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## CONTROL TECHNIQUES FOR THERMO-ACOUSTIC TOMOGRAPHY

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**ABSTRACT.** Thermo-acoustic and photo-acoustic tomography are imaging techniques that combine high electromagnetic absorption contrast between two media with ultrasound high resolution. Both techniques lead to an ill-posed inverse problem of the same form which entails inversion (in the wide sense) of the spherical Radon transform. One currently has a choice between three main types of reconstruction procedures namely the filtered backprojection formulae, eigenfunction expansion methods and time reversal method. In this paper we propose to investigate this inverse problem with an alternative control formulation : in our model the function to be recovered is the control function while the (acoustic) pressure is the state function which satisfies a wave equation. We stress that our objective is to give modelling hints, and not to provide new results. Otherwise expressed, the originality of this paper lies essentially in the way the problem is stated.

**1. Introduction.** Thermo-acoustic and photo-acoustic tomography are imaging techniques that combine high electromagnetic absorption contrast between two media with ultrasound high resolution. These hybrid systems use an electromagnetic pulse as an input and record ultrasound waves as an output. The electromagnetic energy is distributed at a given time as uniformly as possible through the object. The induced increase of temperature depends on the local absorption properties (for example, cancerous tissues absorb more energy than healthy ones). This opens the way to the detection of heterogeneities via measurements of the pressure field. Heterogeneities behave like internal acoustic sources, and the signals recorded by pressure detectors outside the medium under study provide information on their distribution. One speaks of thermo-acoustic tomography (TAT) when the heating is realized by means of microwaves, and of photo-acoustic tomography (PAT) when optical heating is used. While in TAT waves of radio frequency range are used to trigger the ultrasound signal, in the PAT the frequency lies in the visual or near infra-red ranges. In brief, TAT and PAT are two hybrid techniques using

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electromagnetic waves as an excitation (input) and acoustic waves as an observation (output). For the mathematical purpose there is no distinction between these methods. Both techniques lead to an ill-posed inverse problem of the same form which, under simplifying assumptions, entails inversion (in the wide sense) of the spherical Radon transform.

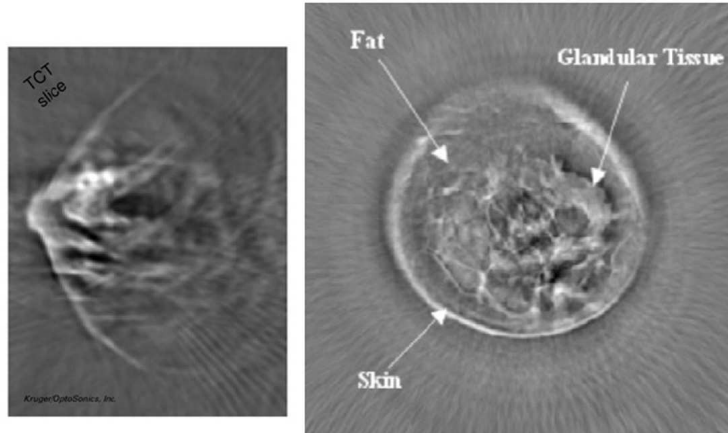


FIGURE 1. Realization of a tomograph with integrating transducers, from Patch and Scherzer [23]

More precisely, the main problem can be formulated as follows [13, 14, 15]: given the sound speed  $c(x)$  and measured data  $p_{obs}$  on  $S \subset \mathbb{R}^n$  ( $n = 2, 3$ ), find the initial value  $u_o(x)$  of the pressure  $p(x, t)$  where  $y$  is solution to the problem

$$\left\{ \begin{array}{l} \left( \frac{\partial^2 p}{\partial t^2} - \Delta p \right) (t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \\ p(0, x) = u_o(x), \quad x \in \mathbb{R}^n, \\ \frac{\partial p}{\partial t}(0, x) = 0, \quad x \in \mathbb{R}^n, \\ p(t, x) = p_{obs}(t, x), \quad x \in S, t \in [0, T]. \end{array} \right.$$

We shall justify this model in the sequel. The initial value  $u_o$  is the TAT (or PAT) image. This problem is highly ill-posed. In most reconstruction methods in TAT, additional assumptions are performed as for example conditions on the support of the function to recover and/or the observation surface, or a constant sound speed. A nice overview of the state of art has been done in [15]. We summarize the introduction (for more details one can refer to the whole paper). One currently has a choice between three main types of reconstruction procedures for closed observation surfaces, namely the filtered backprojection formulae, eigenfunction expansion methods and time reversal method.

The filtered backprojection method is the most popular [7, 9, 10, 13, 16]. However, it is not clear that backprojection-type formulae could be written for any closed observation surface  $S$ . In [10], inversion formulae are provided assuming odd dimensions and constant sound speed. Indeed, in this case the Huygens principle holds : for any initial source with a compact support, the wave leaves any bounded

domain in a finite time. This is not true any longer if the spatial dimension is even and/or the sound speed is not constant any longer. All known formulae of the filtered backprojection type assume constant sound speed and thus are not available for acoustically inhomogeneous media. In addition, the only closed bounded surface  $S$  for which such formulae are known is a sphere.

