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# Robust Path Tracking using Flatness for Fractional Linear MIMO Systems: a Thermal Application

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## Abstract

This paper deals with robust path tracking using flatness principles extended to fractional linear MIMO systems. As soon as the path has been obtained by fractional flatness, a robust path tracking based on CRONE control is presented. Flatness in path planning is used to determine the controls to apply without integrating any differential equations when the trajectory is fixed (in space and in time). Several developments have been made for fractional linear SISO systems using a transfer function approach. On fractional systems, especially in MIMO, developments are still to be made. Along this paper, flatness principles are applied using polynomial matrices for fractional linear MIMO systems. To illustrate the robustness performances, a third multi-scalar generation CRONE controller is compared to a PID one.

*Key words:* Flatness, path tracking, fractional differentiation, state-space-like representation, fractional MIMO systems, robust control.

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## 1. Introduction

Theoretical and experimental studies point out that thermal [1? ], electro-chemical [2] and viscoelastic [3] systems are defined by non integer differential equations. Integer models are not appropriated to identify that type of systems, unless applying very complex models.

A new property called flatness characterizing a certain class of nonlinear systems, allowing the achievement of a simple and robust control, has been introduced by M. Fliess, J. Levine, Ph. Martin and P. Rouchon (see [4, 5, 6, 7]). If we restrict to linear time-invariant systems (LTI), it turns out that such systems are flat if and only if they are controllable and that a particular flat output may be obtained as a by-product of the Brunowsky controllability canonical form. Flat outputs summarize, on its own, all the system dynamics. All system variables will be deduced from them without integrating differential equations.

Expressing the flat output  $z$  with the system variables and their time derivatives occurs to be more difficult. Flatness is well-suited for trajectory planning and offers the possibility of solving straight-forwardly path tracking. One interest of flatness relies on computing directly the command trajectory enabling to generate the desired trajectory without disturbance. In order to reject those disturbances a robust control strategy is required. Here a third generation multi-scalar CRONE approach is applied. Then, a comparison with a PID is given through the simulations.

In previous works, the extension of the flatness principle to fractional systems was established in [8, 9, 10]. In this paper, the study involves controllable fractional LTI systems control. The originality of this work is in applying the flatness extensions to a multivariable (MIMO, Multiple Input Multiple Output) fractional

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system (see [8]), in performing a control strategy in closed-loop with a third generation CRONE controller design and in generating a robust path tracking.

The paper is organized as follows. In section 2, first, some basic facts on fractional differentiation and state-space-like representation are detailed, then, the fractional flatness principles are recalled. The characterization of the so-called defining matrices of linear flat outputs and the dynamic inversion of fractional system is detailed in section 3. Finally, in section 4, an application is detailed: the diffusion of two flux density throughout a metallic rod represents a MIMO fractional system. The next section presents the controller design, and simulations and comparisons are carried out on section 6 in presence of disturbances. A conclusion is given in section 7 reminding the major outlines of this study.

## 2. State-Space-Like Representation of Fractional Systems and flatness principles

### 2.1. Recalls on fractional systems

The concept of differentiation to an arbitrary order (non-integer),

$$\mathbf{D}^\nu \triangleq \left( \frac{d}{dt} \right)^\nu \quad \forall \nu \in \mathbb{R}_+^*,$$

was defined in the 19<sup>th</sup> century by Riemann and Liouville. The  $\nu$ -order fractional derivative of  $x(t)$  is defined as being an integer derivative of order  $[\nu] + 1$  ( $[\cdot]$  stands for the floor operator) of a non-integer integral of order  $\nu - [\nu]$  [11]:

$$\mathbf{D}^\nu x(t) = \mathbf{D}^{[\nu]+1} \left( \mathbf{I}^{[\nu]+1-\nu} x(t) \right) \triangleq \left( \frac{d}{dt} \right)^{[\nu]+1} \left( \frac{1}{\Gamma([\nu] + 1 - \nu)} \int_0^t \frac{x(\tau) d\tau}{(t - \tau)^{\nu - [\nu]}} \right), \quad (1)$$

where  $t > 0$ ,  $\forall \nu \in \mathbb{R}_+^*$ , and the Euler's  $\Gamma$  function is defined as:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad \forall x \in \mathbb{R}^* \setminus \mathbb{N}^-. \quad (2)$$

In its most general form, a monovariate fractional model can be defined by a differential equation characterized with its derivation orders that can be real (integer or fractional):

$$a_0 \mathbf{D}^{\nu_{a_0}} y(t) + a_1 \mathbf{D}^{\nu_{a_1}} y(t) + \dots + a_L \mathbf{D}^{\nu_{a_L}} y(t) = b_0 \mathbf{D}^{\nu_{b_0}} u(t) + \dots + b_M \mathbf{D}^{\nu_{b_M}} u(t), \quad (3)$$

where  $\mathbf{D}$  is the derivative operator. Let us consider that  $\nu_{a_i} < \nu_{a_j}$  for  $0 \leq i < j \leq L$  and that  $\nu_{a_p} < \nu_{b_q}$  for  $0 \leq p < q \leq M$ .

