On the size of identifying codes in triangle-free graphs

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Abstract

In an undirected graph $G = (V, E)$, a subset $C \subseteq V$ such that C is a dominating set of G, and each vertex in V is dominated by a distinct subset of vertices from C, is called an identifying code of G. The concept of identifying codes was introduced by Karpovsky et al. in 1998. Because of the variety of its applications, for example for fault-detection in networks or the location of fires in facilities, it has since been widely studied.

For a given graph G, let $\gamma^{\text{ID}}(G)$ be the minimum cardinality of an identifying code in G. In this paper, we show that for any connected triangle-free graph G on n vertices having maximum degree $\Delta \geq 2$, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{3(\Delta+1)}$.

Keywords: Identifying code, Dominating set, Triangle-free graph, Maximum degree

1. Introduction

Identifying codes, which have been introduced in [16], are dominating sets having the additional property that each vertex of the graph can be uniquely identified using its neighbourhood within the identifying code. They have found numerous applications, such as fault-diagnosis in multiprocessor networks [16], the placement of networked fire detectors in complexes of rooms and corridors [20], compact routing [17], or the analysis of secondary RNA structures [14]. Identifying codes are a variation on the earlier concept of locating-dominating sets (cf. e.g. [7, 21, 22]). For a given graph, the problem of finding a minimum identifying code is known to be NP-hard [6]. Identifying codes have been studied in specific graph classes such as cycles [1, 13], trees [2, 3], grids [16] or hypercubes [15, 19]. Extremal problems regarding the minimum size of an identifying code have been studied in [5, 8, 9, 10, 12, 18].

Herein, we further investigate these extremal questions by giving new upper bounds on the size of minimum identifying codes for triangle-free graphs using their maximum degree.

Let us first give some general notations and definitions which will be used throughout the paper. Let $G = (V, E)$ be a simple undirected graph, and denote by $n = |V|$ the order of G and by Δ its maximum vertex degree. For a vertex v of G, the *ball* $B_r(v)$ is the set of all vertices of V which are at distance at most r from v. Let us also denote by $N[v] = B_1(v)$ the *closed neighbourhood* of v, and by $N(v) = N[v] \setminus \{v\}$, the *open neighbourhood* of v. For a set X of vertices of G, we define $N(X)$ to be the union of the open neighbourhoods of all vertices of X. Whenever we find it necessary to emphasize on which graph G is considered, we write $N_G[u]$, $N_G(u)$ and $N_G(X)$. For a subset S of vertices of G, we denote by $G[S]$ the subgraph of G induced by S. A *matching* M of a graph G is a subset of edges of G such that no two edges of M have a common vertex. If within the set of all endpoints of the edges of M no other edges than the ones of M exist, we call M an *induced matching*. A graph is called *triangle-free* if it contains no cycles of length 3.

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Given a set S of vertices of G, we say that a vertex x of G is *S-isolated* if $x \in S$ and no neighbour of x belongs to S. A set S of vertices is called an *independent set* if for all $x, y \in S$, $xy \notin E$. We say that vertex u *dominates* vertex v if $v \in N[u]$. For two subsets C, U of vertices, we say that C *dominates* U if each vertex of U is dominated by some vertex of C. Set $C \subseteq V$ is called a *dominating set* of G if C dominates V. A pair $\{u, v\}$ of vertices of V are *separated* by some vertex $x \in V$ if x dominates exactly one of the vertices u and v. We call $C \subseteq V$ an *identifying code* of G if it is a dominating set of G, and for all pairs of vertices $u, v \in V$, u and v are separated by some vertex of C. The latter condition can be equivalently stated as $N[u] \cap C \neq N[v] \cap C$, or as $(N[u] \oplus N[v]) \cap C \neq \emptyset$ (denoting by $\oplus$ the symmetric difference of sets). In the following, we might simply call an identifying code a *code*. Given a graph G and a subset S of its vertices, we say that a set $C \subseteq S$ is an *S-identifying code* of G if C is an identifying code of $G[S]$.

A graph is said to be *identifiable* if it admits an identifying code. This is the case if and only if it does not contain a pair of so-called *twins*, that is, a pair u, v of vertices such that $N[u] = N[v]$ [4]. An example of a graph which is not identifiable is the complete graph K_n. For an identifiable graph G, we denote by $\gamma^{\text{ID}}(G)$ the cardinality of a minimum identifying code of G.

For any graph G on n vertices, the tight lower bound of $\gamma^{\text{ID}}(G) \geq \lceil \log_2(n+1) \rceil$ was given in [16], and all graphs reaching it have been described in [18]. In [16], it was also shown that the bound $\gamma^{\text{ID}}(G) \geq \frac{2n}{\Delta+2}$ holds, and all graphs reaching this bound have been described in [8].

For all identifiable graphs having at least one edge, an upper bound on this parameter is $\gamma^{\text{ID}}(G) \leq n - 1$ [5, 12]. This bound is tight, in particular for the star $K_{1,n-1}$ and other graphs which have been recently classified in [9].

When considering graphs of maximum degree Δ, there exist examples of specific graphs such that $\gamma^{\text{ID}}(G) = n - \frac{n}{\Delta}$ (e.g. the complete bipartite graph $K_{\Delta,\Delta}$, Sierpiński graphs [11] and other classes of graphs described in [8]), as well as classes of graphs with slightly smaller values of parameter γ^{ID} (for instance, it is shown in [2] that $\gamma^{\text{ID}}(T) = \lceil n - \frac{n}{\Delta - 1 + \frac{1}{\Delta}} \rceil$ for any complete $(\Delta - 1)$-ary tree T).

Note that for any connected graph G of maximum degree 2 (i.e. when G is either a path or a cycle), the exact value of $\gamma^{\text{ID}}(G)$ is known (see [1, 13]). In fact in that case, the bound $\gamma^{\text{ID}}(G) \leq \frac{n}{2} + \frac{3}{2} = n - \frac{n}{2} + \frac{3}{2}$ holds and is reached for infinitely many values of n (more precisely, this is the case when G is a cycle of odd order $n \geq 7$). It was shown in [9] that for any connected identifiable graph G of maximum degree Δ, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Theta(\Delta^5)}$. We conjecture that the following stronger bound on $\gamma^{\text{ID}}(G)$ holds.