Expansion series are useful in the case where the Huyghens principle is valid. This approach was extended to the constant speed and arbitrary closed observation surface and modified by the usage of the eigenfunctions of the Laplacian with Dirichlet conditions on  $S$  [3, 25]. It theoretically works for any closed surface and for variable sound speeds.

The time reversal method [14, 15] can be used for approximating the initial pressure when the sound speed inside the object is variable. It works for arbitrary geometry of the closed observation surface  $S$ . The sound speed can be variable.

In this paper we propose to investigate this inverse problem with an alternative formulation. We propose a *control* approach: in our model the function to be recovered is the control function while the pressure is the state function which satisfies a wave equation. We stress that our objective is to give modelling hints, and not to provide new results. Otherwise expressed, the originality of this paper lies essentially in the way the problem is stated.

There are many ways to define the state equation and the cost functional (observation) in the optimal control or controllability approach. We state a few open problems for future investigation, which cover various fields such as controllability theory, optimal control both from the theoretical and numerical viewpoints, the relationship with Fourier analysis and the control approach (Fourier aliasing effects and numerical approximations for the wave equation), shape optimization and so on.

The paper is organized as follows: in the next section, we present the physical problem and the model, together with the *integral* formulation (involving the spherical Radon transform). In Section 3, we propose an optimal control approach, give various models and derive general optimality conditions. In Section 4, we explain why a controllability approach is not easy. We end with remarks on numerical aspects.

## 2. Background.

**2.1. From physics to the TAT model.** We describe here the physical model of TAT (or PAT). Our exposition follows that of [24].

We denote by  $v(x, \tau)$  the fluid velocity at position  $x$  and time  $\tau$ , and make the following physical assumptions:

1.  $v(x, \tau)$  is small;
2. the fluid is non viscous and non turbulent;
3. there is no external force.

We also assume that the density  $\rho = \rho(x, \tau)$  and pressure  $p = p(x, \tau)$  undergo small variations:

$$\rho = \rho_0 + \delta\rho \quad \text{with} \quad |\delta\rho| \ll \rho_0 \quad \text{and} \quad p = p_0 + \delta p \quad \text{with} \quad |\delta p| \ll p_0.$$

This allows for linearizing the *mass conservation* and *momentum conservation equations*, which then read

$$\frac{\partial \rho}{\partial \tau} = -\rho_0 \nabla \cdot v \quad \text{and} \quad \rho_0 \frac{\partial v}{\partial \tau} = -\nabla p,$$

respectively. In the latter equations,  $\rho$  and  $p$  now stand the variations  $\delta\rho$  and  $\delta p$ . Combining these equations yields

$$\frac{\partial^2 \rho}{\partial \tau^2} - \Delta p = 0. \quad (1)$$

In order to link  $\rho$  and  $p$  with the absorbed electromagnetic radiation, one makes use of the *thermal expansion equation*, whose linearized form reads

$$\frac{\partial \rho}{\partial \tau} = \frac{1}{v^2} \frac{\partial p}{\partial \tau} - \frac{\beta}{c_p} r. \quad (2)$$

Here,  $\beta$ ,  $c_p$  and  $v$  denote respectively the thermal expansion coefficient, the specific heat capacity and the adiabatic speed of sound, while  $r = r(x, \tau)$  denotes the absorbed electromagnetic power. From (1) and (2), we easily obtain

$$\left( \frac{1}{v^2} \frac{\partial^2}{\partial \tau^2} - \Delta \right) p = \frac{\beta}{c_p} \frac{\partial r}{\partial \tau}. \quad (3)$$

Now, the absorbed electromagnetic power  $r(x, \tau)$  is related to the *absorption coefficient*  $\psi = \psi(x)$  by the equation  $r(x, \tau) = I(x, \tau)\psi(x)$ , in which  $I(x, \tau)$  is the radiation intensity. Finally, due to the high magnitude of the speed of light, one can assume that  $I(x, \tau)$  takes the separated form  $I(x) = J(x)j(\tau)$ . Equation (3) then reads

$$\left( \frac{1}{v^2} \frac{\partial^2}{\partial \tau^2} - \Delta \right) p = \frac{\beta J \Psi}{c_p} \frac{\partial j}{\partial \tau}.$$

The change of variable  $t = v\tau$  yields the following reformulation:

$$\left( \frac{\partial^2}{\partial t^2} - \Delta \right) p = \frac{\beta I \Psi}{c_p} \frac{\partial j}{\partial t}$$

The function  $f(x) := \beta J(x)\Psi(x)/c_p$  is referred to as the *energy deposition function*. Finally, we can assume that the pressure increment, together with its time derivative, is zero at the initial time (right before illumination). We are then led to consider the following system:

$$(TAT) \left\{ \begin{array}{l} \left( \frac{\partial^2}{\partial t^2} - \Delta \right) p(x, t) = f(x) \frac{\partial j}{\partial t}(t), \\ p(x, 0) = 0, \\ \frac{\partial p}{\partial t}(x, 0) = 0. \end{array} \right.$$

