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# Locally identifying coloring of graphs\*

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## Abstract

Let  $G = (V, E)$  be a graph. Let  $c : V \rightarrow \mathbb{N}$  be a vertex-coloring of the vertices of  $G$ . For any vertex  $u$ , we denote by  $N[u]$  its closed neighborhood ( $u$  and its adjacent vertices), and for any  $S \subseteq V$ , let  $c(S)$  be the set of colors that appear on the vertices of  $S$ . A proper vertex-coloring  $c$  is said to be *locally identifying*, if for any edge  $uv$ ,  $N[u] \neq N[v] \Rightarrow c(N[u]) \neq c(N[v])$ . Let  $\chi_{lid}(G)$  be the minimum number of colors used by a locally identifying proper vertex-coloring of  $G$ . In this paper, we give several bounds on  $\chi_{lid}$  for different families of graphs (planar graphs, some subclasses of perfect graphs, graphs with bounded maximum degree) and prove that deciding whether  $\chi_{lid}(G) = 3$  for a subcubic bipartite graph with large girth is an NP-complete problem.

## 1 Introduction

In this paper we focus on colorings of graphs that allow to distinguish the vertices of a graph. In [14], Hornák and Soták considered edge-coloring of a graph such that (i) the edge-coloring is proper (i.e. no adjacent edges receive the same color) and (ii) for any vertices  $u, v$  (with  $u \neq v$ ) the set of colors assigned to the edges incident to  $u$  differs from the set of colors assigned to the edges incident to  $v$ . Such a coloring is called a *vertex-distinguishing proper edge-coloring*. The minimum number of colors required in any vertex-distinguishing proper edge-coloring of  $G$  is called the *observability* of  $G$  and was studied for different families of graphs [3, 6, 8, 11, 12, 14, 15]. This notion was then extended to *adjacent vertex-distinguishing edge-coloring* where Property (ii) must be true only for pairs of adjacent vertices; see [1, 13, 21].

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In the present paper we introduce the notion of *identifying colorings*: a vertex-coloring is said to be *identifying* if (i) the vertex-coloring is proper (i.e. no adjacent vertices receive the same color), and (ii) for any pair  $u, v$  (with  $u \neq v$ ) the set of colors assigned to the closed neighborhood of  $u$  differs from the set of colors assigned to the closed neighborhood of  $v$  whenever these neighborhoods are distinct. A *locally identifying vertex-coloring* of  $G$  is an identifying coloring of  $G$  where Property (ii) is only required to hold for pairs of adjacent vertices. The *locally identifying chromatic number* of the graph  $G$  (or lid-chromatic number, for short), denoted by  $\chi_{lid}(G)$ , is the smallest number of colors required in any locally identifying coloring of  $G$ . In the following we study the parameter  $\chi_{lid}$  for different families of graphs, such as bipartite graphs,  $k$ -trees, interval graphs, split graphs, cographs, graphs with bounded maximum degree, planar graphs with high girth, and outerplanar graphs.

Let  $G = (V, E)$  be a graph. For any vertex  $u$ , we denote by  $N(u)$  its neighborhood and by  $N[u]$  its *closed neighborhood* ( $u$  together with its adjacent vertices). Let  $c$  be a vertex-coloring of  $G$ . For any  $S \subseteq V$ , let  $c(S)$  be the set of colors that appear on the vertices of  $S$ . More formally, a locally identifying coloring of  $G$  is proper vertex-coloring  $c$  of  $G$  such that for any edge  $uv$ ,  $N[u] \neq N[v] \Rightarrow c(N[u]) \neq c(N[v])$ . A graph  $G$  is  *$k$ -lid-colorable* if it admits a locally identifying coloring using at most  $k$  colors.

First of all, observe that this coloring is not hereditary. For instance, if  $P_n$  denotes the path on  $n$  vertices, then  $\chi_{lid}(P_5) = 3$  whereas  $\chi_{lid}(P_4) = 4$ .

In Section 2, we prove that every bipartite graph has lid-chromatic number at most 4. Moreover, deciding whether a bipartite graph is 3-lid-colorable is an NP-complete problem, whereas it can be decided in linear time whether a tree is 3-lid-colorable.

In general,  $\chi_{lid}$  is not bounded by a function of the usual chromatic number. Nevertheless it turns out that for several nice classes of graphs such a function exists: we study  $k$ -trees (Section 3), interval graphs (Section 4), split graphs (Section 5), cographs (Section 6), and give tight bounds in each of these cases. We also conjecture that every chordal graph  $G$  with clique number  $\omega(G)$  has a lid-coloring with  $2\chi(G) = 2\omega(G)$  colors.

Section 7 is dedicated to graphs with bounded maximum degree. We prove that the lid-chromatic number of graphs with maximum degree  $\Delta$  is  $O(\Delta^3)$  and that there are examples with lid-chromatic number  $\Omega(\Delta^2)$ .

In Section 8, we study graphs with a topological structure. Our result on 2-trees does not give any information on outerplanar graphs, since lid-coloring is not monotone under taking subgraphs. So we use a completely different strategy to prove that outerplanar graphs and planar graphs with large girth have lid-colorings using a constant number of colors.

Finally, in Section 9, we propose a tool that allows to extend the lid-colorings of the 2-connected components of a graph to the whole graph.

## 2 Bipartite graphs

This section is dedicated to bipartite graphs. The main interest of the study of bipartite graphs here comes from the following two lemmas:

**Lemma 1** *If a connected graph  $G$  satisfies  $\chi_{lid}(G) \leq 3$ , then  $G$  is either a triangle or a bipartite graph.*

PROOF. Consider a 3-lid-coloring  $c$  of  $G$  with colors 1, 2, 3, and assume first that  $G$  does not contain any edge  $uv$  with  $N[u] = N[v]$ . Then the coloring  $c'$  defined by  $c'(x) = |c(N[x])|$  for any vertex  $x$  is a proper 2-coloring of  $G$  (with colors 2 and 3, if we assume that  $G$  is not a single vertex): If two vertices  $u, v$  satisfy  $c'(u) = c'(v) = 3$ , then  $c(N[u]) = c(N[v]) = \{1, 2, 3\}$  and  $u, v$  are not adjacent by definition of  $c$ . If  $c'(u) = c'(v) = 2$  and  $u, v$  are adjacent, then without loss of generality  $c(N[u]) = \{1, 2\}, c(N[v]) = \{1, 3\}$ , and we must have  $c(u) = c(v) = 1$ , a contradiction. It follows that  $G$  is bipartite.

Assume now that there exists an edge  $uv$  with  $N[u] = N[v]$ , and that  $G$  does not consist of the single edge  $uv$ . Then there exists a vertex  $w$  adjacent to  $u$  and  $v$ . But in this case  $c(N[u]) = c(N[v]) = c(N[w]) = \{1, 2, 3\}$ , which implies that  $N[u] = N[v] = N[w]$  by definition of  $c$ . This is only possible if  $G$  contains a  $K_4$  (which would imply that  $\chi_{lid}(G) > 3$ ) or if  $G$  consists only of the triangle  $uvw$ .  $\square$

Indeed, more can be said about the color classes in a 3-lid-coloring of a (bipartite) graph:

**Lemma 2** *Let  $G$  be a 3-lid-colorable connected bipartite graph on at least 3 vertices, with bipartition  $(U, V)$ , and let  $c$  be a 3-lid-coloring of  $G$  with colors 1, 2, 3. If for some vertex  $u \in U$ ,  $c(N[u]) = \{1, 2, 3\}$ , then  $c(U) = \{c(u)\}$  and  $c(V) = \{1, 2, 3\} \setminus \{c(u)\}$ .*

PROOF. Without loss of generality, assume that  $c(u) = 1$ . Then all the neighbors of  $u$  must be colored 2 or 3, and the vertices at distance two from  $u$  must be colored 1 (otherwise there would be a neighbor  $v$  of  $u$  with  $c(N[v]) = \{1, 2, 3\}$ ). Iterating this observation, we remark that all the vertices at even distance from  $u$  must be colored 1, while the vertices at odd distance from  $u$  must be colored either 2 or 3, which yields the conclusion.  $\square$

As a corollary we obtain a precise description of 3-lid-colorable trees.

**Corollary 1** *A tree  $T$  with at least 3 vertices is 3-lid-colorable if and only if the distance between every two leaves is even.*

PROOF. Observe that for each leaf  $u$  of  $T$ , we have  $|c(N[u])| = 2$  in any proper coloring  $c$  of  $T$ , so by Lemma 2 the distance between every two leaves is even.

