$$\mathcal{N}[y - y^n; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, T_1 \rrbracket_n; \mathcal{B}_p)] \leq c_{y,x,f,\psi} T_1^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] + \frac{c_y}{(2^n)^{\mu-1}}$$

and

$$\begin{aligned}
\mathcal{N}[y - y^n; \mathcal{C}_1^0(\llbracket 0, T_1 \rrbracket_n; \mathcal{B}_{\gamma',p})] & \leq c_{y,x,f,\psi} T_1^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] \\
& \quad + c_y \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\}.
\end{aligned}$$

Then, since  $K_{ts}^y - K_{ts}^{y^n} = X_{ts}^{xx,ij} [F_{ij}(y_s) - F_{ij}(y_s^n)] + [J_{ts}^y - J_{ts}^{y^n}]$ ,

$$\mathcal{N}[K^y - K^{y^n}; \mathcal{C}_2^{2\gamma}(\llbracket 0, T_1 \rrbracket_n; \mathcal{B}_p)] \leq c_{y,x,f,\psi} T_1^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] + \frac{c_y}{(2^n)^{\mu-1}}$$

and we have thus proved that

$$\mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] \leq c_{y,x,f,\psi}^1 T_1^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] + c_y^1 \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\}.$$

Choose  $T_1$  such that  $c_{y,x,f,\psi}^1 T_1^\gamma = \frac{1}{2}$  to get

$$\mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] \leq 2c_y^1 \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\}.$$

By using the same arguments on  $\llbracket kT_1, (k+1)T_1 \rrbracket_n$ , we get

$$\mathcal{N}[y - y^n; \mathcal{Q}(\llbracket kT_1, (k+1)T_1 \rrbracket_n)] \leq 2c_y^1 \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\} + c_{x,f} \|y_{kT_1} - y_{kT_1}^n\|_{\mathcal{B}_{\gamma',p}},$$

and it is now easy to establish that

$$\mathcal{N}[y - y^n; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] + \mathcal{N}[y - y^n; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{y,x,f,\psi} \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\}. \tag{47}$$

This inequality clearly proves the uniqueness of the solution and therefore enables to identify  $y$  with the solution constructed in Section 3. This identification allows in turn to choose  $\mu$  and  $\varepsilon$  as in Proposition 3.4 and to assert that  $\mathcal{N}_y \leq c_{x,f,\psi}$ , which completes the proof of the result.  $\square$

## 5. CONTINUITY OF THE SOLUTION

It remains to prove Theorem 2.12. Following the statement of this result, we suppose that  $p > n$  and that Assumptions (A1), (A2), (X) $_\gamma$  and (F) $_3$  are satisfied. We fix  $\gamma' \in (1 - \gamma, \gamma + 1/2)$  and the two initial conditions  $\psi, \tilde{\psi} \in \mathcal{B}_{\gamma',p}$ . We denote by  $X = (X^x, X^{ax}, X^{xx})$  (resp.  $\tilde{X} = (\tilde{X}^x, \tilde{X}^{ax}, \tilde{X}^{xx})$ ) the path constructed from  $(x, \mathbf{x}^2)$  (resp.  $(\tilde{x}, \tilde{\mathbf{x}}^2)$ ) through Definition 2.2. With this notation, we define  $y^n$  as the process described by the scheme (26) and  $\tilde{y}^n$  as the process obtained by replacing  $(\psi, X^x, X^{xx})$  with  $(\tilde{\psi}, \tilde{X}^x, \tilde{X}^{xx})$  in the latter scheme.