Conjecture 1. *Let G be a connected identifiable graph of maximum degree $\Delta \geq 2$. Then $\gamma^{ID}(G) \leq n - \frac{n}{\Delta} + O(1)$.*

In this paper, we give a bound close to the one of our conjecture when G is a connected identifiable triangle-free graph by showing that in this case, $\gamma^{\text{ID}}(G) \leq n - \frac{n}{3(\Delta+1)}$ (Theorem 9). For triangle-free graphs of minimum degree 3, this bound is then improved to $\gamma^{\text{ID}}(G) \leq n - \frac{n}{2(\Delta+1)}$ (Theorem 11). Note that the proofs of these results are constructive and the corresponding codes can be constructed in polynomial time.

2. The upper bound

In this section, we first give some useful definitions and prove some preliminary results. These preliminaries will be used in a second part to prove our main results.

2.1. Preliminaries

We first introduce some specific definitions which will be used throughout this section.

The first definition introduces a notion which plays a central role in the proof of our main theorem.

Definition 2. *Given a graph G together with two disjoint subsets of vertices L and R, an induced matching M of G is called an* (L, R)-matching *if the following holds:*

- *R is the set of end-vertices of the edges of M and $L = N(R) \setminus R$,*

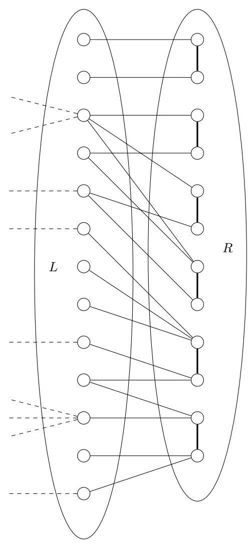

Figure 1: Example of an (L, R)-matching (thick edges)

- *L is an independent set in G, and*
- *every vertex x of R has degree at least 2 in G (i.e. $N(x) \cap L \neq \emptyset$).*

An illustration of an (L, R)-matching is given in Figure 1. Note that in some graphs, one cannot necessarily find two sets L and R such that there exists an (L, R)-matching M in the graph. Indeed, each edge of M must belong to at least some induced path on four vertices.

By noting that for any sets of vertices L, R of a graph G such that there exists an (L, R)-matching in G, $G[L \cup R]$ has no triangle and no isolated edge (i.e. two adjacent vertices of degree 2), we can make the following observation.

Observation 3. *Let G be a graph and L, R be subsets of vertices of G such that there exists an (L, R)-matching M of G. Then $G[L \cup R]$ is identifiable.*

In order to construct small identifying codes, given sets of vertices L, R of a graph G such that G admits an (L, R)-matching, we will construct special codes for the graph induced by the set $(L \cup R)$, defined as follows.

Definition 4. *Let G be an identifiable graph and L, R be subsets of vertices of G such that there exists an (L, R)-matching M of G, and let $G' = G[L \cup R]$. We say that $C \subseteq (L \cup R)$ is an (L, R)-*quasi-identifying code *of G if:*

1. *each vertex of $L \cup R$ is dominated by some vertex of C,*
2. *for each pair of vertices u, v of $L \cup R$, $C \cap N_{G'}[u] \neq C \cap N_{G'}[v]$, unless both u and v belong to L and $N_{G'}(u) = N_{G'}(v)$,*
3. *no vertex of C is C-isolated, and*
4. *for each edge e of M, at least one of the vertices of e belongs to C.*

To prove our main result, given an identifiable connected triangle-free graph $G = (V, E)$, we will construct a (possibly empty) (L, R)-matching of G for some sets L and R, and partition V into the set $L \cup R$ and its complement $V \setminus (L \cup R)$. We show in the lemmas of this section how to build an (L, R)-quasi-identifying code of bounded size (Lemmas 6 and 7), and how to compute a special independent set of G (Lemma 8). This independent set will be used to build a $V \setminus (L \cup R)$-identifying code of G. But first, the following lemma shows that we will be able to merge these two codes to form a valid identifying code of G.

Lemma 5. *Let G be an identifiable triangle-free graph and L, R be subsets of vertices of G such that there exists an (L, R)-matching M of G, and $G[V \setminus (L \cup R)]$ is identifiable. Suppose that for all vertices l, l' of L, $N_G(l) \neq N_G(l')$. Let C_1 be an (L, R)-quasi-identifying code of G and C_2 a $(V \setminus (L \cup R))$-identifying code of G where all the neighbours of vertices of L within $V \setminus (L \cup R)$ belong to C_2. Then $C_1 \cup C_2$ is an identifying code of G.*

Proof First note that by Observation 3 C_1 exists, and C_2 exists by assumption. Let us show that all pairs of vertices of G are separated. Since C_2 is a $(V \setminus (L \cup R))$-identifying code, all pairs of vertices of $V \setminus (L \cup R)$ are separated. Since C_1 is (L, R)-quasi-identifying, each vertex x of $L \cup R$ is adjacent to at least one vertex of $R \cap C_1$ (this follows from points number 1, 3 and 4 of Definition 4), which we denote $f_{C_1}(x)$. Moreover, by definition of the sets L and R, no vertex of $V \setminus (L \cup R)$ is adjacent to a vertex of R. Therefore all pairs of vertices x, y with $y \in L \cup R$ and $x \in V \setminus (L \cup R)$ are separated by $f_{C_1}(x)$. It remains to check the pairs of vertices of $L \cup R$. By contradiction, suppose there are two vertices u, v of $L \cup R$ which are not separated. By point number 2 of the definition of an (L, R)-quasi-identifying code, u and v belong to L and have the same neighbourhood within $L \cup R$. But since we assumed that they have distinct sets of neighbours in G and all their neighbours in $V \setminus (L \cup R)$ are in C_2, u and v are separated by the neighbours they do not have in common, a contradiction. □

Given two sets of vertices L, R of a graph G such that G admits an (L, R)-matching, the next two lemmas show how to build (L, R)-quasi-identifying codes in G whose size is bounded. We first deal with the special case where all vertices of R have degree 2.

