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Locally identifying coloring of graphs*

Louis Esperet[†], Sylvain Gravier[‡], Mickaël Montassier[§], Pascal Ochem[¶], Aline Parreau[‡]

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Abstract

We introduce the notion of locally identifying coloring of a graph. A proper vertex-coloring c of a graph G is said to be *locally identifying*, if for any adjacent vertices u and v with distinct closed neighborhood, the sets of colors that appear in the closed neighborhood of u and v are distinct. Let $\chi_{\text{lid}}(G)$ be the minimum number of colors used in a locally identifying vertex-coloring of G. In this paper, we give several bounds on χ_{lid} for different families of graphs (planar graphs, some subclasses of perfect graphs, graphs with bounded maximum degree) and prove that deciding whether $\chi_{\text{lid}}(G) = 3$ for a subcubic bipartite graph G with large girth is an NP-complete problem.

1 Introduction

In this paper we focus on colorings allowing to distinguish the vertices of a graph. In [15], Horňák and Soták considered edge-coloring of a graph such that (i) the edge-coloring is proper (i.e. no adjacent edges receive the same color) and (ii) for any vertices u, v (with $u \neq v$) the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v. Such a coloring is called a *vertex-distinguishing proper edge-coloring*. The minimum number of colors required in any vertex-distinguishing proper edge-coloring of G is called the *observability* of G and was studied for different families of graphs [3, 6, 8, 11, 12, 15, 16]. This notion was then extended to *adjacent vertex-distinguishing edge-coloring* where Property (ii) must be true only for pairs of adjacent vertices; see [1, 14, 22].

In the present paper we introduce the notion of *locally identifying colorings*: a vertex-coloring is said to be *locally identifying* if (i) the vertex-coloring is proper (i.e. no adjacent

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vertices receive the same color), and (*ii*) for any pair of adjacent vertices u, v the set of colors assigned to the closed neighborhood of u differs from the set of colors assigned to the closed neighborhood of v whenever these neighborhoods are distinct. The *locally identifying chromatic number* of the graph G (or lid-chromatic number, for short), denoted by $\chi_{\text{lid}}(G)$, is the smallest number of colors required in any locally identifying coloring of G. In the following we study the parameter χ_{lid} for different families of graphs, such as bipartite graphs, k-trees, interval graphs, split graphs, cographs, graphs with bounded maximum degree, planar graphs with high girth, and outerplanar graphs.

Let $G = (V, E)$ be a graph. For any vertex u, we denote by $N(u)$ its neighborhood and by $N[u]$ its *closed neighborhood* (u together with its adjacent vertices) and by $d(u)$ its degree. Let c be a vertex-coloring of G. For any $S \subseteq V$, let $c(S)$ be the set of colors that appear on the vertices of S. More formally, a locally identifying coloring of G (or a lid-coloring, for short) is proper vertex-coloring c of G such that for any edge uv, $N[u] \neq N[v] \Rightarrow c(N[u]) \neq c(N[v])$. Observe that the lid-chromatic number of a graph G is the maximum of the lid-chromatic numbers of its connected components. Hence, in the proofs of most of our results it will be enough to restrict ourselves to connected graphs. A graph G is *k-lid-colorable* if it admits a locally identifying coloring using at most k colors. Notice the following:

Observation 1 *A connected graph G is 2-lid-colorable if and only if G has at most two vertices.*

Proof. Let G be a connected graph with a 2-lid-coloring c and at least 3 vertices. Consider an edge uv. Then we have $N[u] \neq N[v]$, since otherwise G would contain a triangle and then we would have $\chi_{\text{lid}}(G) \geq \chi(G) \geq 3$. Since c is a 2-coloring and $N[u]$ and $N[v]$ both contain u and v, we have $c(N[u]) = c(N[v]) = \{c(u), c(v)\}$, a contradiction.

The other implication is trivial. □

Note that locally identifying coloring is not hereditary. For instance, if P_n denotes the path on n vertices, then $\chi_{\text{lid}}(P_5) = 3$ whereas $\chi_{\text{lid}}(P_4) = 4$.

In Section 2, we prove that every bipartite graph has lid-chromatic number at most 4. Moreover, deciding whether a bipartite graph is 3-lid-colorable is an NP-complete problem, whereas it can be decided in linear time whether a tree is 3-lid-colorable.

In general, χ_{lid} is not bounded by a function of the usual chromatic number χ. Nevertheless it turns out that for several nice classes of graphs such a function exists: we study k-trees (Section 3), interval graphs (Section 4), split graphs (Section 5), cographs (Section 6), and give tight bounds in each of these cases. We also conjecture that every chordal graph G has a lid-coloring with $2\chi(G)$ colors.

Section 7 is dedicated to graphs with bounded maximum degree. We prove that the lid-chromatic number of graphs with maximum degree Δ is $O(\Delta^3)$ and that there are examples with lid-chromatic number $\Omega(\Delta^2)$.

In Section 8, we study graphs with a topological structure. Our result on 2-trees does not give any information on outerplanar graphs, since lid-coloring is not monotone under taking subgraphs. So we use a completely different strategy to prove that outerplanar graphs and planar graphs with large girth have lid-colorings using a constant number of colors.

Finally, in Section 9, we propose a tool allowing to extend the lid-colorings of the 2-connected components of a graph to the whole graph.

2 Bipartite graphs

This section is dedicated to bipartite graphs. The main interest of the study of bipartite graphs here comes from the following lemmas:

Lemma 2 *If a connected graph G satisfies $\chi_{lid}(G) \leq 3$, then G is either a triangle or a bipartite graph.*

Proof. Consider a 3-lid-coloring c of G with colors $1, 2, 3$. By Observation 1, we can assume that G has at least three vertices.

Define the coloring c' by $c'(x) = |c(N[x])|$ for any vertex x. Since G is connected, $c'(x) \in \{2, 3\}$ for any vertex x. If two adjacent vertices u, v satisfy $c'(u) = c'(v) = 3$, then $c(N[u]) = c(N[v]) = \{1, 2, 3\}$, and if $c'(u) = c'(v) = 2$, then $c(N[u]) = c(N[v]) = \{c(u), c(v)\}$. It follows that c' is a proper 2-coloring of G unless $N[u] = N[v]$ for some edge uv. In this case, since G does not consist of the single edge uv, there exists a vertex w adjacent to u and v. But then $c(N[u]) = c(N[v]) = c(N[w]) = \{1, 2, 3\}$, which implies that $N[u] = N[v] = N[w]$. This is only possible if G is a triangle. □

Indeed, more can be said about the color classes in a 3-lid-coloring of a (bipartite) graph:

Lemma 3 *Let G be a 3-lid-colorable connected bipartite graph on at least three vertices, with bipartition $\{U, V\}$, and let c be a 3-lid-coloring of G with colors $1, 2, 3$. Then G has a vertex u with $c(N[u]) = \{1, 2, 3\}$ and if $u \in U$, then $c(U) = \{c(u)\}$ and $c(V) = \{1, 2, 3\} \setminus \{c(u)\}$.*

Proof. Let uv be an edge of G. We have $N[u] \neq N[v]$ because G is a bipartite connected graph on at least three vertices. Then $c(N[u]) = \{1, 2, 3\}$ or $c(N[v]) = \{1, 2, 3\}$. Without loss of generality, assume that $c(N[u]) = \{1, 2, 3\}$ and $c(u) = 1$. Then all the neighbors of u must be colored 2 or 3, and the vertices at distance two from u must be colored 1 (otherwise there would be a neighbor w of u with $c(N[w]) = \{1, 2, 3\}$ and $N[u] \neq N[w]$). Iterating this observation, we remark that all the vertices at even distance from u must be colored 1, while the vertices at odd distance from u must be colored either 2 or 3, which yields the conclusion. □

As a corollary we obtain a precise description of 3-lid-colorable trees.

Corollary 4 *A tree T with at least 3 vertices is 3-lid-colorable if and only if the distance between every two leaves is even.*

Proof. Observe that for each leaf u of T, we have $|c(N[u])| = 2$ in any proper coloring c of T, so by Lemma 3 the distance between every two leaves is even.