Now assume that the distance between every two leaves of  $T$  is even, and fix a leaf  $u$  of  $T$ . Let  $c$  be the 3-coloring of  $T$  defined by  $c(v) = 2$  if  $d(u, v)$  is odd,  $c(v) = 1$  if  $d(u, v) \equiv 0 \pmod{4}$ , and  $c(v) = 3$  if  $d(u, v) \equiv 2 \pmod{4}$ . The coloring  $c$  is clearly proper, and we have  $c(N[v]) = \{1, 2\}$  if  $d(u, v) \equiv 0 \pmod{4}$ , and  $c(N[v]) = \{2, 3\}$  if  $d(u, v) \equiv 2 \pmod{4}$ . If

$v$  is a vertex at odd distance from  $u$ , then  $v$  is not a leaf and  $c(N[v]) = \{1, 2, 3\}$ . As a consequence,  $c$  is a 3-lid-coloring of  $T$ .  $\square$

Another class of bipartite graphs that behaves nicely with regards to locally identifying coloring is the class of graphs obtained by taking the Cartesian product of two bipartite graphs. For two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , the Cartesian product of  $G_1$  and  $G_2$ , denoted by  $G_1 \square G_2$ , is the graph with vertex set  $V_1 \times V_2$ , in which two vertices  $(u_1, u_2)$  and  $(v_1, v_2)$  are adjacent whenever  $u_2 = v_2$  and  $u_1 v_1 \in E_1$ , or  $u_1 = v_1$  and  $u_2 v_2 \in E_2$ .

**Theorem 1** *Let  $G_1$  and  $G_2$  be two bipartite graphs without isolated vertices. Then  $\chi_{lid}(G_1 \square G_2) = 3$ .*

PROOF. Let  $(U_1, V_1)$  and  $(U_2, V_2)$  be the partite sets of  $G_1$  and  $G_2$ , respectively. Then  $G_1 \square G_2$  is a bipartite graph with partition  $((U_1 \times U_2) \cup (V_1 \times V_2), (U_1 \times V_2) \cup (V_1 \times U_2))$  and because there are no isolated vertices in  $G_1$  and  $G_2$ , each vertex of  $(U_1 \times U_2) \cup (V_1 \times V_2)$  has a neighbor in  $U_1 \times V_2$  and a neighbor in  $V_1 \times U_2$ .

We define  $c$  by  $c(u) = 1$  if  $u \in (U_1 \times U_2) \cup (V_1 \times V_2)$ ,  $c(u) = 2$  if  $u \in U_1 \times V_2$ , and  $c(u) = 3$  if  $u \in V_1 \times U_2$ . Then  $c$  is a lid-coloring of  $G_1 \square G_2$ :  $c(N[u]) = \{1, 2, 3\}$  for vertices of  $(U_1 \times U_2) \cup (V_1 \times V_2)$ ,  $c(N[u]) = \{1, 2\}$  for vertices of  $U_1 \times V_2$  and  $c(N[u]) = \{1, 3\}$  for vertices of  $V_1 \times U_2$ .  $\square$

As a corollary, we obtain that hypercubes and grids in any dimension are 3-lid-colorable. We now focus on bipartite graphs that are not 3-lid-colorable.

**Theorem 2** *If  $G$  is a bipartite graph, then  $\chi_{lid}(G) \leq 4$ .*

PROOF. If  $G$  is disconnected, then we can color each component independently and if  $G$  contains at most four vertices, then the theorem is trivially true. So we can assume that  $G$  is connected and has at least five vertices. Then there exists a vertex  $u$  of  $G$  that is not adjacent to a vertex of degree one. For any vertex  $v$  of  $G$ , set  $c(v)$  to be the element of  $\{0, 1, 2, 3\}$  congruent with  $d(u, v)$  modulo 4. We claim that  $c$  is a lid-coloring of  $G$ . Since  $G$  is bipartite,  $c$  is clearly a proper coloring. Let  $v, w$  be two adjacent vertices in  $G$ . We may assume that they are at distance  $k \geq 0$  and  $k + 1$  from  $u$ , respectively. If  $k = 0$ , then  $v = u$  and  $w$  has a neighbor at distance two from  $u$ , so  $c(N[v]) = \{0, 1\}$  and  $c(N[w]) = \{0, 1, 2\}$ . If  $k \geq 1$ , then  $(k - 1) \bmod 4$  is in  $c(N[v])$  but not in  $c(N[w])$ , so  $c(N[v]) \neq c(N[w])$ .  $\square$

Moreover,

**Theorem 3** *For any fixed integer  $g$ , deciding whether a bipartite graph with girth at least  $g$  and maximum degree 3 is 3-lid-colorable is an NP-complete problem.*

PROOF. We recall that a 2-coloring of a hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  is a partition of its vertex set  $\mathcal{V}$  into two color classes such that no edge in  $\mathcal{E}$  is monochromatic. We reduce our problem to the NP-complete problem of deciding the 2-colorability of 3-uniform hypergraphs [16].

Let  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  be a hypergraph with at least one hyperedge. We construct the bipartite graph  $G = (V, E)$  in the following way. A vertex  $v \in \mathcal{V}$  is associated to a path  $v_0, \dots, v_{4t}$  in  $G$  (where  $t$  will depend on the degree of  $v$  in  $\mathcal{H}$  and  $g$ ). A hyperedge  $e \in \mathcal{E}$  is associated to a vertex  $w_e$  in  $G$ . If a hyperedge  $e$  contains a vertex  $v$  in  $\mathcal{H}$ , then we add an edge in  $G$  between  $w_e$  and a vertex  $v_i$  with some index  $i \equiv 2 \pmod{4}$ . We require that a vertex  $v_i$  is adjacent to at most one vertex corresponding to a hyperedge. The graph  $G$  is clearly bipartite with maximum degree 3. Moreover, we can construct  $G$  in polynomial time such that it has girth at least  $g$  by putting enough space between the vertices of degree 3 lying on a same path  $v_0, \dots, v_{4t}$ .

Then  $\chi_{\text{lid}}(G) = 3$  if and only if  $\mathcal{H}$  is 2-colorable.

Assume first that  $\mathcal{H}$  admits a 2-coloring  $\mathcal{C} : V \rightarrow \{1, 2\}$ . We define the following 3-coloring  $c$  of  $G$  such that  $c(v_{i \equiv 2 \pmod{4}}) = \mathcal{C}(v)$ ,  $c(v_{i \equiv 0 \pmod{4}}) = 3 - \mathcal{C}(v)$ ,  $c(v_{i \equiv 1 \pmod{2}}) = 3$  if  $v \in V$ , and  $c(w) = 3$  if  $e \in \mathcal{E}$ . Let us check that  $c$  is a lid-coloring of  $G$ . We have  $c(N[w]) = \{1, 2, 3\}$  since  $c(w) = 3$  and  $w$  is adjacent to a vertex colored 1 and to a vertex colored 2 because of the 2-coloring of  $\mathcal{H}$ . Also,  $c(N[v_{i \equiv 1 \pmod{2}}]) = \{1, 2, 3\}$ ,  $c(N[v_{i \equiv 2 \pmod{4}}]) = \{\mathcal{C}(v), 3\}$ , and  $c(N[v_{i \equiv 0 \pmod{4}}]) = \{3 - \mathcal{C}(v), 3\}$ . So, for every edge  $uv$  in  $G$ , we have  $c(N[u]) \neq c(N[v])$ .

Conversely, assume that  $G$  (with bipartition  $(U, V)$ ) admits a lid-coloring  $c$  using colors 1, 2, 3. By Lemma 2, we can assume that  $c(U) = \{1, 2\}$  and  $c(V) = \{3\}$ , and that the vertices of degree one in  $G$  are in  $U$ . This implies that  $c(v_{i \equiv 2 \pmod{4}}) \in \{1, 2\}$ ,  $c(v_{i \equiv 0 \pmod{4}}) = 3 - c(v_{i \equiv 2 \pmod{4}})$ , and  $c(v_{i \equiv 1 \pmod{2}}) = c(w) = 3$ . Hence, this coloring restricted to the vertices  $v_{i \equiv 2 \pmod{4}}$  gives a 2-coloring of the hypergraph  $\mathcal{H}$ .  $\square$

It turns out that the connection between 3-lid-coloring and hypergraph 2-coloring highlighted in the proof of Theorem 3 has further consequences. For a connected bipartite graph  $G$  with bipartition  $(U, V)$ , let  $\mathcal{H}_U$  be the hypergraph with vertex set  $U$  and hyperedge set  $\{N(v), v \in V\}$ . A direct consequence of Lemmas 1 and 2 is that a connected graph  $G$  distinct from a triangle is 3-lid-colorable if and only if it is bipartite (say with bipartition  $(U, V)$ ) and at least one of  $\mathcal{H}_U$  and  $\mathcal{H}_V$  is 2-colorable.