*Definition* : The commensurate order is defined as the greatest real positive divisor of the derivative orders  $(\nu_{a_1}, \dots, \nu_{a_n}, \nu_{b_0}, \dots, \nu_{b_m})$  in (3) (the highest number is generally chosen).

For integer systems, the commensurate order is equal to 1. For real non commensurable differentiation order, using an order approximation by rational numbers, it is possible to determine a common divisor order  $\nu$ .

In the following, all fractional systems considered are supposed to be of commensurable order.

### 2.2. State-space-like representation

Elementary differential equations, in which the order is subjected to two levels of generalization (see [12] for details), step in the extension of state space representation for fractional linear systems (see [12, 13, 14, 15]). In a classic state-space representation, the state needs only its previous value in order to predict the next step; however, in fractional calculus, the fractional derivative needs the whole past of the state to predict its next value [16, 17]. Consequently, in fractional calculus, this representation is called "fractional"

or state-space "like" representation. In this paper, only the first level of generalization is considered<sup>1</sup>; the derivation orders of each elementary differential equation are the same and given by the commensurate order, which is a scalar,

$$\begin{cases} x^{(\nu)}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \quad (5)$$

where  $A$ ,  $B$ ,  $C$  and  $D$  are respectively a  $n \times n$ ,  $n \times m$ ,  $r \times n$  and  $r \times m$  constant matrices.

### 2.3. Temporal simulation of fractional differentiation systems

The proposed approach works with any simulation algorithm in the time domain. However, only the simulation algorithm used in the example of this paper is explained here, as the objective is not to discuss different simulation algorithms<sup>2</sup>. Due to the consideration that real physical systems generally have bandlimited fractional behavior and due to the practical limitations of input and output signals (Shannon's cut-off frequency for the upper band and the spectrum of the input signal for the lower band), fractional operators are usually approximated by high integer order models. As a result, a fractional model and its integer approximation have the same dynamics within a frequency bandlimit. The most commonly used approximation for  $s^\nu$  in the frequency band  $[\omega_A, \omega_B]$  is the recursive distribution of zeros and poles proposed in [12] (see Figure 1):

$$s^\nu \rightarrow s_{[\omega_A, \omega_B]}^\nu = C_0 \left( \frac{1 + \frac{s}{\omega_A}}{1 + \frac{s}{\omega_B}} \right)^\nu \approx C_0 \prod_{k=1}^N \frac{1 + \frac{s}{\omega_k'}}{1 + \frac{s}{\omega_k}} \quad (6)$$

where  $\omega_i = \alpha\omega_i'$ ,  $\omega_{i+1}' = \eta\omega_i'$  and

$$\gamma = 1 - \frac{\log \alpha}{\log \alpha\eta}, \quad (7)$$

$\alpha$  and  $\eta$  define the differentiation order  $\gamma$ . The bigger  $N$  the better the approximation of the differentiator  $s^\nu$ .

## 3. Fractional flat output of a fractional linear system stepping from polynomial form

All along this paper,  $u^{(k)}$  stands for the  $k$ -th order derivative of  $u$  with respect to time,  $k$  being a non integer scalar and each component of  $u$  is derived at order  $k$ . Let the system be given by the following differential equation:

$$x^{(\nu)} = f(x, u), \quad (8)$$

with  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  being the input command,  $\nu \in [0, 2]$ <sup>3</sup> and  $f = (f_1, \dots, f_n)$  a regular function of  $x$  and  $u$  where the rank of  $\frac{\partial^\nu f}{\partial u^\nu}$  is equal to  $m$ .

This system is said differentially flat (see [4] for the rational case and [9] for the extension to fractional systems and its algebra of  $s^\nu$ -polynomials) if one can find a set of independent variables referred to as the flat output  $z$  such that each system variable, including the inputs, is a function of  $z$  and a finite number of its time derivatives. The fractional flat output  $z$  is such that

$$z = h(x, u, u^{(\nu)}, u^{(2\nu)}, \dots, u^{(\beta\nu)}), \quad z \in \mathbb{R}^m, \quad (9)$$

with  $\beta$  a finite  $m$ -tuple of integers<sup>4</sup> and  $\nu$  stands for the commensurable order.

<sup>1</sup>In the second level of generalization, the derivation orders of the elementary differential equations are different, and the system takes the following representation:

$$\begin{cases} x^{(\underline{\nu})}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \quad (4)$$

where  $\underline{\nu} = (\nu_1, \dots, \nu_K)^T$ ,  $\nu_{i=1, \dots, K} \in \mathbb{R}$ . The state variable of rank  $i$  is differentiated at the order  $\nu_i$ . For simplicity reasons, the study will be proceeded for a first level of generalization, with a scalar  $\nu$ .

<sup>2</sup>Refer to [18], for an extended discussion on time-domain simulation of fractional differentiation systems.