Now assume that the distance between every two leaves of T is even, and fix a leaf u of T. Let c be the 3-coloring of T defined by $c(v) = 2$ if $d(u, v)$ is odd, $c(v) = 1$ if $d(u, v) \equiv 0 \bmod 4$, and $c(v) = 3$ if $d(u, v) \equiv 2 \bmod 4$. The coloring c is clearly proper, and we have $c(N[v]) = \{1, 2\}$ if $d(u, v) \equiv 0 \bmod 4$, and $c(N[v]) = \{2, 3\}$ if $d(u, v) \equiv 2 \bmod 4$. If v is a vertex at odd distance from u, then v is not a leaf and $c(N[v]) = \{1, 2, 3\}$. As a consequence, c is a 3-lid-coloring of T. □

Another class of bipartite graphs that behaves nicely with regards to locally identifying coloring is the class of graphs obtained by taking the Cartesian product of two bipartite graphs. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the Cartesian product of G_1 and G_2, denoted by $G_1 \square G_2$, is the graph with vertex set $V_1 \times V_2$, in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent whenever $u_2 = v_2$ and $u_1v_1 \in E_1$, or $u_1 = v_1$ and $u_2v_2 \in E_2$.

Theorem 5 *If G_1 and G_2 are bipartite graphs without isolated vertices, then $G_1 \square G_2$ is 3-lid-colorable.*

Proof. Let $\{U_1, V_1\}$ and $\{U_2, V_2\}$ be the partite sets of G_1 and G_2, respectively. Then $G_1 \square G_2$ is a bipartite graph with partition $\{(U_1 \times U_2) \cup (V_1 \times V_2), (U_1 \times V_2) \cup (V_1 \times U_2)\}$ and because there are no isolated vertices in G_1 and G_2, each vertex of $(U_1 \times U_2) \cup (V_1 \times V_2)$ has a neighbor in $U_1 \times V_2$ and a neighbor in $V_1 \times U_2$.

We define c by $c(u) = 1$ if $u \in (U_1 \times U_2) \cup (V_1 \times V_2)$, $c(u) = 2$ if $u \in U_1 \times V_2$, and $c(u) = 3$ if $u \in V_1 \times U_2$. Then c is a lid-coloring of $G_1 \square G_2$: $c(N[u]) = \{1, 2, 3\}$ for vertices of $(U_1 \times U_2) \cup (V_1 \times V_2)$, $c(N[u]) = \{1, 2\}$ for vertices of $U_1 \times V_2$ and $c(N[u]) = \{1, 3\}$ for vertices of $V_1 \times U_2$.

By Observation 1, $G_1 \square G_2$ does not have a 2-lid-coloring. □

As a corollary, we obtain that hypercubes and grids in any dimension are 3-lid-colorable. We now focus on bipartite graphs that are not 3-lid-colorable.

Theorem 6 *If G is a bipartite graph, then $\chi_{lid}(G) \leq 4$.*

Proof. We can assume that G is a connected graph with at least five vertices. Then there exists a vertex u of G that is not adjacent to a vertex of degree one. For any vertex v of G, set $c(v)$ to be the element of $\{0, 1, 2, 3\}$ congruent with $d(u, v)$ modulo 4. We claim that c is a lid-coloring of G. Since G is bipartite, c is clearly a proper coloring. Let v, w be two adjacent vertices in G. We may assume that they are at distance $k \geq 0$ and $k + 1$ from u, respectively. If $k = 0$, then $v = u$ and w has a neighbor at distance two from u, so $c(N[v]) = \{0, 1\}$ and $c(N[w]) = \{0, 1, 2\}$. If $k \geq 1$, then $(k - 1) \bmod 4$ is in $c(N[v])$ but not in $c(N[w])$, so $c(N[v]) \neq c(N[w])$. □

We now prove that deciding whether a bipartite graph is 3 or 4-lid-colorable is a hard problem.

Theorem 7 *For any fixed integer g, deciding whether a bipartite graph with girth at least g and maximum degree 3 is 3-lid-colorable is an NP-complete problem.*

Proof. We recall that a 2-coloring of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a partition of its vertex set $\mathcal{V}$ into two color classes such that no edge in $\mathcal{E}$ is monochromatic. We reduce our problem to the NP-complete problem of deciding the 2-colorability of 3-uniform hypergraphs [17].

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with at least one hyperedge. We construct the bipartite graph $G = (V, E)$ in the following way. To each vertex $v \in \mathcal{V}$, we associate a path P_v with vertices $\{v_0, \ldots, v_{4t}\}$ in G (where t will depend on the degree of v in $\mathcal{H}$ and the girth g we want for G). All the paths P_v are built on disjoint sets. To each hyperedge $e \in \mathcal{E}$, we associate a vertex w_e in G. If a hyperedge e contains a vertex v in $\mathcal{H}$, then we add an edge in G between w_e and a vertex v_i of P_v for some index $i \equiv 2 \bmod 4$. We require that a vertex v_i on a path P_v is adjacent to at most one vertex corresponding to a hyperedge containing v. It follows that the graph G is bipartite with maximum degree 3. Moreover, we can construct G in polynomial time and ensure that the girth of G is at least g by leaving enough space (at least $g/2$ vertices of degree two) between any two consecutive vertices of degree 3 on the paths P_v.

We shall prove that $\mathcal{H}$ is 2-colorable if and only if $\chi_{\text{lid}}(G) = 3$.

Assume first that $\mathcal{H}$ admits a 2-coloring $\mathcal{C} : V \to \{1, 2\}$. We define the following 3-coloring c of G such that $c(v_{i \equiv 2 \bmod 4}) = \mathcal{C}(v)$, $c(v_{i \equiv 0 \bmod 4}) = 3 - \mathcal{C}(v)$, $c(v_{i \equiv 1 \bmod 2}) = 3$ if $v \in V$, and $c(w_e) = 3$ for all vertices w_e with $e \in \mathcal{E}$. Let us check that c is a lid-coloring of G. We have $c(N[w_e]) = \{1, 2, 3\}$ since $c(w_e) = 3$ and w_e is adjacent to a vertex colored 1 and to a vertex colored 2 because of the 2-coloring of $\mathcal{H}$. Also, $c(N[v_{i \equiv 1 \bmod 2}]) = \{1, 2, 3\}$, $c(N[v_{i \equiv 2 \bmod 4}]) = \{\mathcal{C}(v), 3\}$, and $c(N[v_{i \equiv 0 \bmod 4}]) = \{3 - \mathcal{C}(v), 3\}$. So, for every edge uv in G, we have $c(N[u]) \neq c(N[v])$.

Conversely, assume that G (with bipartition $\{U, V\}$) admits a lid-coloring c using colors $1, 2, 3$. By Lemma 3, we can assume that $c(U) = \{1, 2\}$ and $c(V) = \{3\}$, and that the vertices of degree one in G are in U. This implies that $c(v_{i \equiv 2 \bmod 4}) \in \{1, 2\}$, $c(v_{i \equiv 0 \bmod 4}) = 3 - c(v_{i \equiv 2 \bmod 4})$, and $c(v_{i \equiv 1 \bmod 2}) = c(w_e) = 3$. Hence, the vertex-coloring of $\mathcal{V}$, in which each vertex v receives the color $c(v_{i \equiv 2 \bmod 4})$, is 2-coloring of the hypergraph $\mathcal{H}$. $\square$

It turns out that the connection between 3-lid-coloring and hypergraph 2-coloring highlighted in the proof of Theorem 7 has further consequences. For a connected bipartite graph G with bipartition $\{U, V\}$, let $\mathcal{H}_U$ be the hypergraph with vertex set U and hyperedge set $\{N(v), v \in V\}$. A direct consequence of Lemmas 2 and 3 is that a connected graph G distinct from a triangle is 3-lid-colorable if and only if it is bipartite (say with bipartition $\{U, V\}$) and at least one of $\mathcal{H}_U$ and $\mathcal{H}_V$ is 2-colorable.