A consequence of a result of Moret [17] (see also [2] for further details) is that if  $G$  is a subcubic bipartite planar graph with bipartition  $(U, V)$ , then we can check in polynomial time whether  $\mathcal{H}_U$  (or  $\mathcal{H}_V$ ) is 2-colorable. As a counterpart of Theorem 3, this implies:

**Theorem 4** *It can be checked in polynomial time whether a planar graph  $G$  with maximum degree three is 3-lid-colorable.*

It was proved by Burstein [7] and Penaud [18] that every planar hypergraph in which all hyperedges have size at least three is 2-colorable, and Thomassen [19] proved that for any  $k \geq 4$  any  $k$ -regular  $k$ -uniform hypergraph is 2-colorable. As a consequence, we obtain the following two results:

**Theorem 5** *Let  $G$  be a bipartite planar graph with bipartition  $(U, V)$  such that all vertices in  $U$  or all vertices in  $V$  have degree at least three. Then  $G$  is 3-lid-colorable.*

**Theorem 6** For  $k \geq 4$ , a  $k$ -regular graph is 3-lid-colorable if and only if it is bipartite.

Since bipartite graphs have bounded lid-chromatic number, a natural question is whether  $\chi_{lid}$  is upper-bounded by a function of the (usual) chromatic number. However, this is not true, since the graph  $G$  obtained from a clique on  $n$  vertices by subdividing each edge exactly twice has  $\chi_{lid}(G) = n$  (it suffices to observe that two vertices of the initial clique cannot have the same color in the subdivided graph), whereas it is 3-colorable. This example also shows that if the edges of a graph  $G$  are partitioned into two sets  $E_1$  and  $E_2$ , and the subgraphs of  $G$  induced by  $E_1$  and  $E_2$  have bounded lid-chromatic number, then  $\chi_{lid}(G)$  is not necessarily bounded.

We propose the following conjecture relating  $\chi_{lid}$  and  $\chi$  for highly structured graphs. A graph is *chordal* if it does not contain an induced cycle of length at least four.

**Conjecture 1** For any chordal graph  $G$ ,  $\chi_{lid}(G) \leq 2\chi(G)$ .

The next three sections are dedicated to important subclasses of chordal graphs on which we are able to verify Conjecture 1.

### 3 $k$ -trees

This section is devoted to the study of  $k$ -trees. A  $k$ -tree is a chordal graph in which every maximal clique is a  $(k+1)$ -clique (a clique on  $k+1$  vertices). A  $k$ -tree can be constructed inductively from a  $(k+1)$ -clique by adding at each step a vertex whose neighborhood is a  $k$ -clique in the previous graph.

**Theorem 7** If  $G$  is a  $k$ -tree, then  $\chi_{lid}(G) \leq 2k+2$ .

PROOF. In this proof the colors are the integers modulo  $2k+2$ . For instance,  $x \mapsto x+k+1$  is an involution.

First observe that every  $k$ -clique of a  $k$ -tree is contained in a  $(k+1)$ -clique. Let  $v_0, \dots, v_{n-1}$  be the  $n$  vertices of  $G$  ordered in such way that  $G$  is constructed from a  $(k+1)$ -clique induced by  $v_0, \dots, v_k$  by adding at each step  $i$ , with  $k+1 \leq i \leq n-1$ , a vertex  $v_i$  whose neighborhood is a  $k$ -clique in  $G_{i-1}$ , the subgraph of  $G$  induced by  $v_0, \dots, v_{i-1}$  (which is also a  $k$ -tree). We construct the following coloring  $c$  of  $G$  iteratively for  $0 \leq i \leq n-1$ . If  $i \leq k$ , then we set  $c(v_i) = i$ . Suppose  $i \geq k+1$ . Let  $C$  be the neighborhood of  $v_i$  in  $G_i$ . Since  $G_{i-1}$  is a  $k$ -tree, the clique  $C$  is contained in a  $(k+1)$ -clique  $C'$  of  $G_{i-1}$ . Let  $\{v_j\} = C' \setminus C$ . We set  $c(v_i) = c(v_j) + k+1$  (we may have several choices for  $C'$  and thus for  $j$ ).

We now prove that  $c$  is a lid-coloring of  $G$ . Throughout the procedure, the following two properties remain trivially satisfied: (i)  $c$  is a proper coloring of  $G$ , and (ii) no vertex colored  $i$  has a neighbor colored  $i+k+1$ . Consider an edge  $v_i v_j$  of  $G$  with  $N[v_i] \neq N[v_j]$ . We may assume without loss of generality that some neighbors of  $v_i$  are not adjacent to  $v_j$ . If  $i, j \leq k+1$ , then consider the minimum index  $\ell$  such that  $v_\ell$  is a neighbor of  $v_i$

not adjacent to  $v_j$ . By definition of  $c$ , we have  $c(v_j) = c(v_\ell) + k + 1$ . Otherwise we can assume that  $j > i$  and  $j > k + 1$ . Let  $C$  be the neighborhood of  $v_j$  in  $G_j$ . By definition of  $c$ , there exists a  $(k + 1)$ -clique  $C'$  of  $G_{j-1}$  containing  $C$  such that  $c(v_j) = c(v_\ell) + k + 1$ , where  $C' \setminus C = \{v_\ell\}$ . In both cases,  $c(v_\ell) \in c(N[v_i])$  while  $c(v_\ell) \notin c(N[v_j])$  by Property (ii). Hence,  $c$  is a lid-coloring of  $G$ .  $\square$

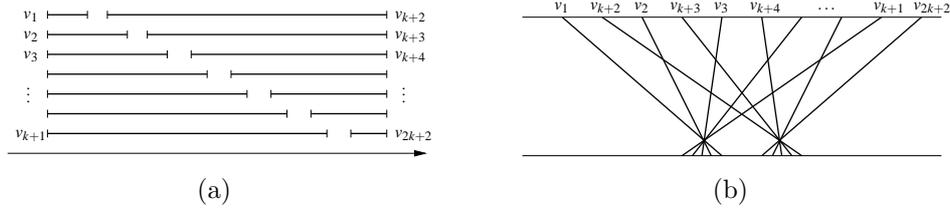


Figure 1: The graph  $P_{2k+2}^k$  as an interval graph (a) and as a permutation graph (b).

Since for fixed  $t$ , the fact that a graph admits a lid-coloring with at most  $t$  colors can be easily expressed in monadic second-order logic, Theorem 7 together with [10] imply that for fixed  $k$ , the lid-chromatic number of a  $k$ -tree can be computed in linear time. Another remark is that for trees, Theorem 7 provides the same 4-lid-coloring as Theorem 2.

For any two integers  $k, \ell \geq 1$ , we define  $P_\ell^k$  as the graph with vertex set  $v_1, \dots, v_\ell$  in which  $v_i$  and  $v_j$  are adjacent whenever  $|i - j| \leq k$ . The graph  $P_{2k+2}^k$  is clearly a  $k$ -tree: it can be constructed from the clique formed by  $v_1, \dots, v_{k+1}$  by adding at each step  $k+2 \leq i \leq 2k+2$  a vertex  $v_i$  adjacent to  $v_{i-k}, \dots, v_{i-1}$ . The graph  $P_{2k+2}^k$  is also an interval graph (see Figure 1(a)) and a permutation graph (see Figure 1(b)). We now prove that the graph  $P_{2k+2}^k$  also provides an example showing that Theorem 7 is best possible.

**Proposition 1** *For any  $k \geq 1$ , we have  $\chi_{lid}(P_{2k+2}^k) = 2k + 2$ .*

**PROOF.** Let  $c$  be a lid-coloring of  $P_{2k+2}^k$ . Without loss of generality we have  $c(v_i) = i$  for each  $1 \leq i \leq k + 1$ . Observe that for any  $1 \leq i \leq k$ , the symmetric difference between  $N[v_i]$  and  $N[v_{i+1}]$  is precisely  $\{v_{i+k+1}\}$ . Therefore,  $c(v_i) > k + 1$  for any  $i \geq k + 2$ . And we can assume that  $c(v_i) = i$  for any  $1 \leq i \leq 2k + 1$ .