**2.2. Integral formulation.** The *direct problem* consists in determining  $p(x, t)$  of all  $(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+$  from the knowledge of  $f$  and  $j$ . It is well known (see e.g. [13]) that the solution is given by

$$p(x, t) = \left( \frac{dj}{dt} \otimes (tRf) \right) (x, t). \quad (4)$$

Here, the *convolution* operation  $\otimes$  is defined by

$$(g \otimes h)(x, t) = \int_0^t g(t-s)h(x, s) ds$$

and the operator  $R$ , referred to as the *spherical Radon transform*, is defined by

$$(Rf)(x, t) := \frac{1}{4\pi} \int_{S^2} f(x + t\omega) d\omega.$$

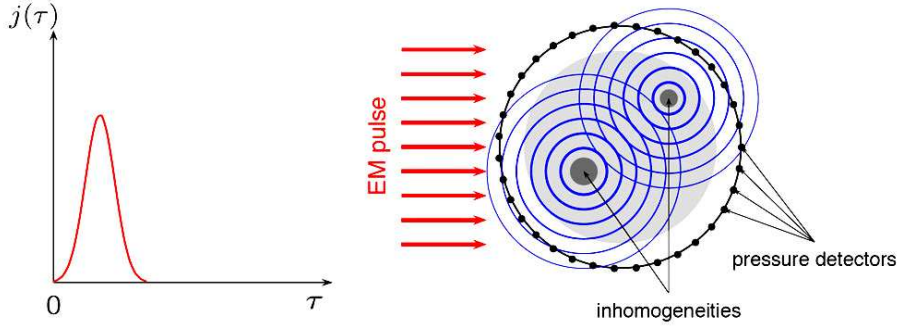


FIGURE 2. Physical principle of Thermo-acoustic Tomography: intensity time profile of the EM pulse (left) and scheme showing the pressure wave generation and propagation (right).

In TAT, one is rather confronted the *inverse problem*:

*Recover the energy deposition function  $f(x)$  from measurements of  $p(x, t)$  for  $x$  running over a surface  $S$  outside the illuminated fluid.*

We now show that additional approximations give rise to an integral formulation of the latter inverse problem, under the assumption that the *intensity profile*  $j(t)$  is nearly a Dirac centered at the origin. Assuming that  $j$  has support in  $[0, T]$  (see Figure 2 (left)), one may write, for every smooth function  $h(x, t)$ ,

$$\begin{aligned}
 \frac{d}{dt}(j \otimes h)(x, t) &= \frac{d}{dt} \int_0^t j(t-s)h(x, s) ds \\
 &= \frac{d}{dt} \int_0^\infty j(t-s)h(x, s) ds \quad (\text{for } t \geq T) \\
 &= \int_0^\infty \frac{dj}{dt}(t-s)h(x, s) ds \\
 &= \left( \frac{dj}{dt} \otimes h \right)(x, t).
 \end{aligned}$$

If  $j \approx \delta$ , then

$$\frac{dj}{dt} \otimes h \approx \frac{dh}{dt},$$

so that

$$\left( \frac{dj}{dt} \otimes (tRf) \right)(x, t) \approx \frac{d}{dt}(tRf).$$

Consequently, Equation (4) can be approximated by

$$p(x, t) = \frac{d}{dt}(tRf). \quad (5)$$

We see that, after integrating and dividing by  $t$ , measurements of  $p(x, t)$  on  $S \times \mathbb{R}_+$  give access to approximate values of  $(Rf)(x, t)$  on the same domain.

**2.3. Regularization by mollification.** The integral formulation of the problem leads to a linear inverse problem of standard form, that is:

*Recover  $f$  from an approximate knowledge of  $Rf$ , where  $R$  is a compact linear operator (modelling the data acquisition process) from a norm space into another.*

In TAT,  $R$  is the spherical Radon transform.

We call *regularization by mollification* a set of reconstruction methodologies deriving from the idea that the original ill-posed problem of reconstructing the unknown object  $f$  should be replaced by that of recovering a smooth *version* of it. In this approach, the target object is no longer  $f$  but  $\varphi * f$ , where  $\varphi$  is a *convolution kernel* and  $*$  denotes the convolution operation.

This idea has been developed in two different ways, independently. One of them, which bears the name of *approximate inverse*, was introduced by Louis and Maaß [20, 21, 22]. In the context of TAT, it was developed by Altmeier, Schuster and Scherzer [13]. The other way goes back to Lannes *et al.* [17], who introduced a particular *variational* regularization in the context of deconvolution and aperture synthesis. This approach was further analyzed and developed more recently by Alibaud *et al.* [1] and Bonnefond *et al.* [5] (see also [2]).

**2.3.1. Approximate inverses.** Given a continuous and nonnegative function  $\phi$  with  $\int \phi = 1$ , one defines the family

$$\phi_\beta(x, y) := \frac{1}{\beta^n} \phi\left(\frac{x - y}{\beta}\right), \quad \beta > 0,$$

called an *approximation of unity*. The mollified version of  $f$ , denoted  $f_\beta$ , can be expressed as the scalar product of  $f$  by  $\phi_\beta(\cdot, y)$ :

$$f_\beta(y) := \int f(x) \phi_\beta(x, y) dx = \langle f, \phi_\beta(\cdot, y) \rangle.$$

Now, it is well-known that, in several senses,

$$\int f(x) \phi_\beta(x, \cdot) dx \rightarrow f \quad \text{as } \beta \downarrow 0.$$