Besides, we define  $\tilde{J}$  and  $\tilde{K}$  by replacing  $(X^x, X^{xx})$  with  $(\tilde{X}^x, \tilde{X}^{xx})$  in Formulas (14) and (15). For sake of clarity, we also set  $J^n := J^{y^n}$ ,  $K^n := K^{y^n}$ ,  $\tilde{J}^n := \tilde{J}^{\tilde{y}^n}$ ,  $\tilde{K}^n = \tilde{K}^{\tilde{y}^n}$ , and as in the previous section, we introduce the intermediate quantity

$$\begin{aligned} & \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket s, t \rrbracket_n)] \\ & := \mathcal{N}[y^n - \tilde{y}^n; \hat{\mathcal{C}}_1^\gamma(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] + \mathcal{N}[y^n - \tilde{y}^n; \mathcal{C}_1^0(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma',p})] + \mathcal{N}[K^n - \tilde{K}^n; \mathcal{C}_2^{2\gamma}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)]. \end{aligned}$$

Remember that owing to the results of Section 3, we can rely on the uniform control

$$\mathcal{N}[y^n; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] + \mathcal{N}[y^n; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] + \mathcal{N}[K^n; \mathcal{C}_2^{2\gamma}(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] \leq c_{x,\psi},$$

with an equivalent result for  $\tilde{y}^n$ . The proof of Theorem 2.12 now leans on the two following lemmas:

**Lemma 5.1.** *For all  $\tilde{\mu} > 1$  and  $\kappa > 0$ , there exists a constant  $c = c_{\tilde{\mu},\kappa}$  such that if  $s < t \in \Pi^n$ ,*

$$\begin{aligned} \|J_{ts}^n - \tilde{J}_{ts}^n\|_{\mathcal{B}_{\gamma',p}} & \leq c \left\{ |t - s|^\kappa + |t - s|^{\tilde{\mu} - \gamma'} \right\} \\ & \quad \left\{ \mathcal{N}[\hat{\delta}(J^n - \tilde{J}^n); \mathcal{C}_3^\kappa(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma',p})] + \mathcal{N}[\hat{\delta}(J^n - \tilde{J}^n); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] \right\} \end{aligned}$$

and

$$\|J_{ts}^n - \tilde{J}_{ts}^n\|_{\mathcal{B}_p} \leq c |t - s|^{\tilde{\mu}} \mathcal{N}[\hat{\delta}(J^n - \tilde{J}^n); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)].$$

*Proof.* It suffices to follow the lines of the proof of Lemma 3.2.  $\square$

**Lemma 5.2.** *Set  $\tilde{\mu} := \inf(\gamma + \gamma', 3\gamma)$ . Then for any  $s < t \in \Pi^n$ ,*

$$\mathcal{N}[\hat{\delta}(J^n - \tilde{J}^n); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] \leq c_{x,\tilde{x},\psi,\tilde{\psi}} \left\{ \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket s, t \rrbracket_n)] + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\} \quad (48)$$

and

$$\mathcal{N}[\hat{\delta}(J^n - \tilde{J}^n); \mathcal{C}_3^\gamma(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{x,\tilde{x},\psi,\tilde{\psi}} \left\{ \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket s, t \rrbracket_n)] + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\}. \quad (49)$$

*Proof.* This is the same type of arguments as in the proof of Lemma 4.2. For (48), we resort to the decomposition (30) and notice for instance that

$$\begin{aligned}
& \|X_{tu}^{x,i} \left( \int_0^1 dr f'_i(y_s^n + r(\delta y^n)_{us}) \cdot K_{us}^n \right) - \tilde{X}_{tu}^{x,i} \left( \int_0^1 dr f'_i(\tilde{y}_s^n + r(\delta \tilde{y}^n)_{us}) \cdot \tilde{K}_{us}^n \right)\|_{\mathcal{B}_p} \\
& \leq c \|X_{tu}^{x,i} - \tilde{X}_{tu}^{x,i}\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} \|K_{us}^n\|_{\mathcal{B}_p} + \|\tilde{X}_{tu}^{x,i}\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} \\
& \quad \left\| \int_0^1 dr f'_i(y_s^n + r(\delta y^n)_{us}) \cdot K_{us}^n - \int_0^1 dr f'_i(\tilde{y}_s^n + r(\delta \tilde{y}^n)_{us}) \cdot \tilde{K}_{us}^n \right\|_{\mathcal{B}_p} \\
& \leq c_{x, \tilde{x}, \psi} |t-s|^{3\gamma} \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma + c_{\tilde{x}} |t-u|^\gamma \\
& \quad \left\{ \left\| \int_0^1 dr [f'_i(y_s^n + r(\delta y^n)_{us}) - f'_i(\tilde{y}_s^n + r(\delta \tilde{y}^n)_{us})] \cdot K_{us}^n \right\|_{\mathcal{B}_p} \right. \\
& \quad \left. + \left\| \int_0^1 dr f'_i(\tilde{y}_s^n + r(\delta \tilde{y}^n)_{us}) \cdot [K_{us}^n - \tilde{K}_{us}^n] \right\|_{\mathcal{B}_p} \right\} \\
& \leq c_{x, \tilde{x}, \psi}^1 |t-s|^{3\gamma} \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma + c_{x, \tilde{x}, \psi}^2 |t-s|^{3\gamma} \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket s, t \rrbracket_n)],
\end{aligned}$$