Let  $\alpha = c(v_{2k+2})$ , and assume for the sake of contradiction that  $\alpha \neq 2k + 2$ . Since vertices  $v_{k+2}, \dots, v_{2k+2}$  induce a clique, we have  $\alpha \leq k + 1$ . The symmetric difference between  $N[v_{\alpha+k}]$  and  $N[v_{\alpha+k+1}]$  is precisely  $\{v_\alpha\}$  if  $\alpha \geq 2$  and is  $\{v_1, v_{2k+2}\}$  if  $\alpha = 1$ . In both cases,  $c(v_{2k+2}) = c(v_\alpha) = \alpha$  would imply that  $c(N[v_{\alpha+k}]) = c(N[v_{\alpha+k+1}])$ , a contradiction.  $\square$

## 4 Interval graphs

In this section, we prove that the previous example is also extremal for the class of interval graphs.

**Theorem 8** For any interval graph  $G$ ,  $\chi_{lid}(G) \leq 2\omega(G)$ .

PROOF. In this proof the colors are the integers modulo  $2k$ . Let  $G$  be a connected interval graph on  $n$  vertices. We identify the vertices  $v_1, \dots, v_n$  of  $G$  with a family of intervals  $(I_i = [a_i, b_i])_{1 \leq i \leq n}$  such that  $v_i v_j$  is an edge of  $G$  precisely if  $I_i$  and  $I_j$  intersect. We may assume that  $a_1 \leq a_2 \leq \dots \leq a_n$ . Without loss of generality, we can assume that if  $a_i < a_j$  and  $I_i \cap I_j \neq \emptyset$ , then there exists an interval  $I_\ell$  such that  $a_i \leq b_\ell < a_j$ ; otherwise, we can change  $I_j$  to the interval  $[a_i, b_j]$  and the intersection graph remains the same. By a similar argument, we can also assume that if  $b_j < b_i$  and  $I_i \cap I_j \neq \emptyset$ , then there exists an interval  $I_\ell$  such that  $b_j < a_\ell \leq b_i$ .

Let  $\{a_1 = a_{t_1} < a_{t_2} < \dots < a_{t_s}\}$  be the set of starting endpoints. At each step  $i = 1, \dots, s$ , we color all the intervals starting at time  $a_{t_i}$ . We first color the intervals starting at  $a_{t_1}$  with distinct colors in  $\{0, \dots, k-1\}$ . Assume we have colored all the intervals starting strictly before  $a_{t_i}$ . Now, we color all the intervals  $\mathcal{I}(t_i)$  starting at  $a_{t_i}$ . First, we define the following subsets of intervals:

- $\mathcal{V}(t_i)$ : intervals  $I_j$  such that  $a_j < a_{t_i} \leq b_j$ ,
- $\mathcal{U}(t_i)$ : intervals  $I_j$  such that  $a_{t_{i-1}} \leq b_j < a_{t_i}$ ,
- $\mathcal{T}(t_i)$ : intervals  $I_j$  of  $\mathcal{U}(t_i)$  such that there is an interval  $I_\ell$  in  $\mathcal{V}(t_i)$  with  $a_j = a_\ell$ .

Note that  $\mathcal{V}(t_i)$  is the set of intervals that are already colored and intersect  $\mathcal{I}(t_i)$ . The set  $\mathcal{U}(t_i)$  is a subset of intervals already colored that intersect all the intervals of  $\mathcal{V}(t_i)$ . It is not empty (take any interval with rightmost end finishing before  $a_{t_i}$ ). Necessarily, all the intervals of  $\mathcal{U}(t_i)$  have the same end because no interval begins between  $a_{t_{i-1}}$  and  $a_{t_i}$ . Finally, if  $\mathcal{T}(t_i) \neq \emptyset$ , then let  $I_0$  be an interval of  $\mathcal{T}(t_i)$  with leftmost beginning, and otherwise let  $I_0$  be any interval of  $\mathcal{U}(t_i)$ . Let  $c_0$  be the color of  $I_0$ . Note that any interval of  $\mathcal{U}(t_i)$  and  $\mathcal{V}(t_i)$  intersects  $I_0$ , and thus has color  $c_0$  in its neighborhood. We can now color the intervals of  $\mathcal{I}(t_i)$ . We color with color  $c_0 + k$  one of the intervals having the latest end. We color the other intervals with colors in  $\{0, \dots, 2k-1\}$  such that no edge with colors  $(j, j)$  or  $(j, j+k)$  appears (this is always possible since intervals of  $\mathcal{V}(t_i) \cup \mathcal{I}(t_i)$  induce a clique of size at most  $k$ ). This coloring  $c$  is clearly a proper  $2k$ -coloring.

We now show that  $c$  is a lid-coloring of  $G$ . Let  $I_i$  and  $I_j$  be two intersecting intervals with  $N[I_i] \neq N[I_j]$ . Assume first that  $a_i \neq a_j$ . Without loss of generality,  $a_i < a_j$ . During the process, when  $I_j$  is colored, an interval  $I_\ell$  also starting at  $a_j$  is colored with a color  $c_0 + k$  such that  $c_0 \in c(N[I_i])$ . Necessarily,  $I_j \subseteq I_\ell$  since  $I_\ell$  has the rightmost end among all intervals starting at  $a_j$ . So  $c_0 + k \in c(N[I_j])$  but  $c_0 \notin c(N[I_\ell])$  and so  $c_0 \notin c(N[I_j])$ . Hence,  $c(N[I_i]) \neq c(N[I_j])$ . Assume now that  $a_i = a_j$ . Without loss of generality,  $b_j < b_i$  and so  $I_j \subseteq I_i$ . Let  $a_{t_\ell}$  be the leftmost beginning such that  $b_j < a_{t_\ell} \leq b_i$  (it exists because  $N[I_i] \neq N[I_j]$ ). Then we have  $I_i \in \mathcal{V}(t_\ell)$  and  $I_j \in \mathcal{T}(t_\ell)$ . By construction, one of the intervals of  $\mathcal{I}(t_\ell)$ , say  $I$ , will receive color  $c_0 + k$  where  $c_0$  is the color of an interval  $I_0 \in \mathcal{T}(t_\ell)$ . Necessarily,  $I_j \subseteq I_0$  and  $c_0 \in c(N[I_j]) \subset c(N[I_i])$ . We also have  $c_0 + k \in c(N[I_i])$

because  $I_i$  is a neighbor of  $I$ . But  $c_0 + k \notin c(N[I_j])$  since  $c_0 + k \notin c(N[I_0])$  and  $I_j \subseteq I_0$ . Hence,  $c(N[I_i]) \neq c(N[I_j])$ .  $\square$

## 5 Split graphs

A *split graph* is a graph  $G = (K \cup S, E)$  whose vertex set can be partitioned into a clique  $K$  and an independent set  $S$ . In the following, we will always consider partitions  $K \cup S$  with  $K$  of maximum size. A split graph is a chordal graph with clique number and chromatic number  $|K|$ . We prove that it is lid-colorable with  $2|K| - 1$  colors.

We say that a set  $S' \subseteq S$  *discriminates* a set  $K' \subseteq K$  if for any  $u, v \in K'$  with  $N[u] \neq N[v]$ , we also have  $N[u] \cap S' \neq N[v] \cap S'$ . The following theorem is due to Bondy:

**Theorem 9** ([4],[9]) *If  $A_1, A_2, \dots, A_n$  is a family of  $n$  distinct subsets of a set  $\mathcal{A}$  with at least  $n$  elements, then there is a subset  $\mathcal{A}'$  of  $\mathcal{A}$  of size  $n - 1$  such that all the sets  $A_i \cap \mathcal{A}'$  are distinct.*

**Corollary 2** *Let  $G = (K \cup S, E)$  be a split graph. For any  $K' \subseteq K$ , there is a subset  $S'$  of  $S$  of size at most  $|K'| - 1$  such that  $S'$  discriminates  $K'$ .*

PROOF. We apply Theorem 9 with the (at most)  $|K'|$  pairwise distinct sets among  $\{N[v] \cap S \mid v \in K'\}$ .  $\square$

One can easily show that every split graph  $G$  has lid-chromatic number at most  $2|K|$  by giving colors  $1, \dots, |K|$  to the vertices of  $K$ , colors  $|K| + 1, \dots, |K| + k - 1$ , for some  $k \leq |K|$ , to the vertices of a discriminating set  $S' \subseteq S$  of  $K$ , and finally color  $|K| + k$  to the vertices of  $S \setminus S'$ .