Lemma 6. *Let G be an identifiable triangle-free graph and L, R be subsets of vertices of G such that there exists an (L, R)-matching M of G where all vertices of R have degree exactly 2. There is an $(L \cup R)$-identifying code $C_{L,R}$ of G having the following properties:*

1. $|C_{L,R}| \leq |L| + \frac{|R|}{2}$,
2. *no vertex of R is $C_{L,R}$-isolated,*
3. *for each edge of M, at least one of its endpoints belongs to $C_{L,R}$, and*
4. *at most half of the vertices of L do not belong to $C_{L,R}$.*

Proof Let us construct an $(L \cup R)$-identifying code of G. In order to simplify its construction, let us first define the graph $G_{L,R} = (L, E)$ with vertex set L and in which there is an edge between two vertices l_1 and l_2 if there exist in G two adjacent vertices r_1, r_2 of R, such that l_1 is a neighbour of r_1, and l_2 is a neighbour of r_2. In other words, we contract every path of length 3 of $G[L \cup R]$ having both endpoints in L into one edge.

From $G_{L,R}$ we build the oriented graph $\overrightarrow{G}_{L,R}$: we first orient the arcs arbitrarily, and will modify these orientations later on. Given an orientation of $\overrightarrow{G}_{L,R}$, we define the subset $S(\overrightarrow{G}_{L,R})$ of vertices of $L \cup R$ in the following way: all the vertices of L belong to $S(\overrightarrow{G}_{L,R})$, and for each arc $\overrightarrow{l_1 l_2}$ of $\overrightarrow{G}_{L,R}$ corresponding to the path $l_1 r_1 r_2 l_2$ in G, r_2 belongs to $S(\overrightarrow{G}_{L,R})$. An illustration is given in Figure 2, where the gray vertices belong to $S(\overrightarrow{G}_{L,R})$.

Note that for any orientation of $\overrightarrow{G}_{L,R}$, the set $S(\overrightarrow{G}_{L,R})$ has size $|L| + \frac{|R|}{2}$. Our aim is to get an orientation for which $S(\overrightarrow{G}_{L,R})$ is an $(L \cup R)$-identifying code of G. Note that every vertex of $L \cup R$ is dominated by $S(\overrightarrow{G}_{L,R})$. However, there might be some vertices which are not separated. Since L is an independent set and G is triangle-free, this is the case if and only if a code vertex l of L is adjacent to a code vertex r of R, but none of the other neighbours of l or r is in $S(\overrightarrow{G}_{L,R})$ (see Figure 3 for an illustration). This is equivalent to the case where l is of in-degree 1 in $\overrightarrow{G}_{L,R}$. In this case we modify the orientation of $\overrightarrow{G}_{L,R}$ in possibly two steps, in order to avoid this.

Let us describe the first step. Construct an arbitrary rooted spanning forest of $\overrightarrow{G}_{L,R}$. Go through all vertices of each tree of this forest, level by level in a bottom-up order. Modify the orientation of the arcs of the tree so that in $\overrightarrow{G}_{L,R}$, the in-degree of all vertices (except, eventually, for the root) becomes even —

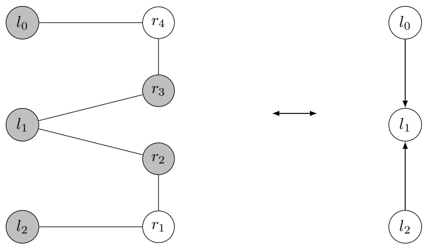

Figure 2: Correspondance between a subset of $L \cup R$ and $\overrightarrow{G_{L,R}}$

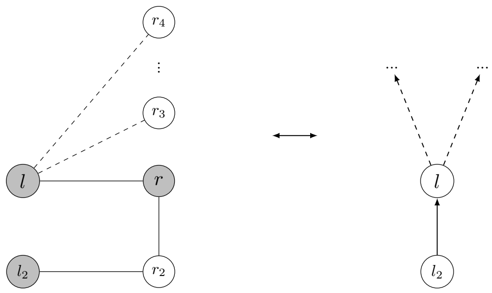

Figure 3: Vertices l and r are not separated

and therefore different from 1: if the in-degree of a vertex v is odd, the arc pointing from v to its parent is reversed, making the in-degree of v even, and adding 1 to the in-degree of the parent. Since the algorithm considers the vertices level by level from the bottom up to the root, at the end of the process only the roots of the trees of the spanning forest may have an odd in-degree. If this in-degree is equal to 1, they are in the situation depicted in Figure 3, where l is one of these roots, and l and r are not separated.

Now, let $C = S(\overrightarrow{G}_{L,R})$. If C is not yet an $(L \cup R)$-identifying code, then some roots of the previously constructed spanning trees are not separated. We fix this problem as follows.

Let $l \in L$ be such a root, with r its unique neighbour in $C \cap R$, and r_2, the neighbour of r in R. It is sufficient to take out l from C and to replace it by r_2 in order to get an $(L \cup R)$-identifying code of G (see Figure 4 for an illustration), without changing the cardinality of C. Indeed, we know from the previous discussion that in $G[L \cup R]$, the only vertices from the same connected component as l which are not separated by C, are l and r. Now, they are separated by r_2. Moreover, all the neighbours of l are still separated from the other vertices because they are all in $R \setminus C$ and therefore have a neighbour in $R \cap C$, which themselves have at least one neighbour in $L \cap C$. Since this process did not change the cardinality of C, we get the first property of the claim of the lemma.

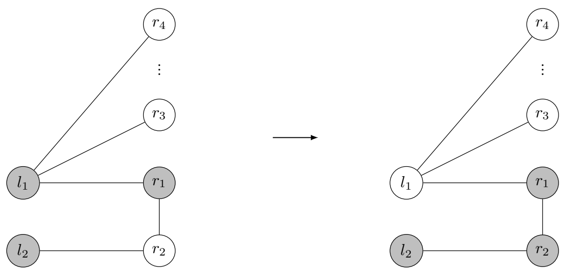

Figure 4: Local modification of the constructed code

Notice that there are at most $\frac{|L|}{2}$ such roots since every connected component of $G[L \cup R]$ contains at

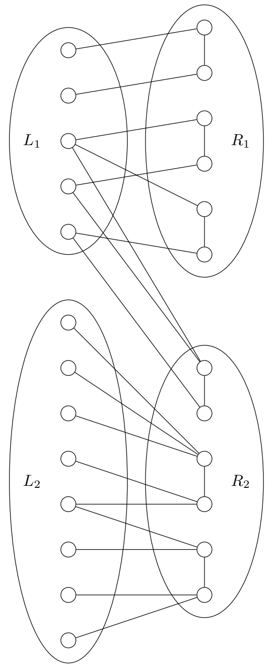

Figure 5: Illustration of the sets L_1, L_2, R_1, and R_2

least two vertices of L. Thus property number 4 of the claim of the lemma follows.