A consequence of a result of Moret [18] (see also [2] for further details) is that if G is a subcubic bipartite planar graph with bipartition $\{U, V\}$, then we can check in polynomial time whether $\mathcal{H}_U$ (or $\mathcal{H}_V$) is 2-colorable. As a counterpart of Theorem 7, this implies:

Theorem 8 *It can be checked in polynomial time whether a planar graph G with maximum degree three is 3-lid-colorable.*

It was proved by Burstein [7] and Penaud [19] that every planar hypergraph in which all hyperedges have size at least three is 2-colorable, and Thomassen [20] proved that for any $k \geq 4$ any k-regular k-uniform hypergraph is 2-colorable. As a consequence, we obtain the following two results:

Theorem 9 *Let G be a bipartite planar graph with bipartition $\{U, V\}$ such that all vertices in U or all vertices in V have degree at least three. Then G is 3-lid-colorable.*

Theorem 10 *For $k \geq 4$, a k-regular graph is 3-lid-colorable if and only if it is bipartite.*

Since bipartite graphs have bounded lid-chromatic number, a natural question is whether χ_{lid} is upper-bounded by a function of the (usual) chromatic number. However, this is not true, since the graph G obtained from a clique on n vertices by subdividing each edge exactly twice has $\chi_{\text{lid}}(G) = n$ (it suffices to observe that two vertices of the initial clique cannot have the same color in the subdivided graph), whereas it is 3-colorable. This example also shows that if the edges of a graph G are partitioned into two sets E_1 and E_2, and the subgraphs of G induced by E_1 and E_2 have bounded lid-chromatic number, then $\chi_{\text{lid}}(G)$ is not necessarily bounded.

We propose the following conjecture relating χ_{lid} and χ for highly structured graphs. A graph is *chordal* if it does not contain an induced cycle of length at least four.

Conjecture 11 *For any chordal graph G, $\chi_{lid}(G) \leq 2\chi(G)$.*

The next three sections are dedicated to important subclasses of chordal graphs for which we are able to verify Conjecture 11.

3 k-trees

This section is devoted to the study of k-trees. A *k-tree* is a graph whose vertices can be ordered $v_1, v_2, \ldots, v_n$ in such a way that the vertices v_1 up to v_{k+1} induce a $(k+1)$-clique and for each $k+2 \leq i \leq n$, the neighbors of v_i in $\{v_j \mid j < i\}$ induce a k-clique. By definition, for every $k+1 \leq i \leq n$ the graph G_i induced by $\{v_j \mid j \leq i\}$ is a k-tree and every k-clique in a k-tree is contained in a $(k+1)$-clique.

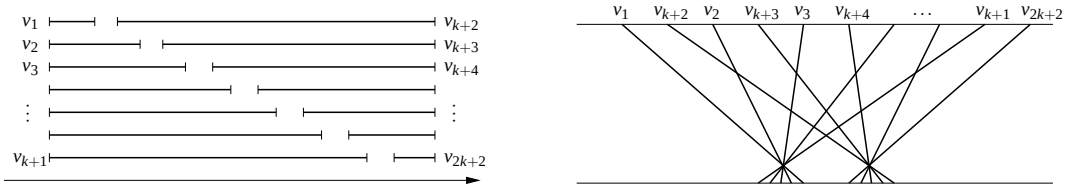
Theorem 12 *If G is a k-tree, then $\chi_{lid}(G) \leq 2k+2$.*

Proof. In this proof the colors are the integers modulo $2k+2$. In particular, this implies that the function on integers $x \mapsto x+k+1$ is an involution.

Let $v_1, \ldots, v_n$ be the n vertices of G ordered as above.

We construct the following coloring c of G iteratively for $1 \leq i \leq n$. If $i \leq k+1$, then we set $c(v_i) = i$. Suppose $i \geq k+2$. Let C be the neighborhood of v_i in G_i. Since G_{i-1} is a k-tree, the clique C is contained in a $(k+1)$-clique C' of G_{i-1}. Let $\{v_j\} = C' \setminus C$. We set $c(v_i) = c(v_j) + k + 1$ (we may have several choices for C' and thus for j).

We now prove that c is a lid-coloring of G. Throughout the procedure, the following two properties remain trivially satisfied: (i) c is a proper coloring of G, and (ii) no vertex colored i has a neighbor colored $i+k+1$. Consider an edge v_iv_j of G with $N[v_i] \neq N[v_j]$. We may assume without loss of generality that some neighbors of v_i are not adjacent to v_j. If $i, j \leq k+1$, then consider the minimum index ℓ such that v_ℓ is a neighbor of v_i not adjacent to v_j. By definition of c and minimality of ℓ, we have $c(v_j) = c(v_\ell) + k + 1$. Otherwise we can assume that $j > i$ and $j > k+1$. Let C be the neighborhood of v_j in G_j. By definition of c, there exists a $(k+1)$-clique C' of G_{j-1} containing C such that $c(v_j) = c(v_\ell) + k + 1$, where $C' \setminus C = \{v_\ell\}$. In both cases, $c(v_\ell) \in c(N[v_i])$ while $c(v_\ell) \notin c(N[v_j])$ by Property (ii). Hence, c is a lid-coloring of G. $\square$



(a) (b)

Figure 1: The graph P^k_{2k+2} as an interval graph (a) and as a permutation graph (b).

For fixed t, the fact that a graph admits a lid-coloring with at most t colors can be easily expressed in monadic second-order logic. Thus Theorem 12 together with [10] imply that for fixed k, the lid-chromatic number of a k-tree can be computed in linear time. Another remark is that for trees, Theorem 12 provides the same 4-lid-coloring as Theorem 6.

For any two integers $k, \ell \geq 1$, we define P^k_ℓ as the graph with vertex set $v_1, \ldots, v_\ell$ in which v_i and v_j are adjacent whenever $|i-j| \leq k$. The graph P^k_{2k+2} is clearly a k-tree: it can be constructed from the clique formed by $v_1, \ldots, v_{k+1}$ by adding at each step $k+2 \leq i \leq 2k+2$ a vertex v_i adjacent to $v_{i-k}, \ldots, v_{i-1}$. The graph P^k_{2k+2} is also an interval graph (see Figure 1(a)) and a permutation graph (see Figure 1(b)). We now prove that the graph P^k_{2k+2} also provides an example showing that Theorem 12 is best possible.

Proposition 13 *For any $k \geq 1$, we have $\chi_{lid}(P^k_{2k+2}) = 2k+2$.*

Proof. Let c be a lid-coloring of P^k_{2k+2}. Without loss of generality we have $c(v_i) = i$ for each $1 \leq i \leq k+1$. Observe that for any $1 \leq i \leq k$, the symmetric difference between $N[v_i]$ and $N[v_{i+1}]$ is precisely $\{v_{i+k+1}\}$. In addition, $N[v_i] = \{v_1, \ldots, v_{i+k}\}$ and so $c(N[v_i])$ contains colors 1 up to $k+1$. Therefore, $c(v_i) > k+1$ whenever $k+2 \leq i \leq 2k+1$. And we can assume that $c(v_i) = i$ for any $1 \leq i \leq 2k+1$.

Let $\alpha = c(v_{2k+2})$, and assume for the sake of contradiction that $\alpha \neq 2k+2$. Since vertices $v_{k+2}, \ldots, v_{2k+2}$ induce a clique, we have $\alpha \leq k+1$. The symmetric difference between $N[v_{\alpha+k}]$ and $N[v_{\alpha+k+1}]$ is precisely $\{v_\alpha\}$ if $\alpha \geq 2$ and is $\{v_1, v_{2k+2}\}$ if $\alpha = 1$.

In both cases, $c(v_{2k+2}) = c(v_\alpha) = \alpha$ would imply that $c(N[v_{\alpha+k}]) = c(N[v_{\alpha+k+1}])$, a contradiction. $\square$

4 Interval graphs

In this section, we prove that the previous example is also extremal for the class of interval graphs.