Assuming that  $\phi_\beta \in \text{ran } R^*$ , let  $\psi_\beta$  be defined by

$$\psi_\beta(\cdot, y) = (R^*)^{-1} \phi_\beta(\cdot, y). \tag{6}$$

Then, one has:

$$f_\beta(y) = \langle f, \phi_\beta(\cdot, y) \rangle = \langle f, R^* \psi_\beta(\cdot, y) \rangle = \langle Rf, \psi_\beta(\cdot, y) \rangle.$$

We see that if one can compute explicitly  $\psi_\beta$ ,  $f_\beta$  is readily obtained from the data  $g = Rf$  by taking the scalar product with the so-called *reconstruction kernel*  $\psi_\beta$ . In the case where  $R^*$  is not invertible, one may replace  $(R^*)^{-1}$  in equation (6) by the pseudo-inverse of  $R^*$  (which corresponds to the least-square solution of the equation  $R^* \psi(\cdot, y) = \phi_\beta(\cdot, y)$ ). Clearly, the difficulty in this method lies in general in equation (6) which is an ill-posed problem as well. However, as mentioned above this nice duality trick has been successfully applied in the context of TAT by Haltmeier *et al.* in [13].

2.3.2. *Regularization by mollification.* We assume here that the original object  $f_0$  has support in a bounded domain  $\Omega_0$  in  $\mathbb{R}^d$ , and that the *mollified* object  $\varphi * f_0$  has *most* of its support in  $\Omega \supset \Omega_0$ . Notice that, if  $\varphi$  is chosen to have compact support, the definition of  $\Omega$  is straightforward.

We also assume that  $R$  maps  $L^2(\Omega)$  into some infinite dimensional separable Hilbert space  $G$ , and that  $R$  is compact and injective.

The convolution kernel is regarded as a member of the family

$$\phi_\beta(x) := \frac{1}{\beta^d} \phi\left(\frac{x}{\beta}\right), \quad \text{with } \phi \in L^1(\mathbb{R}^d) \quad \text{and} \quad \int_{\mathbb{R}^d} \phi(x) dx = 1. \quad (7)$$

The latter family is an *approximation of unity*, and the choice of  $\beta$  determines the target resolution. Clearly, in this approach, the relevant regularization parameter is  $\beta$ .

Let  $C_\beta$  denote the convolution by  $\varphi_\beta$ . We now aim at approximating the new *target object*  $C_\beta f = \phi_\beta * f_0$ , which is performed by solving the following optimization problem:

$$(\mathcal{P}_\beta) \quad \left| \begin{array}{l} \text{Minimize} \quad \frac{1}{2} \|\Phi_\beta g - Rf\|_G^2 + \frac{\alpha}{2} \|(I - C_\beta)f\|_{L^2}^2 \\ \text{s.t.} \quad f \in L^2(\Omega), \end{array} \right.$$

in which  $I$  denotes the identity and  $\Phi_\beta: G \rightarrow G$  is itself a solution to:

$$(\mathcal{Q}_\beta) \quad \left| \begin{array}{l} \text{Minimize} \quad \frac{1}{2} \|RC_\beta - XR_{|E}\|_{L(E,G)}^2 \\ \text{s.t.} \quad X \in L(G), \quad X = 0 \text{ on } (\text{ran } R_{|E})^\perp. \end{array} \right.$$

Here,  $E$  is a subspace of  $L^2(\mathbb{R}^d)$ ,  $L(E, G)$  denotes as usual the space of continuous linear mappings from  $E$  to  $G$  and  $L(G) := L(G, G)$ .

The choice of the functional to be minimized in  $(\mathcal{P}_\beta)$  can be explained as follows. Any object  $f$  is the sum of its *low frequency component*  $C_\beta f$  and of its *high frequency component*  $(I - C_\beta)f$ . The functional to be minimized acts on each component as independently as possible: the *fit term* introduces constraints on the low frequency component while the *regularization term* deals with the high frequency component. The complementarity of the *filters*  $\hat{\phi}_\beta$  et  $1 - \hat{\phi}_\beta$  allows for a smooth transition between the experimental and regularization constraints.

We emphasize that, unlike most regularization techniques, the fit term requests adequacy to the *regularized data*  $\Phi_\beta g$ . The rationale for this is that, in some sense,  $\Phi_\beta g$  is optimally consistent with the new target object. Ideally,  $\Phi_\beta$  should be such that

$$\Phi_\beta R = RC_\beta, \quad (8)$$

which may not be feasible. Therefore,  $\Phi_\beta$  is chosen to minimize the discrepancy in Equation (8).

**3. A control formulation.** In this section, the pressure  $p$  will be denoted  $y$ , in order to fit the traditional notation in optimal control.



**3.1. The state equation.** We have seen in the previous section that the equation that describes the behavior of the system is (in  $\mathbb{R}^n$  with  $n = 3$ ):

$$\begin{cases} \left( \frac{\partial^2 y}{\partial t^2} - \Delta y \right) (t, x) = u_o(x) \frac{\partial j}{\partial t}(t), & (t, x) \in [0, T] \times \mathbb{R}^n, \\ y(0, x) = 0, & x \in \mathbb{R}^n, \\ \frac{\partial y}{\partial t}(0, x) = 0, & x \in \mathbb{R}^n. \end{cases} \quad (9)$$

where  $j$  is the electromagnetic pulse intensity profile.