Properties number 2 and 3 are fulfilled by the construction of C since in every adjacent pair of R, at least one of its elements belongs to the code, and either it has a code neighbour in L if there was no modification done, or in R if a switch of two elements of L and R was necessary. This completes the proof. $\square$

We now deal with the general case where the vertices of R have minimum degree 2.

Lemma 7. *Let G be an identifiable triangle-free graph and L, R be subsets of vertices of G such that there exists an (L, R)-matching M of G. There exists a set L' of vertices of $L \cup R$ such that $|L'| \geq \frac{|L|}{3}$, and $C_{L,R} = (L \cup R) \setminus L'$ is an (L, R)-quasi-identifying code of G.*

Proof Let us first divide the sets L and R into the following subsets: let $R_1 \subseteq R$ be such that $r \in R_1$ if both r and its unique neighbour in R are of degree 2. Let $L_1 \subseteq L$ be the set of all neighbours of vertices of R_1, let $R_2 = R \setminus R_1$, and let $L_2 = L \setminus L_1$ (see Figure 5 for an illustration).

We can now use Lemma 6 to construct an $(L_1 \cup R_1)$-identifying code C_1 of G. We can choose C_1 such that the four properties described in Lemma 6 are fulfilled - in particular $|C_1| \leq |L_1| + \frac{|R_1|}{2}$.

Let us now describe the construction of two distinct (L, R)-quasi-identifying codes C_a and C_b (see the Appendix for a proof of the validity of the constructions).

1. Construct code C_a as the union of C_1, the set L_2, and a subset of R_2 such that for each pair r, r' of adjacent vertices of R_2, exactly one of them (say r) is in the code and it has at least two neighbours in L (such vertices exist by definition of R_2). In order to make sure that r and r' will be separated, if r has less than two neighbours within $C_a \cap L$ (this may happen if some of its neighbours are in L_1, and they are not in the code), we pick an additional one — which exists since r has at least two neighbours in L — and put it into C_a. Note that this will be done for at most $\frac{|R_2|}{2}$ vertices of R_2. Moreover, by property number 4 of Lemma 6, there are at most $\frac{|L_1|}{2}$ vertices of L_1 out of C_1. Finally, for each C_a-isolated vertex l of L, take it out of C_a and put an arbitrary neighbour of l into C_a (this operation does not affect the size of C_a).
   The size of C_a is therefore at most $|L_1| + \frac{|R_1|}{2} + |L_2| + \frac{|R_2|}{2} + \min\left\{\frac{|L_1|}{2}, \frac{|R_2|}{2}\right\}$.

2. Construct C_b as the union of C_1, the set R_2, and, for every pair $\{r, r'\}$ of adjacent vertices of R_2, one neighbour in L of either r or r'. Finally, similarly to the construction of C_a, we get rid of each C_b-isolated vertex l of L by taking l out of C_b and putting an arbitrary neighbour of l into C_b instead. The size of C_b is at most $3\frac{|R_2|}{2} + |L_1| + \frac{|R_1|}{2}$.

Let us now determine a lower bound on the cardinality of $(L \cup R) \setminus C_x$ for $x \in \{a, b\}$. Taking into account that $|L_1| \le |R_1|$, we obtain:

- $|(L \cup R) \setminus C_a| \ge \frac{|R_1|}{2} + \frac{|R_2|}{2} - \min\left\{\frac{|L_1|}{2}, \frac{|R_2|}{2}\right\}$. Thus $|(L \cup R) \setminus C_a| \ge \frac{|R_1|}{2} + \frac{|R_2|}{2} - \frac{|L_1|}{2} \ge \frac{|R_2|}{2}$ and $|(L \cup R) \setminus C_a| \ge \frac{|R_1|}{2} \ge \frac{|L_1|}{2}$, and
- $|(L \cup R) \setminus C_b| \ge |L_2| + \frac{|R_1|}{2} - \frac{|R_2|}{2} \ge |L_2| + \frac{|L_1|}{2} - \frac{|R_2|}{2}$.

Let $C_{L,R} \in \{C_a, C_b\}$ be the code having the minimum cardinality. Then, denoting $b = \frac{\max\left\{|L_1|,\ |R_2|\right\}}{|L|}$ we have:

$$\begin{aligned}
|(L \cup R) \setminus C_{L,R}| &\ge \tfrac{1}{2} \cdot \max\left\{|R_2|,\ |L_1|,\ 2|L_2| + |L_1| - |R_2|\right\} \\
&= \tfrac{|L|}{2} \cdot \max\left\{\tfrac{|R_2|}{|L|},\ \tfrac{|L_1|}{|L|},\ \tfrac{2|L_2|+|L_1|-|R_2|}{|L_1|+|L_2|}\right\} \\
&= \tfrac{|L|}{2} \cdot \max\left\{\tfrac{\max\left\{|L_1|,\ |R_2|\right\}}{|L|},\ \tfrac{2|L_1|+2|L_2|-|L_1|-|R_2|}{|L_1|+|L_2|}\right\} \\
&\ge \tfrac{|L|}{2} \cdot \max\left\{\tfrac{\max\left\{|L_1|,\ |R_2|\right\}}{|L|},\ 2 - 2 \cdot \tfrac{\max\left\{|L_1|,\ |R_2|\right\}}{|L|}\right\} \\
&= \tfrac{|L|}{2} \cdot \max\left\{b,\ 2-2b\right\} \ \ge \tfrac{|L|}{2} \cdot \min_{b \ge 0}\left\{\max\left\{b,\ 2-2b\right\}\right\} \ge \tfrac{|L|}{2} \cdot \tfrac{2}{3} = \tfrac{|L|}{3}
\end{aligned}$$

Note that equality in the previous inequality is achieved when $|L_1| = |R_1| = |R_2| = 2|L_2|$.

