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# Global well-posedness and instability of travelling-waves for a nonlocal conservation law

Afaf Bouharguane \*

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## Abstract

We are interested in a nonlocal conservation law which describes the morphodynamics of sand dunes sheared by a fluid flow, recently proposed by Andrew C. Fowler and studied by [1, 2]. We begin by proving the global-in-time well-posedness in the neighbourhood of travelling-waves. We next show the instability of travelling-waves solutions of the Fowler equation.

**Keywords:** nonlocal evolution equation, fractional anti-diffusive operator, Duhamel formulation, travelling-wave, instability.

**Mathematics Subject Classification:** 35L65, 45K05, 35G25, 35C07, 35B35.

## 1 Introduction

The study of mechanisms that allow the formation of structures such as sand dunes and ripples at the bottom of a fluid flow plays a crucial role in the understanding of coastal dynamics. The modeling of these phenomena is particularly complex since we must not only solve the Navier-Stokes or Saint-Venant equations coupled with an equation for sediment transport, but also take into account the evolution of the bottom. Instead of solving the whole system fluid flow, free surface and free bottom, nonlocal models of fluid flow interacting with the bottom were introduced in [6, 7, 10]. Among these models, we are interested in the following nonlocal conservation law [6, 7, 8]:

$$\begin{cases} \partial_t u(t, x) + \partial_x \left( \frac{u^2}{2} \right) (t, x) + \mathcal{I}[u(t, \cdot)](x) - \partial_{xx}^2 u(t, x) = 0 & t \in (0, T), x \in \mathbb{R}, \\ u(0, x) = u_0(x) & x \in \mathbb{R}, \end{cases} \quad (1)$$

where  $T$  is any given positive time,  $u = u(t, x)$  represents the dune height (see Fig. 1) and  $\mathcal{I}$  is a nonlocal operator defined as follows: for any Schwartz function  $\varphi \in \mathcal{S}(\mathbb{R})$  and any  $x \in \mathbb{R}$ ,

$$\mathcal{I}[\varphi](x) := \int_0^{+\infty} |\xi|^{-\frac{1}{3}} \varphi''(x - \xi) d\xi. \quad (2)$$

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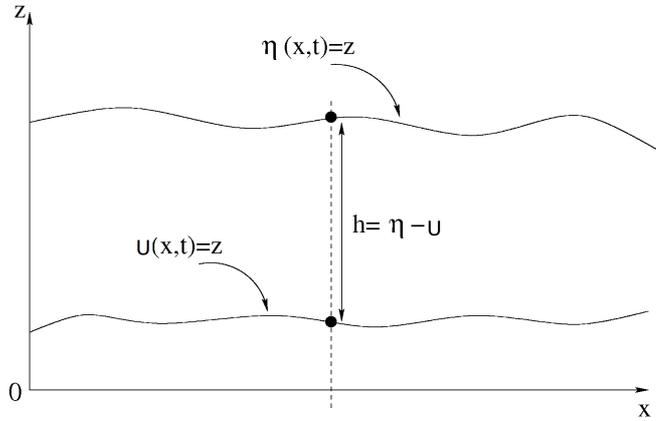


Figure 1: Domain considered for the Fowler model:  $h$  is the depth water,  $\eta$  the free surface and  $u$  the sandy bottom.

Equation (1) is valid for a river flow over an erodible bottom  $u(t, x)$  with slow variation and describes both accretion and erosion phenomena [1].

The nonlocal term  $\mathcal{I}$  can be seen as a fractional power of order  $2/3$  of the Laplacian with the bad sign. Indeed, it has been proved [1] that

$$\mathcal{F}(\mathcal{I}[\varphi] - \varphi'')(\xi) = \psi_{\mathcal{I}}(\xi)\mathcal{F}\varphi(\xi) \quad (3)$$

where

$$\psi_{\mathcal{I}}(\xi) = 4\pi^2\xi^2 - a_{\mathcal{I}}|\xi|^{\frac{4}{3}} + i b_{\mathcal{I}}\xi|\xi|^{\frac{1}{3}}, \quad (4)$$

with  $a_{\mathcal{I}}, b_{\mathcal{I}}$  positive constants,  $\mathcal{F}$  denotes the Fourier transform defined in (7) and  $\Gamma$  denotes the Euler function. One simple way to establish this fact is the derivation of a new formula for the operator  $\mathcal{I}$ , see Proposition 2.

**Remark 1.** For causal functions (i.e.  $\varphi(x) = 0$  for  $x < 0$ ), this operator is, up to a multiplicative constant, the Riemann-Liouville fractional derivative operator of order  $4/3$  which is defined as follows [11]

$$\frac{1}{\Gamma(2/3)} \int_0^{+\infty} \frac{\varphi''(x - \xi)}{|\xi|^{1/3}} d\xi = \frac{d^{-2/3}}{dx^{-2/3}} \varphi''(x) = \frac{d^{4/3}}{dx^{4/3}} \varphi(x). \quad (5)$$

Therefore, the Fowler model has two antagonistic terms: a usual diffusion and a nonlocal fractional anti-diffusive term of lower order. This remarkable feature enabled to apply this model for signal processing. Indeed, the diffusion is used to reduce the noise whereas the nonlocal anti-diffusion is used to enhance the contrast [4].

Recently, some results regarding this equation have been obtained, namely, existence of travelling-waves  $u_{\phi}(t, x) = \phi(x - ct)$  where  $\phi \in C_b^1(\mathbb{R})$  and  $c \in \mathbb{R}$  represents wave velocity, the global well-posedness for  $L^2$ -initial data, the failure of the maximum principle and the local-in-time well-posedness in a subspace of  $C_b^1$  [1, 2]. Notice that the travelling-waves are not necessarily of solitary type (see [2]) and therefore may not belong to  $L^2(\mathbb{R})$ , the space where a global well-posedness result is available. In [2], the authors prove local well-posedness in a subspace of  $C_b^1(\mathbb{R})$  but fail to obtain global existence.

One main focus of this paper is the study of travelling-wave solutions of (1). Indeed, an interesting topic is to know if the shape of this travelling-wave is maintained when it is perturbed. This raises the question of the stability of travelling-waves. In our paper, we first establish global well-posedness for an initial *perturbation* in  $L^2(\mathbb{R})$  and then prove the nonlinear instability of travelling-waves solutions to (1). More precisely, we exhibit a disturbance which grows without bound meaning that the original travelling-wave is unstable. To begin with, we prove the global well-posedness in a  $L^2$ -neighbourhood of  $C_b^1$ , namely  $u = u_\phi + v$ , where  $u_\phi \neq 0$  is a nontrivial solution. To prove this result, we consider the following Cauchy problem:

$$\begin{cases} \partial_t v(t, x) + \partial_x \left( \frac{v^2}{2} + u_\phi v \right)(t, x) + \mathcal{I}[v(t, \cdot)](x) - \partial_{xx}^2 v(t, x) = 0 & t \in (0, T), x \in \mathbb{R}, \\ v(0, x) = v_0(x) & x \in \mathbb{R}, \end{cases} \quad (6)$$

where  $v_0 \in L^2(\mathbb{R})$  is an initial perturbation and  $T$  is any given positive time.

To prove the existence and uniqueness results, we begin by introducing the notion of *mild* solution (see Definition 1) based on Duhamel's formula (8), in which the kernel  $K$  of  $\mathcal{I} - \partial_{xx}^2$  appears. The use of this formula allows to prove the local-in-time existence with the help of contracting fixed point theorem. The global existence is obtained thanks to an energy estimate (a priori estimate) (30). This approach is quite classical: we refer for instance to [1, 5]. On the other hand, the Duhamel formulation also allows to prove that travelling-waves are unstable.

The plan of this paper is organised as follows. In the next section, we define the notion of mild solution of (6) and we give some properties on the kernel  $K$  of  $\mathcal{I} - \partial_{xx}^2$  that will be needed in the sequel. Section 3 and 4 are, respectively, devoted to the proof of the uniqueness and the existence of a mild solution of (6). Section 5 contains the proof of the regularity of the solution. The proof of the instability of travelling-waves is given in section 6.

Here are our main results.

**Theorem 1.** *Let  $T > 0$  and  $v_0 \in L^2(\mathbb{R})$ . There exists a unique mild solution  $v \in L^\infty((0, T); L^2(\mathbb{R}))$  of (6) (see Definition 1). Moreover,*

1.  $v \in C([0, T]; L^2(\mathbb{R}))$  and  $v(0, \cdot) = v_0$  almost everywhere,
2.  $v \in C^{1,2}((0, T] \times \mathbb{R})$ ,
3.  $v$  satisfies  $\partial_t v + \partial_x \left( \frac{v^2}{2} + u_\phi v \right) + \mathcal{I}[v] - \partial_{xx}^2 v = 0$ , on  $(0, T] \times \mathbb{R}$ , in the classical sense or equivalently,  $u = u_\phi + v$  is a classical solution of equation (1).

**Theorem 2** (Instability of travelling-waves). *Let  $T > 0$  and  $\phi \in C_b^1(\mathbb{R})$  be a travelling-wave solution to (1). Assume that  $v_0 \in L^2(\mathbb{R})$  is such that  $\mathcal{F}(v_0) = \varepsilon 1_{[a,b]}$  with  $\varepsilon > 0$ ,  $0 < a < b$  satisfying  $\operatorname{Re}(\psi_{\mathcal{I}})(a) < \operatorname{Re}(\psi_{\mathcal{I}})(b) = -\alpha < 0$  and for all  $\xi \in [a, b]$ ,  $\operatorname{Re}(\psi_{\mathcal{I}})(\xi) \leq -\alpha$  (see Figure 3). Then the unique mild solution of (6) satisfies*

$$\|v\|_{C([0,t]; L^2(\mathbb{R}))} \geq C e^{\beta t},$$

where  $0 < \beta < \frac{\alpha}{2}$  and  $C$  is a positive constant.