More generally we set  $u(t, x) := u_o(x) \frac{\partial j}{\partial t}(t)$ . The function to be recovered  $u$  is regarded as a *control function* while the pressure  $y$  is regarded as the *state function*. Therefore Equation (9) is the so-called *state equation*.

**Remark 1.** By using Duhamel's principle [7], one may rewrite this state equation as

$$\begin{cases} \left( \frac{\partial^2 y}{\partial t^2} - \Delta y \right) (t, x) = 0 & \text{in } [0, T] \times \mathbb{R}^n, \\ y(0, x) = u_o(x), & \text{in } \mathbb{R}^n, \\ \frac{\partial y}{\partial t}(0, x) = 0 & \text{in } \mathbb{R}^n. \end{cases} \quad (10)$$

So, in this context, the control function can be a distributed one or an initial one. However, this equivalence is only valid in the linear case and we focus on the distributed case.

Recall that, if  $V$  is a norm space endowed with  $\|\cdot\|_V$ , then for every  $p \in [1, \infty]$

$$L^p(0, T; V) := \{y : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \mid y(t, \cdot) \in V \text{ a.e. } t \in (0, T) \\ \text{and } \int_0^T \|y(t, \cdot)\|_V^p dt < +\infty\}.$$

and

$$\mathcal{C}(0, T; V) := \{y : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \mid \forall t \in (0, T) y(t, \cdot) \in V \\ \text{and } t \mapsto \|y(t, \cdot)\|_V \text{ is continuous}\}.$$

The original problem domain is  $\mathbb{R}^n$ . This is consistent with the fact that one usually uses classical Fourier analysis (Radon transform, spherical means,...). The control viewpoint requires the definition of boundary conditions (to infinity if the domain is  $\mathbb{R}^n$ ) to use duality methods and Green formulae. By contrast, the advantage of this approach is that we may choose an open bounded subset  $\mathcal{B}$  as the domain and use finite differences or finite elements to compute the solution. A challenging application is to set a model which allows for using both spectral (Fourier) analysis and finite difference methods.

We recall a regularity and existence abstract result which gives indication about the functional framework to be used:

**Theorem 3.1** ([19] Vol. 1 p 286). *If  $u \in L^2([0, T] \times \mathbb{R}^n)$ , then (9) has a unique solution in  $\{y \in L^2(0, T, H^1(\mathbb{R}^n))\}$ ,  $y_t \in L^2([0, T] \times \mathbb{R}^n)$ . Moreover*

$$y \in \mathcal{C}(0, T; H^1(\mathbb{R}^n)) \text{ and } y_t \in \mathcal{C}(0, T; L^2(\mathbb{R}^n)),$$

*and the solution  $y$  continuously depends on  $u$ .*

In the sequel, we shall avoid using the whole space  $\mathbb{R}^n$ . Therefore, we consider that infinity means *far enough* and that the problem domain can be an open ball  $\mathcal{B} \subset \mathbb{R}^n$  with large radius. Indeed the pressure  $y$  quickly vanishes and we may consider that it is equal to 0 on the boundary  $\partial\mathcal{B}$  of the ball. The ball  $\mathcal{B}$  contains  $\Omega$  as well as the sensors location  $\omega_\varepsilon$  (we shall justify the notation later on). If we impose Dirichlet conditions on the boundary, we get:

$$\left\{ \begin{array}{l} \left( \frac{\partial^2 y}{\partial t^2} - \Delta y \right) (t, x) = u(t, x), \text{ in } ]0, T[ \times \mathcal{B} := Q, \\ y(0, x) = 0, \text{ in } \mathcal{B} \\ \frac{\partial y}{\partial t}(0, x) = 0 \text{ in } \mathcal{B} \\ y = 0 \text{ on } S \end{array} \right. \quad (11)$$

where  $S := ]0, T[ \times \partial\mathcal{B}$  is the lateral boundary of  $Q$ .

**Remark 2.** We may of course consider Neumann boundary conditions or mixed conditions: this requires additional investigation on the physics. We may also consider  $\mathbb{R}^n$  instead of the ball: the techniques are quite similar.

We have a generic classical result (see [8] p. 632 for example) which can be sharpened of course.

**Theorem 3.2.** *Assume that  $u \in L^2(0, T; H^{-1}(\mathcal{B}))$ . Then, Equation (11) has a unique solution  $y[u] \in L^2(0, T; H_o^1(\mathcal{B})) \cap C^0([0, T]; L^2(\mathcal{B}))$ .*

In the sequel we denote

$$W := L^2(0, T; H_o^1(\mathcal{B})).$$

**Open problem 1.** *We have set  $u(t, x) := u_o(x) \frac{\partial j}{\partial t}(t)$  where  $j$  is a smooth approximation of the Dirac measure. This gives the natural regularity of the control function. However, the following analysis can be performed using less smooth control functions.*