Putting $L' = (L \cup R) \setminus C_{L,R}$, we obtain the claim of the lemma. □

Finally, we prove the following lemma on the existence of an independent set having some special properties in any graph on at least four vertices.

Lemma 8. *Let $G = (V, E)$ be a connected graph on $n \ge 4$ vertices and of maximum degree $\Delta \ge 2$. There exists an independent set S in G such that $S = S_1 \cup S_2$, S_1 and S_2 are disjoint, and the following properties hold:*

1. *for each vertex u of S_2, there exists another vertex v in G such that $N_G(u) = N_G(v)$*
2. *for each pair u, v of vertices of S, $N_G(u) \ne N_G(v)$*
3. *for each vertex $u \in V$ of degree 1, there exists a vertex at distance 2 of u which does not belong to S*
4. *$|S_1| \ge \frac{\alpha n}{\Delta+1}$ and $|S_2| \ge \frac{(1-\alpha)n}{2\Delta}$ for some $\alpha \in [0, 1]$*

Proof The proof of this lemma is algorithmic, in the sense that we propose an algorithm (Algorithm 1) which builds S, S_1 and S_2. Algorithm 1 is a variant of the simple greedy algorithm which computes an independent set (by choosing an arbitrary vertex, putting it into the independent set, removing it and all its neighbours from the set of candidates, and reiterating that process until there is no candidate vertex left). Let us describe Algorithm 1 in more detail.

At each step, the set X contains the set of candidate vertices, i.e. the potential vertices to be put into either S_1 or S_2. In the beginning of the algorithm $X = V$. When picking a candidate vertex x from X to put it either in S_1 or S_2, we remove (at least) $B_1(x)$ from X. This ensures that S is an independent set. Moreover, since we pick a vertex at most once and add it either to S_1 or to S_2 but not to both, these two sets are disjoint.

The algorithm first handles vertices of degree 1 in G (lines 2 to 6): while there exists such a vertex x in the set of candidates X, it is picked and put into the set S_1, while $B_2(x)$ is removed from X.

After this step and if $X \ne \emptyset$, we continue to pick candidate vertices from X. For a candidate x, if there exists a vertex y such that $N_G(x) = N_G(y)$, we put x into S_2 (this ensures that the first property is

Algorithm 1 Greedy construction of the special independent set $S = S_1 \cup S_1$

Input: a connected graph $G = (V, E)$ on at least four vertices

1: $X \leftarrow V$, $S_1 \leftarrow \emptyset$, $S_2 \leftarrow \emptyset$, $R_1 \leftarrow \emptyset$, $R_2 \leftarrow \emptyset$
2: **while** there exists a vertex $s \in X$ of degree 1 in G **do**
3: $\quad S_1 \leftarrow S_1 \cup \{s\}$
4: $\quad R_1 \leftarrow R_1 \cup (B_2(s) \cap X)$
5: $\quad X \leftarrow X \setminus B_2(s)$
6: **end while**
7: **while** $X \neq \emptyset$ **do**
8: $\quad$ arbitrarily choose $s \in X$
9: $\quad$ **if** there exists a vertex t having the same set of neighbours as s in G **then**
10: $\qquad S_2 \leftarrow S_2 \cup \{s\}$
11: $\qquad Q \leftarrow \Big(B_1(s) \cup \{t \in V \mid N_G(s) = N_G(t)\}\Big) \cap X$
12: $\qquad R_2 \leftarrow R_2 \cup Q$
13: $\qquad X \leftarrow X \setminus Q$
14: $\quad$ **else**
15: $\qquad S_1 \leftarrow S_1 \cup \{s\}$
16: $\qquad R_1 \leftarrow R_1 \cup (B_1(s) \cap X)$
17: $\qquad X \leftarrow X \setminus B_1(x)$
18: $\quad$ **end if**
19: **end while**
20: **return** S_1 and S_2 $(S = S_1 \cup S_2)$

fulfilled), and remove $B_1(x)$ from X, as well as all vertices y such that $N_G(x) = N_G(y)$, therefore ensuring the property number 2. Otherwise, we put x into S_1 and remove $B_1(x)$ from X.

Let us show property number 3. Let u be a vertex of degree 1. If $u \in S$, when u is picked from X in the algorithm, its ball of radius 2 is removed from the set X of candidates. Since $n \geq 4$ and G is connected, this ball contains at least one vertex at distance 2 of u, which is removed from X and therefore does not belong to S. Now if $u \notin S$, it means it has been removed from X when another vertex, say v, was added to S. Note that this has necessarily happened in the first step of the algorithm (lines 2 to 6), otherwise the algorithm would have put u into S. Therefore v is also a degree 1-vertex itself, and thus $B_2(v)$ was removed from X when adding v. Again since $n \geq 4$ and G is connected, $B_2(v)$ contains at least one vertex at distance 2 of both u and v, which therefore does not belong to S.

It remains to prove property number 4. We denote by $R_1 \subseteq V$ the set of vertices removed from X when adding a vertex to S_1, and by $R_2 \subseteq V$, the set of vertices removed from X when adding a vertex to S_2. Since every element removed from X gets added to either R_1 or R_2 (but not to both), and the algorithm runs until $X = \emptyset$, we have $V = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$. Hence, R_1 and R_2 form a partition of V and there exists some $\alpha \in [0, 1]$ such that $|R_1| = \alpha n$ and $|R_2| = (1 - \alpha)n$.

Now we claim that $|R_1| \leq (\Delta + 1)|S_1|$. Indeed, let s be a vertex which is put into S_1. If s is of degree 1 in G, it has a unique neighbour s'. When s is put into S_1, all vertices of $B_2(s)$ are removed from the set X and added to R_1. But since $B_2(s) = B_1(s')$, $|B_2(s)| \leq \Delta + 1$. Now if s has degree at least 2, at most $|B_1(s)| \leq \Delta + 1$ vertices are put into R_1 as well.