Theorem 14 *For any interval graph G, $\chi_{lid}(G) \leq 2\omega(G)$.*

Proof. Let $k = \omega(G)$. In this proof the colors are the integers modulo $2k$. Let G be a connected interval graph on n vertices. We identify the vertices $v_1, \ldots, v_n$ of G with a family of intervals $(I_i = [a_i, b_i])_{1 \leq i \leq n}$ such that $v_i v_j$ is an edge of G precisely if I_i and I_j intersect. We may assume that $a_1 \leq a_2 \leq \ldots \leq a_n$. Without loss of generality, we can assume that if $a_i < a_j$ and $I_i \cap I_j \neq \varnothing$, then there exists an interval I_ℓ such that $a_i \leq b_\ell < a_j$; otherwise, we can change I_j to the interval $[a_i, b_j]$ and the intersection graph remains the same. By a similar argument, we can also assume that if $b_j < b_i$ and $I_i \cap I_j \neq \varnothing$, then there exists an interval I_ℓ such that $b_j < a_\ell \leq b_i$.

Let $\{a_1 = a_{t_1} < a_{t_2} < \ldots < a_{t_s}\}$ be the set of left ends. At each step $i = 1, \ldots, s$, we color all the intervals starting at a_{t_i}. We first color the intervals starting at a_{t_1} with distinct colors in $\{0, \ldots, k-1\}$. Assume we have colored all the intervals starting before a_{t_i}. Now, we color all the intervals $\mathcal{I}(t_i)$ starting at a_{t_i}. First, we define the following subsets of intervals:

- $\mathcal{V}(t_i)$: intervals I_j such that $a_j < a_{t_i} \leq b_j$,
- $\mathcal{U}(t_i)$: intervals I_j such that $a_{t_{i-1}} \leq b_j < a_{t_i}$,
- $\mathcal{T}(t_i)$: intervals I_j of $\mathcal{U}(t_i)$ such that there is an interval I_ℓ in $\mathcal{V}(t_i)$ with $a_j = a_\ell$.

Note that $\mathcal{V}(t_i)$ is the set of intervals that are already colored and intersect $\mathcal{I}(t_i)$. The set $\mathcal{U}(t_i)$ is a subset of intervals already colored that intersect all the intervals of $\mathcal{V}(T_i)$. It is not empty (take any interval finishing before a_{t_i} with rightmost right end). Necessarily, all the intervals of $\mathcal{U}(t_i)$ have the same right end because no interval starts between $a_{t_{i-1}}$ and a_{t_i}. Finally, if $\mathcal{T}(t_i) \neq \varnothing$, then let I_0 be an interval of $\mathcal{T}(t_i)$ with leftmost left end, and otherwise let I_0 be any interval of $\mathcal{U}(t_i)$. Let c_0 be the color of I_0. Note that any interval of $\mathcal{U}(t_i)$ and $\mathcal{V}(t_i)$ intersects I_0, and thus has color c_0 in its neighborhood. We can now color the intervals of $\mathcal{I}(t_i)$. We color with color $c_0 + k$ one of the intervals having the rightmost right end. We color the other intervals with colors in $\{0, \ldots, 2k-1\}$ such that no vertex with color j is adjacent to a vertex with color j or $j+k$ (this is always possible since intervals of $\mathcal{V}(t_i) \cup \mathcal{I}(t_i)$ induce a clique of size at most k). This coloring c is clearly a proper $2k$-coloring and there is no vertex with color j, $0 \leq j \leq k-1$, adjacent to a vertex with color $j+k$.

We now show that c is a lid-coloring of G. Let I_i and I_j be two intersecting intervals with $N[I_i] \neq N[I_j]$. Assume first that $a_i \neq a_j$. Without loss of generality, $a_i < a_j$. During the process, when I_j is colored, an interval I_ℓ also starting at a_j is colored with a color c_0+k such that $c_0 \in c(N[I_i])$. Necessarily, $I_j \subseteq I_\ell$ since I_ℓ has the rightmost right end among all intervals starting at a_j. So $c_0 + k \in c(N[I_j])$ but $c_0 \notin c(N[I_\ell])$ and so $c_0 \notin c(N[I_j])$. Hence, $c(N[I_i]) \neq c(N[I_j])$. Assume now that $a_i = a_j$. Without loss of generality, $b_j < b_i$ and so $I_j \subseteq I_i$. Let a_{t_ℓ} be the leftmost left end such that $b_j < a_{t_\ell} \leq b_i$ (it exists because $N[I_i] \neq N[I_j]$). Then we have $I_i \in \mathcal{V}(t_\ell)$ and $I_j \in \mathcal{T}(t_\ell)$. By construction, one of the intervals of $\mathcal{I}(t_\ell)$, say I, will receive the color $c_0 + k$ where c_0 is the color of an interval $I_0 \in \mathcal{T}(t_\ell)$. Necessarily, $I_j \subseteq I_0$ and $c_0 \in c(N[I_j]) \subseteq c(N[I_i])$. We also have $c_0+k \in c(N[I_i])$ because I_i is a neighbor of I. But $c_0 + k \notin c(N[I_j])$ since $c_0 + k \notin c(N[I_0])$ and $I_j \subseteq I_0$. Hence, $c(N[I_i]) \neq c(N[I_j])$. □

5 Split graphs

A *split graph* is a graph $G = (K \cup S, E)$ whose vertex set can be partitioned into a clique K and an independent set S. In the following, we will always consider partitions $K \cup S$ with K of maximum size. A split graph is a chordal graph and its clique number and chromatic number are equal to $|K|$. We prove that it is lid-colorable with $2|K| - 1$ colors.

We say that a set $S' \subseteq S$ *discriminates* a set $K' \subseteq K$ if for any $u, v \in K'$ with $N[u] \neq N[v]$, we also have $N[u] \cap S' \neq N[v] \cap S'$. The following theorem is due to Bondy:

Theorem 15 ([4, 9]) *If $A_1, A_2, \ldots, A_n$ is a family of n distincts subsets of a set $\mathcal{A}$ with at least n elements, then there is a subset $\mathcal{A}'$ of $\mathcal{A}$ of size $n-1$ such that all the sets $A_i \cap \mathcal{A}'$ are distinct.*

Corollary 16 *Let $G = (K \cup S, E)$ be a split graph. For any $K' \subseteq K$, there is a subset S' of S of size at most $|K'| - 1$ such that S' discriminates K'.*

Proof. We apply Theorem 15 to the (at most) $|K'|$ pairwise distinct sets among $\{N[v] \cap S \mid v \in K'\}$. □

One can easily show that every split graph G has lid-chromatic number at most $2|K|$ by giving colors $1, \ldots, |K|$ to the vertices of K, colors $|K| + 1, \ldots, |K| + k - 1$, for some $k \leq |K|$, to the vertices of a smallest discriminating set $S' \subseteq S$ of K, and finally color $|K| + k$ to the vertices of $S \setminus S'$.

We now prove the following stronger result:

Theorem 17 *Let $G = (K \cup S, E)$ be a split graph. If $\omega(G) \geq 3$ or if G is a star, then $\chi_{lid}(G) \leq 2\omega(G) - 1$.*

Proof. Assume that $|K| = k$ and denote the vertices of K by $v_1, \ldots, v_k$. If $k = 1$, then G has no edges and it is clear that $\chi_{\text{lid}}(G) \leq 1$. If $G = K_{1,n}$, then $\chi_{\text{lid}}(G) \leq 3$ by Corollary 4. So we can assume that $k \geq 3$. If $|S| \leq k-1$ or if S contains a set of size at most $k-2$ that discriminates K, then the result is trivial. Therefore, we assume that $|S| \geq k$ and consider a minimal set S_1 that discriminates K. We can assume that the set S_1 has size precisely $k-1$ and there is no edge uv with $N[u] = N[v]$. Indeed, if $N[u] = N[v]$ for an edge uv, then any set discriminating $K \setminus \{v\}$ discriminates also k. We consider two cases.

Case 1. There is a vertex $x \in S \setminus S_1$ of degree $k-1$ and a neighbor $v_i \in K$ of x such that $N[v_i] \cap S_1 = \varnothing$. Without loss of generality, we can assume that $v_i = v_{k-1}$ and that $K \setminus N(x) = \{v_k\}$. Let $S_x = \{y \in S, N(y) = N(x) = K \setminus \{v_k\}\}$. We have $S_x \cap S_1 = \varnothing$ (recall that v_{k-1} has no neighbor in S_1) and by definition of S_1, for each vertex $v_i \neq v_{k-1}$, $N[v_i] \cap S_1 \neq \varnothing$ (S_1 is a discriminating set).

