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## Batch sizing with controllable production rates

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### Abstract

In this paper, we focus on a production system where a single product is manufactured on a single facility and delivered to the subsequent stage in batch shipments. In contrast to earlier works, we assume that the inventory on the producing stage is depleted at discrete time intervals, and analyse the effect of a variable production rate on the inventory build-up and the total costs of the system. We develop formal models for the case of equal- and unequal-sized batch shipments and propose solution procedures for the models. In a numerical study, we illustrated that deviating from the “design production rate” of a production system may reduce inventory carrying costs and thus lead to lower total costs.

**Keywords:** inventory control; production; variable production rate; batch sizing

## Introduction

The SCM-related literature contains a broad variety of models that deal with the control of inventories in different application areas. This is an important field of research, since it has been shown that the total inventory in a supply chain may exceed 100 days of supply (see Lee et al. 1997), wherefore the reduction of excess inventories may lead to many benefits. Lot-sizing models, in this context, aim on balancing lot-proportional costs (i.e. costs that depend on the lot size, such as inventory carrying costs) and lot-fixed costs (i.e. fixed costs incurred with each lot produced, such as setup costs) by determining a production quantity for the system under study.

One major drawback we identified when studying the literature on lot-sizing is that the production rate of a facility is often treated as a constant and not as a decision variable. In contrast, the mean output per unit of time of a production system can be influenced in many practical situations, either by directly varying the rate of performance of the production equipment, or by inserting idle time between task elements and by thus varying the time which is effectively spent on manufacturing (see for example Buzacott and Ozkarahan, 1983; Schweitzer and Seidmann, 1991). Since the production rate can, for example, influence inventory build-up or the life-span of the tools used in the manufacturing process, controlling the production rate may help to reduce the total costs of the system under study.

In recent years, several authors have studied the effect of a variable production rate on the performance of inventory systems. Thereby, some authors permit the production rate to be changed during the production process, while other authors assume that the production rate has to be fixed before a production run starts. The first case, which is commonly referred to as the “flexible case”, may occur if setup costs are only incurred when switching to a different item, but not when adjusting the speed of the production equipment, while the latter, which is termed the “rigid case”, may occur if changing the setup of a machine during a production run is technically impossible or associated with prohibitively high costs.

One of the first models dealing with the rigid case is due to Buzacott and Ozkarahan (1983), who consider the problem of scheduling two products on a single or two machines. The authors assume that the production rate can be varied by inserting idle time over the production run, and show that it is optimal to choose the production rates for both models in such a way that the machines are always fully utilised, i.e. that no inserted idle time occurs between runs of the products.

A variation of the model is introduced by Silver (1990), who analyzes the production of a family of items on a single facility. The author assumes that the products are produced once in

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3 a common production cycle, and permits the production rate to be varied between prescribed  
4 limits. In contrast to Buzacott and Ozkarahan (1983), operating costs which occur per unit  
5 operating time are considered in the model formulation. Thus, since running the production  
6 equipment is no longer free of cost, it is not generally optimal to avoid idle time on the  
7 machine.  
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11 The effect of a variable production rate on the production of items with a limited shelf life is  
12 first analyzed by Sarker and Babu (1993). The authors show that in case certain products can  
13 be stored in the inventory only for a limited time span, it may be beneficial either to reduce  
14 the production cycle or the production rate to shorten the storage time of the products and to  
15 avoid spoilage. Extensions to the model can be found in Silver (1995), Goyal (1994),  
16 Viswanathan (1995), and Viswanathan and Goyal (1997), who permit a simultaneous  
17 reduction of both model parameters or a different scheduling strategy, among others.  
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21 Larsen (1997) and Sharma (2008) study both a variable demand and a variable production rate  
22 in an EPQ-model. Both authors assume that the demand rate may fluctuate, and analyze how  
23 the production rate should be adjusted to compensate these fluctuations. It is shown that the  
24 appropriate response to an increased demand rate is a reduction of the production rate, which  
25 leads to reduced inventory build-up and consequently lower inventory carrying costs.  
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29 The effect of a variable production rate on product quality is analyzed by Khouja and Mehrez  
30 (1994) and Khouja (1999). Khouja and Mehrez (1994) assume that product quality  
31 deteriorates with an increased production rate and integrate different production cost and  
32 quality functions into an EPQ-model. The authors show that in case an increased production  
33 rate leads to a significant decrease in quality, the optimal production rate may be smaller than  
34 the rate that minimises unit production cost. Similarly, Khouja (1999) assumes that the  
35 production process may shift out of control with a probability depending on the production  
36 rate. The author extends the model proposed by Silver (1990) and shows that incorporating  
37 quality leads to a reduction in the optimal lot sizes and cycle time.  
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41 The flexible case is first analyzed by Moon et al. (1991), who extend the model due to Silver  
42 (1990) to allow production rates to be varied during the production run. The authors show that  
43 it is optimal first to reduce the production rate of a particular item to meet the demand rate in  
44 order to avoid inventory build-up, and then to produce the item at its maximum rate to build  
45 enough inventory to satisfy demand until the item is produced again. Since the authors do not  
46 consider production costs, there is no idle time in the optimal schedule. An extension to the  
47 model is provided by Gallego (1993), who applies a power-of-two-policy, instead of a  
48 common cycle-policy, and shows that in minimising a lower bound on the average cost it is  
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3 optimal to reduce the production rate of at most one item. A further extension is introduced by  
4 Elhafsi and Bai (1997), who allow backordering in their model and show that varying the  
5 production rate might again lead to a substantial reduction in average cost.  
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8 An EPQ-model which considers the flexible case is finally provided by Larsen (2005), who  
9 assumes that the production rates and their corresponding run times are decision variables.  
10 The author shows that the production rates should be chosen in the interval between the  
11 demand rate and the production rate which minimises unit production costs, and that they  
12 should be used in an increasing order.  
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17 In this paper, we focus on the case where a single product is manufactured on a single facility  
18 and delivered to the subsequent stage in batch shipments. In contrast to earlier works, we  
19 assume that the inventory on the producing stage is depleted at discrete time intervals, and  
20 analyze the effect of a variable production rate on the inventory build-up and the total costs of  
21 the system. Thereby, we illustrated that deviating from the “design production rate” of a  
22 production system may reduce inventory in the system and thus lead to lower total costs. In  
23 conformity with the literature, we analyze both the “rigid case” and the “flexible case” in the  
24 following and study their impact on the system for the cases of equal- and unequal-sized batch  
25 shipments.  
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33 Our research is especially important for companies facing high in-process inventories or  
34 bottleneck-problems. As will be shown, varying the production rate of a machine may  
35 significantly influence waiting time-related inventory and may thus help to control queues in  
36 front of subsequent production stages. This is consistent with the OPT-Philosophy (see for  
37 example Goldratt 1988), which aims on maximizing the throughput of the production system  
38 as a whole, instead of the throughput of a single machine, and the JIT-philosophy, which aims  
39 on reducing in-process inventories.  
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46 The remainder of the paper is organised as follows: in the next section, the article describes  
47 the problem we focus and outlines the assumptions and definitions which will be used in the  
48 remaining sections of the paper. Accordingly, we develop formal models for the rigid and the  
49 flexible case and the case of equal- and unequal-sized batch shipments and propose solution  
50 procedures for the models. Section 4 contains a numerical study and section 5 concludes the  
51 article.  
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## 59 **Problem description**

