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SPECTRAL TRIPLES FOR FINITELY GENERATED GROUPS

SÉBASTIEN PALCOUX

ABSTRACT. Using Cayley graphs and Clifford algebras, we are able to give, for every finitely generated groups, a uniform construction of spectral triples with a generically non-trivial phase for the Dirac operator. We are naturally led to a classification of such groups into three types.

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1. INTRODUCTION

In this paper, we define spectral triples for every finitely generated groups such that the phase of the Dirac operator is generically non-trivial. We just generalize the unbounded version of the construction of the Fredholm module for the free group given by Connes [1] and M. Pimsner-Voiculescu [5]. For so, we use the Clifford algebra in the same spirit that Julg-Valette do in [4]. We also use topics in geometric group theory (see [3]) as the Cayley graph. We start giving a first construction running for a particular class of group called \mathcal{C} ; then for the general case, we homogenize the construction. The idea is that the more we need to homogenize, the more it increases the chances that the Dirac phase is trivial. We are then led to classify the finitely generated groups into three types A_0 , A_λ and A_1 according to the need of homogenization; \mathcal{C} is a particular class of type A_0 .

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2. BASIC DEFINITIONS

Definition 2.1. A spectral triple (\mathcal{A}, H, D) is given by a unital \star -algebra \mathcal{A} represented on the Hilbert space H , and an unbounded operator D , called the Dirac operator, such that:

- (1) D is self-adjoint.
- (2) $(D^2 + I)^{-1}$ is compact.
- (3) $\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$ is dense in \mathcal{A} .

See the article [6] of G. Skandalis, dedicated to A. Connes and spectral triple.

Definition 2.2. A group Γ is finitely generated if it exists a finite generating set $S \subset \Gamma$. We always take S equals to S^{-1} and the identity element $e \notin S$.

We can also defined a group by generators and relations: $\Gamma = \langle S \mid R \rangle$.

3. GEOMETRIC CONSTRUCTION

Definition 3.1. Let $\ell : \Gamma \rightarrow \mathbb{N}$ be the word length related to S . Let \mathcal{G} be the Cayley graph with the distance $d(g, h) = \ell(gh^{-1})$, see [3].

Definition 3.2. Let $p : \Gamma \rightarrow \mathbb{N}$ with $p(g)$ the number of geodesic paths from g to the identity element e , on the Cayley graph \mathcal{G} .

Definition 3.3. Let $p_s : \Gamma \rightarrow \mathbb{N}$ with $p_s(g)$ the number of geodesic paths from g to e , starting by $s \in S$. Then $\sum_{s \in S} p_s = p$.

Remark 3.4. If $p_s(g) \neq 0$ then $p_s(g) = p(gs)$.

Definition 3.5. Let $\ell^2(\Gamma)$ be the canonical Hilbert space of base $(e_g)_{g \in \Gamma}$ and let ∂_s ($s \in S$) be the unbounded operator defined by:

$$\partial_s \cdot e_g = \frac{p_s(g)}{p(g)} \ell(g) e_g.$$

∂_s is a diagonal positive operator, so $\partial_s^* = \partial_s$ and $\partial_s \partial_{s'} = \partial_{s'} \partial_s$.

Example 3.6. Let $\Gamma = \mathbb{Z}^2$, $S = \{a, a^{-1}, b, b^{-1}\}$, $R = \{aba^{-1}b^{-1}\}$.

Let $g = a^n b^m = (n, m)$ with $n, m \in \mathbb{N}$, then $\ell(g) = n + m$, $p(g) = C_{n+m}^n$, $p_{a^{-1}}(g) = C_{n+m-1}^{n-1}$, $p_{b^{-1}}(g) = C_{n+m-1}^{m-1}$, and so $\frac{C_{n+m-1}^{n-1}}{C_{n+m}^n}(n+m) = n$.