Let $K_1 = K \setminus \{v_{k-1}, v_k\}$, and let S_2 be a subset of S_1 of size at most $|K_1| - 1 = k-3$ that discriminates K_1. Let $S' = S \setminus (S_1 \cup S_x)$. We define a coloring c as follows:

- for $1 \leq i \leq k$, $c(v_i) = i$;
- assign pairwise distinct colors from $k+1, \ldots, 2k-3$ to the vertices of S_2;
- for $u \in S_1 \setminus S_2$, $c(u) = 2k-2$;
- for $u \in S_x$, $c(u) = 2k-1$;
- for $u \in S'$, take $v_i \in K \setminus N(u)$ (v_i exists by maximality of K), and set $c(u) = c(v_i)$.

Then c is a proper coloring of G. We show that c is a lid-coloring of G. First observe that for each vertex v_i of K, $c(N[v_i])$ contains one color of $\{k+1, \ldots, 2k-1\}$. Indeed $2k-1 \in c(N[v_{k-1}])$ and if $v_i \neq v_{k-1}$, then $N[v_i] \cap S_1 \neq \varnothing$ and therefore $c(N[v_i]) \cap \{k+1, \ldots, 2k-2\} \neq \varnothing$. This implies that for each $v_i \in K$, $c(N[v_i])$ is distinct from all $c(N[y])$, $y \in S$. In fact, either $c(y) \in c(K)$ and then $c(N[y]) \subseteq c(K)$, or $c(y) \notin c(K)$ but then there is at least one color of $c(K)$ that $c(N[y])$ does not contain. Furthermore, $c(N[v_k])$ is different from all the sets $c(N[v_i])$ with $i \neq k$ because $2k-1 \in c(N[v_i])$ and $2k-1 \notin c(N[v_k])$. The set $c(N[v_{k-1}])$ is different from all the sets $c(N[v_i])$ with $i \neq k-1$ because $c(N[v_{k-1}])$ contains no color of $c(S_1)$ whereas $c(N[v_i])$ contains at least one color of this set. Finally, $c(N[v_i]) \neq c(N[v_j])$ for $i, j \leq k-2$ because there is a vertex in S_2 that separates them and its color is used only once. Hence, for each edge uv of G such that $N[u] \neq N[v]$, we have $c(N[u]) \neq c(N[v])$.

Case 2. For each vertex x of $S \setminus S_1$, either x has degree at most $k-2$ or x has degree $k-1$ and each vertex of $N(x)$ has a neighbor in S_1. We define a coloring c as follows: vertices of K are assigned colors $1, \ldots, k$, and vertices of S_1 are assigned (pairwise distinct) colors within $k+1, \ldots, 2k-1$. For any vertex x in $S \setminus S_1$, take a vertex v_i in $K \setminus N(x)$ (such a vertex exists by the maximality of K) and set $c(x) = c(v_i)$. We claim that c is a lid-coloring of G. It is clear that c is a proper coloring of G. Let uv be an edge of G with $N[u] \neq N[v]$. If $u, v \in K$, then without loss of generality there is a vertex w of S_1

such that, $w \in N[u]$ and $w \notin N[v]$. Then, $c(w) \in c(N[u])$ and $c(w) \notin c(N[v])$. Otherwise, without loss of generality, $u \in K$ and $v \in S$. If $v \in S_1$, then S_1 does not contain the whole set $c(K)$ and so $c(N[u]) \neq c(N[v])$. Otherwise, $v \notin S_1$. If the degree of v is $k-1$, then u has a neighbor w in S_1 and $c(w) \in c(N[u])$, $c(w) \notin c(N[v])$. If the degree of v is at most $k-2$, then there is a color $1 \leq i \leq k$ such that $i \in c(N[u])$ and $i \notin c(N[v])$. In all cases, $c(N[u]) \neq c(N[v])$. Hence, c is a lid-coloring of G. □

Observe that this bound is sharp: the graph obtained from a k-clique by adding a pendant vertex to each of the vertices of the clique is a split graph and requires $2k-1$ colors in any lid-coloring.

6 Cographs

A *cograph* is a graph that does not contain the path P_4 on 4 vertices as an induced subgraph. Cographs are a subclass of permutation graphs, and so they are perfect (however, they are not necessarily chordal). It is well-known that the class of cographs is closed under disjoint union and complementation [5]. Let $G \cup H$ denote the disjoint union of G and H, and let $G+H$ denote the complete join of G and H, i.e. the graph obtained from $G \cup H$ by adding all possible edges between a vertex from G and a vertex from H. A consequence of the previously mentioned facts is that any cograph G is of one of the three following types:

(S) G is a single vertex.

(U) $G = \bigcup_{i=1}^{k} G_i$ with $k \geq 2$ and every G_i is a cograph of type S or J.

(J) $G = \sum_{i=1}^{k} G_i$ with $k \geq 2$ and every G_i is a cograph of type S or U.

We will use this property to prove the following theorem:

Theorem 18 *If G is a cograph, then $\chi_{lid}(G) \leq 2\omega(G) - 1$.*

Proof. A *universal* vertex of G is a vertex adjacent to all the vertices of G. Observe that if a cograph G has a universal vertex, then G must be of type S or J. Let $\widetilde{\chi}_{\text{lid}}(G)$ be the least integer k such that G has a lid-coloring c with colors $1, \ldots, k$ such that for any vertex v that is not universal, $c(N[v]) \neq \{1, \ldots, k\}$ (in other words, if a vertex sees all the colors, then it is universal). Such a coloring is called a *strong lid-coloring* of G. We will prove the following result by induction:

Claim. For any cograph G, $\chi_{lid}(G) \leq 2\omega(G) - 1$ and $\widetilde{\chi}_{lid}(G) \leq 2\omega(G)$.

If G is a single vertex, then it is universal and therefore $\widetilde{\chi}_{\text{lid}}(G) = \chi_{\text{lid}}(G) = 1 = 2 \times 1 - 1$ and the assumption holds.

Assume now that G is of type J. There exist $G_1, \ldots, G_k$, $k \geq 2$, each of type S or U, such that $G = \sum_{i=1}^{k} G_i$. Let $G_1, \ldots, G_s$ $(0 \leq s \leq k)$ be of type S and $G_{s+1}, \ldots, G_k$ be of type U. Consider a lid-coloring c_1 of G_1 and a strong lid-coloring c_i of G_i for $2 \leq i \leq k$,

such that the sets of colors within G_i and G_j, $i \neq j$, are disjoint. Then the coloring c of G defined by $c(v) = c_i(v)$ for any $v \in G_i$ is a lid-coloring of G. To see this, assume two adjacent vertices u and v such that $N[u] \neq N[v]$ and $c(N[u]) = c(N[v])$. Since every c_i is a lid-coloring of G_i the vertices u and v must be in different G_i's, say $u \in G_i$ and $v \in G_j$, $i < j$. But then in order to have $c(N[u]) = c(N[v])$, u and v must see all the colors in c_i and c_j, respectively. Since c_j is a strong lid-coloring of G_j, v is universal in G_j. This means that G_j (and therefore G_i) is of type S. Hence, u and v are universal in G. This contradicts the fact that $N[u] \neq N[v]$. As a consequence c is a lid-coloring of G.

If c_1 is a strong lid-coloring of G_1, then c is a strong lid-coloring of G: take a vertex $v \in G_i$ that sees all the colors in c. Then it also sees all the colors in c_i, so it is universal in G_i and G.