**3.2. The control problem.** Following Scherzer [13] we assume that  $u$  has a compact support contained in  $[0, T] \times \Omega$ , where  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$  containing the support of the object to be recovered. The pressure  $y$  is measured (observed) on  $\Sigma = [0, T] \times \Gamma$  where  $\Gamma \subset \partial\Omega$  is contained (most of time strictly) in  $\partial\Omega$ . The inverse problem which consists in recovering  $u$  from measurements of  $y$  on  $\Sigma$  can be formulated as the following minimization problem:

$$\left| \begin{array}{l} \text{Minimize } \int_{\Sigma} (y(u) - y_d)^2 dt d\gamma \\ \text{s.t. } u \in \mathcal{U}, \end{array} \right.$$

where  $y_d$  is the measured pressure and  $\mathcal{U}$  is the admissible control function space. Many difficulties then arise:

- The problem is known to be ill-posed: we cannot even ensure the existence of a solution. One must add a regularization term for  $u$ . A standard way to perform the regularization consists in adding to the functional a Tychonov term:

$$\alpha \int_Q u^2 dx dt \quad \text{with } \alpha > 0.$$

**Open problem 2.** *A crucial point is to investigate the physical model and the qualitative properties of the control in order to choose an appropriate regularization term: we may choose the solution with minimal energy, but it may not be the best choice.*

**Open problem 3.** *A natural question is that of describing the asymptotic behavior of the solution as the regularization parameter  $\alpha$  goes to 0 (this is well-known for Tychonov regularization process for example).*

- In order to define the observation term in the cost functional, the trace of  $y[u]$  on  $\Sigma$  has to belong to  $L^2(\Sigma)$ . This depends on the regularity of  $u$ . However, in the case of point detectors, the observation term would take the form

$$\sum_{i=1}^N \int_{[0,T]} (y[u] - y_d)^2(t, x_i) dt = \sum_{i=1}^N \int_{[0,T] \times \mathbb{R}^n} (y[u] - y_d)^2(t, x) dt d\delta_{x_i}(x),$$

in which  $d\delta_{x_i}$  denotes the Dirac measure at  $x_i$ . Notice that one may *relax* the above functional and write instead

$$\sum_{i=1}^N \int_{[0,T] \times \mathbb{R}^n} (y[u] - y_d)^2(t, x) dt d\mu_i(x),$$

in which  $d\mu_i$  is a relaxed version of  $d\delta_{x_i}$ , *e.g.* a gaussian measure centered at  $x_i$ . Clearly, the *width* of each  $\mu_i$  should be less than, say, one half of the expected wavelength of the pressure wave, so that the requested fit is physically acceptable.

**Open problem 4.** *Study the asymptotic behavior of the solution as the fit measures  $\mu_i$  approach the Dirac measures  $\delta_{x_i}$ .*

- Even if the cost functional makes sense, using duality methods and integrating by parts is not possible since  $\Gamma$  is not a closed surface (roughly speaking, there is no *inside* and *outside*).

To overcome the last difficulty we regard  $\Gamma$  as an open subset of  $\omega_\varepsilon$ , with

$$\lim_{\varepsilon \rightarrow 0} \omega_\varepsilon = \Gamma.$$

This convergence must be clarified of course (using tools from shape optimization, for example). A simple way to perform this approximation is to set

$$\omega_\varepsilon = \bigcup_{x \in \Gamma} \mathcal{B}(x, \varepsilon), \tag{12}$$

where  $\mathcal{B}(x, \varepsilon)$  is the open ball of radius  $\varepsilon$  centered at  $x$  (see Figure 3).

In the case where there is a finite number of sensors located at  $x_i, 1 \leq i \leq N, x_i \in \Gamma$  one may choose

$$\omega_\varepsilon = \bigcup_{i=1}^N \mathcal{B}(x_i, \varepsilon).$$

So, we replace a boundary observation by a distributed observation. We set  $\Omega_\varepsilon = \Omega \cup \omega_\varepsilon$  and  $\Gamma_\varepsilon = \partial\omega_\varepsilon$ .

**Open problem 5.** • *More generally  $\Omega_\varepsilon$  is defined as a perturbation of  $\Omega$ : there are many questions related to the asymptotic behavior of the  $\varepsilon$ -solution as  $\varepsilon \rightarrow 0$ . One may use tools as the  $\Gamma$ -convergence for example.*

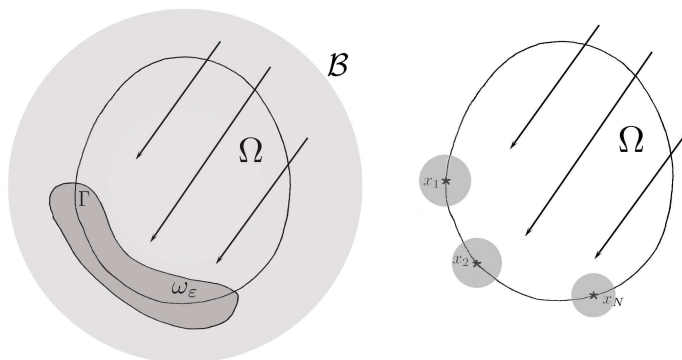


FIGURE 3. Observation surface relaxation

- Another interesting question is to look for an optimal  $\omega_\varepsilon$  with respect to a criterion given by the experimental setup for example.