60 In the following, we study a system where a single product is manufactured on a single  
facility and assume that partial shipments to the subsequent stage can be made before the

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3 entire production lot is completed. This problem has frequently been discussed in the  
4 literature, but thus far under the assumption that production rates are predefined and constant  
5 during the production process.  
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9 In principle, two alternative types of transporting batch shipments to the subsequent stage are  
10 discussed in the literature. Szendrovits (1975) proposed the first alternative, which is to ship  
11 batches of equal size to the subsequent stage. This allows an overlap between operations on  
12 the producing and the consuming stage, which reduces the manufacturing cycle time and the  
13 inventory in the system. The second alternative is to ship batches to the next stage that  
14 increase by a fixed factor, which is equal to the production rate divided by the demand rate  
15 (see Goyal, 1977). Under the assumption that the maximum size of the batch shipments is not  
16 restricted, Goyal's policy leads to lower total system inventory and lower total costs than the  
17 one proposed by Szendrovits. In the next section, we analyze both alternatives under the  
18 assumption that the production rate can be varied.  
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26 The following assumptions and notations are used to formulate the problem:  
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30 *Assumptions:*

- 31 1. All parameters are deterministic and constant over time.
- 32 2. The production rate can be varied between given limits  $p_{min}$  and  $p_{max}$ .
- 33 3. We exclude synchronous production, where the facility is constantly producing and  
34 where there is no setup, from the analysis and assume that  $p_{min} > d$ .
- 35 4. The unit production costs are a function of the production rate.
- 36 5. The unit production cost function is convex in  $p$  and  $p_0$  is the "design production rate"  
37 that minimises unit production costs.
- 38 6. Batches are only sent to the consuming stage when the inventory level at this stage  
39 reaches zero.
- 40 7. We explicitly focus on a time weighted cost function to enhance the practical  
41 applicability of our model. As Higgins et al. (1996) note, production plans are often  
42 made with planning horizons rolling through time as time progresses. Thus, companies  
43 who wish to implement our model as a heuristic planning tool may specify a planning  
44 period and repeat the planning process after the planning time has elapsed. Note that  
45 this assumption is not uncommon in the literature, cf. for example Bogaschewsky et  
46 al. (2001). However, normalizing the planning period to 1 time unit results in the  
47 common money-per-unit-time cost function.  
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8. Shortages are not allowed.

*Notations:*

$c_p(p)$  = unit production cost function

$c_S$  = setup costs per setup

$c_T$  = transportation costs per shipment

$d$  = demand rate in units per unit time

$D$  = total demand in the planning period

$h$  = inventory carrying charges per unit per unit of time

$q_j$  = size of the  $j$ th batch shipment

$\lambda$  = the proportional increase in successive shipments within a batch production run

$m$  = number of batch shipments in a production cycle

$p$  = production rate in units per unit time

$p_0$  = design production rate

$Q$  = production lot size

*Definitions:*

$I^{reg}$  = "regular" inventory

$I^{wait}$  = inventory due to waiting times

$IC$  = inventory carrying costs

$PC$  = production costs

$TC$  = total costs in the planning period

$TWI$  = time weighted inventory

**Model formulation and solution****The rigid case***a) Equal-sized batch shipments*

The first scenario we analyze is the case where equal-sized batch shipments are transported to the subsequent stage and where the production rate may be varied only before the start of a production run. The corresponding inventory time plots are illustrated in figure 1 (cf. the bold lines. Note that the dashed lines represent the case of a reduced production rate). It is shown that the producing stage manufactures a lot of size  $Q$  in  $mt_p$  time units, and that the first batch is shipped to the subsequent stage directly after its completion. Due to  $p > d$ , the second batch is finished before the first batch is completely used up, wherefore it has to be kept in stock for  $t_{w,1} = t_v - t_p$  time units.



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 Figure 1  
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The time weighted inventory in this case consists of a “regular” inventory, which corresponds to the triangles shown in figure 1, and inventory due to waiting times. The “regular” inventory can be expressed as follows:

$$(1) \quad I^{reg} = \frac{Q^2}{2m} \left( \frac{1}{p} + \frac{1}{d} \right)$$

Inventory due to waiting times can be calculated as:

$$(2) \quad I^{wait} = \frac{Q^2}{m^2} \sum_{i=2}^m i \left( \frac{1}{d} - \frac{1}{p} \right) = \frac{Q^2}{2m} (m-1) \left( \frac{1}{d} - \frac{1}{p} \right)$$

The time weighted inventory is given as the sum of (1) and (2). The inventory carrying costs in the planning period thus equal:

$$(3) \quad IC = \frac{DQ}{2m} \left( \frac{1}{p} (2-m) + \frac{m}{d} \right) h$$

Apart from the inventory carrying costs, the system encounters production costs which vary with a varying production rate. In the following, we assume that the unit production cost function is of the form specified by Eiamkanchanalai and Banerjee (1999), wherefore we can conclude that:

$$(4) \quad PC = Dc_p(p) = D(a_0p^2 - a_1p + a_2)$$

The total costs consist of the sum of (3) and (4) and the setup and transportation costs in the planning period, which amount to  $(c_s + mc_T)D/Q$ . Thus, it follows that:

$$(5) \quad TC = \frac{DQ}{2m} \left( \frac{m}{d} + \frac{2-m}{p} \right) h + (c_s + mc_T) \frac{D}{Q} + Dc_p(p)$$

As might be easily shown, (5) is convex in  $Q$  for given values for  $m$  and  $p$ . Thus, the optimal solution for  $Q$  is given as:

$$(6) \quad Q^* = \sqrt{\frac{2m(c_s + mc_T)}{\left( \frac{m}{d} + \frac{2-m}{p} \right) h}}$$

Substituting (6) in (5), the total cost function reduces to:

$$(7) \quad TC = D \sqrt{\frac{2}{m} \left( \frac{m}{d} + \frac{2-m}{p} \right) h (c_s + mc_T)} + Dc_p(p)$$

As could be shown by Szendrovits and Drezner (1980), (7) is quasi-convex in  $m$ . Thus, for a given value for  $p$ , an optimal solution for  $m$  may be calculated with the help of the following optimality condition:

$$(8) \quad TC(m^*-1) \geq TC(m^*) \leq TC(m^*+1)$$

However, since (7) is not necessarily convex in  $p$ , we state the following theorems to derive a solution:

*Theorem 1:* For  $m = 1$ , producing with  $p < p_0$  cannot be optimal.

Proof: see appendix 1.

*Theorem 2:* For  $m = 2$ , it is optimal to produce with  $p = p_0$ .