So we have $\chi_{\text{lid}}(G) \leq \chi_{\text{lid}}(G_1) + \sum_{i=2}^{k} \widetilde{\chi}_{\text{lid}}(G_i)$ and $\widetilde{\chi}_{\text{lid}}(G) \leq \sum_{i=1}^{k} \widetilde{\chi}_{\text{lid}}(G_i)$. Since $\omega(G) = \sum_{i=1}^{k} \omega(G_i)$ we have by induction:

$$\chi_{\text{lid}}(G) \leq 2\omega(G_1) - 1 + \sum_{i=2}^{k} 2\omega(G_i) = 2 \times \sum_{i=1}^{k} \omega(G_i) - 1 = 2\omega(G) - 1$$

and

$$\widetilde{\chi}_{\text{lid}}(G) \leq \sum_{i=1}^{k} 2\omega(G_i) = 2\omega(G).$$

Assume now that G is of type U. There exist $G_1, \ldots, G_k$, $k \geq 2$, each of type S or J, such that $G = \bigcup_{i=1}^{k} G_i$. Consider a lid-coloring c_i of G_i with colors $1, \ldots, \chi_{\text{lid}}(G_i)$. Without loss of generality we have $\chi_{\text{lid}}(G_1) = \max_{i=1}^{k} \chi_{\text{lid}}(G_i)$. The coloring c of G defined by $c(v) = c_i(v)$ for any $v \in G_i$ is clearly a lid-coloring of G, and so $\chi_{\text{lid}}(G) = \max_{i=1}^{k} \chi_{\text{lid}}(G_i)$.

To obtain a strong lid-coloring, assign a new color $\chi_{\text{lid}}(G_1)+1$ to all the vertices colored 1 in G_1, and color all the other vertices of G as they were colored in c. The obtained coloring c' is still a lid-coloring of G. Since no vertex u satisfies $c(N[u]) = \{1, \ldots, \chi_{\text{lid}}(G_1)+1\}$ (the vertices in G_1 miss the color 1, while the others miss the color $\chi_{\text{lid}}(G_1)+1$), c' is also a strong lid-coloring of G. Therefore $\widetilde{\chi}_{\text{lid}}(G) \leq \max_{i=1}^{k} \chi_{\text{lid}}(G_i) + 1$. Since $\omega(G) = \max_{i=1}^{k} \omega(G_i)$ we have by induction

$$\chi_{\text{lid}}(G) \leq \max_{i=1}^{k}(2\omega(G_i) - 1) = 2\omega(G) - 1$$

and

$$\widetilde{\chi}_{\text{lid}}(G) \leq \max_{i=1}^{k}(2\omega(G_i) - 1) + 1 = 2\omega(G).$$

□

The bound of Theorem 18 is tight. The following construction gives an example of cographs of clique number ω requiring $2\omega - 1$ colors in any lid-coloring. For any $k \geq 1$, take a complete graph with vertex set $v_1, \ldots, v_k$ and for each $2 \leq i \leq k$ add a vertex u_i such that $N(u_i) = \{v_i, v_{i+1}, \ldots, v_k\}$. This graph is a cograph with clique number k, the vertices u_i form an independent set U, and every vertex v_i satisfies $N(v_i) \cap U = \{u_2, \ldots, u_i\}$. Let c be a lid-coloring of this graph, then for any $3 \leq i \leq k$ the vertex u_i must be assigned

a color distinct from $c(u_2), \ldots, c(u_{i-1})$ and $c(v_1), \ldots, c(v_k)$ since otherwise we would have $c(N[v_i]) = c(N[v_{i-1}])$. Hence, at least $k + (k-1) = 2k - 1$ distinct colors are required.

As mentionned in Section 3, for fixed t, the fact that a graph admits a lid-coloring with at most t colors can be expressed in monadic second-order logic. It is well known that the class of cographs is exactly the class of graphs with clique-width at most two. It follows from [10] and Theorem 18 that, for a fixed k, the lid-chromatic number of a cograph of clique number at most k can be computed in linear time.

Given the results in Sections 2 to 6, it seems natural to conjecture that every perfect graph G has lid-chromatic number at most $2\chi(G)$. This is not true, however, as the following example shows. Take three stable sets S_1, S_2, S_3, each of size k ($k \geq 2$), add all possible edges between S_1 and S_2, add a perfect matching between S_1 and S_3, and add the complement of a perfect matching between S_2 and S_3. The obtained graph G_k is perfect: since the subgraph of G_k induced by S_1 and S_2 is a complete bipartite graph, an induced subgraph of G_k is bipartite if and only if it does not have a triangle, and is 3-colorable otherwise.

Consider a lid-coloring c of G_k, and a vertex x_2 of S_2. Let x_3 be the only vertex of S_3 that is not adjacent to x_2, and x_1 be the unique neighbor of x_3 in S_1. Observe that $N[x_1] = N[x_3] \cup \{x_2\}$. Since $c(N[x_1]) \neq c(N[x_3])$, the color of x_2 appears only once in S_2. Hence, all the vertices of S_2 have distinct colors and it follows that $\chi_{\text{lid}}(G_k) \geq k + 2$, whereas $\chi(G_k) = \omega(G_k) = 3$.

7 Graphs with bounded maximum degree

Proposition 19 *If a graph G has maximum degree Δ, then $\chi_{lid}(G) \leq \Delta^3 - \Delta^2 + \Delta + 1$.*

Proof. Let c be a coloring of G so that vertices at distance at most three in G have distinct colors. Since every vertex has at most $\Delta^3 - \Delta^2 + \Delta$ vertices at distance at most three, such a coloring using at most $\Delta^3 - \Delta^2 + \Delta + 1$ colors exists. Let uv be an edge of G. Let N_u be the set of neighbors of u not in $N[v]$ and N_v be the set of neighbors of v not in $N[u]$. Using that vertices at distance at most two in G have distinct colors, we obtain that all the elements of N_u (resp. N_v) have distinct colors. Since vertices at distance at most three have distinct colors, the sets of colors of N_u and N_v are disjoint. If $N[u] \neq N[v]$, then $N_u \cup N_v \neq \varnothing$, and $c(N[u]) \neq c(N[v])$ by the previous remark. □

We believe that this result is not optimal, and that the bound should rather be quadratic in Δ:

Question 20 *Is it true that for any graph G with maximum degree Δ, we have $\chi_{lid}(G) = O(\Delta^2)$?*

If true, then this result would be best possible. Take a projective plane P of order n, for some prime power n. Let G_{n+1} be the graph obtained from the complete graph on $n+1$ vertices by adding, for every vertex v of the clique, a vertex v' adjacent only to v. Note

that in any lid-coloring of G_{n+1}, all vertices v' must receive distinct colors. For any line l of the projective plane P, consider a copy G^l_{n+1} of G_{n+1} in which the new vertices v' are indexed by the $n+1$ points of l. For any point p of P, identify the $n+1$ vertices indexed p in the graphs G^l_{n+1}, where $p \in l$, into a single vertex p^*. The resulting graph H_{n+1} is $(n+1)$-regular and has $(n^2+n+1)(n+2)$ vertices. By construction, all the vertices p^*, $p \in P$, have distinct colors in any lid-coloring. Hence, at least $n^2+n+1 = \Delta^2 - \Delta + 1$ colors are required in any lid-coloring of this Δ-regular graph. The 3-regular graph H_3 with $\chi_{\text{lid}}(H_3) \geq 7$ is depicted in Figure 2.

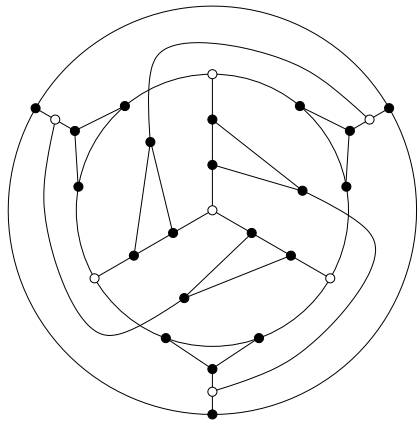


Figure 2: In any lid-coloring of the 3-regular graph H_3, the seven white vertices must receive pairwise distinct colors.

We saw that the lid-chromatic number cannot be upper-bounded by the chromatic number. For a graph G, the square of G, denoted by G^2, is the graph with the same vertex set as G, in which two vertices are adjacent whenever they are at distance at most two in G. The following question is somehow related to the previous one (depending on the possible linearity of f).

Question 21 *Does there exist a function f so that for any graph G, we have $\chi_{lid}(G) \leq f(\chi(G^2))$?*

8 Planar and outerplanar graphs

This section is devoted to graphs embeddable in the plane. A maximal outerplanar graph is a 2-tree and so is 6-lid colorable by Theorem 12. However, χ_{lid} is not monotone under taking subgraphs and so this result does not extend to all outerplanar graphs. So we have to use a different strategy to give an upper bound of the lid-chromatic number on the class of outerplanar graphs.