We now prove the following stronger result:

**Theorem 10** *Let  $G = (K \cup S, E)$  be a split graph. If  $|K| \neq 2$  or  $G = K_{1,n}$ , then  $\chi_{lid}(G) \leq 2\omega(G) - 1$ .*

PROOF. Assume that  $|K| = k$  and denote the vertices of  $K$  by  $v_1, \dots, v_k$ . If  $k = 1$ , then  $G$  has no edges and it is clear that  $\chi_{lid}(G) \leq 1$ . If  $G = K_{1,n}$ , then  $\chi_{lid}(G) \leq 3$  by Corollary 1. So we can assume that  $k \geq 3$ . If  $|S| \leq k - 1$  or if  $S$  contains a set of size at most  $k - 2$  that discriminates  $K$ , then the result is trivial, so we assume that  $|S| \geq k$  and consider a minimal set  $S_1$  that discriminates  $K$  (therefore  $S_1$  has size precisely  $k - 1$  and there is no edges  $uv$  with  $N[u] = N[v]$ ). We consider two cases.

**Case 1.** There is a vertex  $x \in S \setminus S_1$  of degree  $k - 1$  and a neighbor  $v_i \in K$  of  $x$  such that  $N[v_i] \cap S_1 = \emptyset$ . Without loss of generality, we can assume that  $v_i = v_{k-1}$  and that  $K \setminus N(x) = \{v_k\}$ . Let  $S_x = \{y \in S, N(y) = N(x) = K \setminus \{v_k\}\}$ . We have  $S_x \cap S_1 = \emptyset$  (recall that  $v_{k-1}$  has no neighbor in  $S_1$ ) and by definition of  $S_1$ , for each vertex  $v_i \neq v_{k-1}$ ,  $N[v_i] \cap S_1 \neq \emptyset$  ( $S_1$  is a discriminating set).

Let  $K_1 = K \setminus \{v_{k-1}, v_k\}$ , and let  $S_2$  be a subset of  $S_1$  of size at most  $|K_1| - 1 = k - 3$  that discriminates  $K_1$ . Let  $S' = S \setminus (S_1 \cup S_x)$ . We define a coloring  $c$  as follows:

- for  $1 \leq i \leq k$ ,  $c(v_i) = i$ ;
- assign pairwise distinct colors from  $k + 1, \dots, 2k - 3$  to the vertices of  $S_2$ ;
- for  $u \in S_1 \setminus S_2$ ,  $c(u) = 2k - 2$ ;
- for  $u \in S_x$ ,  $c(u) = 2k - 1$ ;
- for  $u \in S'$ , take  $v_i \in K \setminus N(u)$  ( $v_i$  exists by maximality of  $K$ ), and set  $c(u) = c(v_i)$ .

Then  $c$  is a proper coloring of  $G$ . We show that  $c$  is a lid-coloring of  $G$ . First observe that for each vertex  $v_i$  of  $K$ ,  $c(N[v_i])$  contains one color of  $\{k + 1, \dots, 2k - 1\}$ . Indeed  $2k - 1 \in c(N[v_{k-1}])$  and if  $v_i \neq v_{k-1}$ , then  $N[v_i] \cap S_1 \neq \emptyset$  and therefore  $c(N[v_i]) \cap \{k + 1, \dots, 2k - 2\} \neq \emptyset$ . This implies that for each  $v_i \in K$ ,  $c(N[v_i])$  is distinct from all  $c(N[y])$ ,  $y \in S$ . In fact, either  $c(y) \in c(K)$  and then  $c(N[y]) \subseteq c(K)$ , or  $c(y) \notin c(K)$  but then there is at least one color of  $c(K)$  that  $c(N[y])$  does not contain. Furthermore,  $c(N[v_k])$  is different of all the sets  $c(N[v_i])$  with  $i \neq k$  because  $2k - 1 \in c(N[v_i])$  and  $2k - 1 \notin c(N[v_k])$ . The set  $c(N[v_{k-1}])$  is different of all the sets  $c(N[v_i])$  with  $i \neq k - 1$  because  $c(N[v_{k-1}])$  contains no color of  $c(S_1)$  whereas  $c(N[v_i])$  contains at least one color of this set. Finally,  $c(N[v_i]) \neq c(N[v_j])$  for  $i, j \leq k - 2$  because there is a vertex in  $S_2$  that separates them and its color is used only once. Hence, for each edge  $uv$  of  $G$  such that  $N[u] \neq N[v]$ , we have  $c(N[u]) \neq c(N[v])$ .

**Case 2.** For each vertex  $x$  of  $S \setminus S_1$ , either  $x$  has degree at most  $k - 2$  or  $x$  has degree  $k - 1$  and each vertex of  $N(x)$  has a neighbor in  $S_1$ . We define a coloring  $c$  as follows: vertices of  $K$  are assigned colors  $1, \dots, k$ , and vertices of  $S_1$  are assigned (pairwise distinct) colors within  $k + 1, \dots, 2k - 1$ . For any vertex  $u$  in  $S \setminus S_1$ , take a vertex  $v_i$  in  $K \setminus N(u)$  (such a vertex exists by the maximality of  $K$ ) and set  $c(u) = c(v_i)$ . We claim that  $c$  is a lid-coloring of  $G$ . It is clear that  $c$  is a proper coloring of  $G$ . Let  $uv$  be an edge of  $G$  with  $N[u] \neq N[v]$ . If  $u, v \in K$ , then without loss of generality there is a vertex  $w$  of  $S_1$  such that,  $w \in N[x]$  and  $w \notin N[y]$ . Then,  $c(w) \in c(N[x])$  and  $c(w) \notin c(N[y])$ . Otherwise, without loss of generality,  $u \in K$  and  $v \in S$ . If  $v \in S_1$ , then  $S_1$  does not contain the whole set  $c(K)$  and so  $c(N[u]) \neq c(N[v])$ . Otherwise,  $v \notin S_1$ . If the degree of  $v$  is  $k - 1$ , then  $u$  has a neighbor  $w$  in  $S_1$  and  $c(w) \in c(N[u])$ ,  $c(w) \notin c(N[v])$ . If the degree of  $v$  is at most  $k - 2$ , then there is a color  $1 \leq i \leq k$  such that  $i \in c(N[u])$  and  $i \notin c(N[v])$ . In all cases,  $c(N[u]) \neq c(N[v])$ . Hence,  $c$  is a lid-coloring of  $G$ .  $\square$

Observe that this bound is sharp: the graph obtained from a  $k$ -clique by adding a pendent vertex to each of the vertices of the clique is a split graph and requires  $2k - 1$  colors in any lid-coloring.

## 6 Cographs

A *cograph* is a graph that does not contain the path  $P_4$  on 4 vertices as an induced subgraph. Cographs are a subclass of permutation graphs, and so they are perfect (however, they are

not necessarily chordal). It is well-known that the class of cographs is closed under disjoint union and complementation [5]. Let  $G \cup H$  denote the disjoint union of  $G$  and  $H$ , and let  $G + H$  denote the complete join of  $G$  and  $H$ , i.e. the graph obtained from  $G \cup H$  by adding all possible edges between a vertex from  $G$  and a vertex from  $H$ . A consequence of the previously mentioned facts is that any cograph  $G$  is of one of the three following types:

(S)  $G$  is a single vertex.

(U)  $G = \bigcup_{i=1}^k G_i$  with  $k \geq 2$  and every  $G_i$  is a cograph of type S or J.

(J)  $G = \sum_{i=1}^k G_i$  with  $k \geq 2$  and every  $G_i$  is a cograph of type S or U.

We will use this property to prove the following theorem:

**Theorem 11** *If  $G$  is a cograph, then  $\chi_{lid}(G) \leq 2\omega(G) - 1$ .*

PROOF. Let  $\widetilde{\chi}_{lid}(G)$  be the least integer  $k$  such that  $G$  has a lid-coloring  $c$  with colors  $1, \dots, k$  such that for any vertex  $v$  that is not universal,  $c(N[v]) \neq \{1, \dots, k\}$  (in other words, if a vertex sees all the colors, then it is universal). Such a coloring is called a *strong lid-coloring* of  $G$ . Observe that if  $G$  has a universal vertex, then  $G$  must be of type S or J. We will prove the following result by induction:

*Claim. For any cograph  $G$ ,  $\chi_{lid}(G) \leq 2\omega(G) - 1$  and  $\widetilde{\chi}_{lid}(G) \leq 2\omega(G)$ .*

If  $G$  is a single vertex, then it is universal and therefore  $\widetilde{\chi}_{lid}(G) = \chi_{lid}(G) = 1 = 2 \times 1 - 1$  and the assumption holds.