**Remark 2.** The shape of this perturbation is given by

$$v_0(x) = \varepsilon \begin{cases} 2 \frac{\sin(\frac{b-a}{2}x)}{x} e^{-i\frac{a+b}{2}x} & \text{if } x \neq 0, \\ b - a & \text{otherwise.} \end{cases}$$

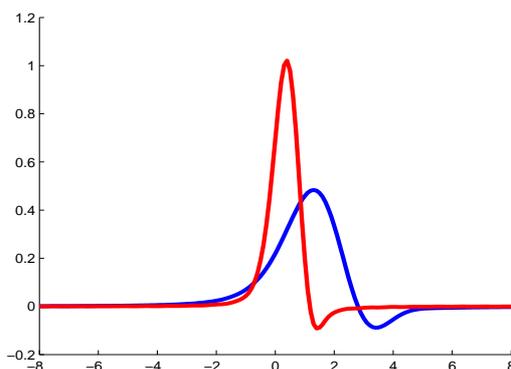


Figure 2: Evolution of the kernel  $K$  for  $t = 0.1$  (red) and  $t = 0.5$  s (blue)

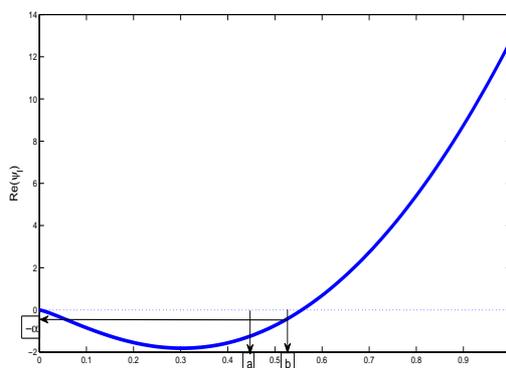


Figure 3: Behaviour of  $\text{Re}(\psi_{\mathcal{I}})$

**Remark 3.** An interesting property for the kernel  $K$  is the non-positivity (see Figure 2) and the main consequence of this feature is the failure of maximum principle [1]. We use again this property to establish the instability of travelling-waves, see proof of Theorem 2.

**Notations.**

- The norm of a measurable function  $f \in L^p(\mathbb{R})$  is written  $\|f\|_{L^p(\mathbb{R})}^p = \int_{\mathbb{R}} |f(x)|^p dx$  for  $1 \leq p < \infty$ .
- We denote by  $\mathcal{F}$  the Fourier transform of  $f$  which is defined by: for all  $\xi \in \mathbb{R}$

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}} e^{-2i\pi x\xi} f(x) dx, \tag{7}$$

and  $\mathcal{F}^{-1}$  denotes the inverse of Fourier transform.

- The Schwartz space of rapidly decreasing functions on  $\mathbb{R}$  is denoted by  $\mathcal{S}(\mathbb{R})$ .

- We write  $C^k(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{C}; f, f', \dots, f^{(k)} \text{ are continuous on } \mathbb{R}\}$ .
- We denote by  $C_b(\mathbb{R})$  the space of all bounded continuous real-valued functions on  $\mathbb{R}$  endowed with the norm  $\|\cdot\|_{L^\infty} = \sup_{\mathbb{R}} |f|$ .
- We write for any  $T > 0$ ,

$$C^{1,2}((0, T] \times \mathbb{R}) := \{u \in C((0, T] \times \mathbb{R}); \partial_t u, \partial_x u, \partial_{xx}^2 u \in C((0, T] \times \mathbb{R})\}.$$

- We denote by  $\mathcal{D}(U)$  the space of test functions on  $U$  and  $\mathcal{D}'(U)$  denotes the distribution space.

## 2 Duhamel formula and main properties of $K$

**Definition 1.** Let  $T > 0$  and  $v_0 \in L^2(\mathbb{R})$ . We say that  $v \in L^\infty((0, T); L^2(\mathbb{R}))$  is a *mild solution* of (6) if for any  $t \in (0, T)$ :

$$v(t, x) = K(t, \cdot) * v_0(x) - \int_0^t \partial_x K(t-s, \cdot) * \left( \frac{v^2}{2} + u_\phi v \right) (s, \cdot)(x) ds \quad (8)$$

where  $K(t, x) = \mathcal{F}^{-1} \left( e^{-t\psi_{\mathcal{I}}(\cdot)} \right) (x)$  is the kernel of the operator  $\mathcal{I} - \partial_{xx}^2$  and  $\psi_{\mathcal{I}}$  is defined in (4).

The expression (8) is the Duhamel formula and is obtained using the spatial Fourier transform.

**Proposition 1** (Main properties of  $K$ , [1]). *The kernel  $K$  satisfies:*

1.  $\forall t > 0, K(t, \cdot) \in L^1(\mathbb{R})$  and  $K \in C^\infty((0, \infty) \times \mathbb{R})$ ,
2.  $\forall s, t > 0, K(s, \cdot) * K(t, \cdot) = K(s+t, \cdot)$ ,  
 $\forall u_0 \in L^2(\mathbb{R}), \lim_{t \rightarrow 0} K(t, \cdot) * u_0 = u_0$  in  $L^2(\mathbb{R})$ ,
3.  $\forall T > 0, \exists K_0$  such that  $\forall t \in (0, T], \|\partial_x K(t, \cdot)\|_{L^2(\mathbb{R})} \leq K_0 t^{-3/4}$ ,
4.  $\forall T > 0, \exists K_1$  such that  $\forall t \in (0, T], \|\partial_x K(t, \cdot)\|_{L^1(\mathbb{R})} \leq K_1 t^{-1/2}$ .

**Remark 4.** Using Plancherel formula, we have for any  $v_0 \in L^2(\mathbb{R})$  and any  $t \in (0, T]$

$$\|K(t, \cdot) * v_0\|_{L^2(\mathbb{R})} \leq e^{\alpha_0 t} \|v_0\|_{L^2(\mathbb{R})}$$

where  $\alpha_0 = -\min \operatorname{Re}(\psi_{\mathcal{I}}) > 0$ .

**Proposition 2** (Integral formula for  $\mathcal{I}$ ). *For all  $\varphi \in \mathcal{S}(\mathbb{R})$  and all  $x \in \mathbb{R}$ ,*

$$\mathcal{I}[\varphi](x) = \frac{4}{9} \int_{-\infty}^0 \frac{\varphi(x+z) - \varphi(x) - \varphi'(x)z}{|z|^{7/3}} dz. \quad (9)$$

*Proof.* The proof is based on simple integrating by parts. The regularity and the rapidly decreasing of  $\varphi$  ensure the validity of the computations that follow. We have

$$\begin{aligned}
\int_0^{+\infty} \varphi''(x-\xi)|\xi|^{-1/3} d\xi &= \int_0^{+\infty} \frac{d}{d\xi} (\varphi'(x) - \varphi'(x-\xi)) |\xi|^{-1/3} d\xi, \\
&= \frac{1}{3} \int_0^{+\infty} |\xi|^{-4/3} (\varphi'(x) - \varphi'(x-\xi)) d\xi, \\
&= \frac{1}{3} \int_0^{+\infty} |\xi|^{-4/3} \frac{d}{d\xi} (\varphi'(x)\xi + \varphi(x-\xi) - \varphi(x)) d\xi, \\
&= \frac{4}{9} \int_0^{+\infty} \frac{\varphi(x-\xi) - \varphi(x) + \varphi'(x)\xi}{|\xi|^{7/3}} d\xi, \\
&= \frac{4}{9} \int_{-\infty}^0 \frac{\varphi(x+\xi) - \varphi(x) - \varphi'(x)\xi}{|\xi|^{7/3}} d\xi.
\end{aligned}$$

There is no boundary term at infinity (resp. at zero) because  $\varphi$  is a rapidly decreasing function on  $\mathbb{R}$  (resp.  $\varphi$  is smooth). ■

**Remark 5.** Using integral formula (9), [1, 2] proved that

$$\mathcal{F}(\mathcal{I}[\varphi])(\xi) = 4\pi^2\Gamma\left(\frac{2}{3}\right)|\xi|^{4/3} \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \operatorname{sgn}(\xi)\right) \mathcal{F}\varphi(\xi).$$

Notice that  $\mathcal{F}(\mathcal{I}[\varphi])(\xi) = 4\pi^2\Gamma\left(\frac{2}{3}\right)(i\xi)^{4/3}$  which is consistent with Remark 1: up to a multiplicative constant  $\mathcal{I}[\varphi]$  is  $\frac{d^{4/3}\varphi}{dx^{4/3}}$ .

**Proposition 3.** Let  $s \in \mathbb{R}$  and  $\varphi \in H^s(\mathbb{R})$ . Then  $\mathcal{I}[\varphi] \in H^{s-4/3}(\mathbb{R})$  and we have

$$\|\mathcal{I}[\varphi]\|_{H^{s-4/3}(\mathbb{R})} \leq 4\pi^2\Gamma\left(\frac{2}{3}\right)\|\varphi\|_{H^s(\mathbb{R})}. \quad (10)$$

*Proof.* For all  $s \in \mathbb{R}$  and all  $\varphi \in H^s(\mathbb{R})$ , we have, using remark 5

$$\begin{aligned}
\|\mathcal{I}[\varphi]\|_{H^{s-4/3}(\mathbb{R})} &= \left( \int_{\mathbb{R}} (1+|\xi|^2)^{s-4/3} |\mathcal{F}(\mathcal{I}[\varphi])(\xi)|^2 d\xi \right)^{1/2}, \\
&= 4\pi^2\Gamma\left(\frac{2}{3}\right) \left( \int_{\mathbb{R}} (1+|\xi|^2)^{s-4/3} \left| \frac{1}{2} - i \operatorname{sgn}(\xi) \frac{\sqrt{3}}{2} \right| |\xi|^{8/3} |\mathcal{F}(\varphi)(\xi)|^2 d\xi \right)^{1/2}, \\
&= 4\pi^2\Gamma\left(\frac{2}{3}\right) \left( \int_{\mathbb{R}} \left( \frac{|\xi|^2}{1+|\xi|^2} \right)^{4/3} (1+|\xi|^2)^s |\mathcal{F}(\varphi)(\xi)|^2 d\xi \right)^{1/2}, \\
&\leq 4\pi^2\Gamma\left(\frac{2}{3}\right) \left[ \int_{\mathbb{R}} (1+|\xi|^2)^s |\mathcal{F}(\varphi)(\xi)|^2 d\xi \right]^{1/2}, \\
&= 4\pi^2\Gamma\left(\frac{2}{3}\right)\|\varphi\|_{H^s(\mathbb{R})}.
\end{aligned}$$
■

**Remark 6.** From the previous Proposition, we deduce that for all  $s \in \mathbb{R}$  and all  $\varphi \in H^s(\mathbb{R})$ ,  $\mathcal{I}[\varphi] \in H^{s-4/3}(\mathbb{R})$ . In particular, using the Sobolev embedding  $H^{2/3} \hookrightarrow C_b(\mathbb{R}) \cap L^2(\mathbb{R})$ , we deduce that  $\mathcal{I} : H^2 \rightarrow C_b(\mathbb{R}) \cap L^2(\mathbb{R})$  is a bounded linear operator.

**Proposition 4** (Duhamel formula (8) is well-defined). Let  $T > 0$ ,  $v_0 \in L^2(\mathbb{R})$  and  $w \in L^\infty((0, T); L^1(\mathbb{R})) \cup L^\infty((0, T); L^2(\mathbb{R}))$ . Then, the function

$$v : t \in (0, T] \rightarrow K(t, \cdot) * v_0 - \int_0^t \partial_x K(t-s, \cdot) * w(s, \cdot) ds \quad (11)$$

is well-defined and belongs to  $C([0, T]; L^2(\mathbb{R}))$  ( being extended at  $t = 0$  by the value  $v(0, \cdot) = v_0$  ).