Theorem 22 *Every outerplanar graph is 20-lid-colorable.*

Proof. Let G be a connected outerplanar graph, and let H be any maximal outerplanar graph containing G (that is, H is obtained by adding edges to G). The graph H is 2-connected, and has minimum degree two. Consider a drawing of H in the plane, such that all the vertices lie on the outerface, and take the clockwise ordering $x_1, \ldots, x_n$ of the vertices around the outerface, starting at some vertex x_1 of degree two in H (and thus at most two in G). This ordering has the following properties:

- For any four integers $i, j, k, \ell \in \{1, \ldots, n\}$ with $i < j < k < \ell$, at most one of the pairs $\{x_i, x_k\}$ and $\{x_j, x_\ell\}$ corresponds to an edge of G.
- Let x_{i_0} be a vertex and $x_{i_1}, \ldots, x_{i_k}$ be its neighbors in G such that $x_{i_0}, x_{i_1}, \ldots, x_{i_k}$ appear in clockwise order around the outerface of H. The previous property implies that, for $1 \leq j \leq k$, the neighbors of x_{i_j} distinct from x_{i_0} appear (in clockwise order around the outerface of H) between $x_{i_{j-1}}$ and $x_{i_{j+1}}$ (if $j \neq k$) and between $x_{i_{k-1}}$ and x_{i_0} (if $j = k$). Moreover, two distinct vertices x_{i_j} and x_{i_ℓ} have at most one common neighbor outside $N[x_{i_0}]$. If such a common neighbor exists, then we have $|j - \ell| = 1$.

For any $i \geq 1$, let $L_i = \{x_{i_1}, \ldots, x_{i_{k_i}}\}$ be the set of vertices at distance i from x_1 in G, with $i_1 < \cdots < i_{k_i}$, and let L_s be the last nonempty L_i-set. For the sake of clarity, we write $x^i_1, \ldots, x^i_{k_i}$ instead of $x_{i_1}, \ldots, x_{i_{k_i}}$, and we say that two vertices x^i_j and x^i_{j+1} are *consecutive* in L_i. Observe the following:

- A vertex in L_{i+1} has at most two neighbors in L_i.
- Two vertices of L_i have at most one common neighbor in L_{i+1}.
- If two vertices of L_i have a common neighbor in L_{i+1}, they are consecutive in L_i.
- If two vertices of L_i are adjacent, then they are consecutive in L_i. This implies that the graph induced by L_i is a disjoint union of paths.

Indeed, if one of the two first facts was not true, there would be a subdivision of $K_{2,3}$ in G. The two last facts are due to the embedding of G and H and to the previous properties. From now on, we forget about H and consider G only (the sole purpose of H was to give a clean definition of the order $x_1, \ldots, x_n$). With the facts above, we can notice that in the ordering of L_{i+1}, we find first the neighbors of x^i_1, then the neighbors of x^i_2, and so on...

We will color the vertices of G with 20 colors partitioned in four classes of colors C_0, C_1, C_2 and C_3 with $C_j = \{5j, \ldots, 5j + 4\}$. Vertices in L_i will be colored with colors from $C_{i \bmod 4}$, almost as we did for bipartite graphs in Theorem 6. We will slightly modify this coloring by using *marked vertices*.

We start by coloring x_1 with color 0, and mark the last vertex $x^1_{k_1}$ of L_1. We then apply Algorithm 1.

Let us describe this algorithm. Sets L_i are colored one after the other (line 3). When we color L_i, we first mark some vertices in L_{i+1} (the last neighbors in L_{i+1} of vertices in L_i, see lines 4 to 6). Then we color vertices of L_i in the order they appear. There are

Algorithm 1 Lid-coloring of outerplanar graphs

1: $c(x_1) = 0$
2: Mark vertex $x^1_{k_1}$
3: **for** $i = 1$ **to** s **do**
4: **for** $j = 1$ **to** k_i **do**
5: Mark, if it exists, the last neighbor of x^i_j in L_{i+1}.
6: **end for**
7: **for** $k = 0$ **to** 3 **do**
8: $c_k \leftarrow k + 5 \times (i \bmod 4)$
9: **end for**
10: $c_\varnothing \leftarrow 4 + 5 \times (i \bmod 4)$
11: **for** $j = 1$ **to** k_i **do**
12: $c(v^i_j) = c_{j \bmod 4}$
13: **if** v^i_j is marked **then**
14: $\text{tmp} \leftarrow c_{(j+1) \bmod 4}$
15: $c_{(j+1) \bmod 4} \leftarrow c_\varnothing$
16: $c_\varnothing \leftarrow c_{(j-1) \bmod 4}$
17: $c_{(j-1) \bmod 4} \leftarrow \text{tmp}$
18: **end if**
19: **end for**
20: **end for**
21: **return** c

four current colors of $C_{i \bmod 4}$ which are used, c_0 to c_3 and one forbidden color $c_\varnothing$, that are originally set to $5 \times (i \bmod 4)$, $1 + 5 \times (i \bmod 4)$, $2 + 5 \times (i \bmod 4)$, $3 + 5 \times (i \bmod 4)$, and $4 + 5 \times (i \bmod 4)$, respectively. The vertices of L_i are then colored with the pattern $c_0c_1c_2c_3c_0...$ (line 12), but every time a marked vertex v^i_j is colored, we perform a cyclic permutation on the values of $c_{(j+1) \bmod 4}$, $c_\varnothing$, and $c_{(j-1) \bmod 4}$ (lines 13 to 18). This is done in such a way that:

- The coloring is proper.
- Four consecutive vertices in L_i receive four different colors.
- Two consecutives vertices of L_{i-1} do not have the same set of colors in their neighborhood in L_i, when these neighborhoods differ.

Thus, this algorithm provides a proper coloring c of G with 20 colors such that for any i, $c(L_i) \subseteq C_{i \bmod 4}$.

Let us prove that the coloring given by the algorithm is locally identifying. Let uv be an edge of G such that $N[u] \neq N[v]$. If uv is not an edge of a layer L_i, then we can assume that $u \in L_i$ and $v \in L_{i+1}$. If $u \neq x_1$, then there is a neighbor t of u in L_{i-1} and then $c(t) \notin c(N[v])$. So we may assume that $u = x_1$. If the vertex v has degree 1, then u has

degree 2 and has an other neighbor, t, and $c(t) \notin c(N[v])$. Otherwise, the vertex v has degree at least 2, so there is a neighbor $t \neq u$ of v. If $t \in L_1$ then there is another neighbor t' of v in L_2 (because $N[u] \neq N[v]$). So we can assume that $t \in L_2$ and then $c(t) \notin c(N[u])$. So in any case, $c(N[u]) \neq c(N[v])$.

Assume now that $u, v \in L_i$ for some i. Without loss of generality, we may assume that $u = x^i_j$, $v = x^i_{j+1}$ for some j and that there is a vertex t adjacent to exactly one vertex among $\{u, v\}$. If $t \in L_i$, then we are done because four consecutive vertices have different colors in L_i. If $t \in L_{i-1}$, and $t \in N(u) \setminus N(v)$, then v has at most two neighbors in L_{i-1}. Those neighbors (if any) are just following t in the layer L_{i-1} and so $c(t) \notin c(N[v])$. Otherwise, $t \in L_{i+1}$, the vertices u and v are consecutive and have distinct neighborhoods in L_{i+1}, so the sets of colors in their neighborhoods in L_{i+1} are distinct. □

We believe that this bound is far from tight.

Question 23 *Is it true that every outerplanar graph G satisfies $\chi_{lid}(G) \leq 6$?*

We now prove that sparse enough planar graphs have low lid-chromatic number.

Theorem 24 *If G is a planar graph with girth at least 36, then $\chi_{lid}(G) \leq 5$.*

Proof. Let us call *nice* a lid-coloring c using at most 5 colors such that every vertex v with degree at least 2 satisfies $|c(N[v])| = 3$. We show that every planar graph with girth at least 36 admits a nice lid-coloring.