where we have used the continuous inclusion  $\mathcal{B}_{\gamma', p} \subset \mathcal{B}_\infty$ . (49) can be proved likewise, with the same kind of estimates as in the proof (46).  $\square$

*Proof of Theorem 2.12.* Following the same procedure as in the proof of Theorem 2.11, we first deduce

$$\begin{aligned}
& \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket 0, T_2 \rrbracket_n)] \\
& \leq c_{x, \tilde{x}, \psi, \tilde{\psi}}^1 \left\{ T_2^\gamma \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket 0, T_2 \rrbracket_n)] + \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\}.
\end{aligned}$$

Indeed, one has for instance, if  $0 \leq s < t \leq T_2$ ,

$$\begin{aligned}
\|X_{ts}^{x,i} [f_i(y_s^n) - f_i(\tilde{y}_s^n)]\|_{\mathcal{B}_p} & \leq c_x |t-s|^\gamma \|y_s^n - \tilde{y}_s^n\|_{\mathcal{B}_p} \\
& \leq c_x |t-s|^\gamma \left\{ \|\hat{\delta}(y^n - \tilde{y}^n)_{s0}\|_{\mathcal{B}_p} + \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} \right\} \\
& \leq c_x |t-s|^\gamma \left\{ T_2^\gamma \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket 0, T_2 \rrbracket_n)] + \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} \right\}.
\end{aligned}$$

Then we take of course  $T_2$  such that  $c_{x, \tilde{x}, \psi, \tilde{\psi}}^1 T_2^\gamma = \frac{1}{2}$  so as to retrieve

$$\mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket 0, T_2 \rrbracket_n)] \leq 2 c_{x, \tilde{x}, \psi, \tilde{\psi}}^1 \left\{ \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\}.$$

Repeating the procedure on  $[T_2, 2T_2]$ ,  $[2T_2, 3T_2]$ , ..., leads to the uniform control

$$\begin{aligned}
& \mathcal{N}[y^n - \tilde{y}^n; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] + \mathcal{N}[y^n - \tilde{y}^n; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma', p})] \\
& \leq c_{x, \tilde{x}, \psi, \tilde{\psi}} \left\{ \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\}. \quad (50)
\end{aligned}$$

To conclude with, let us introduce, for all  $s < t \in [0, 1]$ , two sequences  $s_n < t_n \in \Pi^n$  such that  $s_n$  decreases to  $s$  and  $t_n$  increases to  $t$ , and write (for instance) successively

$$\|\hat{\delta}(y - \tilde{y})_{ts}\|_{\mathcal{B}_p} \leq \|\hat{\delta}(y - \tilde{y})_{tt_n}\|_{\mathcal{B}_p} + \|\hat{\delta}(y - \tilde{y})_{t_n s_n}\|_{\mathcal{B}_p} + \|\hat{\delta}(y - \tilde{y})_{s_n s}\|_{\mathcal{B}_p},$$

$$\|\hat{\delta}(y - \tilde{y})_{t_n s_n}\|_{\mathcal{B}_p} \leq \|\hat{\delta}(y - y^n)_{t_n s_n}\|_{\mathcal{B}_p} + \|\hat{\delta}(y^n - \tilde{y}^n)_{t_n s_n}\|_{\mathcal{B}_p} + \|\hat{\delta}(\tilde{y} - \tilde{y}^n)_{t_n s_n}\|_{\mathcal{B}_p}.$$

The control (50), together with the approximation result (47), then provides (24).