Proof: see appendix 2.

*Theorem 3:* For  $m > 2$ , producing with  $p > p_0$  cannot be optimal.

Proof: see appendix 3.

To calculate a good, but not necessarily optimal, value for  $p$ , we can apply a one-dimensional line search algorithm. The following procedure can be used to find a solution for  $m^*$ ,  $p^*$ , and  $Q^*$ :

Step 1: Set  $m = 1$  and search  $p \in [p_0; p_{max}]$  that minimises  $TC$ . Set  $TC^* = TC$ ,  $m^* = m$ , and  $p^* = p$ .

Step 2: Set  $m = 2$  and  $p = p_0$ . If  $TC > TC^*$ , Goto Step 4.

Set  $TC^* = TC$ ,  $m^* = m$ ,  $p^* = p$ , and  $m = m+1$

Step 3: Search  $p \in [p_{min}; p_0]$  that minimises  $TC$ . If  $TC > TC^*$ , Goto Step 4.

Set  $TC^* = TC$ ,  $m^* = m$ ,  $p^* = p$ , and  $m = m+1$ . Goto Step 3.

Step 4: Find  $Q^*$  from (6).

#### b) Unequal-sized batch shipments

In the case where unequal-sized batches are shipped to the subsequent stage, the size of the  $j$ th batch can be calculated as follows (see Goyal, 1977):

$$(9) \quad q_j = q_1 \lambda^{j-1} = q_1 (p/d)^{j-1} \quad \text{with} \quad Q = \sum_{i=1}^m q_i = q_1 \sum_{i=1}^m \lambda^{i-1}$$

The corresponding inventory time plots that result if  $t_{p,j} = t_{v,j-1}$  are illustrated in figure 2 (cf. the bold lines). The dashed lines again illustrate the case of a reduced production rate). As can be seen, there is no inventory due to waiting times if subsequent batches increase by  $\lambda = p/d$ .

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Figure 2  
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The time weighted inventory can be calculated as follows:

$$(10) \quad \begin{aligned} TWI &= \frac{q_1}{2} \left( \frac{q_1}{p} + \frac{q_1}{d} \right) + \frac{q_2}{2} \left( \frac{q_2}{p} + \frac{q_2}{d} \right) + \dots + \frac{q_m}{2} \left( \frac{q_m}{p} + \frac{q_m}{d} \right) = \\ &= \frac{q_1^2}{2} \left( \frac{1}{p} + \frac{1}{d} \right) \sum_{i=1}^m (\lambda^{i-1})^2 = \frac{q_1^2}{2} \left( \frac{1}{p} + \frac{1}{d} \right) \frac{\lambda^{2m} - 1}{\lambda^2 - 1} \end{aligned}$$

The total costs can now be derived if the time weighted inventory is multiplied with the number of lots in the planning period ( $D/Q$ ) and the unit inventory carrying charges per unit of time  $h$ , and if the setup costs per order and the transportation costs per batch are considered in addition:

$$(11) \quad TC = \frac{Dq_1}{2} \left( \frac{1}{p} + \frac{1}{d} \right) \frac{\lambda^m + 1}{\lambda + 1} h + (c_s + mc_T) \frac{D(\lambda - 1)}{q_1(\lambda^m - 1)} + Dc_p(p)$$

Note that (11) equals (5) for  $\lambda = 1$  and  $q_1 = Q/m$ . Since (11) is convex in  $q_1$  for given values for  $m$  and  $p$ , a solution for the optimal first batch size can be calculated with the help of differential calculus. It follows:

$$(12) \quad q_1^* = \sqrt{\frac{2(\lambda - 1)(\lambda + 1)(c_s + mc_T)}{(\lambda^m - 1)(\lambda^m + 1) \left( \frac{1}{p} + \frac{1}{d} \right) h}}$$

Substituting (12) in (11), the total cost function reduces to:

$$(13) \quad TC = D \sqrt{2h(c_s + mc_T) \left( \frac{1}{d} - \frac{1}{p} \right) \frac{\left( \frac{p}{d} \right)^m + 1}{\left( \frac{p}{d} \right)^m - 1}} + Dc_p(p)$$

Since (13) has a unique minimum in  $m$  for given values of  $p$  (see Goyal, 1977), an optimal solution may be derived with the help of the following optimality condition:

$$(14) \quad TC(m^*-1) \geq TC(m^*) \leq TC(m^*+1)$$

However, (13) is not necessarily convex in  $p$ . To derive a solution for the production rate, we state the following theorem and proposition:

*Theorem 4:* For  $m = 1$ , producing with  $p < p_0$  cannot be optimal.

Proof: follows directly from theorem 1.

*Proposition 1:* Assume that  $\hat{p}$  is the production rate that minimises inventory carrying cost. If  $\hat{p}|_{m=2} > p_0$ , the optimal solution for  $p$  is contained in the interval  $[p_{min}; d(1 + \sqrt{2})]$ , and otherwise in the interval  $[p_{min}; p_0]$

Explanation: see appendix 4.

To calculate a good, but not necessarily optimal, value for  $p$ , we can again apply a one-dimensional line search algorithm over the intervals identified above. The following procedure can be used to find a solution for  $m^*$ ,  $p^*$ , and  $q_1^*$ :

Step 1: Set  $m = 1$  and search  $p \in [p_0; p_{max}]$  that minimises  $TC$ . Set  $TC^* = TC$ ,  $m^* = m$ , and  $p^* = p$ . Set  $m = m+1$ .

Step 2: Search  $p \in [p_{min}; \text{Max}[p_0, d(1 + \sqrt{2})]]$  that minimises  $TC$ . If  $TC > TC^*$ , Goto Step 3. Set  $TC^* = TC$ ,  $m^* = m$ ,  $p^* = p$ , and  $m = m+1$ . Goto Step 2.

Step 3: Find  $q_1^*$  from (12).

## The flexible case

### a) Equal-sized batch shipments

So far, we assumed that the production rate has to be fixed before a production run starts. In the following, we relax this assumption and suppose that each batch may be produced with a separate production rate. In compliance with earlier work, we assume that the costs of changing the production rate are negligible.