*Proof.* From Proposition 1, it is easy to see that  $v$  is well-defined and that for any  $t \in (0, T]$ ,  $v(t, \cdot) \in L^2(\mathbb{R})$ . Indeed,  $\forall t > 0$ ,  $\partial_x K(t, \cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  so by Young inequality  $\partial_x K(t, \cdot) * w(t, \cdot)$  exists and using the estimates on the gradient (item 3 and 4 of Proposition 1) we deduce that  $v$  is well-defined and  $v(t, \cdot) \in L^2(\mathbb{R})$ .

Let us prove the continuity of  $v$ . By the second item of Proposition 1, we have that the function  $t \in (0, T] \rightarrow K(t, \cdot) * v_0$  is continuous and it is extended continuously up to  $t = 0$  by the value  $v(0, \cdot) = v_0$ . We define the function

$$F : t \in [0, T] \rightarrow \int_0^t \partial_x K(t-s, \cdot) * w(s, \cdot) ds.$$

Now, we are going to prove that  $F$  is uniformly continuous. For any  $h > 0$ , Young inequalities imply

$$\begin{aligned} \|F(t+h, \cdot) - F(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \int_0^t \|\partial_x K(t+h-s, \cdot) - \partial_x K(t-s, \cdot)\|_{L^i(\mathbb{R})} ds \|w\|_{L^\infty((0, T); L^j(\mathbb{R}))} \\ &\quad + \int_t^{t+h} \|\partial_x K(t+h-s, \cdot)\|_{L^i(\mathbb{R})} ds \|w\|_{L^\infty((0, T); L^j(\mathbb{R}))}, \end{aligned} \quad (12)$$

where  $i, j \in \mathbb{N}^*$  are such that  $i + j = 3$ . Since  $\partial_x K(t, \cdot) = \mathcal{F}^{-1}(\xi \rightarrow 2i\pi\xi e^{-t\psi_{\mathcal{I}}(\xi)})$ , the dominated convergence theorem implies that

$$\|\partial_x K(t-s+h, \cdot) - \partial_x K(t-s, \cdot)\|_{L^i(\mathbb{R})} \rightarrow 0, \text{ as } h \rightarrow 0.$$

Moreover, using the estimates on the gradient (item 3 and 4 of Proposition 1), we have the following inequality

$$\int_t^{t+h} \|\partial_x K(t-s+h, \cdot)\|_{L^j(\mathbb{R})} ds \leq c_j h^{\alpha_j},$$

where  $c_j$  is a positive constant and  $\alpha_j = \begin{cases} 1/2 & \text{if } j = 1 \\ 1/4 & \text{if } j = 2 \end{cases}$ .

Using (12), we obtain that  $\|F(t+h, \cdot) - F(t, \cdot)\|_{L^2(\mathbb{R})} \rightarrow 0$ , as  $h \rightarrow 0$ . Hence, the function  $F$  is continuous and this completes the proof of the continuity of  $v$ .  $\blacksquare$

**Remark 7.** Using the semi-group property, one can easily check [1] that for all  $t_0 \in (0, T)$  and all  $t \in [0, T - t_0]$ ,

$$v(t+t_0, \cdot) = K(t, \cdot) * v(t_0, \cdot) - \int_0^t \partial_x K(t-s, \cdot) * w(t_0+s, \cdot) ds.$$

### 3 Uniqueness of a solution

Let us first establish the following Lemma.

**Lemma 1.** *Let  $T > 0$  and  $v_0 \in L^2(\mathbb{R})$ . For  $i = 1, 2$ , let  $w_i \in L^\infty((0, T); L^1(\mathbb{R})) \cup L^\infty((0, T); L^2(\mathbb{R}))$  and define  $v_i$  as in Proposition 4 by:*

$$v_i(t, \cdot) = K(t, \cdot) * v_0 - \int_0^t \partial_x K(t-s, \cdot) * w_i(s, \cdot) ds.$$

Then,

$$\|v_1 - v_2\|_{C([0, T]; L^2(\mathbb{R}))} \leq \begin{cases} 4K_0 T^{1/4} \|w_1 - w_2\|_{L^\infty((0, T); L^1(\mathbb{R}))} & \text{if } w_i \in L^\infty((0, T); L^1(\mathbb{R})), \\ 2K_1 \sqrt{T} \|w_1 - w_2\|_{L^\infty((0, T); L^2(\mathbb{R}))} & \text{if } w_i \in L^\infty((0, T); L^2(\mathbb{R})). \end{cases}$$

*Proof.* For all  $t \in [0, T]$ , we have

$$v_1(t, \cdot) - v_2(t, \cdot) = - \int_0^t \partial_x K(t-s, \cdot) * (w_1 - w_2)(s, \cdot) ds.$$

Hence with the help of Young inequalities, we get

$$\|v_1(t, \cdot) - v_2(t, \cdot)\|_{L^2(\mathbb{R})} \leq \begin{cases} \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^2(\mathbb{R})} \|(w_1 - w_2)(s, \cdot)\|_{L^1(\mathbb{R})} ds & \text{if } w_i \in L^\infty((0, T); L^1(\mathbb{R})), \\ \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^1(\mathbb{R})} \|(w_1 - w_2)(s, \cdot)\|_{L^2(\mathbb{R})} ds & \text{if } w_i \in L^\infty((0, T); L^2(\mathbb{R})). \end{cases}$$

It then follows that

$$\|v_1(t, \cdot) - v_2(t, \cdot)\|_{L^2(\mathbb{R})} \leq \begin{cases} \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^2(\mathbb{R})} ds \|w_1 - w_2\|_{L^\infty((0, T); L^1(\mathbb{R}))} & \text{if } w_i \in L^\infty((0, T); L^1(\mathbb{R})), \\ \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^1(\mathbb{R})} ds \|w_1 - w_2\|_{L^\infty((0, T); L^2(\mathbb{R}))} & \text{if } w_i \in L^\infty((0, T); L^2(\mathbb{R})). \end{cases}$$

Using again the estimates of the gradient of  $K$  (see Proposition 1), we conclude the proof of this Lemma. ■

**Proposition 5.** *Let  $T > 0$  and  $v_0 \in L^2(\mathbb{R})$ . There exists at most one  $v \in L^\infty((0, T); L^2(\mathbb{R}))$  which is a mild solution to (6).*

*Proof.* Let  $v_1, v_2 \in L^\infty((0, T); L^2(\mathbb{R}))$  be two mild solutions to (6) and  $t \in [0, T]$ . Using the previous Lemma, we get

$$\|v_1 - v_2\|_{C([0, t]; L^2(\mathbb{R}))} \leq 2K_0 t^{1/4} \|v_1^2 - v_2^2\|_{L^\infty((0, t); L^1(\mathbb{R}))} + 2K_1 \sqrt{t} \|u_\phi v_1 - u_\phi v_2\|_{L^\infty((0, t); L^2(\mathbb{R}))}.$$

Since,

$$\|v_1^2 - v_2^2\|_{L^\infty((0, t); L^1(\mathbb{R}))} \leq M \|v_1 - v_2\|_{C([0, t]; L^2(\mathbb{R}))} \tag{13}$$

with  $M = \|v_1\|_{C([0, T]; L^2(\mathbb{R}))} + \|v_2\|_{C([0, T]; L^2(\mathbb{R}))}$ ,

then

$$\|v_1 - v_2\|_{C([0, t]; L^2(\mathbb{R}))} \leq (2MK_0 t^{1/4} + 2K_1 t^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})}) \|v_1 - v_2\|_{C([0, t]; L^2(\mathbb{R}))}.$$

Therefore,  $v_1 = v_2$  on  $[0, t]$  for any  $t \in (0, T]$  satisfying  $2MK_0t^{1/4} + 2K_1t^{1/2}\|u_\phi\|_{C_b^1(\mathbb{R})} < 1$ . Since  $v_1$  and  $v_2$  are continuous with values in  $L^2(\mathbb{R})$ , we have that  $v_1 = v_2$  on  $[0, T_*]$  where  $T_*$  is the positive solution of the following equation

$$2MK_0t^{1/4} + 2K_1t^{1/2}\|u_\phi\|_{C_b^1(\mathbb{R})} = 1,$$

$$\text{i.e. } T_* = \left( \frac{-2MK_0 + \sqrt{4M^2K_0^2 + 8K_1\|u_\phi\|_{C_b^1(\mathbb{R})}}}{4K_1\|u_\phi\|_{C_b^1(\mathbb{R})}} \right)^4.$$

To prove that  $v_1 = v_2$  on  $[0, T]$ , let us define

$$t_0 := \sup\{t \in [0, T] \text{ s.t. } v_1 = v_2 [0, t]\}$$

and we assume that  $t_0 < T$ . By continuity of  $v_1$  and  $v_2$ , we have that  $v_1(t_0, \cdot) = v_2(t_0, \cdot)$ . Using the semi-group property, see remark 7, we deduce that  $v_1(t_0 + \cdot, \cdot) = v_2(t_0 + \cdot, \cdot)$  are mild solutions to (6) with the same initial data  $v_1(t_0, \cdot) = v_2(t_0, \cdot)$ , which implies that  $v_1(t, \cdot) = v_2(t, \cdot)$  for  $t \in [t_0, T_* + t_0]$ . Finally, we get a contradiction with the definition of  $t_0$  and we infer that  $t_0 = T$ . This completes the proof of this proposition.  $\blacksquare$

## 4 Global-in-time existence of a mild solution

**Proposition 6** (local-in-time existence). *Let  $v_0 \in L^2(\mathbb{R})$ . There exists  $T_* > 0$  that only depends on  $\|v_0\|_{L^2(\mathbb{R})}$  and  $\|u_\phi\|_{C_b^1(\mathbb{R})}$  such that (6) admits a unique mild solution  $v \in C([0, T_*]; L^2(\mathbb{R})) \cap C((0, T_*]; H^1(\mathbb{R}))$ . Moreover,  $v$  satisfies*

$$\sup_{t \in (0, T_*)} t^{1/2} \|\partial_x v(t, \cdot)\|_{L^2(\mathbb{R})} < +\infty.$$

*Proof.* The proof of this proposition is based on the Banach fixed point theorem.