□

6. APPENDIX A: A USEFUL ALGORITHM

We give here the description and a brief analysis of the algorithm used in the proofs of Lemmas 3.2, 4.1 and 5.1.

Consider a generic partition  $\{0, 1, 2, \dots, N\}$ . We remove the inner points of this partition ( $\{1, 2, \dots, N - 1\}$ ) one by one according to the following procedure (see Figure 6):

- At step 1, we successively remove, from the right to the left, every two points, starting from  $N$  (excluded) until 0 (also excluded). Then, still at step 1, we take off the point of the (updated) partition between 0 (excluded) and the last removed point, if such a middle point exists.
- We repeat the procedure with the remaining points (steps 2,3,...) until the partition is empty.

We denote by:

- $M$  the number of steps necessary to empty the partition.
- $(k_m)_{m \in \{1, \dots, N-1\}}$  the sequence of successively removed points.
- $k_m^+$  the point of the partition (at 'time'  $m$  of the algorithm) that follows  $k_m$  (when reading from the left to the right),  $k_m^-$  the point that precedes it.
- $A_r$  the total number of points that have been taken off at the end of step  $r$ . We also set  $A_0 := 0$ .

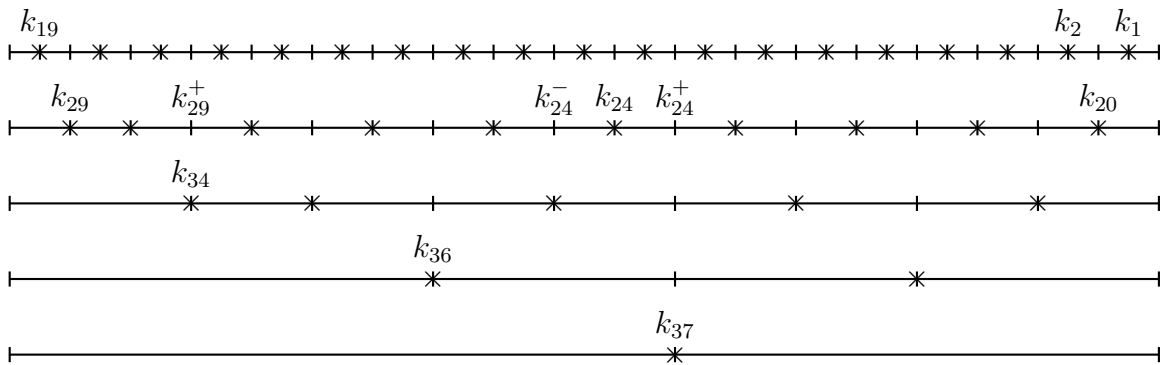


FIGURE 1. The algorithm for  $N = 38$ . Each line corresponds to one step. Thus,  $M = 5$ ,  $A_1 = 19$ ,  $A_2 = 29$ ,  $A_3 = 34$ ,  $A_4 = 36$ .

**Lemma 6.1.** For every  $r \in \{0, 1, \dots, M\}$ ,

$$0 \leq A_r - N \left(1 - \frac{1}{2^r}\right) \leq 1.$$

In particular,  $|A_r - A_{r-1} - \frac{N}{2^r}| \leq 1$  et  $2^{M-1} \leq N \leq 2^{M+1}$ .