If each batch is manufactured with a separate production rate, the “regular” inventory given in (1) has to be reformulated as follows:

$$(15) \quad I^{reg} = \frac{Q^2}{2m^2} \left( \frac{m}{d} + \sum_{i=1}^m \frac{1}{p_i} \right)$$

Similarly, inventory due to waiting times can be calculated as:

$$(16) \quad I^{wait} = \frac{Q}{m} \left( \sum_{i=2}^m (i-1) \frac{Q}{d} - \sum_{j=2}^i \frac{Q}{p_j} \right) = \frac{Q^2}{2m^2} \left( m(m-1) \frac{1}{d} - 2 \sum_{i=2}^m \sum_{j=2}^i \frac{1}{p_j} \right)$$

The time weighted inventory is again given as the sum of (15) and (16). The inventory carrying costs in the planning period thus equal:

$$(17) \quad IC = \frac{Q^2}{2m^2} \left( \frac{m^2}{d} + \sum_{i=1}^m \frac{1}{p_i} - 2 \sum_{i=2}^m \sum_{j=2}^i \frac{1}{p_j} \right) h = \frac{Q^2}{2m^2} \left( \frac{m^2}{d} + \frac{1}{p_1} - \sum_{i=2}^m \left( \sum_{j=2}^{i-1} \frac{1}{p_j} + \sum_{j=2}^i \frac{1}{p_j} \right) \right) h$$

Considering production costs and setup and transportation costs leads to the total cost function:

$$(18) \quad TC = \frac{DQ}{2m^2} \left( \frac{m^2}{d} + \frac{1}{p_1} - \sum_{i=2}^m \left( \sum_{j=2}^{i-1} \frac{1}{p_j} + \sum_{j=2}^i \frac{1}{p_j} \right) \right) h + (c_s + mc_T) \frac{D}{Q} + \frac{D}{m} \sum_{i=1}^m c_p(p_i)$$

Since (18) is convex in  $Q$  for given values for  $m$  and a given  $p$ -vector, the optimal solution for  $Q$  is given as follows:

$$(19) \quad Q^* = \sqrt{\frac{2m^2(c_s + mc_T)}{\left( \frac{m^2}{d} + \frac{1}{p_1} - \sum_{i=2}^m \left( \sum_{j=2}^{i-1} \frac{1}{p_j} + \sum_{j=2}^i \frac{1}{p_j} \right) \right) h}}$$

Substituting (19) in (18) leads to:

$$(20) \quad TC = \frac{D}{m} \left( \sqrt{2(c_s + mc_T) \left( \frac{m^2}{d} + \frac{1}{p_1} - \sum_{i=2}^m \left( \sum_{j=2}^{i-1} \frac{1}{p_j} + \sum_{j=2}^i \frac{1}{p_j} \right) \right) h} + \sum_{i=1}^m c_p(p_i) \right)$$

Since (20) is not necessarily convex in  $p_i$  and  $m$ , we state the following theorems to derive a good solution:

*Theorem 5:* Producing the first batch with  $p_1 < p_0$  cannot be optimal. Producing the batches 2 to  $m$  with  $p_i > p_0$  cannot be optimal.

Proof: see appendix 5.

*Theorem 6:* For  $j > k$ ,  $p_j < p_k$  cannot be optimal.

Proof: see appendix 5.

To calculate a good, but not necessarily optimal, value for  $p_i$ , we can again apply a line search algorithm over the intervals identified above. Further, we calculate a solution for  $m$  with the help of the following condition:

$$(21) \quad TC(m^*-1) \geq TC(m^*) \leq TC(m^*+1)$$

The following procedure can be applied to find a solution for  $m^*$ ,  $Q^*$ , and the  $p$ -vector:

Step 1: Set  $m = 1$  and search  $p_1 \in [p_0; p_{max}]$  that minimises  $TC$ . Set  $TC^* = TC$ ,  $m^* = m$ , and  $p_1^* = p_1$ . Set  $m = m+1$ .

Step 2: Search  $p_j \in [p_{\min}; \text{Max}[p_0, d(1 + \sqrt{2})]] \forall j \in [2, \dots, m]$  that minimises  $TC$ . If  $TC > TC^*$ ,  
Goto Step 3.

Set  $TC^* = TC$ ,  $m^* = m$ ,  $p_j^* = p_j \forall j \in [2, \dots, m]$ , and  $m = m+1$ . Goto Step 2.

Step 3: Find  $q_1^*$  from (12).

### b) Unequal-sized batch shipments

In the case where unequal-sized batches are transported to the subsequent stage, the size of the  $j$ th batch depends on the size of the first shipment and the production rate of batches 2 to  $j$  (cf. figure 3). Thus, it follows that

$$(22) \quad q_j = q_1 \prod_{i=2}^j \frac{p_i}{d} \quad \text{and} \quad \sum_{j=1}^m q_j = q_1 \sum_{j=1}^m \prod_{i=2}^j \frac{p_i}{d}$$

Figure 3

The time weighted inventory can be calculated as follows:

$$(23) \quad \begin{aligned} TWI &= \frac{q_1}{2} \left( \frac{q_1}{p_1} + \frac{q_1}{d} \right) + \frac{q_2}{2} \left( \frac{q_2}{p_2} + \frac{q_2}{d} \right) + \dots + \frac{q_m}{2} \left( \frac{q_m}{p_m} + \frac{q_m}{d} \right) = \\ &= \frac{q_1^2}{2} \left( \frac{1}{p_1} + \frac{1}{d} \right) + \frac{q_1^2}{2} \left( \frac{p_2}{d} \right)^2 \left( \frac{1}{p_2} + \frac{1}{d} \right) + \dots + \frac{q_1^2}{2} \left( \prod_{i=2}^m \frac{p_i}{d} \right)^2 \left( \frac{1}{p_m} + \frac{1}{d} \right) = \\ &= \frac{q_1^2}{2} \sum_{i=1}^m \left( \prod_{j=2}^i \frac{p_j}{d} \right)^2 \left( \frac{1}{p_i} + \frac{1}{d} \right) \end{aligned}$$

The total costs are again given if the time weighted inventory is multiplied with the number of lots in the planning period ( $D/\sum q_i$ ) and the unit inventory carrying charges per unit of time  $h$ , and if the setup costs per order and the transportation costs per batch are considered in addition:

$$(24) \quad TC = \frac{D}{\sum_{j=1}^m \prod_{i=2}^j \frac{p_i}{d}} \left( \frac{q_1 h}{2} \sum_{i=1}^m \left( \prod_{j=2}^i \frac{p_j}{d} \right)^2 \left( \frac{1}{p_i} + \frac{1}{d} \right) + c_s + mc_T + \sum_{i=1}^m c_p(p_i) \prod_{j=2}^i \frac{p_j}{d} \right)$$

Since (24) is convex in  $q_1$  for given values for  $m$  and  $p_i$ , the optimal solution for  $q_1$  is given as:

$$(25) \quad q_1^* = \sqrt{\frac{2(c_s + mc_T)}{\sum_{i=1}^m \left( \prod_{j=2}^i \frac{p_j}{d} \right)^2 \left( \frac{1}{p_i} + \frac{1}{d} \right) h}}$$

Substituting (25) in (24) leads to:

$$(26) \quad TC = \frac{D}{\sum_{j=1}^m \prod_{i=2}^j \frac{p_i}{d}} \left( \sqrt{2(c_s + mc_T)h \sum_{i=1}^m \left( \prod_{j=2}^i \frac{p_j}{d} \right)^2 \left( \frac{1}{p_i} + \frac{1}{d} \right)} + \sum_{i=1}^m c_p(p_i) \prod_{j=2}^i \frac{p_j}{d} \right)$$

Since (26) is not necessarily convex in  $p_i$  and  $m$ , we state the following theorems to derive a good solution:

*Theorem 7:* Producing the first batch with  $p_1 < p_0$  cannot be optimal. Producing batches 2 to  $m$  with  $p_i > p_0$  cannot be optimal.