Since the gradient of the kernel with the  $L^2$ -norm explodes with a controlled rate, see item 4 of Proposition 1, we consider for  $v \in C([0, T]; L^2(\mathbb{R})) \cap C((0, T]; H^1(\mathbb{R}))$ , the following norm

$$\|v\| := \|v\|_{C([0, T]; L^2(\mathbb{R}))} + \sup_{t \in (0, T]} t^{\frac{1}{2}} \|\partial_x v(t, \cdot)\|_{L^2(\mathbb{R})} \quad (14)$$

and we define the affine space

$$X := \{v \in C([0, T]; L^2(\mathbb{R})) \cap C((0, T]; H^1(\mathbb{R})) \text{ s.t. } v(0, \cdot) = v_0 \text{ and } \|v\| < +\infty\}.$$

It is readily seen that  $X$  endowed with the distance induced by the norm  $\|v\|$  is a complete metric space. For  $v \in X$ , we define the function

$$\Theta v : t \in [0, T] \rightarrow K(t, \cdot) * v_0 - \frac{1}{2} \int_0^t \partial_x K(t-s, \cdot) * v^2(s, \cdot) ds - \int_0^t \partial_x K(t-s, \cdot) * u_\phi v(s, \cdot) ds.$$

From Proposition 4,  $\Theta v \in C([0, T]; L^2(\mathbb{R}))$  and satisfies  $\Theta v(0, \cdot) = v_0$ .

**First step:**  $\Theta v \in X$ . Since

$$\partial_x(K(t, \cdot) * v_0) = \partial_x K(t, \cdot) * v_0 = \mathcal{F}^{-1}(\xi \mapsto 2i\pi\xi e^{-t\psi_X(\xi)} \mathcal{F}v_0(\xi)),$$

the dominated convergence theorem implies that for any  $t_0 > 0$ ,

$$\int_{\mathbb{R}} 4\pi^2 |\xi|^2 \left| e^{-t\psi_I(\xi)} - e^{-t_0\psi_I(\xi)} \right|^2 |\mathcal{F}v_0(\xi)|^2 d\xi \rightarrow 0, \quad \text{as } t \rightarrow t_0.$$

Therefore, the function  $t > 0 \rightarrow (\xi \mapsto 2i\pi\xi e^{-t\psi_I(\xi)} \mathcal{F}v_0(\xi)) \in L^2(\mathbb{R})$  is continuous and since  $\mathcal{F}$  is an isometry of  $L^2$ , we deduce that  $t > 0 \rightarrow \partial_x K(t, \cdot) * v_0 \in L^2(\mathbb{R})$  is continuous. We then have established that  $t > 0 \rightarrow K(t, \cdot) * v_0 \in H^1(\mathbb{R})$  is continuous. Moreover, from Proposition 1, we have

$$\|\partial_x K(t, \cdot) * v_0\|_{L^2(\mathbb{R})} \leq K_1 t^{-1/2} \|v_0\|_{L^2(\mathbb{R})}. \quad (15)$$

Let  $w$  denote the function

$$w(t, \cdot) = \frac{1}{2} \int_0^t \partial_x K(t-s, \cdot) * v^2(s, \cdot) ds + \int_0^t \partial_x K(t-s, \cdot) * u_\phi v(s, \cdot) ds.$$

Let us now prove that  $w \in C((0, T]; H^1(\mathbb{R}))$ . We first have

$$\partial_x w(t, \cdot) = \int_0^t \partial_x K(t-s, \cdot) * v \partial_x v(s, \cdot) ds + \int_0^t \partial_x K(t-s, \cdot) * \partial_x (u_\phi v)(s, \cdot) ds.$$

Using Young inequalities and Proposition 1, we get

$$\begin{aligned} \|\partial_x w(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \int_0^t \|\partial_x K(t-s, \cdot) * v \partial_x v(s, \cdot)\|_{L^2(\mathbb{R})} ds \\ &\quad + \int_0^t \|\partial_x K(t-s, \cdot) * \partial_x (u_\phi v)(s, \cdot)\|_{L^2(\mathbb{R})} ds, \\ &\leq \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^2(\mathbb{R})} \|v \partial_x v(s, \cdot)\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^1(\mathbb{R})} \|\partial_x (u_\phi v)(s, \cdot)\|_{L^2(\mathbb{R})} ds, \\ &\leq \|v\|_{C([0, T]; L^2(\mathbb{R}))} \int_0^t K_0 (t-s)^{-3/4} s^{-1/2} ds \sup_{s \in (0, T]} s^{1/2} \|\partial_x v(s, \cdot)\|_{L^2(\mathbb{R})} \\ &\quad + \int_0^t K_1 (t-s)^{-1/2} s^{-1/2} ds \sup_{s \in (0, T]} s^{1/2} \|\partial_x (u_\phi v)(s, \cdot)\|_{L^2(\mathbb{R})}, \\ &\leq K_0 I \|v\|_{C([0, T]; L^2(\mathbb{R}))} T^{-1/4} \sup_{s \in (0, T]} s^{1/2} \|\partial_x v(s, \cdot)\|_{L^2(\mathbb{R})} \\ &\quad + K_1 J \sup_{s \in (0, T]} s^{1/2} \|\partial_x (u_\phi v)(s, \cdot)\|_{L^2(\mathbb{R})}, \end{aligned} \quad (16)$$

where  $I = B(\frac{1}{2}, \frac{1}{4})$  and  $J = B(\frac{1}{2}, \frac{1}{2}) = \pi$ ,  $B$  being the beta function defined by

$$B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

As  $\|v\| < \infty$  then  $\sup_{s \in (0, T]} s^{1/2} \|\partial_x v(s, \cdot)\|_{L^2(\mathbb{R})} < \infty$  and  $\sup_{s \in (0, T]} s^{1/2} \|\partial_x (u_\phi v)(s, \cdot)\|_{L^2(\mathbb{R})} < \infty$ . We then deduce that  $\partial_x w(t, \cdot)$  is in  $L^2$  and so  $\partial_x v(t, \cdot) \in L^2(\mathbb{R})$  for all  $t \in (0, T]$ .

Let us now prove that  $\partial_x w$  is continuous on  $(0, T]$  with values in  $L^2$ . For  $\delta > 0$  and  $t \in (0, T]$ , we define

$$\begin{aligned} (\partial_x w)_\delta(t, \cdot) &:= \int_0^t \partial_x K(t-s, \cdot) * (1_{\{s>\delta\}}(v\partial_x v)(s, \cdot)) ds \\ &\quad + \int_0^t \partial_x K(t-s, \cdot) * (1_{\{s>\delta\}}\partial_x(u_\phi v)(s, \cdot)) ds. \end{aligned}$$

Since  $1_{\{s>\delta\}}(v\partial_x v)(s, \cdot) \in L^\infty([0, T]; L^1(\mathbb{R}))$  and  $1_{\{s>\delta\}}\partial_x(u_\phi v)(s, \cdot) \in L^\infty([0, T]; L^2(\mathbb{R}))$  then Proposition 4 implies that  $(\partial_x w)_\delta : [0, T] \rightarrow L^2(\mathbb{R})$  is continuous. Moreover, we have

$$\begin{aligned} \|\partial_x w(t, \cdot) - (\partial_x w)_\delta(t, \cdot)\|_{L^2(\mathbb{R})} &\leq K_0 \int_0^\delta (t-s)^{-3/4} s^{-1/2} ds \|v\|_{C([0, T]; L^2(\mathbb{R}))} \sup_{s \in (0, T]} s^{1/2} \|\partial_x v(s, \cdot)\|_{L^2(\mathbb{R})} \\ &\quad + K_1 \int_0^\delta (t-s)^{-1/2} s^{-1/2} ds \sup_{s \in (0, T]} s^{1/2} \|\partial_x(u_\phi v)(s, \cdot)\|_{L^2(\mathbb{R})}. \end{aligned}$$

It then follows that

$$\sup_{t \in (0, T]} \|\partial_x w(t, \cdot) - \partial_x w_\delta(t, \cdot)\|_{L^2(\mathbb{R})} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

We next infer that  $\partial_x w \in C((0, T]; L^2(\mathbb{R}))$  because it is a local uniform limit of continuous function. Hence, we have established that  $\Theta v \in C([0, T]; L^2(\mathbb{R})) \cap C((0, T]; H^1(\mathbb{R}))$ . To prove that  $\Theta v \in X$ , it remains to show that  $\|\Theta v\| < +\infty$ . Using (15) and (16), we have

$$\begin{aligned} \sup_{t \in (0, T]} t^{1/2} \|\partial_x \Theta v(t, \cdot)\|_{L^2(\mathbb{R})} &\leq K_1 \|v_0\|_{L^2(\mathbb{R})} + K_0 I T^{1/4} \sup_{s \in (0, T]} s^{1/2} \|\partial_x v(s, \cdot)\|_{L^2(\mathbb{R})} \|v\|_{C([0, T]; L^2(\mathbb{R}))} \\ &\quad + K_1 J T^{1/2} \sup_{s \in (0, T]} s^{1/2} \|\partial_x(u_\phi v)(s, \cdot)\|_{L^2(\mathbb{R})}. \end{aligned} \quad (17)$$

Finally, we have  $\Theta : X \rightarrow X$ .

**Second step:** We begin by considering a ball of  $X$  of radius  $R$  centered at the origin

$$B_R := \{v \in X / \|v(t, \cdot)\| \leq R\}$$

where  $R > \|v_0\|_{L^2(\mathbb{R})} + K_1 \|v_0\|_{L^2(\mathbb{R})}$ . Take  $v \in B_R$  and let us now prove that  $\Theta$  maps  $B_R$  into itself. We have

$$\|\Theta(v)(t, \cdot)\|_{L^2(\mathbb{R})} \leq \|K(t, \cdot) * v_0\|_{L^2(\mathbb{R})} + \int_0^t \|\partial_x K(t-s, \cdot) * \left(\frac{v^2}{2} + u_\phi v\right)(s, \cdot)\|_{L^2(\mathbb{R})} ds.$$

By remark 4, we get

$$\|K(t, \cdot) * v_0\|_{L^2(\mathbb{R})} \leq e^{\alpha_0 T} \|v_0\|_{L^2(\mathbb{R})} \quad (18)$$

where  $\alpha_0 = -\min \operatorname{Re}(\psi_{\mathcal{I}}) > 0$ . Moreover, since  $\|v^2\|_{L^\infty((0, T); L^1(\mathbb{R}))} = \|v\|_{L^\infty((0, T); L^2(\mathbb{R}))}^2$  and with the help of properties of Proposition 1, we get

$$\|\Theta(v)(t, \cdot)\|_{L^2(\mathbb{R})} \leq e^{\alpha_0 T} \|v_0\|_{L^2(\mathbb{R})} + 2K_0 T^{1/4} R^2 + 2K_1 T^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})} R. \quad (19)$$

Using (17) and (19), we deduce that

$$\begin{aligned} |||\Theta v||| &\leq e^{\alpha_0 T} \|v_0\|_{L^2(\mathbb{R})} + K_1 \|v_0\|_{L^2(\mathbb{R})} + (2+I)K_0 T^{1/4} R^2 + (2+J)RK_1 T^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})} \\ &\quad + K_1 J \|u_\phi\|_{C_b^1(\mathbb{R})} RT. \end{aligned}$$

Therefore, for  $T > 0$  sufficiently small such that

$$e^{\alpha_0 T} \|v_0\|_{L^2(\mathbb{R})} + K_1 \|v_0\|_{L^2(\mathbb{R})} + (2+I)K_0 T^{1/4} R^2 + (2+J)RK_1 T^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})} + K_1 J \|u_\phi\|_{C_b^1(\mathbb{R})} RT \leq R, \quad (20)$$

we get that  $|||\Theta v||| \leq R$ .