Assume now that  $G$  is of type J. There exist  $G_1, \dots, G_k$ ,  $k \geq 2$ , each of type S or U, such that  $G = \sum_{i=1}^k G_i$ . Let  $G_1, \dots, G_s$  ( $0 \leq s \leq k$ ) be of type S and  $G_{s+1}, \dots, G_k$  be of type U. Consider a lid-coloring  $c_1$  of  $G_1$  and a strong lid-coloring  $c_i$  of  $G_i$  for  $2 \leq i \leq k$ , such that the sets of colors within  $G_i$  and  $G_j$ ,  $i \neq j$ , are disjoint. Then the coloring  $c$  of  $G$  defined by  $c(v) = c_i(v)$  for any  $v \in G_i$  is a lid-coloring of  $G$ . To see this, assume two adjacent vertices  $u$  and  $v$  such that  $N[u] \neq N[v]$  and  $c(N[u]) = c(N[v])$ . Since every  $c_i$  is a lid-coloring of  $G_i$  the vertices  $u$  and  $v$  must be in different  $G_i$ 's, say  $u \in G_i$  and  $v \in G_j$ ,  $i < j$ . But then in order to have  $c(N[u]) = c(N[v])$ ,  $u$  and  $v$  must see all the colors in  $c_i$  and  $c_j$ , respectively. Since  $c_j$  is a strong lid-coloring of  $G_j$ ,  $v$  is universal in  $G_j$ . This means that  $G_j$  (and therefore  $G_i$ ) is of type S. Hence,  $u$  and  $v$  are universal in  $G$  and  $N[u] = N[v]$ , a contradiction. As a consequence  $c$  is a lid-coloring of  $G$ .

If  $c_1$  is a strong coloring of  $G_1$ , then  $c$  is a strong lid-coloring of  $G$ : take a vertex  $v \in G_i$  that sees all the colors in  $c$ . Then it also sees all the colors in  $c_i$ , so it is universal in  $G_i$  and  $G$ .

So we have  $\chi_{lid}(G) \leq \chi_{lid}(G_1) + \sum_{i=2}^k \widetilde{\chi}_{lid}(G_i)$  and  $\widetilde{\chi}_{lid}(G) \leq \sum_{i=1}^k \widetilde{\chi}_{lid}(G_i)$ . Since  $\omega(G) = \sum_{i=1}^k \omega(G_i)$  we have by induction:

$$\chi_{lid}(G) \leq 2\omega(G_1) - 1 + \sum_{i=2}^k (2\omega(G_i)) = 2 \times \sum_{i=1}^k \omega(G_i) - 1 = 2\omega(G) - 1$$

and

$$\widetilde{\chi}_{lid}(G) \leq \sum_{i=1}^k (2\omega(G_i)) = 2\omega(G).$$

Assume now that  $G$  is of type U. There exist  $G_1, \dots, G_k$ ,  $k \geq 2$ , each of type S or J, such that  $G = \bigcup_{i=1}^k G_i$ . Consider a lid-coloring  $c_i$  of  $G_i$  with colors  $1, \dots, \chi_{lid}(G_i)$ . Without loss of generality we have  $\chi_{lid}(G_1) = \max_{i=1}^k \chi_{lid}(G_i)$ . The coloring  $c$  of  $G$  defined by  $c(v) = c_i(v)$  for any  $v \in G_i$  is clearly a lid-coloring of  $G$ , and so  $\chi_{lid}(G) = \max_{i=1}^k \chi_{lid}(G_i)$ .

To obtain a strong lid-coloring, assign a new color  $\chi_{lid}(G_1)+1$  to all the vertices colored 1 in  $G_1$ , and color all the other vertices of  $G$  as they were colored in  $c$ . The coloring  $c'$  obtained is still a lid-coloring of  $G$ . Since no vertex  $u$  satisfies  $c(N[u]) = \{1, \dots, \chi_{lid}(G_1) + 1\}$  (the vertices in  $G_1$  miss color 1, while the others miss color  $\chi_{lid}(G_1) + 1$ ),  $c'$  is also a strong lid-coloring of  $G$ . Therefore  $\widetilde{\chi}_{lid}(G) \leq \max_{i=1}^k \chi_{lid}(G_i) + 1$ . Since  $\omega(G) = \max_{i=1}^k \omega(G_i)$  we have by induction

$$\chi_{lid}(G) \leq \max_{i=1}^k (2\omega(G_i) - 1) = 2\omega(G) - 1$$

and

$$\widetilde{\chi}_{lid}(G) \leq \max_{i=1}^k (2\omega(G_i) - 1) + 1 = 2\omega(G).$$

□

The bound of Theorem 11 is tight. The following construction gives an example of cographs of clique number  $\omega$  requiring  $2\omega - 1$  colors in any lid-coloring. For any  $k \geq 1$ , take a complete graph with vertex set  $v_1, \dots, v_k$  and for each  $2 \leq i \leq k$  add a vertex  $u_i$  such that  $N(u_i) = \{v_i, v_{i+1}, \dots, v_k\}$ . This graph is a cograph with clique number  $k$ , the vertices  $u_i$  form an independent set  $U$ , and every vertex  $v_i$  satisfies  $N(v_i) \cap U = \{u_2, \dots, u_i\}$ . Let  $c$  be a lid-coloring of this graph, then for any  $3 \leq i \leq k$  the vertex  $u_i$  must be assigned a color distinct from  $c(u_2), \dots, c(u_{i-1})$  and  $c(v_1), \dots, c(v_k)$  since otherwise we would have  $c(N[v_i]) = c(N[v_{i-1}])$ . Hence, at least  $k + (k - 1) = 2k - 1$  distinct colors are required.

Since cographs have bounded clique-width (indeed, these are exactly the graphs with clique-width at most two), Theorem 11 together with [10] imply that the lid-chromatic number of a cograph of bounded clique number can be computed in linear time.

Given the results in Sections 2 to 6, it seems natural to conjecture that every perfect graph  $G$  has lid-chromatic number at most  $2\chi(G)$ . This is not true, however, as the following example shows. Take three stable sets  $S_1, S_2, S_3$ , each of size  $k$ , add all possible edges between  $S_1$  and  $S_2$ , add a perfect matching between  $S_1$  and  $S_3$ , and add the complement of a perfect matching between  $S_2$  and  $S_3$ . The obtained graph  $G_k$  is perfect: since the subgraph of  $G_k$  induced by  $S_1$  and  $S_2$  is a complete bipartite graph,  $G_k$  is bipartite if and only if it does not have a triangle, and is 3-colorable otherwise.

Consider a lid-coloring  $c$  of  $G_k$ , and a vertex  $x_2$  of  $S_2$ . Let  $x_3$  be the only vertex of  $S_3$  that is not adjacent to  $x_2$ , and  $x_1$  be the unique neighbor of  $x_3$  in  $S_1$ . Observe that  $N[x_1] = N[x_3] \cup \{x_2\}$ . Since  $c(N[x_1]) \neq c(N[x_3])$ , the color of  $x_2$  appears only once in  $S_2$ . Hence, all the vertices of  $S_2$  have distinct colors and it follows that  $\chi_{lid}(G_k) \geq k + 2$ , whereas  $\chi(G_k) = \omega(G_k) = 3$ .

## 7 Graphs with bounded maximum degree

**Proposition 2** *If a graph  $G$  has maximum degree  $\Delta$ , then  $\chi_{lid}(G) \leq \Delta^3 - \Delta^2 + \Delta + 1$ .*

PROOF. Let  $c$  be a coloring of  $G$  so that vertices at distance at most three in  $G$  have distinct colors. Since every vertex has at most  $\Delta^3 - \Delta^2 + \Delta$  vertices at distance at most three, such a coloring using at most  $\Delta^3 - \Delta^2 + \Delta + 1$  colors exists. Let  $uv$  be an edge of  $G$ . Let  $N_u$  be the set of neighbors of  $u$  not in  $N[v]$  and  $N_v$  be the set of neighbors of  $v$  not in  $N[u]$ . Using that vertices at distance at most two in  $G$  have distinct colors, we obtain that all the elements of  $N_u$  (resp.  $N_v$ ) have distinct colors. Since vertices at distance at most three have distinct colors, the set of colors of  $N_u$  and  $N_v$  are disjoint. If  $N[u] \neq N[v]$ , then  $N_u \cup N_v \neq \emptyset$ , and  $c(N[u]) \neq c(N[v])$  by the previous remark.  $\square$

We believe that this result is not optimal, and that the bound should rather be quadratic in  $\Delta$ :

**Question 1** *Is it true that for any graph  $G$  with maximum degree  $\Delta$ , we have  $\chi_{lid}(G) = O(\Delta^2)$ ?*