Finally, the regularized optimal control problem reads:

$$(\mathcal{P})_\varepsilon \quad \begin{cases} \min J_\varepsilon(y, u) := \frac{1}{2} \int_{[0, T] \times \omega_\varepsilon} (y - y_d)^2(t, x) dt dx + \frac{\alpha}{2} \int_{[0, T] \times \Omega} u^2(t, x) dt dx \\ y = y[u] \text{ solution to (9), } u \in \mathcal{U} \end{cases}$$

where  $\alpha > 0$ ,  $y_d \in L^2(Q)$  and  $\mathcal{U}$  is defined for example as

$$\mathcal{U} := \{u \in L^2(]0, T[ \times \mathcal{B}) \mid u(t, x) = 0 \text{ a.e. } (t, x) \in ]0, T[ \times (\mathcal{B} \setminus \Omega)\} .$$

**Remark 3.** One can choose

$$\mathcal{U} := \{u \in L^2(]0, T[ \times \mathcal{B}) \mid u(t, x) = u_o(x) \frac{\partial j}{\partial t}(t) \text{ a.e. } (t, x) \in ]0, T[ \times \Omega, \text{ supp}(u_o) \subset \Omega\} .$$

In the case where the control is an initial one the regularization term could be

$$\frac{\alpha}{2} \int_{\mathcal{B}} u_o^2(x) dx \left( = \frac{\alpha}{2} \int_{\Omega} u_o^2(x) dx \right) .$$

**Theorem 3.3.** For every  $\varepsilon > 0$  problem  $(\mathcal{P})_\varepsilon$  has a unique solution  $(y_\varepsilon, u_\varepsilon)$

*Proof.* The proof is standard: any minimizing sequence weakly converges to some  $u^*$  (up to a subsequence) in  $L^2(Q)$  and strongly in  $H^{-1}(Q)$ . The sensitivity results for the wave equation implies that the corresponding state sequence converges to  $y[u^*]$ . As  $\mathcal{U}$  is closed and convex ( $\subset L^2(Q)$ )  $u^*$  is admissible. The conclusion comes from the lower-semi continuity of the cost functional.  $\square$

**Open problem 6.** One has to precisely study the asymptotic behavior of  $(y_\varepsilon, u_\varepsilon)$  with respect to  $\varepsilon$  and if possible give error estimates.

**3.3. Optimality conditions.** Once the optimal control problem has been formulated it is easy to use the classical machinery (see [18]) introducing the adjoint

state  $p_\varepsilon$  :

$$\begin{cases} \frac{\partial^2 p_\varepsilon}{\partial t^2} - \Delta p_\varepsilon = \begin{cases} y_\varepsilon - y_d & \text{in } [0, T] \times \omega_\varepsilon \\ 0 & \text{elsewhere} \end{cases} \\ p_\varepsilon(T, x) = 0 \text{ in } \mathcal{B}, \\ \frac{\partial p_\varepsilon}{\partial t}(T, x) = 0, \text{ in } \mathcal{B}, \\ p_\varepsilon = 0, \text{ on } S \end{cases}$$

Since  $y_\varepsilon - y_d \in L^2(Q)$  then  $p_\varepsilon \in L^2(0, T; H_o^1(\mathcal{B})) \cap C^0([0, T]; L^2(\mathcal{B}))$ . As  $(y_\varepsilon, u_\varepsilon)$  is the solution to  $(\mathcal{P})_\varepsilon$  then

$$\forall (y, u) \in W \times \mathcal{U}, \text{ such that } y = y[u] \quad J'_\varepsilon(y_\varepsilon, u_\varepsilon)(y - y_\varepsilon, u - u_\varepsilon) \geq 0,$$

that is

$$\begin{aligned} \int_{[0, T] \times \omega_\varepsilon} (y_\varepsilon - y_d)(y - y_\varepsilon) dt dx + \alpha \int_{[0, T] \times \Omega} u_\varepsilon(u - u_\varepsilon) dt dx &\geq 0 \\ \int_{[0, T] \times \mathcal{B}} \left( \frac{\partial^2 p_\varepsilon}{\partial t^2} - \Delta p_\varepsilon \right) (y - y_\varepsilon) dt dx + \alpha \int_{[0, T] \times \Omega} u_\varepsilon(u - u_\varepsilon) dt dx &\geq 0. \end{aligned}$$

The Green formula gives ([19] Vol. 1)

$$\int_{[0, T] \times \mathcal{B}} \left( \frac{\partial^2 p_\varepsilon}{\partial t^2} - \Delta p_\varepsilon \right) (y - y_\varepsilon) = \int_{[0, T] \times \mathcal{B}} \left( \frac{\partial^2 (y_\varepsilon - y)}{\partial t^2} - \Delta (y_\varepsilon - y) \right) p_\varepsilon.$$

Finally

$$\forall u \in \mathcal{U} \quad \int_{[0, T] \times \mathcal{B}} (p_\varepsilon + \alpha u_\varepsilon)(u - u_\varepsilon) dt dx \geq 0, \quad (13)$$

that is

$$u_\varepsilon = \Pi_{\mathcal{U}} \left( -\frac{p_\varepsilon}{\alpha} \right),$$

where  $\Pi_{\mathcal{U}}$  is the  $L^2$ - projection on  $\mathcal{U}$ . We get the following optimality system