To finish with, we are going to prove that  $\Theta$  is a contraction.

For  $v, w \in B_R$ , we have for any  $t \in (0, T)$

$$\begin{aligned} \|\Theta(v)(t, \cdot) - \Theta(w)(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \frac{1}{2} \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^2(\mathbb{R})} \|(v^2 - w^2)(s, \cdot)\|_{L^1(\mathbb{R})} ds \\ &\quad + \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^1(\mathbb{R})} \|u_\phi(v-w)(s, \cdot)\|_{L^2(\mathbb{R})} ds, \\ &\leq 2K_0 t^{1/4} \|v^2 - w^2\|_{C([0, T]; L^1(\mathbb{R}))} \\ &\quad + 2K_1 t^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})} \|v - w\|_{C([0, T]; L^2(\mathbb{R}))}, \end{aligned}$$

and since,

$$\begin{aligned} \|v^2 - w^2\|_{C([0, T]; L^1(\mathbb{R}))} &\leq (\|v\|_{C([0, T]; L^2(\mathbb{R}))} + \|w\|_{C([0, T]; L^2(\mathbb{R}))}) \|v - w\|_{C([0, T]; L^2(\mathbb{R}))}, \\ &\leq 2R \|v - w\|_{C([0, T]; L^2(\mathbb{R}))}, \end{aligned}$$

we get

$$\|\Theta(v)(t, \cdot) - \Theta(w)(t, \cdot)\|_{L^2(\mathbb{R})} \leq (4RK_0 t^{1/4} + 2K_1 t^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})}) \|v - w\|_{C([0, T]; L^2(\mathbb{R}))}. \quad (21)$$

Moreover

$$\begin{aligned} \|\partial_x(\Theta v - \Theta w)(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \frac{1}{2} \int_0^t \|\partial_x K(t-s, \cdot) * \partial_x(v^2 - w^2)(s, \cdot)\|_{L^2(\mathbb{R})} ds \\ &\quad + \int_0^t \|\partial_x K(t-s, \cdot) * \partial_x(u_\phi(v-w))(s, \cdot)\|_{L^2(\mathbb{R})} ds, \\ &\leq K_0 I t^{-1/4} \sup_{s \in (0, T]} s^{1/2} \|(v \partial_x v - w \partial_x w)(s, \cdot)\|_{L^1(\mathbb{R})} \\ &\quad + K_1 J \sup_{s \in (0, T]} s^{1/2} \|\partial_x(u_\phi(v-w))(s, \cdot)\|_{L^2(\mathbb{R})}. \end{aligned}$$

And since

$$\|(v \partial_x v - w \partial_x w)(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|\partial_x w(t, \cdot)\|_{L^2(\mathbb{R})} \|(v-w)(t, \cdot)\|_{L^2(\mathbb{R})} + \|v(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x(v-w)(t, \cdot)\|_{L^2(\mathbb{R})},$$

then

$$\begin{aligned} t^{1/2} \|(v \partial_x v - w \partial_x w)(t, \cdot)\|_{L^1(\mathbb{R})} &\leq \|(v-w)(t, \cdot)\|_{L^2(\mathbb{R})} \|w\| + \|v\| t^{1/2} \|\partial_x(v-w)(t, \cdot)\|_{L^2(\mathbb{R})}, \\ &\leq 2R \|v-w\|. \end{aligned}$$

Therefore, we obtain

$$\|\partial_x(\Theta v - \Theta w)(t, \cdot)\|_{L^2(\mathbb{R})} \leq 2K_0 I t^{-1/4} R \|v - w\| + K_1 J \|u_\phi\|_{C_b^1(\mathbb{R})} T^{1/2} \|v - w\| + K_1 J \|u_\phi\|_{C_b^1(\mathbb{R})} \|v - w\|. \quad (22)$$

Finally, using (21) and (22), we get

$$\|\Theta v - \Theta w\| \leq [(2 + I)2RK_0 T^{1/4} + (2 + J)\|u_\phi\|_{C_b^1(\mathbb{R})} K_1 T^{1/2} + K_1 J T \|u_\phi\|_{C_b^1(\mathbb{R})}] \|v - w\|.$$

**Last step: conclusion.** For any  $T_* > 0$  sufficiently small such that (20) holds true and

$$[(2 + I)2RK_0 T_*^{1/4} + (2 + J)\|u_\phi\|_{C_b^1(\mathbb{R})} K_1 T_*^{1/2} + K_1 J T_* \|u_\phi\|_{C_b^1(\mathbb{R})}] < 1,$$

$\Theta$  is a contraction from  $B_R$  into itself. The Banach fixed point theorem then implies that  $\Theta$  admits a unique fixed point  $v \in C([0, T_*]; L^2(\mathbb{R})) \cap C((0, T_*]; H^1(\mathbb{R}))$  which is a mild solution to (6). ■

**Lemma 2** (Regularity  $H^2$  of  $v(t, \cdot)$ ). *Let  $v_0 \in L^2(\mathbb{R})$ . There exists  $T'_* > 0$  that only depends on  $\|v_0\|_{L^2(\mathbb{R})}$  and  $\|u_\phi\|_{C_b^1(\mathbb{R})}$  such that (6) admits a unique mild solution  $v \in C([0, T'_*]; L^2(\mathbb{R})) \cap C((0, T'_*]; H^2(\mathbb{R}))$ . Moreover,  $v$  satisfies*

$$\sup_{t \in (0, T'_*]} t^{1/2} \|\partial_x v(t, \cdot)\|_{L^2(\mathbb{R})} < +\infty \quad \text{and} \quad \sup_{t \in (0, T'_*]} t \|\partial_{xx}^2 v(t, \cdot)\|_{L^2(\mathbb{R})} < +\infty.$$

*Proof.* To prove this result, we use again a contracting fixed point theorem. But this time, it is the gradient of the solution  $v$  which is searched as a fixed point.

From Proposition 6, there exists  $T_* > 0$  which depends on  $\|v_0\|_{L^2(\mathbb{R})}$  and  $\|u_\phi\|_{C_b^1(\mathbb{R})}$  such that  $v \in C([0, T_*]; L^2(\mathbb{R})) \cap C((0, T_*]; H^1(\mathbb{R}))$  is a mild solution to (6). Since  $v \in C((0, T_*]; H^1(\mathbb{R}))$ , we can consider the gradient of  $v(t, \cdot)$  for any  $t \in (0, T_*]$ . Let then  $t_0 \in (0, T_*)$  and  $T'_* \in (0, T_* - t_0]$ . We consider the same complete metric space  $X$  defined in the proof of the Proposition 6 and we take the norm  $\|\cdot\|$  defined in (14):

$$X := \left\{ w \in C([0, T'_*]; L^2(\mathbb{R})) \cap C((0, T'_*]; H^1(\mathbb{R})) \text{ s.t. } w(0, \cdot) = w_0 \text{ and } \|w\| < +\infty \right\},$$

with the initial data  $w_0 = \partial_x v(t_0, \cdot)$ .

We now wish to apply the fixed point theorem at the following function

$$\begin{aligned} \Theta w : t \in [0, T'_*] \quad \rightarrow \quad & K(t, \cdot) * w_0 - \int_0^t \partial_x K(t - s, \cdot) * (\bar{v}w)(s, \cdot) ds \\ & - \int_0^t \partial_x K(t - s, \cdot) * (\partial_x(u_\phi)\bar{v})(s, \cdot) ds \\ & - \int_0^t \partial_x K(t - s, \cdot) * (u_\phi w)(s, \cdot) ds, \end{aligned}$$

where  $\bar{v}(t, \cdot) := v(t_0 + t, \cdot)$ . First, we leave to the reader to verify that  $\Theta$  maps  $X$  into itself. The proof is similar to the first step of the proof of the Proposition 6.

For any  $w \in X$ , we have from Young inequalities

$$\begin{aligned} \|\Theta w(t, \cdot)\|_{L^2(\mathbb{R})} &\leq e^{\alpha_0 T'_*} \|w_0\|_{L^2(\mathbb{R})} + \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))} \|w\| \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^2(\mathbb{R})} ds \\ &\quad + \|u_\phi\|_{C_b^1(\mathbb{R})} \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))} \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^1(\mathbb{R})} ds \\ &\quad + \|u_\phi\|_{C_b^1(\mathbb{R})} \|w\| \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^1(\mathbb{R})} ds, \end{aligned}$$

and using item 3 and 4 of the Proposition 1, we get

$$\begin{aligned} \|\Theta w(t, \cdot)\|_{L^2(\mathbb{R})} &\leq e^{\alpha_0 T'_*} \|w_0\|_{L^2(\mathbb{R})} + 4K_0 T_*'^{1/4} \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))} \|w\| \\ &\quad + 2K_1 T_*'^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})} \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))} \\ &\quad + 2K_1 T_*'^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})} \|w\|. \end{aligned} \quad (23)$$

Moreover, we have the following estimate

$$\begin{aligned} \|\partial_x \Theta w(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \|\partial_x K(t, \cdot)\|_{L^1(\mathbb{R})} \|w_0\|_{L^2(\mathbb{R})} + \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x(\bar{v}w)(s, \cdot)\|_{L^1(\mathbb{R})} ds \\ &\quad + \|u_\phi\|_{C_b^1(\mathbb{R})} \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^1(\mathbb{R})} \|\partial_x \bar{v}(s, \cdot)\|_{L^2(\mathbb{R})} ds \\ &\quad + \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^1(\mathbb{R})} \|\partial_x(u_\phi w)(s, \cdot)\|_{L^2(\mathbb{R})} ds, \end{aligned}$$

and using again the estimates for the gradient of the kernel given in Proposition 1, we obtain the next result for any  $t \in (0, T'_*]$

$$\begin{aligned} t^{1/2} \|\partial_x \Theta w(t, \cdot)\|_{L^2(\mathbb{R})} &\leq K_1 \|w_0\|_{L^2(\mathbb{R})} + 4K_0 T_*'^{3/4} \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))} \|w\| \\ &\quad + K_0 I T_*'^{1/4} \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))} \|w\| + 2K_1 T_*' \|u_\phi\|_{C_b^1} \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))} \\ &\quad + 2K_1 T_*' \|u_\phi\|_{C_b^1} \|w\| + K_1 J T_*'^{1/2} \|u_\phi\|_{C_b^1} \|w\|, \end{aligned} \quad (24)$$

where  $I, J$  are two positive constants defined in the proof of the Proposition 6.