Similarly, $|R_2| \leq 2\Delta|S_2|$: when a vertex s is added to S_2, at most its ball of radius one and the vertices having the same neighbourhood as s are removed from X and added to R_2. This set of vertices has at most $\Delta + 1 + \Delta - 1 = 2\Delta$ elements and the claim follows.

Therefore $\alpha n \leq (\Delta + 1)|S_1|$ and $(1 - \alpha)n \leq 2\Delta|S_2|$, which completes the proof of the lemma. □

2.2. The main result

We are now ready to prove the main theorem of this paper.

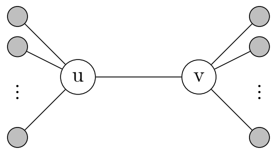

Figure 6: Vertices u, v with $(N(u) \cup N(v)) \setminus \{u, v\} \subseteq S$

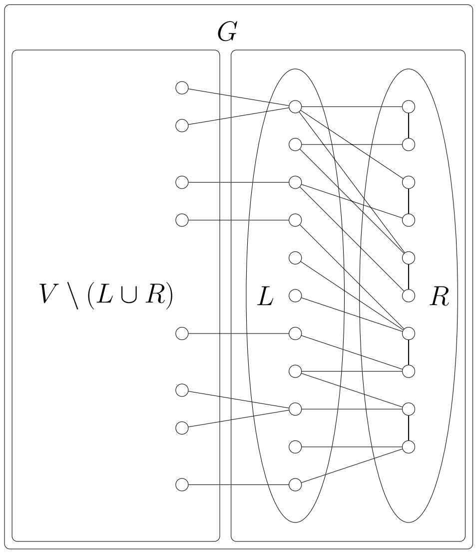

Figure 7: Partition of $V(G)$

Theorem 9. *Let $G = (V, E)$ be a connected identifiable triangle-free graph on n vertices with maximum degree $\Delta \geq 2$. Then $\gamma^{ID}(G) \leq n - \frac{n}{3(\Delta+1)}$.*

Proof We may first assume that $n \geq 4$ since the theorem is true for the only connected triangle-free identifiable graph on at most three vertices and of maximum degree at least 2, i.e. the path on three vertices P_3, for which $\gamma^{\mathrm{ID}}(P_3) = 2$.

Since $n \geq 4$ and G is connected, we can use Lemma 8 to construct an independent set $S = S_1 \cup S_2$ of G having all properties described in this lemma. Consider all pairs u, v of vertices of G such that u and v are adjacent, both u and v are of minimum degree 2, and all the vertices of $N(u) \cup N(v) \setminus \{u, v\}$ belong to S (see Figure 6 for an illustration). Since all neighbours of u and v (except u and v themselves) are in S, these neighbours form an independent set. Let M be the set of all edges uv such that u and v form such a pair. By the previous remark M is an (L, R)-matching of G with $R = \{u, v \mid uv \in M\}$ and $L = \left(\bigcup_{v \in R} N_G(v)\right) \setminus R$. Note that we have $L \subseteq S$. Also note that M might be empty.

Let us now partition V into two subsets of vertices (one of them being possibly empty): $(L \cup R)$ on the one hand, and $V \setminus (L \cup R)$ on the other hand. Such a partition is illustrated in Figure 7. Note that $G[L \cup R]$ is identifiable by Observation 3. Let us show that $G[V \setminus (L \cup R)]$ is also identifiable. By contradiction, suppose it is not the case and let u, v be a pair of vertices such that $N_{G[V \setminus (L \cup R)]}[u] = N_{G[V \setminus (L \cup R)]}[v]$. Vertices u and v are therefore adjacent, and since G is triangle-free, neither u nor v has other neighbours within $G[V \setminus (L \cup R)]$. Since G is identifiable, at least one of them has a neighbour in L. Suppose they both have a neighbour in L. Then by construction of S, u and v both do not belong to S. But then u and v should belong to R, a contradiction. Thus, one of them, say u, has degree 1 in G, and all neighbours of v belong to $L \subseteq S$. But by the third property of Lemma 8 at least one vertex at distance 2 of u does not belong to S, a contradiction.

We will now build two subsets $C_1 \subseteq (L \cup R)$ and $C_2 \subseteq (V \setminus (L \cup R))$ such that $C = C_1 \cup C_2$ is an identifying code of G.

- **Building $C_1 \subseteq (L \cup R)$.**
  If $(L \cup R) = \emptyset$ we take $C_1 = \emptyset$. Otherwise we build C_1 using Lemma 7: applying it to G and M, we know that there exists an (L, R)-quasi-identifying code of G. Let C_1 be such a code. From Lemma 7 we also know that $|L'| \geq \frac{|L|}{3}$, where $L' = (L \cup R) \setminus C_1$.

- **Building $C_2 \subseteq V \setminus (L \cup R)$.**
  Again if $V \setminus (L \cup R) = \emptyset$ we take $C_2 = \emptyset$. Otherwise we take C_2 to be the complement of S in $V \setminus (L \cup R)$: $C_2 = (V \setminus S) \setminus (L \cup R)$. Let us show that C_2 is a $\big(V \setminus (L \cup R)\big)$-identifying code of G.

  Since S is an independent set, each vertex v having at least one neighbour within $G[V \setminus (L \cup R)]$ is dominated. If v is an isolated vertex in $G[V \setminus (L \cup R)]$, since G has no isolated vertices, all its neighbours belong to L. But since $L \subseteq S$, $v \notin S$ and thus $v \in C_2$. Thus C_2 is a dominating set.

  Let us now check the separation condition. Let u, v be an arbitrary pair of vertices of $V \setminus (L \cup R)$. We distinguish the following cases:

  1. If u and v are not adjacent but have the same set of neighbours, by property number 2 of Lemma 8, at least one of u and v is not in S (and thus in C_2) and separates u and v.
  2. If u and v are not adjacent and have distinct sets of neighbours, if u or v belong to C_2, they are separated. Suppose this is not the case: both u and v belong to S. Then all their neighbours belong to C_2 (none of them can belong to $L \subseteq S$), and since they have distinct sets of neighbours they are separated.
  3. If u and v are adjacent and both have degree at least 2 in $G[V \setminus (L \cup R)]$, suppose by contradiction that they are not separated. This means all their neighbours are in S. But then they are both not in S, thus they should belong to the set R, a contradiction.
  4. Finally, if u and v are adjacent and one of them, say u, has degree 1 in $G[V \setminus (L \cup R)]$. Since $G[V \setminus (L \cup R)]$ is identifiable, v has at least one neighbour in $V \setminus (L \cup R)$. Suppose that u has degree at least 2 in G: all its neighbours, except v, belong to L. Thus $u \notin S$: $u \in C_2$. Now we claim that there is a neighbour of v other than u that belongs to C_2, thus separating u and v. By contradiction, if not then $v \notin S$ and $(B_1(u) \cup B_1(v)) \setminus \{u, v\} \subseteq S$. But then u and v belong to R, a contradiction. Now, assume u has degree 1 in G. Then by property number 3 of Lemma 8 there is a vertex at distance 2 of u in C_2, separating u and v.