If true, then this result would be best possible. Take a projective plane  $P$  of order  $n$ , for some prime power  $n$ . Let  $G_{n+1}$  be the graph obtained from the complete graph on  $n + 1$  vertices by adding, for every vertex  $v$  of the clique, a vertex  $v'$  adjacent only to  $v$ . Note that in any lid-coloring of  $G_{n+1}$ , all vertices  $v'$  must receive distinct colors. For any line  $l$  of the projective plane  $P$ , consider a copy  $G_{n+1}^l$  of  $G_{n+1}$  in which the new vertices  $v'$  are indexed by the  $n + 1$  points of  $l$ . For any point  $p$  of  $P$ , identify the  $n + 1$  vertices indexed  $p$  in the graphs  $G_{n+1}^l$ , where  $p \in l$ , into a single vertex  $p^*$ . The resulting graph  $H_{n+1}$  is  $(n + 1)$ -regular and has  $(2n + 1)(n + 2)$  vertices. By construction, all the vertices  $p^*$ ,  $p \in P$ , have distinct colors in any lid-coloring. Hence, at least  $n^2 + n + 1 = \Delta^2 - \Delta + 1$  colors are required in any lid-coloring of this  $\Delta$ -regular graph. The 3-regular graph  $H_3$  with  $\chi_{lid}(H_3) \geq 7$  is depicted in Figure 2.

We saw that the lid-chromatic number cannot be upper-bounded by the chromatic number. For a graph  $G$ , the square of  $G$ , denoted by  $G^2$ , is the graph with the same vertex set as  $G$ , in which two vertices are adjacent whenever they are at distance at most two in  $G$ . The following question is somehow related to the previous one (depending of the possible linearity of  $f$ ).

**Question 2** *Does there exist a function  $f$  so that for any graph  $G$ , we have  $\chi_{lid}(G) \leq f(\chi(G^2))$ ?*

## 8 Planar and outerplanar graphs

This section is devoted to graphs embeddable in the plane. A maximal outerplanar graph is a 2-tree and so is 6-lid colorable by Theorem 7. However,  $\chi_{lid}$  is not monotone under

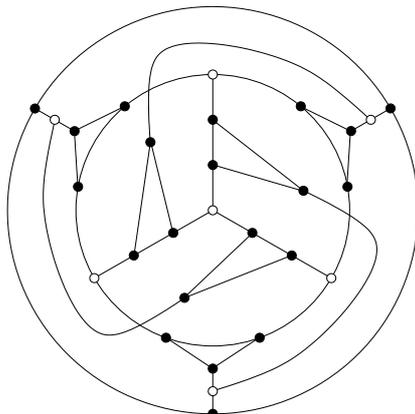


Figure 2: In any lid-coloring of the 3-regular graph  $H_3$ , the seven white vertices must receive pairwise distinct colors.

taking subgraphs and so this result does not extend to all outerplanar graphs. So we have to use a different strategy to give an upper bound of the lid-chromatic number on the class of outerplanar graphs.

**Theorem 12** *Every outerplanar graph is 20-lid-colorable.*

PROOF. Let  $G$  be an outerplanar graph embedded in the plane such that all the vertices lie in the outer face. If  $G$  is not connected, then we can color each connected component independently, so we may assume that  $G$  is connected.

Let  $x_0$  be a vertex of minimum degree in  $G$  ( $x_0$  has degree at most two). For any  $i \geq 1$ , let  $L_i$  be the set of vertices at distance  $i$  from  $x_0$ . The embedding of  $G$  induces a cyclic order on the neighbors of every vertex, and these cyclic orders inductively define an order  $x_i^1, \dots, x_i^{k_i}$  on the vertices of  $L_i$ . In  $L_i$ , there may be only edges between two consecutive vertices  $x_i^j$  and  $x_i^{j+1}$  (this implies that  $L_i$  induces a disjoint union of paths). Furthermore, a vertex of  $L_i$  has at most two neighbors in  $L_{i-1}$  and two vertices  $x_i^j$  and  $x_i^{j'}$  of  $L_i$  have no common neighbor in  $L_{i+1}$  if  $j' \neq j+1$  and at most one if  $j' = j+1$ .

For  $i \in \{1, 2, 3, 4\}$ , let  $C_i$  be the set of 5 integers  $\{1 + 5(i-1), \dots, 5i\}$ . We start by coloring  $x_0$  with color 1, and mark the vertex  $x_1^{k_1}$ . We then apply Algorithm 1.

It is easy to check that this algorithm provides a proper coloring  $c$  of  $G$  with 20 colors such that for any  $i$ ,  $c(L_i) \subseteq C_{i \bmod 4}$ . Furthermore, for any vertex  $x_i^j$ ,  $|c(N(x_i^j)) \cap L_{i+1}| \leq 4$ . Let us call *forbidden color of  $x_i^j$*  the color of  $C_{i+1 \bmod 4}$  that does not appear in  $N(x_i^j) \cap L_{i+1}$ . Then the forbidden color of  $x_i^j$  is in  $c(N[x_i^{j-1}])$  and in  $c(N[x_i^{j+1}])$  if possible (i.e. if those vertices have a neighbor in  $L_{i+1}$  not adjacent to  $x_i^j$ ).

Let us prove that the coloring given by the algorithm is locally identifying. Let  $uv$  be an edge of  $G$  such that  $N[u] \neq N[v]$ . If  $uv$  is not an edge of a layer  $L_i$ , then we can assume that  $u \in L_i$  and  $v \in L_{i+1}$ . If  $u \neq x_0$ , then there is a neighbor  $t$  of  $u$  in  $L_{i-1}$  and

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**Algorithm 1** Lid-coloring of outerplanar graphs

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```
1: for all  $L_i$  do
2:   for all vertex  $x_i^j$  in  $L_i$  do
3:     mark, if it exists, the last neighbor in  $L_{i+1}$  of  $x_i^j$ 
4:   end for
5:    $c_1 \leftarrow 1 + 5 \times (i \bmod 4)$ 
6:    $c_\emptyset \leftarrow 5 \times (i \bmod 4)$ 
7:   repeat
8:     begin with color  $c_1$  and color  $L_i$  in order until the next marked vertex  $x_i^j$  by
       alternating colors of  $C_{i \bmod 4} \setminus \{c_\emptyset\}$  (following the pattern  $c_1 c_2 c_3 c_4 c_1 \dots$ )
9:      $c_1 \leftarrow c_\emptyset$ 
10:     $c_\emptyset \leftarrow c_{x_i^{j-1}}$ 
11:   until  $L_i$  is completely colored
12: end for
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then  $c(t) \notin c(N[v])$ . So we may assume that  $u = x_0$ . The vertex  $v$  has degree at least 2, otherwise  $N[u] = N[v]$ , so there is a neighbor  $t \neq u$  of  $v$ . If  $t \in L_1$  then there is another neighbor  $t'$  of  $v$  in  $L_2$  (because  $N[u] \neq N[v]$ ). So we can assume that  $t \in L_2$  and then  $c(t) \notin c(N[u])$ . So in all cases,  $c(N[u]) \neq c(N[v])$ .

Assume now that  $u, v \in L_i$  for some  $i$ . Without loss of generality, we may assume that  $u = x_i^j$ ,  $v = x_i^{j+1}$  for some  $j$  and that there is a vertex  $t \in N(u) \setminus N(v)$ . If  $t \in L_i$ , then we are done because 4 consecutive vertices have different colors in  $L_i$ . If  $t \in L_{i-1}$ , then  $v$  has at most two neighbors in  $L_{i-1}$ . Those two neighbors are just following  $t$  in the layer  $L_{i-1}$  and so  $c(t) \notin c(N[v])$ . Otherwise,  $t \in L_{i+1}$ , we can choose  $t$  to be the last vertex of  $L_{i+1}$  that is in  $N[u]$  but not in  $N[v]$ . Then  $c(t) \notin c(N[v])$  because it is the forbidden color of  $v$  in  $L_{i+1}$ . So  $c$  is a lid-coloring of  $G$ .  $\square$

We believe that this bound is far from tight.

**Question 3** *Is it true that every outerplanar graph  $G$  satisfies  $\chi_{lid}(G) \leq 6$ ?*

We now prove that sparse enough planar graphs have low lid-chromatic number.

**Theorem 13** *If  $G$  is a planar graph with girth at least 36, then  $\chi_{lid}(G) \leq 5$ .*

**PROOF.** Let us call *nice* a lid-coloring  $c$  using at most 5 colors such that every vertex  $v$  with degree at least 2 satisfies  $|c(N[v])| = 3$ . We show that every planar graph with girth at least 36 admits a nice lid-coloring.