Hence, using (23) and (24), we get

$$\begin{aligned} \|\Theta w\| &\leq e^{\alpha_0 T'_*} \|w_0\|_{L^2(\mathbb{R})} + K_1 \|w_0\|_{L^2(\mathbb{R})} + 2K_1 \|u_\phi\|_{C_b^1(\mathbb{R})} \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))} (T_*' + T_*'^{1/2}) \\ &\quad + C \|w\| (T_*'^{1/4} + T_*'^{1/2} + T_*'^{3/4} + T_*'), \end{aligned}$$

for some positive constant  $C$  which depends on  $K_0, K_1, \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))}$  and  $\|u_\phi\|_{C_b^1(\mathbb{R})}$ .

We leave to reader to verify that: for any  $w_1, w_2 \in X$ ,

$$\|\Theta w_1 - \Theta w_2\| \leq C'(K_0, K_1, \|u_\phi\|_{C_b^1(\mathbb{R})}, \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))}) (T_*'^{1/4} + T_*'^{1/2} + T_*'^{3/4} + T_*') \|w_1 - w_2\|,$$

where  $C'$  is a non-negative constant which also depends on  $K_0, K_1, \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))}$  and  $\|u_\phi\|_{C_b^1(\mathbb{R})}$ .

Let us now take  $R$  such that  $R > \|w_0\|_{L^2(\mathbb{R})} + K_1 \|w_0\|_{L^2(\mathbb{R})}$ . Then, if  $T_*' > 0$  satisfies

$$\begin{aligned} e^{\alpha_0 T'_*} \|w_0\|_{L^2(\mathbb{R})} + K_1 \|w_0\|_{L^2(\mathbb{R})} + 2K_1 \|u_\phi\|_{C_b^1(\mathbb{R})} \|\bar{v}\|_{C([t_0, T'_*]; H^1(\mathbb{R}))} (T_*' + T_*'^{1/2}) \\ + CR(T_*'^{1/4} + T_*'^{1/2} + T_*'^{3/4} + T_*') \leq R, \end{aligned}$$

and

$$C'(T_*'^{1/4} + T_*'^{1/2} + T_*'^{3/4} + T_*') < 1,$$

$\Theta : B_R(X) \rightarrow B_R(X)$  is a contraction, where  $B_R(X)$  is ball of  $X$  of radius  $R$  centered at the origin. Using a contracting point fixed theorem, it exists a unique fixed point, which we denote by  $w$ . But it is easy to see that  $\Theta \partial_x \bar{v} = \partial_x \bar{v}$  taking into account the space derivated of the Duhamel formulation (8). Thanks to an uniqueness argument, we deduce that  $w = \partial_x \bar{v}$  and thus that  $v \in C((0, T_*']; H^2(\mathbb{R}))$ , which completes the proof of this lemma. ■

Let us now prove the global-in-time existence of mild solution  $v$ .

**Proposition 7** (Global-in-time existence ). *Let  $v_0 \in L^2(\mathbb{R})$  and  $T > 0$ . There exists a (unique) mild solution  $v \in C([0, T]; L^2(\mathbb{R})) \cap C((0, T]; H^2(\mathbb{R}))$  of (6). Moreover,  $v$  satisfies the PDE (6) in the distribution sense.*

*Proof. First step:  $v$  is a distribution solution.* Taking the Fourier transform w.r.t the space variable in (8), we get for all  $t \in [0, T]$  and all  $\xi \in \mathbb{R}$ ,

$$\mathcal{F}(v(t, \cdot))(\xi) = e^{-t\psi_{\mathcal{I}}(\xi)} \mathcal{F}v_0(\xi) - \int_0^t i\pi\xi e^{-(t-s)\psi_{\mathcal{I}}(\xi)} \mathcal{F}(v^2(s, \cdot))(\xi) ds - \int_0^t 2i\pi\xi e^{-(t-s)\psi_{\mathcal{I}}(\xi)} \mathcal{F}(u_\phi v(s, \cdot))(\xi) ds. \quad (25)$$

Let us denote  $G$  the following function:

$$G(t, \xi) = - \int_0^t 2i\pi\xi e^{-(t-s)\psi_{\mathcal{I}}(\xi)} \mathcal{F} \left( \frac{v^2}{2} + u_\phi v \right) (s, \cdot)(\xi) ds.$$

Classical results on ODE imply that  $G$  is derivable w.r.t the time with

$$\begin{aligned} \partial_t G(t, \xi) + \psi_{\mathcal{I}}(\xi) G(t, \xi) &= -i\pi\xi \mathcal{F}(v^2(s, \cdot))(\xi) - 2i\pi\xi \mathcal{F}((u_\phi v)(s, \cdot))(\xi), \\ &= -\mathcal{F}(\partial_x(\frac{v^2}{2})(t, \cdot))(\xi) - \mathcal{F}(\partial_x(u_\phi v)(t, \cdot))(\xi). \end{aligned} \quad (26)$$

Let us now prove that all terms in (26) are continuous with values in  $L^2$ . Since,  $v \in C((0, T]; H^1(\mathbb{R}))$  then  $\partial_x(v^2), \partial_x(u_\phi v) \in C((0, T]; L^2(\mathbb{R}))$ . We thus deduce that  $\mathcal{F}(\partial_x(\frac{v^2}{2})(t, \cdot))$  and  $\mathcal{F}(\partial_x(u_\phi v)(t, \cdot))$  are continuous with values in  $L^2(\mathbb{R})$ . Moreover, equation (25) implies that

$$\psi_{\mathcal{I}} G(t, \cdot) = \psi_{\mathcal{I}} \left( \mathcal{F}(v(t, \cdot)) - e^{-t\psi_{\mathcal{I}}} \mathcal{F}v_0 \right),$$

and so  $\psi_{\mathcal{I}} G(t, \cdot)$  is continuous with values in  $L^2$ . Indeed,

$$\begin{aligned}
\int_{\mathbb{R}} |\psi_{\mathcal{I}}(\xi)G(t, \xi)|^2 d\xi &= \int_{-1}^1 |\psi_{\mathcal{I}}(\xi)G(t, \xi)|^2 d\xi + \int_{\mathbb{R} \setminus (-1,1)} |\psi(\xi)G(t, \xi)|^2 d\xi, \\
&\leq \sup_{\xi \in [-1,1]} |\psi_{\mathcal{I}}(\xi)|^2 \|G(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C \int_{\mathbb{R} \setminus (-1,1)} |\xi^2 G(t, \xi)|^2 d\xi, \\
&\leq \sup_{\xi \in [-1,1]} |\psi_{\mathcal{I}}(\xi)|^2 \|G(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\quad + C \int_{\mathbb{R} \setminus (-1,1)} |\mathcal{F}(\partial_{xx}^2 v(t, \cdot)) - \xi^2 e^{-t\psi_{\mathcal{I}}(\xi)} \mathcal{F}v_0|^2 d\xi, \\
&\leq \sup_{\xi \in [-1,1]} |\psi_{\mathcal{I}}(\xi)|^2 \|G(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \tilde{C} \|v(t, \cdot)\|_{H^2(\mathbb{R})}^2 \\
&\quad + \tilde{C} \|v_0\|_{L^2(\mathbb{R})}^2 + C \|v(t, \cdot)\|_{H^2} \|v_0\|_{L^2}, \\
&< \infty,
\end{aligned}$$

with  $C, \tilde{C}$  two positive constants. Hence, we have clearly that the function  $t \rightarrow \psi_{\mathcal{I}}G(t, \cdot) \in L^2(\mathbb{R}, \mathbb{C})$  is continuous. Finally, we have proved that all the terms in (26) are continuous with values in  $L^2$ . Therefore, in particular, from (26), we get that  $G \in C^1([0, T]; L^2(\mathbb{R}, \mathbb{C}))$  and then

$$\frac{d}{dt}(G(t, \cdot)) + \psi_{\mathcal{I}}G(t, \cdot) = -\mathcal{F}(\partial_x(\frac{v^2}{2})(t, \cdot)) - \mathcal{F}(\partial_x(u_{\phi}v)(t, \cdot)).$$

Moreover,  $t \in [0, T] \rightarrow e^{-t\psi_{\mathcal{I}}} \mathcal{F}v_0 \in L^2(\mathbb{R}, \mathbb{C})$  is  $C^1$  with

$$\frac{d}{dt}(e^{-t\psi_{\mathcal{I}}} \mathcal{F}v_0) + \psi_{\mathcal{I}}e^{-t\psi_{\mathcal{I}}} \mathcal{F}v_0 = 0.$$

From equation (25), we infer that  $\mathcal{F}v$  is  $C^1$  on  $[0, T]$  with values in  $L^2$  with

$$\frac{d}{dt}\mathcal{F}(v(t, \cdot)) = -\psi_{\mathcal{I}}\mathcal{F}(v(t, \cdot)) - \mathcal{F}(\partial_x(\frac{v^2}{2})(t, \cdot)) - \mathcal{F}(\partial_x(u_{\phi}v)(t, \cdot)).$$

Since  $\mathcal{F}$  is an isometry of  $L^2$ , we deduce that  $v \in C^1([0, T]; L^2(\mathbb{R}))$  and by (3), we get

$$\begin{aligned}
\frac{d}{dt}(v(t, \cdot)) &= -\mathcal{F}^{-1}(\psi_{\mathcal{I}}\mathcal{F}(v(t, \cdot))) - \partial_x(\frac{v^2}{2})(t, \cdot) - \partial_x(u_{\phi}v)(t, \cdot), \\
&= -\mathcal{I}[v(t, \cdot)] + \partial_{xx}^2 v(t, \cdot) - \partial_x(\frac{v^2}{2})(t, \cdot) - \partial_x(u_{\phi}v)(t, \cdot).
\end{aligned}$$

We are now going to prove that  $v$  satisfies the PDE (6) in the distribution sense. Let us note

$$w(t, \cdot) := -\mathcal{I}[v(t, \cdot)] + \partial_{xx}^2 v(t, \cdot) - \partial_x(\frac{v^2}{2})(t, \cdot) - \partial_x(u_{\phi}v)(t, \cdot)$$

and let us show that

$$\partial_t v = w \quad \text{in} \quad \mathcal{D}((0, T) \times \mathbb{R}).$$

By definition, we have for any  $\varphi \in \mathcal{D}(0, T)$  and  $\psi \in \mathcal{D}(\mathbb{R})$ :

$$\begin{aligned}
\langle \partial_t v, \varphi \psi \rangle_{\mathcal{D}((0, T) \times \mathbb{R})} &= - \int_0^T \int_{\mathbb{R}} v(t, x) \frac{d\varphi}{dt} \psi(x) dt dx, \\
&= - \int_0^T \left( \int_{\mathbb{R}} v(t, x) \psi(x) dx \right) \frac{d\varphi(t)}{dt} dt.
\end{aligned}$$

Therefore, it is enough to prove that

$$\int_0^T \left( \int_{\mathbb{R}} w(t, x) \psi(x) dx \right) \varphi(t) dt = - \int_0^T \left( \int_{\mathbb{R}} v(t, x) \psi(x) dx \right) \varphi'(t) dt,$$

i.e.

$$\frac{d}{dt} \int_{\mathbb{R}} v(t, x) \psi(x) dx = \int_{\mathbb{R}} w(t, x) \psi(x) dx,$$

in the sense of  $\mathcal{D}'(0, T)$ . But, we have that the function

$$t \in (0, T) \mapsto \int_{\mathbb{R}} w(t, x) \psi(x) dx \in \mathbb{R}$$

is  $C^1$  and

$$\frac{d}{dt} \int_{\mathbb{R}} v(t, x) \psi(x) dx = \int_{\mathbb{R}} w(t, x) \psi(x) dx$$

in the classical sense, which proves that the mild solution  $v$  is a distribution solution of (6).