<sup>3</sup>It is proved in [19] that fractional differentiation systems are unstable for a commensurable order  $\nu \notin ]0, 2[$ .

<sup>4</sup>For a  $m$ -tuple of integers  $\beta = (\beta_1, \dots, \beta_m)$  of dimension  $m$  and a vector  $u$  of dimension  $m$ , the notation  $u^{(\beta)}$  denotes  $(u^{(\beta_1)}, \dots, u^{(\beta_m)})$ .

### 3.1. Fractional flat output of a fractional linear system

In the following, the Laplace transformation  $\mathcal{L}$ , which is an algebraic tool generally used to represent fractional differentiation systems, is defined by (see [20, 17]):

$$\mathcal{L}\{\mathbf{D}^\gamma x(t)\} = s^\gamma X(s) \quad \text{if } x(t) = 0 \quad \forall t \leq 0. \quad (10)$$

All variables are in the operational domain; in order to get back to the time domain, the Laplace inverse operator  $\mathcal{L}^{-1}$  should be applied. Fractional linear systems such as (5) can be rewritten in the following polynomial matrix representation:

$$\begin{cases} A(s^\nu)X = BU \\ Y = CX + DU \end{cases}, \quad (11)$$

with  $A(s^\nu) = s^\nu I - A$  an  $n \times n$  matrix whose entries are polynomials of the formal variable  $s^\nu$  ( $I$  being the identity matrix),  $B$ ,  $C$  and  $D$  are constant matrices, and  $X$ ,  $U$  and  $Y$  for the Laplace transform of  $x$ ,  $u$  and  $y$  respectively. System (11) is assumed controllable, i.e.  $A$  and  $B$  are left coprime (see [6]).

A *fractional linear flat output* defines an output defined by (9) with  $h$  being linear; the state variables, the input command and the output can be redefined with  $z$  such as:

$$\begin{aligned} x_i &= \sum_{j=1}^m \sum_{k=0}^{\alpha_j} a_{i,j,k} z_j^{(k\nu)}, \quad i = 1, \dots, n, \\ u_l &= \sum_{j=1}^m \sum_{k=0}^{\alpha_j+1} b_{l,j,k} z_j^{(k\nu)}, \quad l = 1, \dots, m, \\ y_q &= \sum_{j=1}^m \sum_{k=0}^{\sigma_j} c_{q,j,k} z_j^{(k\nu)}, \quad q = 1, \dots, r, \end{aligned} \quad (12)$$

where  $\underline{\alpha}$  and  $\underline{\sigma}$  are  $m$ -tuple of integers. Notice that all differential orders are multiple of the commensurable order  $\nu$ .

In the polynomial matrix language, (12) reads

$$X = P(s^\nu)Z, \quad U = Q(s^\nu)Z, \quad (13)$$

where  $P$  (resp.  $Q$ ) is an  $n \times m$  (resp.  $m \times m$ ) polynomial matrix of  $s^\nu$ , with entries  $P_{i,j}(s^\nu) = \sum_{k=0}^{q_j} a_{i,j,k} s^{\nu i, k}$  (resp.  $Q_{l,j}(s^\nu) = \sum_{k=0}^{q_j+1} b_{l,j,k} s^{\nu l, k}$ ). For an overview of  $s^\nu$ -polynomials algebra, refer to [9].

Matrices  $P$  and  $Q$  satisfying (13) are called defining matrices of the fractional flat output  $z$ .

**Theorem** *The variable  $z = (z_1, \dots, z_m)$  is a linear flat output of system (11) if and only if its defining matrices  $P$  and  $Q$  are given by*

$$\begin{aligned} R^T A(s^\nu) P(s^\nu) &= 0, \\ A(s^\nu) P(s^\nu) &= BQ(s^\nu), \end{aligned} \quad (14)$$

with  $R$  an arbitrary matrix of rank  $n - m$  orthogonal to  $B$  (it means that  $R^T B = 0$ ), and with  $P(s^\nu)$  and  $Q(s^\nu)$  of rank  $m$  for every  $s^\nu$  and right coprime<sup>5</sup>.

In addition, linear flat output  $y$  of the controllable system (11) always exists (and therefore its defining matrices  $P$  and  $Q$  always exist too).

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<sup>5</sup>For the proof for fractional systems, one may refer to [9].

### 3.2. Dynamic inversion of a fractional differentiation system

Once the state variable  $x$  (resp. the input command  $u$ ) being expressed by the flat output  $z$  with its defining matrix  $P$  (resp.  $Q$ ), the desired output  $y_{des}$  can be expressed with the fractional flat output  $z$ . Knowing  $y_{des}$ , the fractional differentiation system is inverted in order to get the fractional flat output.

From (5) and (13), the desired output is expressed with the fractional flat output such as:

$$Y_{des} = (CP(s^\nu) + DQ(s^\nu)) Z. \quad (15)$$

Thus, in order to have a desired output, it is necessary to invert (15) in order to determine the flat output entirely. We note  $W(s^\nu) = CP(s^\nu) + DQ(s^\nu)$  which is of dimension  $r \times m$ .