  Thus C_2 is a $\big(V \setminus (L \cup R)\big)$-identifying code of G.

We showed that C_1 is (L, R)-quasi-identifying, and that C_2 is a $(V \setminus (L \cup R))$-identifying code of G. Moreover, since $L \subseteq S$, by the second property of Lemma 8, no two vertices of L have the same open neighbourhood. Furthermore, since C_2 is the complement of S in $G[V \setminus (L \cup R)]$, all neighbours of L in $G[V \setminus (L \cup R)]$ belong to C_2. Therefore we can apply Lemma 5 and $C = C_1 \cup C_2$ is an identifying code of G.

Let us now upper-bound the size of C. To this end, we lower-bound the size of its complement. From the construction of C_1 and C_2, we have $V \setminus C = S_2 \cup (S_1 \setminus L) \cup L'$.

We now claim that $L \subseteq S_1$. By contradiction, assume there exists a vertex $l \in L$ such that $l \in S_2$. Then by the first property of Lemma 8 there exists another vertex x such that $N_G(l) = N_G(x)$. Let $r \in R$ be a neighbour of l. By the second property of Lemma 8, x is not in S, and since $N_G(l) = N_G(x)$, r and x are adjacent. But then $x \in R$ and l is adjacent to a neighbour of x in L, a contradiction since L is an independent set.

Therefore, since $|L'| \geq \frac{|L|}{3}$, we have $|(S_1 \setminus L) \cup L'| \geq \frac{|S_1|}{3}$.

Using property number 4 of Lemma 8, we get $|V \setminus C| \geq |S_2| + \frac{|S_1|}{3} \geq \frac{(1-\alpha)n}{2\Delta} + \frac{\alpha n}{3(\Delta+1)} \geq \frac{n}{3(\Delta+1)}$ with $\alpha \in [0, 1]$. Hence, $|C| \leq n - \frac{n}{3(\Delta+1)}$. □

Note that when considering graphs of minimum degree 3, the previous proof can be slightly modified to improve this result. Indeed, in this case the statement of Lemma 7 can be changed as follows.

Lemma 10. *Let G be an identifiable triangle-free graph of minimum degree at least 3 and L, R be subsets of vertices of G such that there exists an (L, R)-matching M of G. There exists a set L' of vertices of $L \cup R$, such that $|L'| \geq \frac{|L|}{2}$ and $(L \cup R) \setminus L'$ is an (L, R)-quasi-identifying code in G.*

Proof The proof is very similar to the one of Lemma 7. It follows from the fact that G has minimum degree 3, that the sets R_1 and L_1 do not exist, since all vertices of R_1 should have degree 2. Therefore, C_a has size $|L_2| + \frac{|R_2|}{2} = |L| + \frac{|R|}{2}$ and C_b size $\frac{3|R_2|}{2} = \frac{3|R|}{2}$. So, we can always construct an (L, R)-quasi-identifying code $(L \cup R) \setminus L'$ where the size of L' is at least $\max\left\{\frac{|R|}{2}, |L| - \frac{|R|}{2}\right\}$ vertices. This expression is at its minimum value when $|L| = |R|$: then it has a value of $\frac{|L|}{2}$. Thus, we can leave at least $\frac{|L|}{2}$ vertices out of the (L, R)-quasi-identifying code and the claim of the lemma follows. $\square$

Using the previous lemma, we can now improve the result of Theorem 9 for triangle-free graphs of minimum degree at least 3 and maximum degree Δ. Let G be such a graph, and let us proceed exactly the same way as in the proof of Theorem 9, constructing a special independent set $S = S_1 \cup S_2$ using Lemma 8. We use S and Lemma 10 to build an identifying code C of G in the same way as in the proof of Theorem 9.

Since G has minimum degree at least 3, by Lemma 10 and recalling that $L \subseteq S_1$, the set L' is such that $|(S_1 \setminus L) \cup L'| \geq \frac{|S_1|}{2}$.

Again, using Lemma 8, we get $|V \setminus C| \geq |S_2| + \frac{|S_1|}{2} \geq \frac{(1-\alpha)n}{2\Delta} + \frac{\alpha n}{2(\Delta+1)} \geq \frac{n}{2(\Delta+1)}$.

Hence, the built code C is an identifying code of size at most $n - \frac{n}{2(\Delta+1)}$ and the following result follows.

Theorem 11. *Let G be a connected identifiable triangle-free graph on n vertices having minimum degree at least 3 and maximum degree Δ. Then $\gamma^{ID}(G) \leq n - \frac{n}{2(\Delta+1)}$.*

Due to their applications in networks, Δ-regular graphs have been much studied in the theory of identifying codes. Since for $\Delta \geq 3$, a Δ-regular graph has minimum degree at least 3, the following corollary is immediate.

Corollary 12. *Let G be an identifiable triangle-free Δ-regular graph ($\Delta \geq 3$) on n vertices having no isolated vertex. Then $\gamma^{ID}(G) \leq n - \frac{n}{2(\Delta+1)}$.*

2.3. Remarks

We conclude this section by two remarks.

Remark 1. We note that our proofs provide polynomial-time algorithms to compute the identifying codes of Theorems 9 and 11. Indeed, their construction is based on the codes computed in Lemmas 6, 7 and 10, and the construction of the independent set of Lemma 8. All these constructions are described in the corresponding proofs and can be done in polynomial time.