*Proof.* This stems from a straightforward iteration procedure based on the formula  $A_{r+1} = A_r + \lfloor \frac{N-A_r+1}{2} \rfloor$ ,  $r \in \{0, 1, \dots, M-1\}$ , where  $\lfloor \cdot \rfloor$  stands for the integer part.  $\square$

**Proposition 6.2.** Let  $\mu > 1$ ,  $0 < \gamma' < 1$  and  $\kappa > 0$ . Then

$$\sum_{r=1}^{M-1} \left\{ \left| 1 - \frac{k_{A_{r-1}+1}^-}{N} \right|^\kappa + \frac{1}{N^\mu} \sum_{m=A_{r-1}+2}^{A_r} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} |k_m^+ - k_m^-|^\mu \right\} \leq c_{\kappa, \mu, \gamma'}, \quad (51)$$

for some finite constant  $c_{\kappa, \mu, \gamma'}$  independent of  $N$ .

*Proof.* We actually use the following explicit expressions: at step  $r$  ( $r \in \{1, \dots, M-1\}$ ), if  $N - A_{r-1}$  is even, one has, for every  $m \in \llbracket A_{r-1} + 1, A_r \rrbracket$ ,

$$k_m^+ = N - 2^r(m - A_{r-1}) + 2^r, \quad (52)$$

$$k_m^- = N - 2^r(m - A_{r-1}), \quad (53)$$

while if  $N - A_{r-1}$  is odd, Formulas (52) and (53) remain true for  $m \in \llbracket A_{r-1} + 1, A_r - 1 \rrbracket$ , but  $k_{A_r}^- = 0$  and  $k_{A_r}^+ = k_{A_{r-1}}^+ = N - 2^r(A_r - A_{r-1} - 1) + 2^r$ . From these expressions, we first deduce

$$\sum_{r=1}^{M-1} \left| 1 - \frac{k_{A_{r-1}+1}^-}{N} \right|^\kappa = \frac{1}{N^\kappa} \sum_{r=1}^{M-1} (2^r)^\kappa \leq c_\kappa^1 \left( \frac{2^M}{N} \right)^\kappa \leq c_\kappa^2,$$

according to Lemma 6.1. Then, if  $N - A_{r-1}$  is even, one has

$$\begin{aligned} \sum_{m=A_{r-1}+2}^{A_r} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} |k_m^+ - k_m^-|^\mu &= \frac{(2^r)^{\mu-\gamma'}}{N^{-\gamma'}} \sum_{m=1}^{A_r - A_{r-1} - 1} m^{-\gamma'} \\ &\leq c_{\gamma'}^3 \frac{(2^r)^{\mu-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 1)^{1-\gamma'}. \end{aligned}$$

Besides, if  $N - A_{r-1}$  is odd,

$$\begin{aligned} &\sum_{m=A_{r-1}+2}^{A_r} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} |k_m^+ - k_m^-|^\mu \\ &\leq c_{\gamma'}^3 \frac{(2^r)^{\mu-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 1)^{1-\gamma'} + \left| 1 - \frac{k_{A_r}^+}{N} \right|^{-\gamma'} (k_{A_r}^+)^\mu \\ &\leq c_{\gamma'}^3 \frac{(2^r)^{\mu-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 1)^{1-\gamma'} \\ &\quad + \frac{(2^r)^{-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 2)^{-\gamma'} (N - 2^r(A_r - A_{r-1} - 2))^\mu \\ &\leq c_{\gamma'}^3 \frac{(2^r)^{\mu-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 1)^{1-\gamma'} + \frac{(2^r)^{-\gamma'}}{N^{-\gamma'}} (N - 2^r(A_r - A_{r-1} - 2))^\mu. \end{aligned}$$

since, in that case,  $A_r - A_{r-1} \geq 3$ . Thanks to Lemma 6.1, we now easily deduce

$$\frac{1}{N^\mu} \sum_{r=1}^{M-1} \sum_{m=A_{r-1}+2}^{A_r} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} |k_m^+ - k_m^-|^\mu \leq \frac{C_{\gamma'}^3}{N^{\mu-1}} \sum_{r=1}^{M-1} (2^r)^{\mu-1} + \frac{C_\mu^A}{N^{\mu-\gamma'}} \sum_{r=1}^{M-1} (2^r)^{\mu-\gamma'} \leq c_{\mu, \gamma'}.$$