Proof: see appendix 6.

To calculate a good, but not necessarily optimal, value for  $p_i$ , we can again apply a line search algorithm over the intervals identified above. Further, we calculate a solution for  $m$  with the help of the following condition:

$$(21) \quad TC(m^*-1) \geq TC(m^*) \leq TC(m^*+1)$$

The following procedure can be applied to find a solution for  $m^*$ ,  $q_1^*$ , and the  $p$ -vector:

Step 1: Set  $m = 1$  and search  $p_1 \in [p_0; p_{max}]$  that minimises  $TC$ . Set  $TC^* = TC$ ,  $m^* = m$ , and  $p_1^* = p_1$ . Set  $m = m+1$ .

Step 2: Search  $p_j \in [p_{min}; p_{j-1,max}] \forall j \in [2, \dots, m]$  with  $p_{2,max} = p_0$  that minimises  $TC$ . If  $TC > TC^*$ , Goto Step 3.

Set  $TC^* = TC$ ,  $m^* = m$ ,  $p_j^* = p_j \forall j \in [2, \dots, m]$ , and  $m = m+1$ . Goto Step 2.

Step 3: Find  $q_1^*$  from (12).

## Numerical Examples

To illustrate the effect of a variable production rate on a two-stage production system with discrete inventory depletions, we solved a set of test problems shown in table 1. In addition to the data given below, we assumed that  $D = 1000$  and  $c_s = 250$ . Each of these test problems were solved using the procedures described above. The results for the rigid case are given in tables 2 and 3 and the results for the flexible case in tables 4 and 5.

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Table 1  
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For problem one, reducing the production rate resulted in lower total costs if the rigid case was assumed. As can be seen, the reduction in  $p$  is smaller in case unequal-sized batch

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3 shipments are transported to the subsequent stage. This is due to the fact that unequal-sized  
4 batch shipments reduce the total inventory in the system, wherefore a reduction in  $p$  leads to a  
5 relatively smaller reduction in inventory carrying costs. Providing that each batch may be  
6 produced with a separate production rate, we found a major difference between the two  
7 shipment policies analyzed above: in case equal-sized batch shipments are transported to the  
8 subsequent stage, the batches follow the pattern identified above, i.e.  $p_2^* \leq p_3^* \leq \dots \leq p_m^* \leq$   
9  $p_1^*$ . For the case of unequal-sized batch shipments, we found that the production rate of the  
10 first batch is increased with  $p_1 > p_0$ , and that the production rate of subsequent batches is at  
11 first gradually reduced and then increased again. However,  $p_m < p_0$  holds. This effect can be  
12 explained as follows: If the number of shipments that succeed a particular batch  $i$  with  $i \geq 2$  is  
13 large, a high production rate for batch  $i$  leads to an increase of the respective and all  
14 subsequent batch sizes (cf. expression (22)). This effect is enhanced if the size of batch  $i$  is  
15 large. Thus, a large difference between  $m$  and  $i$  and a large batch size  $q_i$  favour a low  
16 production rate. If the difference  $m-i$  is reduced, the production rate is increased again.  
17 Finally, it can be seen that the flexible case leads to lower total costs than the rigid case for all  
18 problems we tested, which supports the findings of prior studies.

32 -----  
33 Table 2  
34 -----  
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36  
37 Problems two and three illustrate the effect of a variation of the inventory carrying charges  
38 per unit per unit of time on the selection of the production rate. An increase in  $h$  leads to  
39 higher inventory carrying costs, which enhances the advantage that results from varying the  
40 production rate. In contrast, if  $c_T$  is increased, the number of batch shipments and the  
41 proportion of inventory due to waiting times in the total inventory are decreased. This leads to  
42 a lower variation in  $p_i$  in the rigid case and a higher production rate for the first batch in case  
43 different production rates are allowed for each batch shipment.

49 -----  
50 Table 3  
51 -----  
52

53  
54 The impact that results from varying the shape of the unit production cost function is  
55 illustrated in problems five and six. If the function is compressed, increasing or decreasing the  
56 production rate leads to a sharp increase in  $PC$ . Thus, the higher the slope of the unit  
57 production cost function, the lower the variation in  $p_i$  for both the rigid and the flexible case.  
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Finally, problems seven and eight illustrate the effect of a variation in the interval of feasible production rates.

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Table 4  
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Table 5  
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## Conclusion

In this paper, we considered a two-stage production system with equal- and unequal-sized batch shipments. We extended earlier work by treating the production rate as a decision variable and analyzed the resulting problem for both the cases where the production rate is fixed and variable during a production run. In a numerical study, we illustrated that deviating from the “design production rate” of a production system may reduce inventory carrying costs and thus lead to lower total costs. In order to increase the scope of our analysis, the model presented in this article could be extended to include costs of changing the production rate or a different unit production cost function (see for example Khouja (1995) or Moon and Christy (1998)).

## Appendices

### Appendix 1

For  $m = 1$ , (7) reduces to:

$$(A-1) \quad TC = D \sqrt{2 \left( \frac{1}{d} + \frac{1}{p} \right) h (c_S + c_T)} + D c_p (p)$$

As can be seen, the first addend in (A-1) decreases with an increasing production rate, while the second increases for both an increasing and a decreasing production rate. Consequently, it cannot be optimal to produce with a production rate smaller than  $p_0$ , since there is always a  $p \in [p_0, p_{max}]$  that leads to lower total costs.

### Appendix 2

For  $m = 2$ , (7) reduces to:

$$(A-2) \quad TC = D\sqrt{\frac{2}{d}h(c_s + 2c_T)} + Dc_p(p)$$

As can be seen, the first addend in (A-2) is independent of  $p$ , while the second increases for both an increasing and a decreasing production rate. Consequently, (A-2) is minimised with  $p = p_0$ .

To explain this effect, it is first necessary to analyze the variation of  $I^{reg}$  and  $I^{wait}$  for the case that  $p$  is varied to  $p'$ . It follows that:

$$(A-3) \quad \Delta I^{reg} = I^{reg}(p) - I^{reg}(p') = \frac{Q^2}{2m} \left( \frac{1}{p} - \frac{1}{p'} \right)$$

$$(A-4) \quad \Delta I^{wait} = I^{wait}(p) - I^{wait}(p') = \frac{Q^2}{2m} (m-1) \left( \frac{1}{p'} - \frac{1}{p} \right)$$

The sum of (A-3) and (A-4) equals:

$$(A-5) \quad \Delta I = \Delta I^{reg} + \Delta I^{wait} = \frac{Q^2}{2} \left( \frac{2}{m} - 1 \right) \left( \frac{1}{p} - \frac{1}{p'} \right)$$

For  $m = 2$ , (A-5) equals 0. It follows that in case  $m = 2$ , a variation in  $p$  leads to a decrease (increase) in  $I^{reg}$  that is exactly compensated by an increase (decrease) in  $I^{wait}$  (it can be seen in figure 1 by comparing the bold and dashed lines that a decrease in  $p$  leads to a higher regular inventory and a lower inventory due to waiting times). Thus, since  $c_p(p)$  increases with an increasing or a decreasing production rate, it is optimal to produce with  $p = p_0$ .