Remark 2. In this paper, we have considered triangle-free graphs, that is, graphs of girth at least 4 (the girth of a graph is the length of one of its shortest cycles). It is natural to ask wether better bounds on parameter γ^{ID} hold for graphs of larger girth. This question was partially answered in the positive in [8], where the following linear bound (in terms of n) on γ^{ID} is given.

Theorem 13 ([8]). *Let G be a connected identifiable graph on n vertices having minimum degree at least 2 and girth at least 5. Then $\gamma^{ID}(G) \leq \frac{7n}{8} + 1$.*

However it is not known wether this bound is sharp or not. We leave it as an interesting open problem to fully answer the question of the influence of the girth on the upper bound for parameter γ^{ID}.

3. Conclusion

In this paper we have shown that any connected identifiable triangle-free graph on n vertices having maximum degree $\Delta \geq 2$ admits an identifying code of size at most $n - \frac{n}{3(\Delta+1)}$, and of size at most $n - \frac{n}{2(\Delta+1)}$ if it has minimum degree 3. Moreover the corresponding identifying codes can be computed in polynomial time. This gives more insight into the cardinality of minimum identifying codes of graphs of a given maximum degree than the general upper bound of $n-1$ given in [12]. However, we believe that these upper bounds may still be slightly improved towards our conjecture that any connected graph of order n and of maximum degree $\Delta \geq 2$ admits an identifying code of size at most $n - \frac{n}{\Delta} + O(1)$.

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Appendix A. Proof of validity of the (L,R)-quasi-identifying codes C_a and C_b.

First note that C_a and C_b are dominating sets, so point number 1 of Definition 4 holds. Let us now show point number 2. In each case, all distinct vertices u, v of $L_1 \cup R_1$ are separated from each other, since in both constructed codes, we use an $(L_1 \cup R_1)$-identifying code C_1 given by Lemma 6. Thus it remains to check if u and v are separated when $u \in (L_1 \cup R_1)$ and $v \in (L_2 \cup R_2)$, and when both u and v belong to $L_2 \cup R_2$. Note that for the case where $u \in (L_1 \cup R_1)$ and $v \in (L_2 \cup R_2)$, if u and v are adjacent, then necessarily $u \in L_1$ and $v \in R_2$.

Code C_a. Suppose first that $u \in (L_1 \cup R_1)$ and $v \in (L_2 \cup R_2)$. If $u \in R_1$, u and v are separated either by u itself if it belongs to C_1, or by its neighbour in R_1 otherwise. Hence, suppose $u \in L_1$. If $v \in R_2$, by construction, u is not C_a-isolated, hence it has a neighbour r in $R \cap C_a$. If r is distinct from v, u and v are separated by r. Otherwise, by construction v has at least two neighbours in $L \cap C_a$, thus u and v are separated by at least one of them. Now, if $v \in L_2$, note that C_1 is an $(L_1 \cup R_1)$-identifying code, therefore u is covered by some vertex w of $L_1 \cup R_1$, and w separates u and v.

Now, suppose $u, v \in (L_2 \cup R_2)$.

- If both u and $v \in L_2$ and one of them — say u — is in the code, they are separated by u since L is an independent set. If they are both outside of C_a, then they have been taken out in the last step of the construction. But in this case, they are separated by their neighbours of R_2 by whom they have been replaced (these neighbours are distinct since otherwise, after the switch, one of the vertices of L would not have been C_a-isolated, and therefore would still belong to C_a).

- If $u, v \in R_2$ and they are not adjacent, since every vertex of R is either in the code or has its neighbour in the code, u and v are separated. If they are adjacent, we know that at least one of them belongs to C_a, say u. Thus they are separated by at least one of the neighbours of u in L: such a neighbour exists because of the second step of the construction.

- If $u \in L_2$ and $v \in R_2$ and they are not adjacent, they are separated by u if u belongs to the code. If not, this means that they have a common neighbour in R_2 which has been put into the code instead of u in the last step of the construction. But then v is in the code and it separates them. If u and v are adjacent and if v is not in the code, then its neighbour in R_2 is, and u and v are separated by it. If $v \in C_a$, then by the second step of the construction, v has at least one additional neighbour w of L being in the code, and u and v are separated by w.

Code C_b. Suppose first that $u \in (L_1 \cup R_1)$ and $v \in (L_2 \cup R_2)$. If v belongs to R_2, since both v and its neighbour r in R_2 are in C_a, u and v are separated by at least one of v and r. If v belongs to L_2, as in the case of C_a, u and v are separated by some vertex of $L_1 \cup R_1$.

Now, suppose $u, v \in (L_2 \cup R_2)$.

- If $u, v \in R_2$ then u and v are separated by a neighbour of one of them in L. It exists since this has been ensured in the last step of the construction.

- If $u \in R_2$ and $v \in L_2$, then u and v are either separated by u if u and v are not adjacent, or by the neighbour of u in R_2.

- If $u, v \in L_2$, and they have different sets of neighbours in $R_2 \cap C_b$, then they are separated by them. Otherwise, since $R_2 \subseteq C_b$, it means that they have the same set of neighbours within $L \cup R$, but this situation is allowed by definition of an (L, R)-quasi-identifying code (point number 2).

Let us now check points number 3 and 4 of Definition 4. By Lemma 6, C_1 is $(L_1 \cup R_1)$-identifying, there are no C_1-isolated vertices in R_1 (second property of Lemma 6), and each pair of adjacent vertices of R_1 contains at least one code vertex (property number 3 of Lemma 6). But by construction of C_a and C_b there are no C_a-isolated or C_b-isolated vertices in L_1 either. Thus points number 3 and 4 hold for the part $L_1 \cup R_1$.

Now we shall check vertices of $L_2 \cup R_2$. In both codes, by construction, for each pair of adjacent vertices of R_2, at least one of them is in the code. For $x \in \{a, b\}$, we claim that no vertex of R is C_x-isolated: the whole set L is in C_a, thus this is true for C_a. It also holds for C_b since the whole set R_2 belongs to C_b. It now remains to check wether there exist C_x-isolated vertices in L_2 for $x \in \{a, b\}$. This has been explicitly taken care of during the construction of C_a. Furthermore, since $R_2 \subseteq C_b$, this is also true in C_b, which completes the proof. $\square$