Observe first that a cycle of length $n \geq 12$ has a nice lid-coloring that consists of subpaths of length 4 colored 1234 and subpaths of length 5 colored 12345 following the clockwise orientation of G (the number of subpaths of length 5 is exactly n mod 4).

Suppose now that G is a planar graph with girth at least 36 that does not admit a nice lid-coloring and with the minimum number of vertices. Let us first show that G does not contain a vertex of degree at most 1. The case of a vertex of degree 0 is trivial, so suppose that G contains a vertex u of degree 1 adjacent to another vertex v. By minimality of G, the graph $G' = G \setminus u$ admits a nice lid-coloring c. We consider three cases according to the degree of v in G', and in all three cases, we extend c to a nice lid-coloring of G in order to obtain a contradiction. If v has degree at least 2 in G', then we assign to u a color in $c(N[v]) \setminus \{c(v)\}$. So $c(N[v])$ is unchanged, and $c(N[u]) \neq c(N[v])$ since $|c(N[u])| = 2$ and $|c(N[v])| = 3$. We thus have a nice lid-coloring of G. If v has degree 1 in G', then v is adjacent to another vertex w in G' and we assign to u a color that does not belong to $c(N[w])$. Such a color exists since $|c(N[w])| \leq 3$ and the obtained coloring of G is nice: $|c(N[v])| = 3$ and $c(N[v]) \neq c(N[w])$ since $c(u) \in c(N[v])$ but $c(u) \notin c(N[w])$. If v has degree 0 in G', then $N[u] = N[v]$ in G, so u and v need not to be identified.

It follows that G has minimum degree at least 2 and G is not a cycle. It is well-known that if the girth of a planar graph is at least $5k + 1$, then it contains either a vertex of degree at most 1, or a path consisting of k consecutive vertices of degree 2. The graph

G thus contains a path of seven vertices of degree 2. So we can assume that G contains a path $P = x_1x_2\dots x_9$ such that $d(x_1) \geq 3$ (G is not a cycle), $d(x_i) = 2$ for $2 \leq i \leq 8$, and $d(x_9) \geq 2$. By minimality of G, the graph $G' = G \setminus \{x_2, x_3, \dots, x_8\}$ admits a nice lid-coloring c. Without loss of generality, assume that $c(x_1) = 1$ and $c(N[x_1]) = \{1, 2, 3\}$, since the degree of x_1 is at least 2 in G'. We denote $a = c(x_9)$. If the degree of x_9 in G' is at least 2, then we denote $\{b_1, b_2\} = c(N(x_9))$. If the degree of x_9 in G' is 1, then x_9 is adjacent to a vertex x_{10} and we denote $b_1 = c(x_{10})$ and b_2 is any element of $\{1, 2, 3, 4, 5\} \setminus c(N[x_{10}])$.

The following table gives the colors of $x_2, x_3, \dots, x_8$ for all the possible values of $(a; b_1, b_2)$. Note that $c(x_2) \in \{2, 3\}$, $c(x_3) \notin \{2, 3\}$, $c(x_6) \neq a$, $c(x_7) \notin \{a, b_1, b_2\}$, $c(x_8) = b_2$, and four consecutive vertices have different colors. This implies that the coloring c can be extended to a nice lid-coloring of G, a contradiction.

2431243 (1;2,3)	2431543 (2;1,3)	2431542 (3;1,2)	2534152 (4;1,2)	2435142 (5;1,2)
2431254 (1;2,4)	2541354 (2;1,4)	2431254 (3;1,4)	2431253 (4;1,3)	2431243 (5;1,3)
2431245 (1;2,5)	2451345 (2;1,5)	2431245 (3;1,5)	2451235 (4;1,5)	2435124 (5;1,4)
2431254 (1;3,4)	3512354 (2;3,4)	2431254 (3;2,4)	2431253 (4;2,3)	2431243 (5;2,3)
2431245 (1;3,5)	3412345 (2;3,5)	2431245 (3;2,5)	2451235 (4;2,5)	2435214 (5;2,4)
2531425 (1;4,5)	3521435 (2;4,5)	2531425 (3;4,5)	2534125 (4;3,5)	2435124 (5;3,4)

□

We conjecture that planar graphs have bounded lid-chromatic number.

9 Connectivity and lid-coloring

Most of the proofs we gave in this article heavily depend on the structure of the classes of graphs we were considering. We now give a slightly more general tool, allowing us to extend results on the 2-connected components of a graph to the whole graph:

Theorem 25 *Let k be an integer and G be a graph such that every 2-connected component of G is k-lid-colorable. Let H be the subgraph of G induced by the cut-vertices of G. Then $\chi_{lid}(G) \leq k + \chi(H)$.*

Proof. In this proof, we will consider two different colorings of the vertices: the lid-coloring of the vertices of G and the proper coloring of the graph H induced by the cut-vertices. To avoid confusion, we call *type* the color of a cut-vertex in the second coloring. We prove the following stronger result:

Claim: If t is a proper coloring of H with colors $t_1, \dots, t_h$, then G admits a $(k+h)$-lid-coloring c such that for each maximal 2-connected component C of G, $(*)$ there are h colors not appearing in $c(C)$, say $c_1^C, \dots, c_h^C$, such that for every cut-vertex v of G lying in C, if $t(v) = t_i$, then $c(N(v))$ contains c_i^C but none of the c_j^C, $j \neq i$.

We prove the claim by induction on the number of cut-vertices of G. We may assume that G has a cut-vertex, otherwise the property is trivially true.

Let u be a cut-vertex of G and let $C_1, \ldots, C_s$ be the connected components of $G-u$. We can choose u so that at most one of the C_i's, say C_1, contains the remaining cut-vertices. For $1 \leq i \leq s$, let G_i be the graph induced by the set of vertices $C_i \cup \{u\}$. Let C be the maximal 2-connected component of G_1 containing u. Observe that the vertex u is not a cut-vertex in G_1. By the induction, G_1 has a $(k+h)$-lid-coloring c such that, without loss of generality, $c(C) \subseteq \{1, \ldots, k\}$ and every cut-vertex v of C with $t(v) = t_i$ has a neighbor colored $k+i$, but no neighbor colored $k+j$, $1 \leq j \leq h$, $j \neq i$. We can also assume that $t(u) = t_1$ and $1 \in c(N(u))$ (thus $c(u) \neq 1$).

We now extend the coloring c to G by lid-coloring each component $G_2, ...G_s$ with colors $2, 3, \ldots, k+1$ such that $k+1 \in c(N(u))$ (these components share the vertex u but we can assume that u always has the same color in all the lid-colorings of G_2,...,G_s). Let us prove that the coloring obtained is a lid-coloring of G satisfying $(*)$. In order to prove that c is a lid-coloring, by the induction one just needs to check that u has no neighbor v with $c(N[v]) = c(N[u])$. For the sake of contradiction, suppose that such a vertex v exists. Since $1 \in c(N[u])$, v has to lie in C. If v is a cut-vertex of G_1, then $t(v) \neq t_1$ (t is a proper coloring of H) and by the induction, $k+1 \notin c(N[v])$. If v is not a cut-vertex of G_1, then all its neighbors lie in C and again, $k+1 \notin c(N[v])$. Since $k+1 \in c(N[u])$, we obtain a contradiction.

It remains to prove that $(*)$ holds for every maximal 2-connected component of G. It clearly does for G_2,...,G_s, since u is the only cut-vertex of G that they contain and $1 \in c(N[u]) \subseteq \{1, \ldots, k+1\}$, while none of these components contains color 1 or color $k+i$ with $2 \leq i \leq h$. The component C also satisfies $(*)$, since u has a neighbor colored $k+1$ and no neighbor colored $k+i$ with $2 \leq i \leq h$. By the induction, Property $(*)$ trivially holds for the remaining maximal 2-connected components of G. This completes the proof of the claim. $\square$

Among other things, this result can be used to prove that outerplanar graphs without triangles can be 8-lid-colored. We omit the details; we suspect that Theorem 25 can be used to prove results on much wider classes of graphs.

Remark. During the review of the paper, Question 20 has been answered positively, see [13].

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