1) if  $r = m$  and  $\text{rank}(W) = m$ ,  $W$  is invertible, and the flat output is easily determined as:

$$Z = W^{-1}Y_{des}. \quad (16)$$

2) if  $r < m$ ,  $W$  is left invertible (the pseudo-inverse is used here), and the flat output can be expressed such as:

$$Z = (W^T W)^{-1} W^T Y_{des}. \quad (17)$$

3) if  $r > m$ , in this case, the system has more outputs than input commands. Some outputs are linear combination of independent outputs or linear combination of the command inputs. After isolating those outputs, the system can be brought to one of the two previous cases.

## 4. Application: thermal system

### 4.1. Description of the thermal application

In this section, a MIMO fractional system will be handled using fractional state-space-like representation: a finite dimensional thermal system is considered. [1, 21] established that the thermal flux throughout a metallic bar can be defined with fractional differential equations. The thermal system is constituted by a long aluminium rod of  $L = 0.15\text{m}$  heated by two resistors, one on each end of the bar. It is considered as a homogenous finite plane medium with conductivity  $\lambda = 10\text{W}\cdot\text{m}^{-1}\cdot\text{K}^{-1}$ , with diffusivity  $\alpha = 10^{-5}\text{m}^2\cdot\text{s}^{-1}$  and initially at  $0^\circ\text{C}$ . The entire surface of the rod is assumed to be insulated in order to ensure an unidirectional heat transfer. Losses on the surface where the thermal flux is applied are neglected. The aim here is to control the temperature at two specific points of the rod, measured at a distance of  $l_1 = 0.05\text{m}$  and  $l_2 = 0.11\text{m}$  from one of the heated end (Figure 2), knowing that the rod is submitted to two different heat flux density.

The non integer physical behavior proof of this system and the calculation details are not exposed here as they were already developed in [9]. The model linking the flux density applied on the outgoing normal surface of the medium to the temperature measured at an abscissa  $l_n$  ( $n = 1$  for  $l_1$ ,  $n = 2$  for  $l_2$ ) inside the medium is given by the following transfer function (resulting from the theorem of superposition as the system is supposed linear):

$$H(l_n, s) = H_E(l_n, s) + H_L(l_n, s), \quad (18)$$

with the following definitions:

•

$$H_E(l_n, s) = \frac{T_E(l_n, s)}{S\varphi_E(0, s)} = \frac{\sum_{k=0}^K b_k s^{k\nu}}{\sum_{k=1}^K a_k s^{(k+1)\nu}} \quad (19)$$

where  $S = \pi R_{bar}^2$  is the bar section,  $\varphi_E(l = 0, s)$  is the flux density coming from  $l = 0$  (at point E on Figure 2) and  $T_E(l, s)$  the temperature resulting from  $l = 0$  measured at point  $P_n$  ( $l = l_n$ ) (Figure 2),  $\nu = 0.5$  is the commensurable order,  $b_k = \frac{(2L-l_n)^k + l_n^k}{k! \alpha^{k/2}}$  and  $a_k = S\lambda \frac{(2L)^k}{k! \alpha^{\frac{k+1}{2}}}$ .



## 5. Controller design

According to the fractional state-space-like representation (21), as the considered system is decoupled, the regulators can be computed separately in a multi-scalar approach. Several fractional control design exist in the literature [22, 23, 24, 25]; here, a third generation Crone controller is adopted. Two regulators are computed, one for each input command. For a better comparison, two types of controller are given: a PID and a CRONE controller. In the following the indices  $E$  stands for the incoming flux from  $l = 0$ , and  $L$  stands for the incoming flux from  $l = L$  (see Figure 2), and  $X$  being both  $E$  and  $L$ . The controllers should assure a robustness to disturbances and to parametric uncertainties, which bring gain and phase variations. Thus, the control design should be adapted even if the used model is not well identified. A variation on  $R_{bar}$  influences the gain of the plant: the lower  $R_{bar}$  is, the higher the gain will be; a variation of  $\lambda$  brings a phase variation: the lower  $\lambda$  is, the higher the phase variation will be. Figure 5 shows the nominal plant and the envelopes due to uncertain parameters.