Suppose that  $G$  is a planar graph with girth at least 36 that does not admit a nice lid-coloring and with the minimum number of vertices. Let us first show that  $G$  does not contain a vertex of degree at most 1. The case of a vertex of degree 0 is trivial, so suppose that  $G$  contains a vertex  $u$  of degree 1 adjacent to another vertex  $v$ . By minimality of  $G$ , the graph  $G' = G \setminus u$  admits a nice lid-coloring  $c$ . We consider three cases according

to the degree of  $v$  in  $G'$ , and in all three cases, we extend  $c$  to a nice lid-coloring of  $G$  in order to obtain a contradiction. If  $v$  has degree at least 2 in  $G'$ , then we assign to  $u$  a color in  $c(N[v]) \setminus \{c(v)\}$ . So  $c(N[v])$  is unchanged, and  $c(N[u]) \neq c(N[v])$  since  $|c(N[u])| = 2$  and  $|c(N[v])| = 3$ . We thus have a nice lid-coloring of  $G$ . If  $v$  has degree 1 in  $G'$ , then  $v$  is adjacent to another vertex  $w$  in  $G'$  and we assign to  $u$  a color that does not belong to  $c(N[w])$ . Such a color exists since  $|c(N[w])| = 3$  and the obtained coloring of  $G$  is nice:  $|c(N[v])| = 3$  and  $c(N[v]) \neq c(N[w])$  since  $c(u) \in c(N[v])$  but  $c(u) \notin c(N[w])$ . If  $v$  has degree 0 in  $G'$ , then  $N[u] = N[v]$  in  $G$ , so  $u$  and  $v$  do not need to be identified.

It follows that  $G$  has minimum degree at least 2. It is well-known that if the girth of a planar graph is at least  $5k + 1$ , then it contains either a vertex of degree at most 1, or a path consisting of  $k$  consecutive vertices of degree 2. The graph  $G$  thus contains a path of 7 vertices of degree 2. So we can assume that  $G$  contains a path  $P = x_1x_2 \dots x_9$  such that the degree of  $x_1$  is at least 3, the degree of  $x_2, \dots, x_8$  is exactly 2, and the degree of  $x_9$  is at least 2. By minimality of  $G$ , the graph  $G' = G \setminus \{x_2, x_3, \dots, x_8\}$  admits a nice lid-coloring  $c$ . Without loss of generality, assume that  $c(x_1) = 1$  and  $c(N[x_1]) = \{1, 2, 3\}$ , since  $x_1$  has degree at least 2 in  $G'$ . We note  $a = c(x_9)$ . If the degree of  $x_9$  in  $G'$  is at least 2, then we note  $\{b, c\} = c(N(x_9))$ . If the degree of  $x_9$  in  $G'$  is 1, then  $x_9$  is adjacent to a vertex  $x_{10}$  and we note  $\{b, c\} = \{1, 2, 3, 4, 5\} \setminus c(N[x_{10}])$ . The following table gives the colors of  $x_2, x_3, \dots, x_8$  for all the possible values of  $(a; b, c)$ . We can thus extend  $c$  to a nice lid-coloring of  $G$ , a contradiction.

2431243 (1;2,3)	2431543 (2;1,3)	2431542 (3;1,2)	2534152 (4;1,2)	2435142 (5;1,2)
2431254 (1;2,4)	2541354 (2;1,4)	2431254 (3;1,4)	2431253 (4;1,3)	2431243 (5;1,3)
2431245 (1;2,5)	2451345 (2;1,5)	2431245 (3;1,5)	2451235 (4;1,5)	2435124 (5;1,4)
2431253 (1;3,4)	2531453 (2;3,4)	2431254 (3;2,4)	2431253 (4;2,3)	2431243 (5;2,3)
2431243 (1;3,5)	2431543 (2;3,5)	2431245 (3;2,5)	2451235 (4;2,5)	2435214 (5;2,4)
2431524 (1;4,5)	3421534 (2;4,5)	2431524 (3;4,5)	2431523 (4;3,5)	2435123 (5;3,4)

□

We conjecture that planar graphs have bounded lid-chromatic number.

## 9 Connectivity and lid-coloring

Most of the proofs we gave in this article heavily depend on the structure of the classes of graphs we were considering. We now give a slightly more general tool, allowing us to extend results on the 2-connected components of a graph to the whole graph:

**Theorem 14** *Let  $k$  be an integer and  $G$  be a graph such that every 2-connected component of  $G$  is  $k$ -lid-colorable. Let  $H$  be the subgraph of  $G$  induced by the cut-vertices of  $G$ . Then  $\chi_{lid}(G) \leq k + \chi(H)$ .*

PROOF. In this proof, we will consider two different colorings of the vertices: the lid-coloring of the vertices of  $G$  and the proper coloring of the graph induced by the cut vertices. To avoid confusion, we call *type* the color of a cut vertex in the second coloring. We prove the following stronger result:

*Claim: If  $t$  is a proper coloring of  $H$  with colors  $t_1, \dots, t_h$ , then  $G$  admits a  $(k + h)$ -lid-coloring  $c$  such that for each maximal 2-connected component  $C$  of  $G$ , (\*) there are  $h$  colors not appearing in  $c(C)$ , say  $c_1^C, \dots, c_h^C$ , such that for every cut-vertex  $v$  of  $G$  lying in  $C$ , if  $t(v) = t_i$ , then  $c(N(v))$  contains  $c_i^C$  but none of the  $c_j^C$ ,  $j \neq i$ .*

We prove the claim by induction on the number of cut-vertices of  $G$ . We may assume that  $G$  has a cut-vertex, otherwise the property is trivially true.

Let  $t$  be a proper coloring of  $H$  with colors  $t_1, \dots, t_h$ . Let  $u$  be a cut-vertex of  $G$  and let  $C_1, \dots, C_s$  be the connected components of  $G - u$ . We can choose  $u$  so that at most one of the  $C_i$ 's, say  $C_1$ , contains the cut-vertices. Let  $G' = C_1 + u$  and let  $C$  be the maximal 2-connected component of  $G'$  containing  $u$ . Observe that the vertex  $u$  is not a cut-vertex in  $G'$ . By the induction,  $G'$  has a  $(k + h)$ -lid-coloring  $c$  such that  $c(C) \subseteq \{1, \dots, k\}$  and every cut-vertex  $v$  of  $C$  with  $t(v) = t_i$  has a neighbor colored  $k + i$ , but no neighbor colored  $k + j$ ,  $j \neq i$ . We can also assume that  $t(u) = t_1$  and  $1 \in c(N[u])$  but  $c(u) \neq 1$ .

We now extend the coloring  $c$  to  $G$  by lid-coloring each component  $C_2 + u, \dots, C_s + u$  with colors  $2, 3, \dots, k + 1$  such that  $k + 1 \in c(N[u])$ . Let us prove that the coloring obtained is a lid-coloring of  $G$  satisfying (\*). In order to prove that  $c$  is a lid-coloring, by the induction one just needs to check that  $u$  has no neighbor  $v$  with  $c(N[v]) = c(N[u])$ . For the sake of contradiction, suppose that such a vertex  $v$  exists. Since  $1 \in c(N[u])$ ,  $v$  has to lie in  $C$ . If  $v$  is a cut-vertex of  $G'$ , then  $t(v) \neq t_1$  ( $t$  is a proper coloring of  $H$ ) and by the induction,  $k + 1 \notin c(N[v])$ . If  $v$  is not a cut-vertex of  $G'$ , then all its neighbors lie in  $C$  and again,  $k + 1 \notin c(N[v])$ . Since  $k + 1 \in c(N[u])$ , we obtain a contradiction.

It remains to prove that (\*) holds for every maximal 2-connected component of  $G$ . It clearly does for  $C_2 + u, \dots, C_s + u$ , since  $u$  is the only cut-vertex of  $G$  that they contain and  $1 \in c(N[u]) \subseteq \{1, \dots, k + 1\}$ , while none of these components contains color 1 or color  $k + i$  with  $2 \leq i \leq h$ . The component  $C$  also satisfies (\*), since  $u$  has a neighbor colored  $k + 1$  and no neighbor colored  $k + i$  with  $2 \leq i \leq h$ . By the induction, Property (\*) trivially holds for the remaining maximal 2-connected components of  $G$ . This completes the proof of the claim.  $\square$

Among others, this result can be used to prove that outerplanar graphs without triangles can be 8-lid-colored. We omit the details; we suspect that Theorem 14 can be used to prove results on much wider classes of graphs.

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