### Appendix 3

It is obvious that the expression  $(2-m)/p$  in (7) is negative for  $m > 2$ . Thus, the first addend in (7) can be reduced by lowering the production rate.

To explain this effect, we analyze the variation of  $I^{reg}$  and  $I^{wait}$  for the case that  $p$  is varied to  $p'$ . As can be seen in (A-3) and (A-4), an increase in  $p$  leads to a decrease in  $I^{reg}$  and an increase in  $I^{wait}$ . To assess the total effect, we refer to (A-5), which is positive for  $m > 2$  and  $p' < p$ . Thus, we conclude that in case  $m > 2$ , a reduction in  $p$  leads to an increase in  $I^{reg}$ , which is always compensated by a decrease in  $I^{wait}$ . Further, it follows that a decrease in  $p$  is especially beneficial for high values for  $m$ , since the multiplier  $(2/m-1)$  enhances the effect of manufacturing with a reduced production rate for high values for  $m$ . However, the marginal decrease in inventory which results if  $p$  is reduced to  $p'$  decreases with an increase in  $m$ .

### Appendix 4

Minimising inventory carrying costs is equivalent to minimising

$$(A-6) \quad \left( \frac{1}{d} - \frac{1}{p} \right) \frac{\left( \frac{p}{d} \right)^m + 1}{\left( \frac{p}{d} \right)^m - 1} = \alpha\beta$$

As can be seen,  $\alpha$  increases with an increasing production rate and approaches  $1/d$  if  $p$  approaches infinity. In contrast,  $\beta$  is negatively correlated with  $p$  and approaches 1 if  $p$  approaches infinity. If  $m$  approaches infinity,  $\beta$  approaches -1 for  $p < d$  and 1 for  $p > d$ . Further, we conclude that a decrease in  $\beta$ , which occurs if  $p$  is increased to  $p'$ , is larger for  $m$  than  $m'$  with  $m < m'$ .

If  $m = 2$ , (A-6) is convex in  $p$  and  $\hat{p}$  minimises (A-6) with

$$(A-7) \quad \hat{p} = d(1 + \sqrt{2})$$

If  $m > 2$ , it is difficult to derive a closed-form solution for  $\hat{p}$ . However, we identified (A-6) to be quasi-convex in  $p$  with the form shown in figure 4 in numerical studies. Further, we conclude that  $\hat{p}(m) > \hat{p}(m+1)$  and  $\hat{p}(m \rightarrow \infty) = d$  due to the relations identified above.

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Figure 4  
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Figure 5 illustrates the range of possible values for  $p$  that contains the optimal solution. Thus, if  $\hat{p}|_{m=2} > p_0$ , the value for  $p$  that minimises (13) can be found in the interval  $[p_{min}; d(1 + \sqrt{2})]$ , and otherwise in the interval  $[p_{min}; p_0]$ .

-----  
Figure 5  
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## Appendix 5

Minimising inventory carrying costs in  $p_i$  is equivalent to minimising

$$(A-8) \quad \frac{1}{p_1} - \sum_{i=2}^m \left( \sum_{j=2}^{i-1} \frac{1}{p_j} + \sum_{j=2}^i \frac{1}{p_j} \right)$$

As can be easily seen, both minuend and subtrahend decrease with an increasing production rate. Thus, (A-8) can be minimised by increasing  $p_1$  and decreasing  $p_2, \dots, p_m$ . In addition, it can be seen that the sum function in the subtrahend leads to a higher weight for the production rate of earlier batch shipments, which is due to the fact that the production rate of batch  $j$  with  $j > 1$  influences the waiting times of all subsequent batch shipments. Thus, since a variation in

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$p_i$  always leads to higher unit production costs, we conclude that  $p_1^* \in [p_0, p_{max}]$  and  $p_i^* \in [p_{min}; p_0]$  for  $i > 1$ . Further, due to the relations identified above, we conclude that  $p_2^* \leq p_3^* \leq \dots \leq p_m^*$ .

## Appendix 6

As can be seen in figures 2 and 3,  $p_1$  influences the inventory build-up of the first batch, but not the size of the shipments 2 to  $m$ . Thus, a decrease (increase) in  $p_1$  leads to a slower (faster) inventory build-up and consequently higher (lower) inventory carrying costs. Since unit production costs are convex in  $p_i$  with a unique minimum in  $p_0$ , it cannot be optimal to produce with a production rate  $p_1 < p_0$ , which would lead to both an increase in  $IC$  and  $PC$ .

In contrast, a variation in  $p_i$  with  $i > 2$  influences both the inventory build-up of batch  $i$  and the size of the shipments  $i$  to  $m$ . This is due to the fact that the batches 2 to  $m$  are produced in the consumption time of the preceding batch, wherefore a decrease (increase) in  $p_i$  with  $i > 2$  leads to a lower (higher) batch size and consequently lower (higher) inventory carrying costs.

Thus, we can conclude that  $p_1^* \in [p_0; p_{max}]$  and  $p_i^* \in [p_{min}; p_0]$ .

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**Table 1**

#	$h$	$c_T$	$\{a_0, a_1, a_2\}$	$\{p_{min}, p_{max}\}$
1	5	200	$\{1/6000, 0.12, 24\}$	$\{320, 500\}$
2	10	200	$\{1/6000, 0.12, 24\}$	$\{320, 500\}$
3	15	200	$\{1/6000, 0.12, 24\}$	$\{320, 500\}$
4	5	400	$\{1/6000, 0.12, 24\}$	$\{320, 500\}$
5	5	200	$\{1/600, 1.2, 218.4\}$	$\{320, 500\}$
6	5	200	$\{1/60, 12, 2162.4\}$	$\{320, 500\}$
7	5	200	$\{1/6000, 0.12, 24\}$	$\{340, 500\}$
8	5	200	$\{1/6000, 0.12, 24\}$	$\{320, 360\}$

**Table 2**

#	$m^*$	$Q^*$	$p^*$	$TC^*$
1	4	472.07	345.14	6885.26
2	6	481.66	320	8687.46
3	6	393.28	320	10040.60
4	3	493.58	349.92	8292.36
5	4	465.39	358.64	6915.44
6	4	464.82	359.87	6918.18
7	4	472.07	345.14	6885.26
8	4	472.07	345.14	6885.26

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**Table 3**

#	$m^*$	$q_1^*$	$Q^*$	$p^*$	$TC^*$
1	7	71.30	826.66	349.52	6410.29
2	7	52.24	586.10	346.34	8061.51
3	8	42.22	539.88	339.20	9325.47
4	5	114.74	832.24	356.09	7809.66
5	6	73.15	721.59	359.22	6419.91
6	6	72.68	721.26	359.92	6420.84
7	7	71.30	826.66	349.52	6410.29
8	7	71.30	826.66	349.52	6410.29