**Theorem 3.4.** *For every  $\varepsilon > 0$  a necessary and sufficient condition for  $(y_\varepsilon, u_\varepsilon) \in W \times \mathcal{U}$  to be the solution to  $(\mathcal{P})_\varepsilon$  is*

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \Delta y = u, \text{ in } Q, \\ y(0, x) = 0, \text{ in } \mathcal{B}, \\ \frac{\partial y}{\partial t}(0, x) = 0, \text{ in } \mathcal{B}, \\ y = 0, \text{ on } S \end{cases} \quad (14a)$$

$$\begin{cases} \frac{\partial^2 p_\varepsilon}{\partial t^2} - \Delta p_\varepsilon = \begin{cases} y_\varepsilon - y_d & \text{in } [0, T] \times \omega_\varepsilon \\ 0 & \text{elsewhere} \end{cases} \\ p_\varepsilon(T, x) = 0 \text{ in } \mathcal{B}, \\ \frac{\partial p_\varepsilon}{\partial t}(T, x) = 0, \text{ in } \mathcal{B}, \\ p_\varepsilon = 0, \text{ on } S \end{cases} \quad (14b)$$

$$\forall u \in \mathcal{U}, \quad \int_Q (p_\varepsilon + \alpha u_\varepsilon)(u - u_\varepsilon) dt dx \geq 0, \quad (14c)$$

**4. The controllability point of view.** Another way to consider the problem, which seems to be the most natural one is a controllability approach. Roughly speaking, we look for a control function  $u$  with support in  $]0, T[ \times \Omega$  such that  $y[u]$  is given on the set  $\Sigma = ]0, T[ \times \Gamma$ . It is clear that the solution is not unique if there is some. However the existence is not obvious both in the cases of exact and approximate controllability. There is a huge literature on controllability/observability theory and we cannot mention all the papers. However, Bardos, Lebeau and Rauch have shown in a famous paper [4] that for the observation or control of solutions of second-order hyperbolic equation a necessary and sufficient condition for controllability is that the region of control meet every ray of geometric optics that has, at worst, transverse reflection at the boundary. More precisely, for multidimensional problems, the region of control must meet each ray in a non diffractive point. This condition is not ensured in our case since the observability (stabilization) region  $\omega_\varepsilon$  is not necessarily connected to the control region  $\Omega$ .

**Open problem 7.** *What could be a good choice of  $\omega_\varepsilon$  or  $\Gamma$  to give some existence results for the controllability problem:*

*Find  $u$  with compact support in  $\Omega$  such that  $y[u] = y_d$  on  $\Gamma$  or  $\omega_\varepsilon$  ?*

*Which model would be appropriate for using controllability techniques for the wave equation (HUM method, Carleman estimates) ?*

**5. Numerical hints.** Using our model we have obtained necessary and sufficient optimality conditions that we may solve numerically. From now,  $\varepsilon > 0$  is fixed and we may use (for example) a fixed-point algorithm to solve equations (14). This gives (formally)

#### Algorithm

1. Choose  $u_o \in \mathcal{U}$  (for example  $u_o \equiv 0$ ). Set  $k = 1$
2. Iteration  $k$ 
  - (a) Compute  $y_k = y[u_{k-1}]$  solution to equation (14a) with  $u_{k-1}$  instead of  $u_\varepsilon$ .
  - (b) Compute  $p_k$  solution to equation (14b) with  $y_k$  instead of  $y_\varepsilon$ .
  - (c) Set  $u_k = \Pi_{\mathcal{U}}(-p_k/\alpha)$ .
3. Check if a stopping criterion is satisfied. If not  $k = k + 1$  and go back to 2.

There are many classical tools to make this algorithm converge (splitting, relaxation). The more delicate part is the computation of  $\Pi_{\mathcal{U}}$  since the support constraint is a pointwise constraint and we perform a  $L^2$ -projection. An alternative is to keep the constraint as a control constraint and deal with the associate Lagrange multiplier. This will be addressed in future work.

The previous algorithm has been set in a infinite dimensional framework and it is necessary to perform a discretization process, for example with finite difference schemes. However, it is known that for time semi-discrete systems, due to high frequency spurious components, the exponential decay property may be lost as the time step tends to zero (see Zuazua [26, 27]) so that numerical schemes are highly unstable. In [11], Ervedoza and Zuazua use a decoupling argument of low and high frequencies, the low frequency observability property for time semi-discrete approximations of conservative linear systems and the dissipativity of the numerical viscosity on the high frequency components.

**Open problem 8.** *Establish the relationship between the numerical aliasing due to high frequencies in the discretization of the wave equation and the classical aliasing effect in Fourier analysis and signal processing. How can we relate the decoupling method of [11] to Fourier extrapolation and/or mollification ?*

**Open problem 9.** *Find a numerical method (and a model) which use (for example) finite difference schemes for the low frequency part and Fourier expansion for high frequency part, via high frequency penalization.*

**6. Conclusion.** We hope we have clearly presented many open question about a challenging imaging process. We can apply these remarks to photo-acoustic tomography (PAT) as well: the equations are the same (wave equation related to acoustic part) but the frequency scale is different. There are many others questions that arise as for example the optimal location of sensors (shape-optimization problem), variable speed sound and/or non linear equations, filtering effect on  $u$  and so on.

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