□

## 7. APPENDIX B

This section is devoted to the proof of Proposition 2.9. To this end, we will resort to the two following lemmas, respectively borrowed from [13] and [1]:

**Lemma 7.1.** *Fix a time  $T > 0$ . For every  $\alpha, \beta \geq 0$ ,  $p, q \geq 1$ , there exists a constant  $c$  such that for any  $R \in \mathcal{C}_2([0, T]; \mathcal{B}_{\alpha, p})$ ,*

$$\mathcal{N}[R; \mathcal{C}_2^\beta([0, T]; \mathcal{B}_{\alpha, p})] \leq c \left\{ U_{\beta+\frac{2}{q}, \alpha, p}(R) + \mathcal{N}[\hat{\delta}R; \mathcal{C}_3^\beta([0, T]; \mathcal{B}_{\alpha, p})] \right\},$$

where

$$U_{\beta, q, \alpha, p}(R) = \left[ \int_{0 \leq u < v \leq T} \left( \frac{\|R_{vu}\|_{\mathcal{B}_{\alpha, p}}}{|v-u|^\beta} \right)^q dudv \right]^{1/q}.$$

**Lemma 7.2.** *For every  $p \geq 2$ , the Burkholder-Davies-Gundy inequality holds in  $\mathcal{B}_p$ . In other words, for any  $T > 0$ , if  $B$  is a one-dimensional Brownian motion and  $H$  is an adapted process with values in  $L^2([0, T]; \mathcal{B}_p)$ , then for any  $q \geq 2$ , there exists a constant  $c$  independent of  $H$  such that*

$$E \left[ \sup_{0 \leq t \leq T} \left\| \int_0^t H_u dB_u \right\|_{\mathcal{B}_p}^q \right] \leq c E \left[ \left( \int_0^T \|H_u\|_{\mathcal{B}_p}^2 du \right)^{q/2} \right]. \quad (54)$$

*Proof of Proposition 2.9.* On the whole, this is the same identification procedure as in the proof of Proposition 2.8. The only difference lies in the fact that the direct estimates of the integrals at stake (as in (22)) are here replaced with a joint use of Lemmas 7.1 and 7.2.

We denote by  $y$  the (Itô) solution of (21), with initial condition  $\psi \in \mathcal{B}_{\eta, p}$ . Let us fix  $\gamma \in (1/3, 1/2)$  such that  $\gamma + \eta > 1$  and  $2\gamma > \eta$ . If one refers to [14] (for instance), one can assert that  $y \in \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\eta, p})$  a.s, and one even knows that  $\sup_{t \in [0, 1]} E \left[ \|y_t\|_{\mathcal{B}_{\eta, p}}^q \right] < \infty$  for any  $q \in \mathbb{N}$ . Then, since  $(\hat{\delta}y)_{ts} = \int_s^t S_{tu} dx_u^i f_i(y_u)$ , one has, thanks to Lemma 7.2,

$$\begin{aligned} E \left[ \left\| \int_s^t S_{tu} dx_u^i f_i(y_u) \right\|_{\mathcal{B}_p}^q \right] &\leq c E \left[ \left( \int_s^t \|S_{tu} f_i(y_u)\|_{\mathcal{B}_p}^2 du \right)^{q/2} \right] \\ &\leq c |t-s|^{q/2-1} \int_s^t E \left[ \|S_{tu} f_i(y_u)\|_{\mathcal{B}_p}^q \right] du \\ &\leq c |t-s|^{q/2}, \end{aligned} \quad (55)$$