Let $\partial_1 = \partial_{a^{-1}}$ and $\partial_2 = \partial_{b^{-1}}$, then:

$$\partial_1 e_{(n,m)} = n e_{(n,m)} \quad \text{and} \quad \partial_2 e_{(n,m)} = m e_{(n,m)},$$

as for the canonical derivations !

Example 3.7. Let $\Gamma = \mathbb{F}_2$, $S = \{a, a^{-1}, b, b^{-1}\}$, $R = \emptyset$. Let $g \in \mathbb{F}_2$, $g \neq e$, then $p(g) = 1$, let $s \in S$ with $p_s(g) = 1$, then $\partial_s \cdot e_g = \ell(g) \cdot e_g$.

4. CLIFFORD ALGEBRA

We quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [7].

Definition 4.1. Let $\Lambda(\mathbb{R}^S)$ be the exterior algebra equals to $\bigoplus_{k=0}^{2d} \Lambda^k(\mathbb{R}^d)$, with $2d = \text{card}(S)$ and $\Lambda^0(\mathbb{R}^d) = \mathbb{R}\Omega$. We called Ω the vacuum vector. Recall that $v_1 \wedge v_2 = -v_2 \wedge v_1$ so that $v \wedge v = 0$.

Definition 4.2. Let a_v be the creation operator on $\Lambda(\mathbb{R}^S)$ defined by:

$$a_v(v_1 \wedge \dots \wedge v_r) = v \wedge v_1 \wedge \dots \wedge v_r \text{ and } a_v(\Omega) = v$$

Reminder 4.3. The dual a_v^* is called the annihilation operator, then:

$$a_v^*(v_1 \wedge \dots \wedge v_r) = \sum_{i=0}^r (-1)^{i+1} (v, v_i) v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_r \text{ and } a_v^*(\Omega) = 0$$

Reminder 4.4. Let $c_v = a_v + a_v^*$, then $c_v = c_v^*$ and $c_v c_w + c_w c_v = 2(v, w)I$.

Definition 4.5. The operators c_v generate the Clifford algebra $\text{Cliff}(\mathbb{R}^S)$. Note that the operators c_v are bounded and that $\text{Cliff}(\mathbb{R}^S).\Omega = \Lambda(\mathbb{R}^S)$.

Remark 4.6. \mathbb{R}^S admits the orthonormal basis $(v_s)_{s \in S}$. We will write c_s instead of c_{v_s} , so that $[c_s, c_{s'}]_+ = 2\delta_{s,s'}I$.

5. CLASS \mathcal{C} AND DIRAC OPERATOR

Definition 5.1. Let H be the Hilbert space $\Lambda(\mathbb{R}^S) \overline{\otimes} \ell^2(\Gamma)$, we define a self-adjoint operator D on a dense domain of H by:

$$D = \sum_{s \in S} c_s \otimes \partial_s.$$

Lemma 5.2. $D^2 = I \otimes (\sum_{s \in S} \partial_s^2)$.

Proof. $D^2 = \sum_{s,s' \in S} c_s c_{s'} \otimes \partial_s \partial_{s'} = \frac{1}{2} \sum (c_s c_{s'} + c_{s'} c_s) \otimes \partial_s \partial_{s'} = I \otimes (\sum \partial_s^2)$ \square

Lemma 5.3. $(D^2 + I)^{-1}$ is compact.

Proof. First of all $\dim(\Lambda(\mathbb{R}^S)) = 2^{2d} < \infty$.

Next $\sum_{s \in S} \frac{p_s(g)}{p(g)} = 1$, then it exists $s_o \in S$ such that $\frac{p_{s_o}(g)}{p(g)} \geq \frac{1}{2d}$.