### 5.1. The PID controller design

The PID (Proportional Integral Differential) controllers are designed for a desired open-loop gain crossover frequency  $\omega_{cg,X} = 0.3\text{rad/s}$ . A phase margin of  $45^\circ$  is chosen in order to reduce the overshoot. All these specification sheets lead to the PID controller described by the following transfer function explaining the Proportional, Integral, Differential and Filtering (to reduce noise in high frequencies) action parts ( $X = E, L$ ):

$$\begin{aligned}
 C_X(s) &= C_{o,X} \left( \frac{1 + \frac{s}{\omega_{i,X}}}{\frac{s}{\omega_{i,X}}} \right) \left( \frac{1 + \frac{s}{\omega_{a,X}}}{1 + \frac{s}{\omega_{b,X}}} \right) \left( \frac{1}{1 + \frac{s}{\omega_{f,X}}} \right) \\
 C_{o,E} &= 18.07, \quad \omega_{i,E} = 0.2, \quad \omega_{a,E} = 0.86, \\
 \omega_{b,E} &= 0.06, \quad \omega_{f,X} = 2, \\
 C_{o,L} &= 18.85, \quad \omega_{i,L} = 0.23, \quad \omega_{a,L} = 0.86, \\
 \omega_{b,L} &= 0.06, \quad \omega_{f,X} = 2.3
 \end{aligned} \tag{23}$$

Figure 6 shows the open-loop with the PID controller in the Nichols diagram considering the uncertainties.

### 5.2. The CRONE controller design

The CRONE controller (a French acronym on fractional order robust control (see [26])) is defined within the frequency range  $[\omega_A, \omega_B] = [0.001, 1]\text{rad/s}$  around the desired open-loop gain crossover frequency  $\omega_{cg,X}$  in order to ensure a constant phase and more particularly to ensure small variations of the closed-loop system stability-degree. The aim of the CRONE control system design (CSD) is to find a diagonal open-loop transfer matrix whose elements are fractional order transfer functions. It is parameterized to achieve a perfect decoupling for the nominal plant, accuracy specifications at low frequencies, required nominal stability margins of the closed loops (behaviors around the required cut-off frequencies), and specifications of all the control efforts at high frequencies. After an optimization of the diagonal open-loop  $\beta_{0,X}(s)$  transfer matrix, frequency-domain system identification is carried out to obtain the fractional controller. Open-loop transfer functions are used to satisfy the objectives specified above. For a detailed overview of the third generation Crone controller, the readers may refer to [27].

The open-loop transfer function can be described as based on band-limited complex non-integer integration:

$$\begin{aligned}
 \beta_{0,X}(s) &= C_X^{\text{sign}(b)} \left( \frac{1+s/\omega_{h,X}}{1+s/\omega_{l,X}} \right)^a \\
 &\times \left( \Re_{e/i} \left\{ \left( C_g \frac{1+s/\omega_{h,X}}{1+s/\omega_{l,X}} \right)^{ib} \right\} \right)^{-q\text{sign}(b)}
 \end{aligned} \tag{24}$$

with  $C_X = \cosh \left[ b_X \left( \arctan \left( \frac{\omega_{cg,X}}{\omega_{l,X}} \right) - \arctan \left( \frac{\omega_{cg,X}}{\omega_{h,X}} \right) \right) \right]$  and  $C_{g,X} = \left( \frac{1 + \left( \frac{\omega_{cg,X}}{\omega_{l,X}} \right)^2}{1 + \left( \frac{\omega_{cg,X}}{\omega_{h,X}} \right)^2} \right)^{1/2}$ .

The corner frequencies are placed around the extreme frequencies of the frequency range considered such that:  $\omega_{l,X} < \omega_A < \omega_{cg,X} < \omega_B < \omega_{h,X}$

For stable and minimum phase plant, the generalized template is taken into account in the open-loop transfer function as follows:

$$\bar{\beta}_{0,X}(s) = \beta_{l,X}(s) \beta_{0,X}(s) \beta_{h,X}(s) \quad (25)$$

with  $\beta_{l,X}(s) = C_{l,X} \left(\frac{\omega_{l,X}}{s} + 1\right)^{n_{l,X}}$ , the order  $n_{l,X}$  fixing the accuracy of each closed-loop, and  $\beta_{h,X}(s) = \frac{C_{h,X}}{\left(\frac{s}{\omega_{h,X}} + 1\right)^{n_{h,X}}}$ , the order  $n_{h,X}$  enabling the elements of the controller to be proper. Remark that the desired open-loop gain crossover frequency  $\omega_{cg,X}$  are almost the same. This third generation CRONE control open-loop transfer function has been defined using the gain-crossover open-loop frequency and by inverting the plant as follows:

$$\begin{aligned} C_X &= \bar{\beta}_{0,X}(s) H_X^{-1}(l_n, s) \\ \omega_{l,X} &= 10^{-3} \text{rad} \cdot \text{s}^{-1}, \quad \omega_{h,X} = 10^3 \text{rad} \cdot \text{s}^{-1}, \quad \omega_{r,X} = 0.3 \text{rad} \cdot \text{s}^{-1}, \\ \gamma &= a + ib = 1.24 - 0.11i, \quad n_{l,X} = 1.5, \quad n_{h,X} = 2, \end{aligned} \quad (26)$$

They are computed to ensure stability for gain uncertainties between 0.02 and 50, and for phase uncertainties between  $12.5^\circ$  and  $25^\circ$ . Figure 7 presents the open-loop in the Nichols diagram considering the uncertainties.