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**Table 4**

#	$m^*$	$Q^*$	$\{p_1^*, \dots, p_m^*\}$	$TC^*$
1	5	582.63	{366.51, 320, 320, 336.90, 352.99}	6818.53
2	5	414.13	{369.12, 320, 320, 324.63, 349.85}	8591.36
3	5	338.92	{371.08, 320, 320, 320, 347.36}	9945.83
4	3	503.05	{369.23, 324.04, 349.71}	8242.14
5	4	466.20	{360.67, 356.56, 357.95, 359.32}	6911.57
6	4	464.90	{360.07, 359.66, 359.80, 359.93}	6917.80
7	4	475.69	{366.64, 340, 340, 352.84}	6851.92
8	5	582.23	{360, 320, 320, 336.92, 352.99}	6819.89

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**Table 5**

#	$m^*$	$q_1^*$	$Q^*$	$\{p_1^*, \dots, p_m^*\}$	$TC^*$
1	7	73.51	830.09	{364.16, 352.44, 344.86, 341.58, 342.73, 348.26, 358.07}	6401.01
2	8	52.24	663.97	{365.85, 349.28, 337.64, 331.09, 329.79, 333.80, 343.03, 357.36}	8041.99
3	9	43.43	605.92	{367.25, 347.49, 333.11, 324.03, 320.17, 321.48, 328.01, 339.87, 357.18}	9289.88
4	5	116.99	833.84	{366.53, 355.02, 350, 351.78, 360.29}	7804.21
5	6	73.34	721.82	{360.42, 359.27, 358.64, 358.52, 358.91, 359.81}	6419.26
6	6	72.70	721.29	{360.04, 359.93, 359.86, 359.85, 359.89, 359.98}	6420.77
7	7	73.51	830.09	{364.16, 352.44, 344.86, 341.58, 342.73, 348.26, 358.07}	6401.01
8	7	73.47	829.96	{360, 352.49, 344.90, 341.61, 342.74, 348.27, 358.07}	6401.27

## Captions for Figures and Tables

Figure 1: Inventory-time plots for the case of equal-sized batch shipments

Figure 2: Inventory-time plots for the case of unequal-sized batch shipments

Figure 3: Development of batch sizes for the case of a variable production rate

Figure 4: Shape of function (A-6)

Figure 5: Range of values for  $p$  that contains the optimal solution

Table 1: Test problems used for computational experimentation

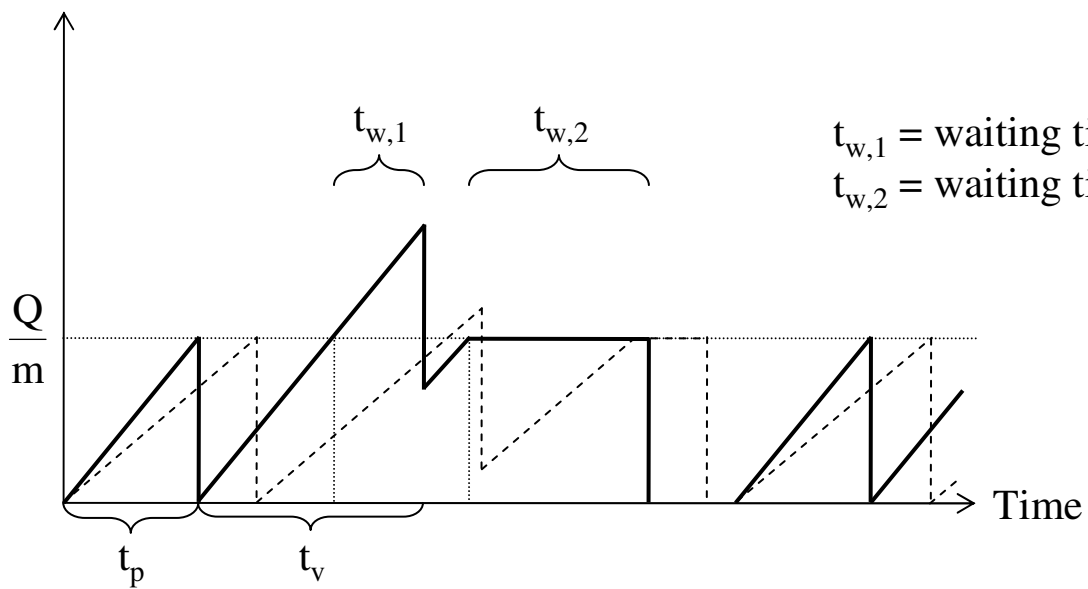
Table 2: Computational results for the rigid case with equal-sized batch shipments

Table 3: Computational results for the rigid case with unequal-sized batch shipments

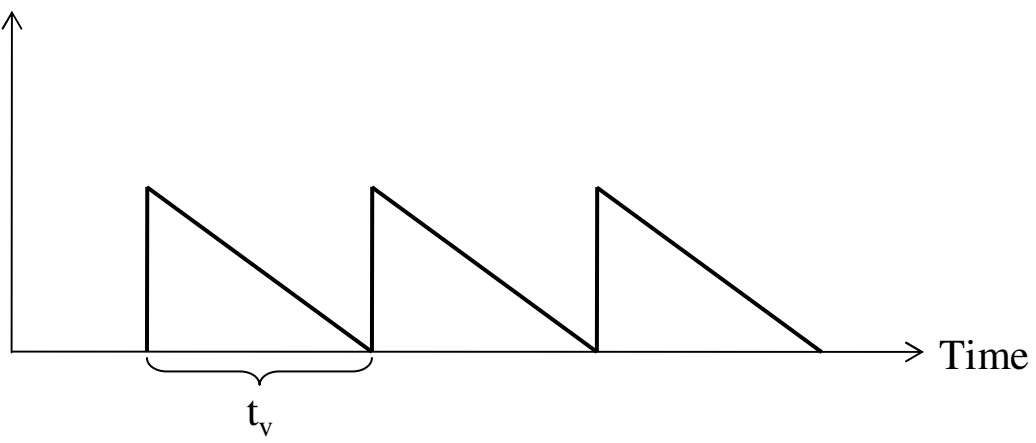
Table 4: Computational results for the flexible case with equal-sized batch shipments

Table 5: Computational results for the flexible case with unequal-sized batch shipments