and consequently, with the notations of Lemma 7.1,

$$\begin{aligned} E \left[ U_{\gamma+\frac{2}{q}, q, 0, p}(\hat{\delta}y) \right] &\leq \left( \iint_{0 \leq u < v \leq 1} \frac{E \left[ \|(\hat{\delta}y)_{vu}\|_{\mathcal{B}_p}^q \right]}{|v-u|^{\gamma q+2}} dudv \right)^{1/q} \\ &\leq \left( \iint_{0 \leq u < v \leq 1} |v-u|^{q(\frac{1}{2}-\gamma)-2} dudv \right)^{1/q} < \infty \end{aligned}$$

by picking  $q > 1/(\frac{1}{2} - \gamma)$ . Together with the result of Lemma 7.1, this yields  $y \in \hat{\mathcal{C}}_1^\gamma([0, 1]; \mathcal{B}_p)$  a.s.

As far as  $K^y$  is concerned, we already know that  $\hat{\delta}K^y = X^{x,i}\delta(f_i(y))$ , which leads to  $\hat{\delta}K^y \in \mathcal{C}_3^{2\gamma}([0, 1]; \mathcal{B}_p)$  a.s. Then, from the expression  $K_{ts}^y = \int_s^t S_{tu} dx_u^i \delta(f_i(y))_{us}$ , we deduce, as in (55),  $E \left[ \|K_{ts}^y\|_{\mathcal{B}_p}^q \right] \leq c|t-s|^q$ , and accordingly, thanks to Lemma 7.1,  $K^y \in \mathcal{C}_2^{2\gamma}([0, 1]; \mathcal{B}_p)$  a.s.

Finally, for  $J^y$ , we first lean on the decomposition (30) of  $\hat{\delta}J^y$  to assert that  $\hat{\delta}J^y \in \mathcal{C}_3^{\gamma+\eta}([0, 1]; \mathcal{B}_p)$  a.s. Then we appeal to the expression of  $J^y$  we have exhibited in the proof of Proposition 2.8, namely  $J_{ts}^y = \int_s^t S_{tu} dx_u^i M_{us}^i$  where  $M^i$  is given by (23), to show that  $E \left[ \|J_{ts}^y\|_{\mathcal{B}_p}^q \right] \leq c|t-s|^{q(\frac{1}{2}+\eta)}$ . Together with Lemma 7.1, those results clearly provide the expected regularity, ie  $J^y \in \mathcal{C}_2^\mu([0, 1]; \mathcal{B}_p)$  a.s, with  $\mu = \gamma + \eta > 1$ .

The control of the regularity of  $J^y$  as a process with values in  $\mathcal{B}_{\eta,p}$  stems from the same reasoning. Indeed, we first deduce from (29) that  $\hat{\delta}J^y \in \mathcal{C}_3^\gamma([0, 1]; \mathcal{B}_{\eta,p})$  a.s, since for instance  $\|X_{tu}^{x,i} f_i(y_u)\|_{\mathcal{B}_{\eta,p}} \leq c_{x,f,y} |t-u|^\gamma$  and

$$\|X_{tu}^{x,i}(\delta x^j)_{us} F_{ij}(y_s)\|_{\mathcal{B}_{\eta,p}} \leq c_x |t-s|^{2\gamma-(\eta-\frac{1}{2})} \|F_{ij}(y_u)\|_{\mathcal{B}_{1/2,p}} \leq c_{x,f,y} |t-s|^\gamma.$$

We can then write  $J^y$  as  $J_{ts}^y = \int_s^t S_{tu} dx_u^i \delta(f_i(y))_{us} - X_{ts}^{xx,ij} F_{ij}(y_s)$  to easily obtain  $E \left[ \|J_{ts}^y\|_{\mathcal{B}_{\eta,p}}^q \right] \leq c_{x,f,y} |t-s|^{q/2}$ , hence  $J^y \in \mathcal{C}_2^\gamma([0, 1]; \mathcal{B}_{\eta,p})$  a.s. □

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(A. Deya) INSTITUT ÉLIE CARTAN NANCY, UNIVERSITÉ DE NANCY, B.P. 239, 54506 VANDŒUVRE-LÈS-NANCY CEDEX, FRANCE

*E-mail address:* deya@iecn.u-nancy.fr