## 6. Simulation Results

The system is studied in closed-loop so as to measure its immunity to different disturbances applied to its input  $\Delta U$  and to its output  $\Delta Y$ . The control scheme is presented in Figure 8, with  $U_{ref}$  the control obtained by the flatness principle using the chosen reference trajectory  $y_{des}$  (the desired output). The PID and Crone controllers are both used in simulation. For this, the disturbances and gain variation influences are studied on path tracking. Two control input disturbances of 35s are applied: one at 625s (during the ascension) and another one at 1910s during the constant phase. Moreover, two output disturbances of 35s are applied: one at 312s (during the ascension) and another one at 1600s during the constant phase. Time responses are given for different gain variations (1, 0.02 and 50 times the gain). It is recalled that the controllers have the same desired open-loop gain crossover frequency. Figure 9 shows a better performance of the CRONE controller compared to the PID one. Simulations with the nominal plant show similar disturbance rejection concerning the time responses (3s), however the overshoot is higher for the PID: almost 2 times higher. The robustness study is presented on figure Figure 10. The CRONE controller show the same sensibility concerning time responses and disturbance rejection. It is 2 times quicker (3.5s instead of 7s) and the output error is 10 times lower. A better path tracking performance with the CRONE controller can be accomplished in presence of disturbances and gain and phase variations. A more detailed synthesis presentation of a CRONE third generation Controller for MIMO systems can be carried out. The authors recommend to the following references: [28] for the CRONE command, [27] for MIMO fractional controller design.

## 7. Conclusion and outlooks

A new path tracking design combining flatness, fractional MIMO systems and CRONE control approaches was presented. First of all, the flatness principle definitions and the fractional state-space-like representation used in control theory are reminded. After that, using the polynomial matrix form, the flat output is computed after the dynamic inversion of the fractional system. Then, this method is applied to a fractional MIMO system: a rod was submitted to two different flux density, one at each end, in order to control the temperature at two specific points of the rod. Variations on  $R_{bar}$  and  $\lambda$  introduce gain and phase uncertainties that the control strategy should consider. The study of robust path tracking via a third generation CRONE control was integrated. To show the performance of the CRONE controller, simulations with two different controllers (PID and CRONE) were implemented. Those simulations pointed out clearly the robustness of the path tracking strategy for the CRONE controller, in terms of sensibility, disturbance rejection and time responses.

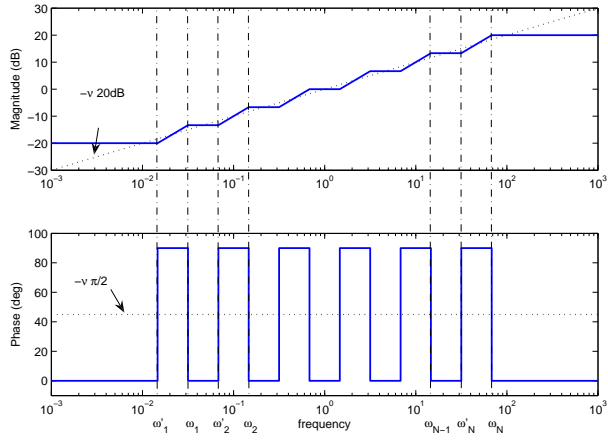


Figure 1: Recursive approximation of a fractional differentiator with poles and zeros

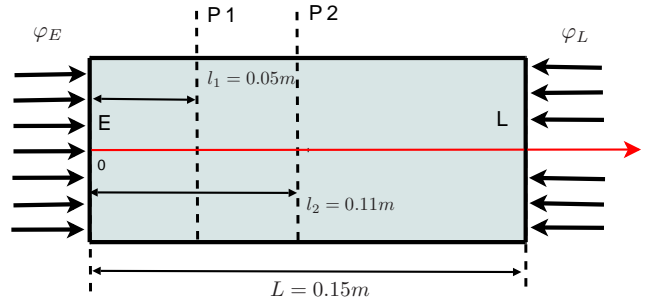


Figure 2: Finite metallic rod, under the influence of two thermal fluxes on each end. Two measures are taken, at  $P_1$  ( $l = l_1$ ) and  $P_2$  ( $l = l_2$ )

Furthermore, this study can be extended for non linear fractional systems. Many more researches on this topic are still to be made for fractional systems, such as non linear fractional systems or the extension of flatness principles for those kind of systems.

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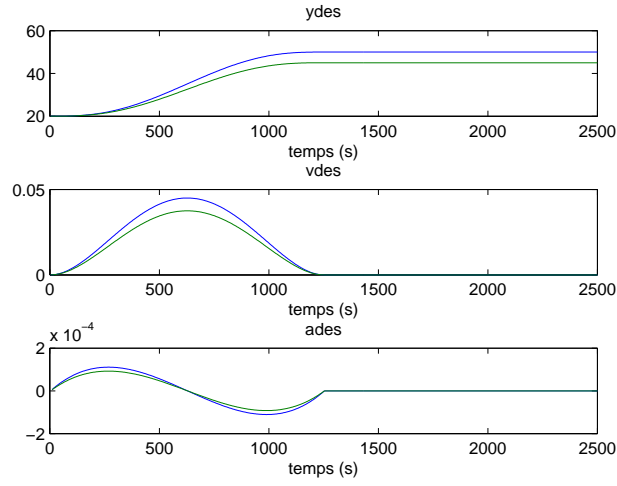


Figure 3: Desired temperatures and its times derivatives.