**Second step: A priori estimate.** By the first step, we have

$$\partial_t v + \partial_x \left( \frac{v^2}{2} + u_\phi v \right) + \mathcal{I}[v] - \partial_{xx}^2 v = 0$$

in the distribution sense. Therefore, multiplying this equality by  $v$  and integrating w.r.t the space variable, we get:

$$\int_{\mathbb{R}} v_t v dx + \int_{\mathbb{R}} (\mathcal{I}[v] - v_{xx}) v dx + \int_{\mathbb{R}} (u_\phi v)_x v dx = 0 \quad (27)$$

because the nonlinear term is zero. Indeed, integrating by parts, we have

$$\int_{\mathbb{R}} \partial_x \left( \frac{v^2}{2} \right) v dx = - \int_{\mathbb{R}} \frac{v^2}{2} \partial_x v dx = - \frac{1}{2} \int_{\mathbb{R}} \partial_x \left( \frac{v^2}{2} \right) v dx.$$

There is no boundary term from the infinity because for all  $t \in (0, T]$ ,  $v(t, \cdot) \in H^2(\mathbb{R})$ . Using (3) and the fact that  $\int_{\mathbb{R}} (\mathcal{I}[v] - \partial_{xx}^2 v) v dx$  is real, we get

$$\int_{\mathbb{R}} (\mathcal{I}[v] - \partial_{xx}^2 v) v dx = \int_{\mathbb{R}} \mathcal{F}^{-1}(\psi_{\mathcal{I}} \mathcal{F} v) v dx = \int_{\mathbb{R}} \psi_{\mathcal{I}} |\mathcal{F} v|^2 d\xi = \int_{\mathbb{R}} \operatorname{Re}(\psi_{\mathcal{I}}) |\mathcal{F} v|^2 d\xi. \quad (28)$$

Moreover, since  $u_\phi v \in H^1(\mathbb{R})$  we have

$$\int_{\mathbb{R}} (u_\phi v)_x v dx = - \int_{\mathbb{R}} u_\phi v v_x dx = - \int_{\mathbb{R}} u_\phi \left( \frac{v^2}{2} \right)_x dx = \int_{\mathbb{R}} (\partial_x u_\phi) \frac{v^2}{2} dx. \quad (29)$$

Using (27), (28) and (29), we obtain

$$\frac{1}{2} \frac{d}{dt} \|v(t, \cdot)\|_{L^2}^2 \leq (\alpha_0 + C_\phi) \|v(t)\|_{L^2}^2$$

where  $\alpha_0 = -\min \operatorname{Re} \psi_{\mathcal{I}} > 0$  and  $C_\phi = \frac{1}{2} \|u_\phi\|_{C_b^1}$ . Finally, we get for all  $t \in [0, T]$  the following estimate

$$\|v(t, \cdot)\|_{L^2(\mathbb{R})} \leq e^{(\alpha_0 + C_\phi)t} \|v_0\|_{L^2(\mathbb{R})}. \quad (30)$$

**Last step: global-in-time existence.** Up to this point, we know thanks to Proposition 6 and Lemma 2 that there exists  $T_* = T_*(\|v_0\|_{L^2(\mathbb{R})}, \|u_\phi\|_{C_b^1(\mathbb{R})}) > 0$  such that  $v \in C([0, T_*]; L^2(\mathbb{R})) \cap C((0, T_*]; H^2(\mathbb{R}))$  is a mild solution of (6) on  $(0, T_*]$ . Let us define

$$t_0 := \sup \{t > 0 / \text{there exists a mild solution of (6) on } (0, t) \text{ with initial condition } v_0\}.$$

To prove the global-in-time existence of a mild solution, we have to prove that  $t_0 \geq T$ , where  $T$  is any positive constant. Assume by contradiction that  $t_0 < T$ . With again the help of Proposition 6, there exists  $T'_* > 0$  such that for any initial data  $w_0$  that satisfy

$$\|w_0\|_{L^2(\mathbb{R})} \leq e^{(\alpha_0 + C_\phi)t} \|v_0\|_{L^2(\mathbb{R})}, \quad (31)$$

it exists a mild solution  $w$  on  $(0, T'_*]$ . Using (30), we have that  $w_0 := v(t_0 - T'_*/2, \cdot)$  satisfies (31). Therefore, using an argument of uniqueness, we deduce that  $v(t_0 - T'_*/2 + t, \cdot) = w(t, \cdot)$  for all  $t \in [0, T'_*/2)$ . To finish with, we define  $\tilde{v}$  by  $\tilde{v} = v$  on  $[0, t_0)$  and  $\tilde{v}(t_0 - T'_*/2 + t, \cdot) = w(t, \cdot)$  for  $t \in [T'_*/2, T'_*]$ . Hence,  $\tilde{v}$  is a mild solution on  $[0, t_0 + T'_*/2]$  with initial datum  $v_0$ , which gives us a contradiction.  $\blacksquare$

## 5 Regularity of the solution

This section is devoted to the proof of the existence of classical solutions  $v$  to (6).

**Proposition 8** (Solution in the classical sense). *Let  $v_0 \in L^2(\mathbb{R})$  and  $T > 0$ . The unique mild solution  $v \in C([0, T]; L^2(\mathbb{R})) \cap C((0, T]; H^2(\mathbb{R}))$  of (6) belongs to  $C^{1,2}((0, T] \times \mathbb{R})$  and satisfies*

$$\partial_t v + \partial_x \left( \frac{v^2}{2} + u_\phi v \right) + \mathcal{I}[v] - \partial_{xx}^2 v = 0,$$

on  $(0, T] \times \mathbb{R}$  in the classical sense.

*Proof.* **First step:  $C^2$ -regularity in space.** Let us take any  $t_0 \in (0, T]$  as initial time and let  $T' \in (0, T - t_0]$ . Differentiating the Duhamel formulation (8) two times w.r.t the space, we get for any  $t \in [0, T']$ ,

$$\begin{aligned} \partial_{xx}^2 v(t + t_0, \cdot) &= K(t, \cdot) * \partial_{xx}^2 v(t_0, \cdot) - \int_0^t \partial_x K(t - s, \cdot) * (u_1 + u_2)(t_0 + s, \cdot) ds \\ &\quad - \int_0^t \partial_{xx}^2 K(t - s, \cdot) * (\partial_x(u_\phi)v)(t_0 + s, \cdot) ds, \end{aligned}$$

where  $u_1 := (\partial_x v)^2 + v \partial_{xx}^2 v$  and  $u_2 := \partial_x(u_\phi \partial_x v)$ . Since  $v \in C((0, T]; H^2(\mathbb{R}))$  then  $u_2 \in C((0, T]; L^2(\mathbb{R}))$ ,  $\partial_x(u_\phi)v \in C((0, T]; L^2(\mathbb{R}))$  and using the Sobolev imbedding  $H^2(\mathbb{R}) \hookrightarrow C_b^1(\mathbb{R})$ , we get that  $u_1 \in C((0, T]; L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$ . Let us now define the following functions

$$F_i(t, x) := \int_0^t \partial_x K(t - s, \cdot) * u_i(t_0 + s, \cdot)(x) ds, \quad \text{for } i = 1, 2,$$

and

$$G(t, x) := \int_0^t \partial_{xx}^2 K(t-s, \cdot) * (\partial_x(u_\phi)v)(t_0+s, \cdot) ds.$$

For all  $x, y \in \mathbb{R}$ , we have thanks to Cauchy-Schwartz inequality

$$\begin{aligned} & |\partial_x K(t-s, \cdot) * u_i(t_0+s, \cdot)(x) - \partial_x K(t-s, \cdot) * u_i(t_0+s, \cdot)(y)| \\ & \leq \int_{\mathbb{R}} |\partial_x K(t-s, z)| |u_i(t_0+s, x-z) - u_i(t_0+s, y-z)| dz, \\ & \leq \|\mathcal{T}_{(x-y)}(u_i(t_0+s, \cdot)) - u_i(t_0+s, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x K(t-s, \cdot)\|_{L^2(\mathbb{R})}, \end{aligned}$$

where  $\mathcal{T}_z\varphi$  denotes the translated function  $x \rightarrow \varphi(x+z)$ .