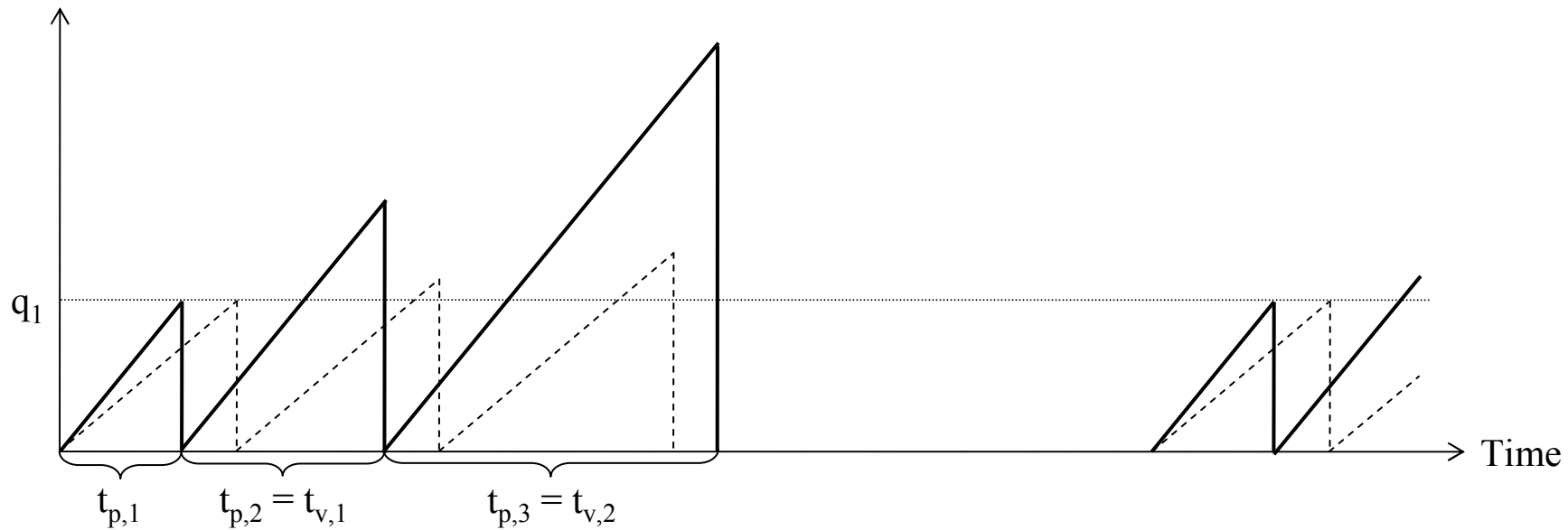
Inventory at the manufacturing stage



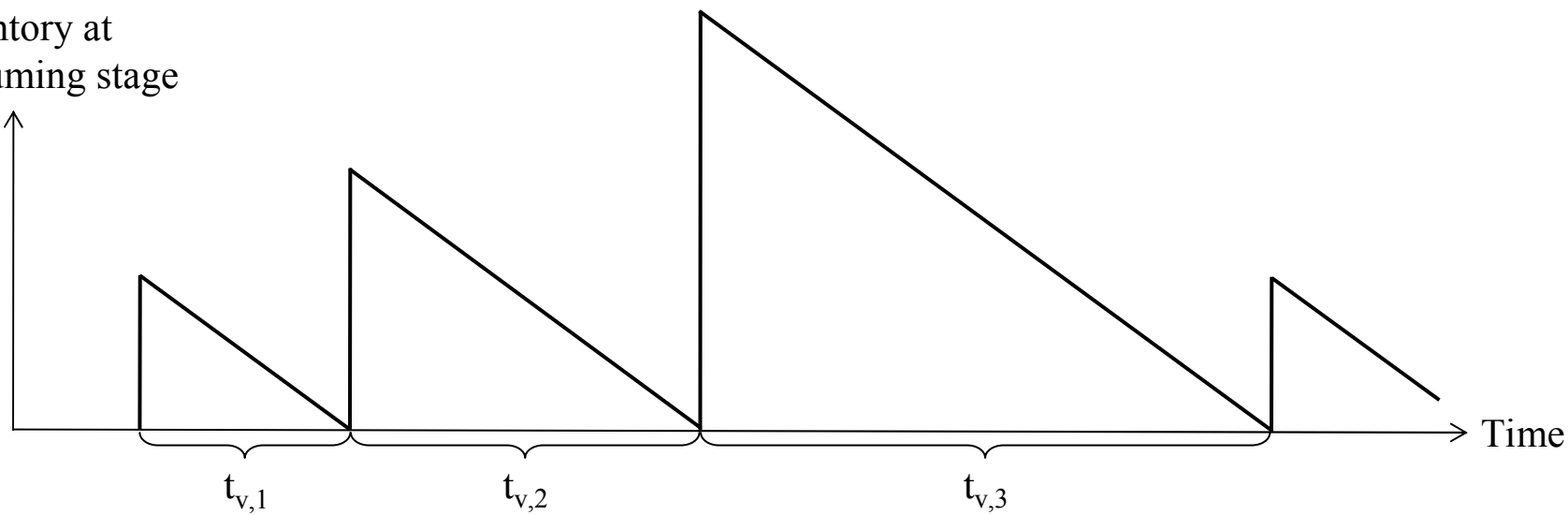
Inventory at the consuming stage



Inventory at the manufacturing stage



Inventory at the consuming stage

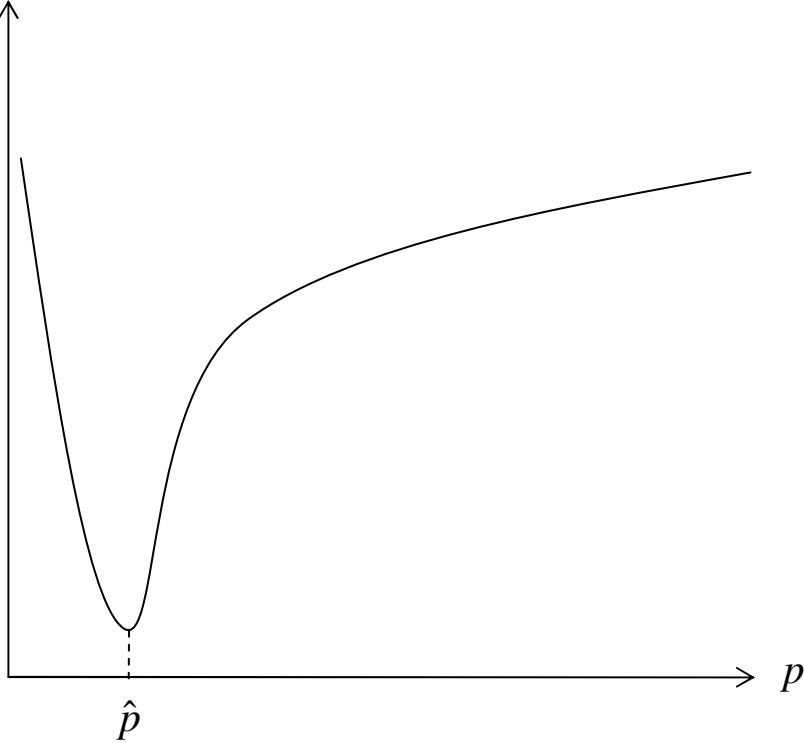


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size of batch	# 1	#2	#3	...	#m
	$q_1$	$q_1 \frac{p_2}{d}$	$q_1 \frac{p_2 p_3}{d^2}$	...	$q_1 \prod_{i=2}^m \frac{p_i}{d}$

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