Now $\sum_{s \in S} [\ell(g) \frac{p_s(g)}{p(g)}]^2 \geq [\ell(g) \frac{p_{s_o}(g)}{p(g)}]^2 \geq [\frac{\ell(g)}{2d}]^2$. \square

Definition 5.4. Let \mathcal{C} be the class of finitely generated group Γ such that it exists a finite generating set $S \subset \Gamma$ (with $S = S^{-1}$ and $e \notin S$) such that $\forall s \in S$ and $\forall g, h \in \Gamma$ (with $h \neq e$):

$$\left| \frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)} \right| \leq \frac{K_g}{\ell(h)}$$

with $K_g \in \mathbb{R}_+$ depending only on g .

Examples 5.5. *The class \mathcal{C} is stable by direct or free product, it contains \mathbb{Z}^n , \mathbb{F}_n , the finite groups, and probably every amenable or automatic groups (containing the hyperbolic groups, see [2]).*

Warning 5.6. *From now, the group Γ is in the class \mathcal{C} .*

Definition 5.7. *Let $\mathcal{A} = C_r^*\Gamma$, acting on H by $I \otimes u_g$, $g \in \Gamma$.*

Proposition 5.8. *$\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$ is dense in \mathcal{A} .*

Proof. $[D, u_g](\Omega \otimes e_h) = \dots = (\sum_{s \in S} [\ell(gh) \frac{p_s(gh)}{p(gh)} - \ell(h) \frac{p_s(h)}{p(h)}] v_s) \otimes e_{gh}$.

Now $|\ell(gh) \frac{p_s(gh)}{p(gh)} - \ell(h) \frac{p_s(h)}{p(h)}| \leq \ell(gh) |\frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)}| + |\ell(gh) - \ell(h)| \frac{p_s(h)}{p(h)}$
 $\leq K_g \frac{\ell(gh)}{\ell(h)} + \ell(g) \leq K_g(\ell(g) + 1) + \ell(g)$. It's bounded in h .

Now let $v \in \Lambda(\mathbb{R}^S)$ then it exists $X \in \text{Cliff}(\mathbb{R}^S)$ with $v = X\Omega$. By linearity we can restrict to $X = c_{s_1} \dots c_{s_r}$ with $s_i \in S$ and $i \neq j$ implies $s_i \neq s_j$.

Warning, the following commutant $[\cdot, X]$ is a graded commutant:

$$[D, u_g](v \otimes e_h) = ([D, u_g], X) + (-1)^r X [D, u_g](\Omega \otimes e_h)$$

$$[[D, u_g], X] = [[D, X], u_g]$$

$$[D, X] = [\sum_{s \in S} c_s \otimes \partial_s, X] = \sum_{s \in S} [c_s, X] \otimes \partial_s$$

$$[c_s, X] = 2 \sum (-1)^{i+1} \delta_{s, s_i} c_{s_1} \dots c_{s_{i-1}} c_{s_{i+1}} \dots c_{s_r}, \quad s_i \neq s_j \text{ if } i \neq j, \text{ then:}$$

$$[D, X] = \sum X_i \otimes \partial_{s_i} \text{ with } X_i = (-1)^{i+1} 2 c_{s_1} \dots c_{s_{i-1}} c_{s_{i+1}} \dots c_{s_r}$$

$$[[D, X], u_g](\Omega \otimes e_h) = \sum [\ell(gh) \frac{p_{s_i}(gh)}{p(gh)} - \ell(h) \frac{p_{s_i}(h)}{p(h)}] X_i \Omega \otimes e_{gh}$$

Now X and X_i are bounded; the result follows. \square

Theorem 5.9. *(\mathcal{A}, H, D) is a spectral triple; the phase of D is generically non-trivial.*

Proof. We use lemma 5.3 and proposition 5.8. The phase is generically non-trivial, because it's non-trivial for $\Gamma = \mathbb{Z}^2$ or \mathbb{F}_2 . \square

6. HOMOGENIZATION AND GENERAL CASE

The class \mathcal{C} doesn't contain every finitely generated group, we have the following non-automatic (see [2]) counterexample:

Counterexample 6.1. *Baumslag-Solitar group $B(2, 1)$, $S = \{a, a^{-1}, b, b^{-1}\}$, $R = \{a^2 b a^{-1} b^{-1}\}$, $a^{2^n} = b^{n-1} a^2 b^{1-n}$, $\ell(a^{2^n}) = 2n$ with $n > 1$,*

$$\left| \frac{p_b(a \cdot a^{2^n})}{p(a \cdot a^{2^n})} - \frac{p_b(a^{2^n})}{p(a^{2^n})} \right| = \left| \frac{2}{4} - \frac{2}{2} \right| = \frac{1}{2}$$

It should rest to prove that such a failure is independant of the presentation...

Then, to obtain a construction in the general case, we need to operate a little homogenization. Let $\Gamma = \langle S \mid R \rangle$ be a finitely generated group.

Definition 6.2. *Let $\mathbb{B}_\Gamma^n = \{g \in \Gamma \mid \ell(g) \leq n\}$ the ball of radius n .*

Definition 6.3. *Let $\mathbb{S}_\Gamma^n = \{g \in \Gamma \mid \ell(g) = n\}$ the sphere of radius n .*

Definition 6.4. Let μ be the probability measure on Γ defined by:

$$\mu(g) = (\#(\mathbb{S}_\Gamma^{\ell(g)}))2^{\ell(g)+1}^{-1}.$$

Definition 6.5. Let $E_{\Gamma,S}$ be the set of smooth functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ with:

(1) $f(0) = 0$, $f \geq 0$, $f' \geq 0$ and $f'' \leq 0$.

(2) $\lambda_f := \lim_{x \rightarrow \infty} (f'(x)) \leq 1$.

(3) $\forall s \in S$ and $\forall g, h \in \Gamma$ (with $h \neq e$):

$$\left| \sum_{\gamma \in \mathbb{B}_\Gamma^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \frac{\mu(\gamma gh)}{\mu(\mathbb{B}_\Gamma^{f(\ell(gh))}.gh)} - \sum_{\gamma \in \mathbb{B}_\Gamma^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \frac{\mu(\gamma h)}{\mu(\mathbb{B}_\Gamma^{f(\ell(h))}.h)} \right| \leq \frac{K_g}{\ell(h)}$$

with $K_g \in \mathbb{R}_+$ depending only on g .

Remark 6.6. The condition (2) is well-defined because f' is decreasing and minorated by 0, so convergent at infinity.

Remark 6.7. Let $f \in E_{\Gamma,S}$, if $\exists \alpha > 0$ with $f(\alpha) = 0$ then $f = 0$.

Lemma 6.8. $E_{\Gamma,S}$ is non empty.

Proof. We will show that $f(x) = x + \log_2(x+1)$ defines a function in $E_{\Gamma,S}$. First of all $\mathbb{B}_\Gamma^{\log_2(\ell(g))} \subset (\mathbb{B}_\Gamma^{f(\ell(g))}.g)$, then $\mu(\mathbb{B}_\Gamma^{f(\ell(g))}.g) \geq \mu(\mathbb{B}_\Gamma^{\log_2(\ell(g))}) \geq 1 - \frac{1}{\ell(g)}$.