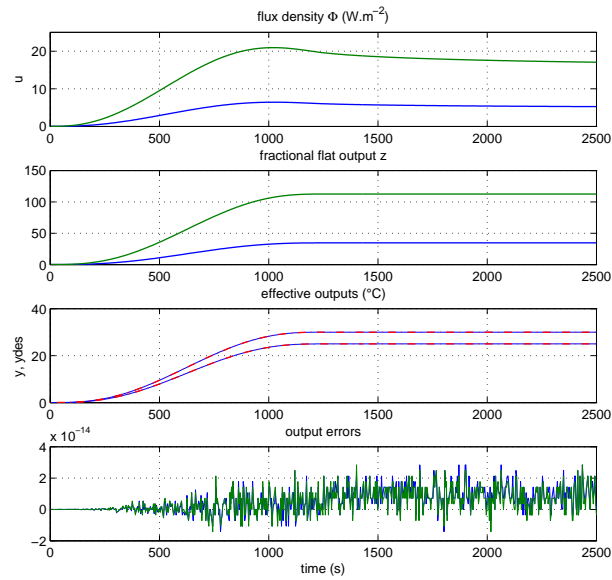


Figure 4: Simulation of the metallic rod in open-loop: input commands ( $u = \Phi$ ), flat outputs, desired (dashed) and effective outputs, and the output errors.

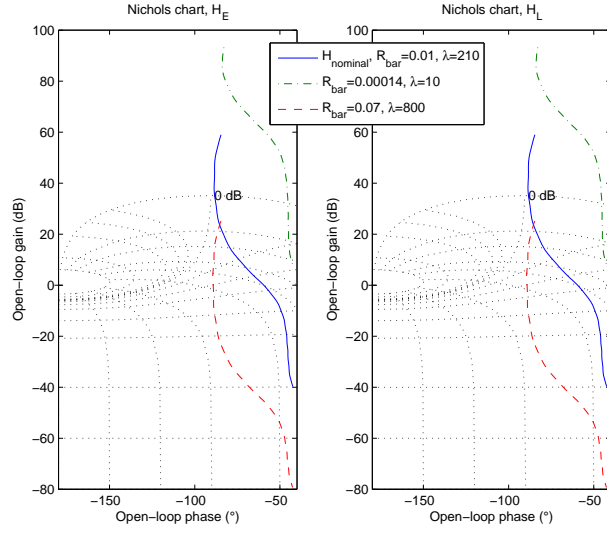


Figure 5: Nichols diagram of thermal plant with its uncertainties (left for  $H_E$ , right for  $H_L$ ), nominal in blue, extreme envelopes (dashed)

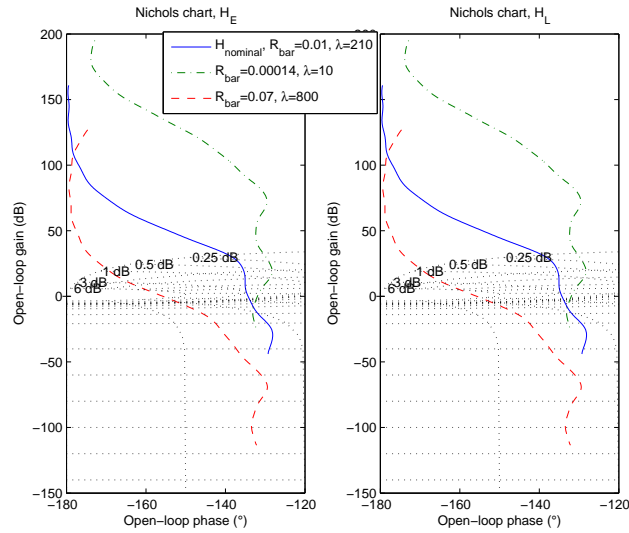


Figure 6: Nichols diagram of the open-loop with PID (left for  $H_E$ , right for  $H_L$ ), nominal in blue, extreme envelopes (dashed)