Therefore, for all  $x, y \in \mathbb{R}$  and all  $t \in [0, T']$ , we deduce that

$$\begin{aligned} |F_i(t, x) - F_i(t, y)| & \leq \int_0^t K_0(t-s)^{-3/4} \|\mathcal{T}_{(x-y)}(u_i(t_0+s, \cdot)) - u_i(t_0+s, \cdot)\|_{L^2(\mathbb{R})} ds, \\ & \leq 4K_0 T'^{1/4} \sup_{s \in [0, T']} \|\mathcal{T}_{(x-y)}(\bar{u}_i(s, \cdot)) - \bar{u}_i(s, \cdot)\|_{L^2(\mathbb{R})}, \end{aligned} \quad (32)$$

with  $\bar{u}_i(s, \cdot) = u_i(t_0+s, \cdot)$ . Then,  $\bar{u}_i$  is uniformly continuous with values in  $L^2$  as a continuous function on a compact set  $[0, T']$ . Therefore, for any  $\epsilon > 0$ , there exists a finite sequence  $0 = s_0 < s_1 < \dots < s_N = T'$  such that for any  $s \in [0, T']$ , there exists  $j \in \{0, \dots, N-1\}$  with

$$\|\bar{u}_i(s, \cdot) - \bar{u}_i(s_j, \cdot)\|_{L^2(\mathbb{R})} \leq \epsilon.$$

Therefore, using (32) we have

$$\begin{aligned} |F_i(t, x) - F_i(t, y)| & \leq 4K_0 T'^{1/4} \sup_{s \in [0, T']} \|\mathcal{T}_{(x-y)}(\bar{u}_i(s, \cdot)) - \mathcal{T}_{(x-y)}(\bar{u}_i(s_j, \cdot))\|_{L^2(\mathbb{R})} \\ & \quad + 4K_0 T'^{1/4} \left\{ \|\mathcal{T}_{(x-y)}(\bar{u}_i(s_j, \cdot)) - \bar{u}_i(s_j, \cdot)\|_{L^2(\mathbb{R})} + \sup_{s \in [0, T']} \|\bar{u}_i(s, \cdot) - \bar{u}_i(s_j, \cdot)\|_{L^2(\mathbb{R})} \right\}. \end{aligned}$$

And since  $\|\mathcal{T}_{(x-y)}(\bar{u}_i(s, \cdot)) - \mathcal{T}_{(x-y)}(\bar{u}_i(s_j, \cdot))\|_{L^2(\mathbb{R})} = \|\bar{u}_i(s, \cdot) - \bar{u}_i(s_j, \cdot)\|_{L^2(\mathbb{R})}$ , we get

$$|F_i(t, x) - F_i(t, y)| \leq 4K_0 T'^{1/4} \left\{ \|\mathcal{T}_{(x-y)}(\bar{u}_i(s_j, \cdot)) - \bar{u}_i(s_j, \cdot)\|_{L^2(\mathbb{R})} + 2\epsilon \right\}.$$

By the continuity of the translation in  $L^2(\mathbb{R})$ , we have

$$\|\mathcal{T}_{(x-y)}(\bar{u}_i(s_j, \cdot)) - \bar{u}_i(s_j, \cdot)\|_{L^2(\mathbb{R})} \rightarrow 0,$$

as  $(x-y) \rightarrow 0$ . Hence,

$$\limsup_{(x-y) \rightarrow 0} |F_i(t, x) - F_i(t, y)| \leq 2\epsilon.$$

Taking the infimum w.r.t  $\epsilon > 0$ , we infer that  $F_i$  is continuous w.r.t the variable  $x$ . And using the same argument as the proof of Proposition 4, we get that  $F_i \in C([0, T']; L^2(\mathbb{R}))$ . From classical results, we then deduce that  $F_i$  is continuous w.r.t the couple  $(t, x)$  on  $[0, T'] \times \mathbb{R}$ .

Let us now consider the function  $G$ . Using again Cauchy-Schwartz inequality, we have for all  $x, y \in \mathbb{R}$ ,

$$\begin{aligned} & |\partial_{xx}^2 K(t-s, \cdot) * (\partial_x(u_\phi)v)(t_0+s, \cdot)(x) - \partial_{xx}^2 K(t-s, \cdot) * (\partial_x(u_\phi)v)(t_0+s, \cdot)(y)| \leq \\ & \|u_\phi\|_{C_b^1(\mathbb{R})} |\partial_x K(t-s, \cdot) * \partial_x v(t_0+s, \cdot)(x) - \partial_x K(t-s, \cdot) * \partial_x v(t_0+s, \cdot)(y)|, \end{aligned}$$

and using the same arguments as previous for  $F_i$ , we get that  $G \in ((0, T] \times \mathbb{R})$ .

Moreover, since  $v(t_0, \cdot) \in H^2(\mathbb{R})$ , we can easily check that  $(t, x) \rightarrow K(t, \cdot) * \partial_{xx}^2 v(t_0, \cdot)(x)$  is continuous on  $(0, T] \times \mathbb{R}$ . Finally, we get that  $\partial_{xx}^2 v \in C([t_0, T] \times \mathbb{R})$  and since  $t_0$  is arbitrary in  $(0, T]$ , we conclude that  $\partial_{xx}^2 v \in C((0, T] \times \mathbb{R})$ .

**Second step:  $C^1$ -regularity in time.** From Proposition 7, we know that the terms  $\partial_t v$  and  $-\partial_x \left( \frac{v^2}{2} + u_\phi v \right) + \partial_{xx}^2 v - \mathcal{I}[v]$  have the same regularity. Moreover, by the first step of this proposition, we have that  $\partial_{xx}^2 v \in C((0, T] \times \mathbb{R})$  and from Sobolev embeddings and Remark 6, we deduce that  $\partial_x \left( \frac{v^2}{2} + u_\phi v \right)$  and  $\mathcal{I}[v]$  belong to  $C((0, T] \times \mathbb{R})$ . Finally, we obtain that  $\partial_t v \in C((0, T] \times \mathbb{R})$  and thus  $v \in C^{1,2}((0, T] \times \mathbb{R})$ . The proof of this Proposition is now complete.  $\blacksquare$

**Remark 8 ( $C^\infty$ -regularity).** Let  $T > 0$  and  $v_0 \in L^2(\mathbb{R})$ . If  $v$  is solution to (6) on  $(0, T)$  in the classical sense, then  $v$  is indefinitely derivable with respect to  $x$  and  $t$ . To get this result, we can use the bootstrap method, already applied in [1, 5] for instance.

## 6 Instability of travelling-waves

In this section, we investigate the proof of the instability of travelling-waves solution stated in Theorem 2, which is an immediate consequence of the Duhamel formula (8).

*Proof of Theorem 2.* To prove the instability of travelling-waves solution of the Fowler equation, we use the Duhamel formula (8) for the perturbation  $v$ :

$$v(t, x) = K(t, \cdot) * v_0(x) - \int_0^t \partial_x K(t-s, \cdot) * \left( \frac{v^2}{2} + u_\phi v \right) (s, \cdot)(x) ds. \quad (33)$$

First, by Plancherel formula, we have

$$\begin{aligned} \|K(t, \cdot) * v_0\|_{L^2(\mathbb{R})}^2 &= \|\mathcal{F}(K(t, \cdot) * v_0)\|_{L^2(\mathbb{R})}^2, \\ &= \int_{\mathbb{R}} |\mathcal{F}(K(t, \cdot))(\xi) \mathcal{F}(v_0)(\xi)|^2 d\xi, \\ &= \int_a^b \varepsilon^2 e^{-2t} \operatorname{Re}(\psi_{\mathcal{I}})(\xi) d\xi, \end{aligned}$$

where  $a, b > 0$  are such that  $\operatorname{Re}(\psi_{\mathcal{I}})(a) < \operatorname{Re}(\psi_{\mathcal{I}})(b) = -\alpha < 0$  and  $\forall \xi \in [a, b]$ ,  $\operatorname{Re}(\psi_{\mathcal{I}})(\xi) \leq -\alpha$  (see Figure 3). We then infer that

$$\|K(t, \cdot) * v_0\|_{L^2(\mathbb{R})}^2 \geq \varepsilon^2 (b-a) e^{2\alpha t} = e^{2\alpha t} \|v_0\|_{L^2(\mathbb{R})}^2. \quad (34)$$

Moreover, by Young inequality

$$\begin{aligned} \left\| \int_0^t \partial_x K(t-s, \cdot) * \left( \frac{v^2}{2} + u_\phi v \right) (s, \cdot) ds \right\|_{L^2(\mathbb{R})} &\leq \\ &\frac{1}{2} \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^2(\mathbb{R})} ds \|v\|_{C([0,t]; L^2(\mathbb{R}))}^2 \\ &+ \int_0^t \|\partial_x K(t-s, \cdot)\|_{L^1(\mathbb{R})} ds \|u_\phi\|_{C_b^1(\mathbb{R})} \|v\|_{C([0,t]; L^2(\mathbb{R}))}, \end{aligned}$$

then from the properties of the gradient of the kernel  $K$  (Proposition 1), (33) and (34) we deduce

$$\|v\|_{C([0,t];L^2(\mathbb{R}))} \geq e^{\alpha t} \|v_0\|_{L^2(\mathbb{R})} - 2K_0 t^{1/4} \|v\|_{C([0,t];L^2(\mathbb{R}))}^2 - 2K_1 t^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})} \|v\|_{C([0,t];L^2(\mathbb{R}))},$$

i.e.

$$2K_0 t^{1/4} \|v\|_{C([0,t];L^2(\mathbb{R}))}^2 + (1 + 2K_1 t^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})}) \|v\|_{C([0,t];L^2(\mathbb{R}))} - e^{\alpha t} \|v_0\|_{L^2(\mathbb{R})} \geq 0. \quad (35)$$

Let us denote  $X = \|v\|_{C([0,t];L^2(\mathbb{R}))}$ . Basic algebra implies that  $X$  solution of the previous inequality (35) satisfies  $X \geq X_0$  where

$$X_0 = \frac{-(1 + 2K_1 \|u_\phi\|_{C_b^1(\mathbb{R})} t^{1/2}) + \sqrt{(1 + 2K_1 \|u_\phi\|_{C_b^1(\mathbb{R})} t^{1/2})^2 + 8K_0 t^{1/4} e^{\alpha t} \|v_0\|_{L^2(\mathbb{R})}}}{4K_0 t^{1/4}}$$

is the positive solution of the following equation

$$2K_0 t^{1/4} X^2 + (1 + 2K_1 t^{1/2} \|u_\phi\|_{C_b^1(\mathbb{R})}) X - e^{\alpha t} \|v_0\|_{L^2(\mathbb{R})} = 0.$$

Therefore, we have

$$\begin{aligned} \|v\|_{C([0,t];L^2(\mathbb{R}))} &\geq \frac{-(1 + 2K_1 \|u_\phi\|_{C_b^1(\mathbb{R})} t^{1/2}) + 2\sqrt{2K_0} t^{1/8} e^{\frac{\alpha}{2}t} \|v_0\|_{L^2(\mathbb{R})}}{4K_0 t^{1/4}}, \\ &= -C_1 t^{-1/4} - C_2 t^{1/4} + C_3 t^{-1/8} e^{\frac{\alpha}{2}t}, \end{aligned}$$

with  $C_1, C_2, C_3$  positive constants which depends on  $K_0, K_1, \|v_0\|_{L^2(\mathbb{R})}$  and  $\|u_\phi\|_{C_b^1(\mathbb{R})}$ . Let  $0 < \beta < \frac{\alpha}{2}$ . Since

$$\frac{-C_1 t^{-1/4} - C_2 t^{1/4} + C_3 t^{-1/8} e^{\frac{\alpha}{2}t}}{e^{\beta t}} \rightarrow +\infty,$$

as  $t \rightarrow +\infty$  then  $\forall M > 0, \exists t_0 > 0$  such that  $\forall t > t_0$ ,

$$\|v\|_{C([0,t];L^2(\mathbb{R}))} \geq M e^{\beta t}.$$

This completes the proof of the instability of travelling-waves. ■

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