$$\begin{aligned} & \left| \sum_{\gamma \in \mathbb{B}_\Gamma^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \frac{\mu(\gamma gh)}{\mu(\mathbb{B}_\Gamma^{f(\ell(gh))}.gh)} - \sum_{\gamma \in \mathbb{B}_\Gamma^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \frac{\mu(\gamma h)}{\mu(\mathbb{B}_\Gamma^{f(\ell(h))}.h)} \right| \leq \\ & \quad |(1/\mu(\mathbb{B}_\Gamma^{f(\ell(gh))}.gh) - 1) \sum_{\gamma \in \mathbb{B}_\Gamma^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \mu(\gamma gh)| + \\ & \quad |(1/\mu(\mathbb{B}_\Gamma^{f(\ell(h))}.h) - 1) \sum_{\gamma \in \mathbb{B}_\Gamma^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \frac{\mu(\gamma h)}{\mu(\mathbb{B}_\Gamma^{f(\ell(h))}.h)}| \\ & \quad + \left| \sum_{\gamma \in \mathbb{B}_\Gamma^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \mu(\gamma gh) - \sum_{\gamma \in \mathbb{B}_\Gamma^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \mu(\gamma h) \right| \leq \\ & 1/\ell(gh) + 1/\ell(h) + \left| \sum_{\gamma \in \Gamma \setminus \mathbb{B}_\Gamma^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \mu(\gamma gh) - \sum_{\gamma \in \Gamma \setminus \mathbb{B}_\Gamma^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \mu(\gamma h) \right| + \\ & \quad \left| \sum_{\gamma \in \Gamma} \frac{p_s(\gamma gh)}{p(\gamma gh)} \mu(\gamma gh) - \sum_{\gamma \in \Gamma} \frac{p_s(\gamma h)}{p(\gamma h)} \mu(\gamma h) \right| \\ & \leq 2/\ell(gh) + 2/\ell(h) \leq 2/(\ell(g)\ell(h) + \ell(h)) + 2/\ell(h) = \frac{1/(\ell(g)+1)}{\ell(h)} \end{aligned}$$

The others conditions for f to be in $E_{\Gamma,S}$ are evidents. \square

Remark 6.9. The class \mathcal{C} is the class of group Γ for which $\exists S$ with $0 \in E_{\Gamma,S}$.

We are led to a classification of finitely generated groups into three types:

Definition 6.10. Let $\lambda_\Gamma := \min_{S \subset \Gamma} \min_{f \in E_{\Gamma, S}} \lambda_f$, with S and λ_f as previously. By definition $\lambda_\Gamma \in [0, 1]$ then:

A₀: Γ is a type A_0 group if $\lambda_\Gamma = 0$ and $\exists f$ with $\lambda_f = 0$.

A _{λ} : Γ is a type A_λ group if it is not A_0 and $0 \leq \lambda_\Gamma < 1$.

A₁: Γ is a type A_1 group if $\lambda_\Gamma = 1$.

Definition 6.11. Let A_0^+ be the subclass of type A_λ groups with $\lambda_\Gamma = 0$.

Remark 6.12. Every groups of the class \mathcal{C} (as \mathbb{Z}^n , \mathbb{F}^n etc...) are of type A_0 .

Conjecture 6.13. Every Baumslag-Solitar groups $B(n, m)$ are of type A_0 .

Problem 6.14. Existence of type A_0^+ , A_λ or A_1 groups.

Definition 6.15. Let $\Gamma = \langle S \mid R \rangle$ be a finitely generated group, let $f \in E_{\Gamma, S}$ be a function with (almost) minimal growth, we define \tilde{D} as D using:

$$\tilde{\partial}_s \cdot e_g = \left[\sum_{\gamma \in \mathbb{B}_f^{\ell(g)}} \frac{p_s(\gamma g)}{p(\gamma g)} \frac{\mu(\gamma g)}{\mu(\mathbb{B}_\Gamma^{f(\ell(g))} \cdot g)} \right] \ell(g) e_g.$$

Remark 6.16. For the class \mathcal{C} , we can take $f = 0$ and so $\tilde{D} = D$.

Theorem 6.17. $(\mathcal{A}, H, \tilde{D})$ is a spectral triple; the phase of \tilde{D} is generically non-trivial.

Proof. The proof runs exactly as for theorem 5.9. □

Definition 6.18. Let \mathcal{C}^\perp be the class of groups Γ such that $\forall S, \forall f \in E_{\Gamma, S}$, the phase of \tilde{D} is always trivial.

Problem 6.19. Existence of groups of class \mathcal{C}^\perp .

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