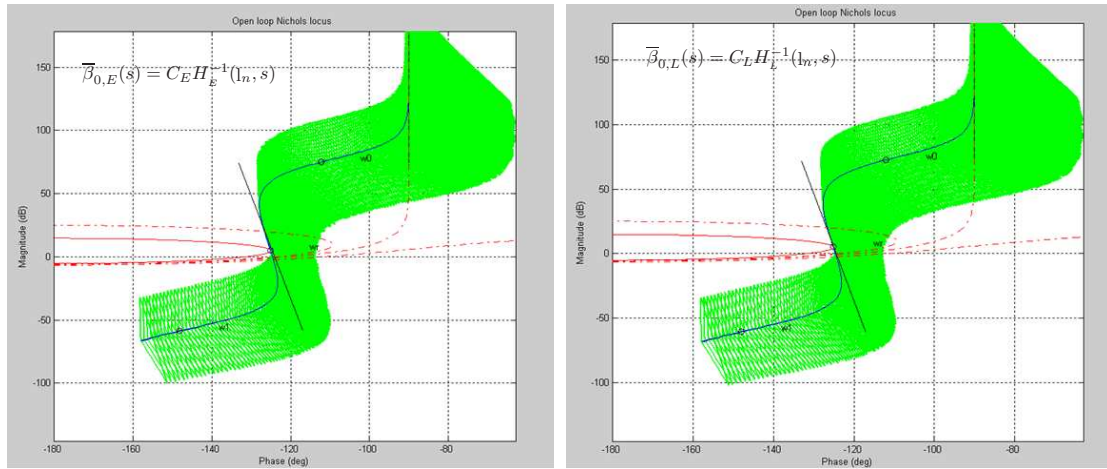


Figure 7: Nichols diagram of the fractional open-loop considering gain  $([0.02 - 50])$  and phase  $(12^\circ - 25^\circ)$  uncertainties.

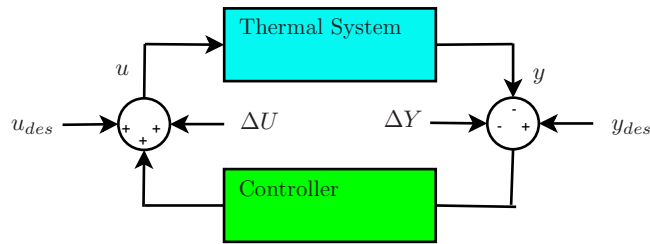


Figure 8: Closed-loop control scheme

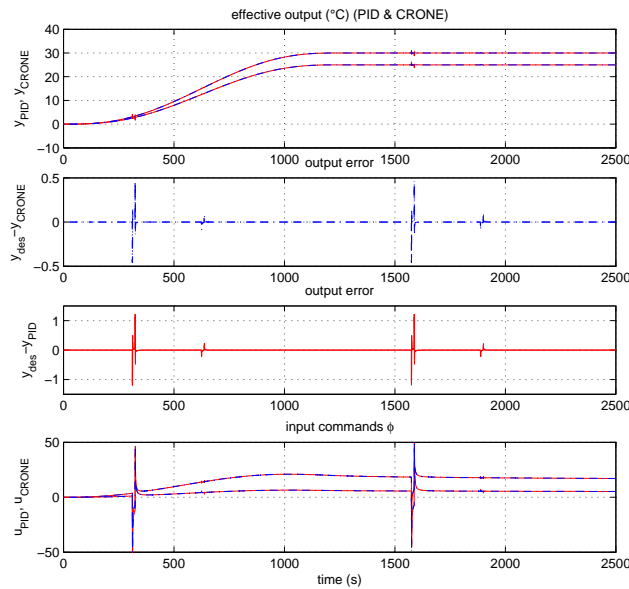


Figure 9: Simulation of the nominal plant; measured temperature (CRONE dashed, PID red), output error  $y_{CRONE} - y_{des}$ , output error  $y_{PID} - y_{des}$ , input commands  $\Phi$

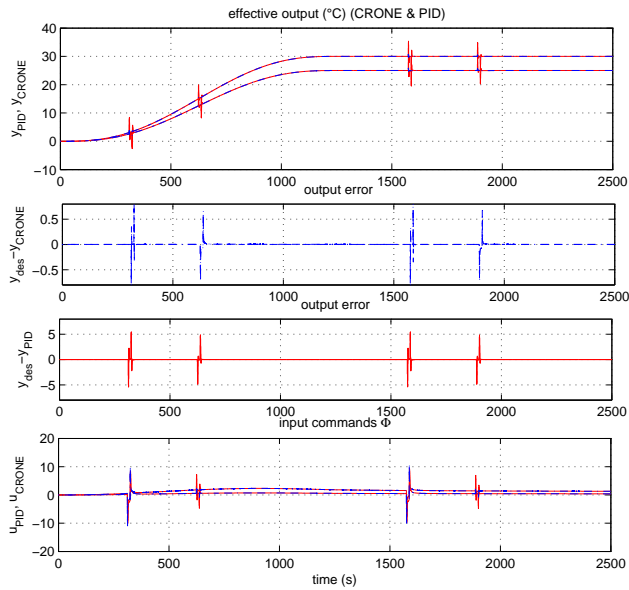


Figure 10: Simulation with  $50 \times H$ ; measured temperature (CRONE dashed, PID red), output error  $y_{CRONE} - y_{des}$ , output error  $y_{PID} - y_{des}$ , input commands  $\Phi$

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