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Sébastien Palcoux

► **To cite this version:**

| Sébastien Palcoux. Spectral triples for finitely generated groups.. 2010. hal-00529553v4

**HAL Id: hal-00529553**

**<https://hal.science/hal-00529553v4>**

Preprint submitted on 8 Nov 2010 (v4), last revised 14 Jan 2011 (v8)

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# Spectral triples for finitely generated groups.

Sébastien Palcoux\*

## Abstract

Using Cayley graphs and Clifford algebras, we are able to give, for every finitely generated groups, a uniform construction of spectral triples with a generically non-trivial phase for the Dirac operator. We are naturally led to a classification of such groups into three types.

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## 1 Introduction

In this paper, we define even spectral triples for every finitely generated groups. We just generalizes the unbounded version of the construction of the

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\*Institut de Mathématiques de Luminy, Marseille, France.  
web: <http://iml.univ-mrs.fr/~palcoux> , mail: [palcoux@iml.univ-mrs.fr](mailto:palcoux@iml.univ-mrs.fr)

Fredholm module for the free group given by Connes [1] and M. Pimsner-Voiculescu [4]. For so, we use the Clifford algebra in the same spirit that Julg-Valette do in [3]. We also use topics in geometric group theory (see [2]) as the Cayley graph. The construction runs for every finitely generated groups and the phase of the Dirac operator is generically non-trivial.

## 2 Basic definitions

**Definition 2.1.** A spectral triple  $(\mathcal{A}, H, D)$  is given by a unital  $\star$ -algebra  $\mathcal{A}$  represented on the Hilbert space  $H$ , and an unbounded operator  $D$ , called the Dirac operator, such that:

1.  $D$  is self-adjoint.
2.  $(D^2 + I)^{-1}$  is compact.
3.  $\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$  is dense in  $\mathcal{A}$ .

See the article [5] of G. Skandalis, dedicated to A. Connes and spectral triple.

**Definition 2.2.** A group  $\Gamma$  is finitely generated if it exists a finite generating set  $S \subset \Gamma$ . We always take  $S$  equals to  $S^{-1}$  and the identity element  $e \notin S$ . We can also defined a group by generators and relations:  $\Gamma = \langle S \mid R \rangle$ .

## 3 Geometric construction

**Definition 3.1.** Let  $\ell : \Gamma \rightarrow \mathbb{N}$  be the word length related to  $S$ . Let  $\mathcal{G}$  be the Cayley graph with the distance  $d(g, h) = \ell(gh^{-1})$ , see [2].

**Definition 3.2.** Let  $p : \Gamma \rightarrow \mathbb{N}$  with  $p(g)$  the number of geodesic paths from  $g$  to the identity element  $e$ , on the Cayley graph  $\mathcal{G}$ .

**Definition 3.3.** Let  $p_s : \Gamma \rightarrow \mathbb{N}$  with  $p_s(g)$  the number of geodesic paths from  $g$  to  $e$ , starting by  $s \in S$ . Then  $\sum_{s \in S} p_s = p$ .

**Remark 3.4.** If  $p_s(g) \neq 0$  then  $p_s(g) = p(gs)$ .

**Definition 3.5.** Let  $\ell^2(\Gamma)$  be the canonical Hilbert space of base  $(e_g)_{g \in \Gamma}$  and let  $\partial_s$  ( $s \in S$ ) be the unbounded operator defined by:

$$\partial_s.e_g = \frac{p_s(g)}{p(g)}\ell(g)e_g.$$

$\partial_s$  is a diagonal positive operator, so  $\partial_s^* = \partial_s$  and  $\partial_s\partial_{s'} = \partial_{s'}\partial_s$ .

**Example 3.6.** Let  $\Gamma = \mathbb{Z}^2$ ,  $S = \{a, a^{-1}, b, b^{-1}\}$ ,  $R = \{aba^{-1}b^{-1}\}$ . Let  $g = a^n b^m = (n, m)$  with  $n, m \in \mathbb{N}$ , then  $\ell(g) = n + m$ ,  $p(g) = C_{n+m}^n$ ,  $p_{a^{-1}}(g) = C_{n+m-1}^{n-1}$ ,  $p_{b^{-1}}(g) = C_{n+m-1}^{m-1}$ , and so  $\frac{C_{n+m-1}^{n-1}}{C_{n+m}^n}(n + m) = n$ . Let  $\partial_1 = \partial_{a^{-1}}$  and  $\partial_2 = \partial_{b^{-1}}$ , then:

$$\partial_1 e_{(n,m)} = n e_{(n,m)} \quad \text{and} \quad \partial_2 e_{(n,m)} = m e_{(n,m)},$$

as for the canonical derivations !

**Example 3.7.** Let  $\Gamma = \mathbb{F}_2$ ,  $S = \{a, a^{-1}, b, b^{-1}\}$ ,  $R = \emptyset$ . Let  $g \in \mathbb{F}_2$ ,  $g \neq e$ , then  $p(g) = 1$ , let  $s \in S$  with  $p_s(g) = 1$ , then  $\partial_s.e_g = \ell(g).e_g$ .

## 4 Clifford algebra

We quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [6].

**Definition 4.1.** Let  $\Lambda(\mathbb{R}^S)$  be the exterior algebra equals to  $\bigoplus_{k=0}^{2d} \Lambda^k(\mathbb{R}^d)$ , with  $2d = \text{card}(S)$  and  $\Lambda^0(\mathbb{R}^d) = \mathbb{R}\Omega$ . We called  $\Omega$  the vacuum vector. Recall that  $v_1 \wedge v_2 = -v_2 \wedge v_1$  so that  $v \wedge v = 0$ .

**Definition 4.2.** Let  $a_v$  be the creation operator on  $\Lambda(\mathbb{R}^S)$  defined by:

$$a_v(v_1 \wedge \dots \wedge v_r) = v \wedge v_1 \wedge \dots \wedge v_r \quad \text{and} \quad a_v(\Omega) = v$$

**Reminder 4.3.** The dual  $a_v^*$  is called the annihilation operator, then:

$$a_v^*(v_1 \wedge \dots \wedge v_r) = \sum_{i=0}^r (-1)^{i+1} (v, v_i) v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_r \quad \text{and} \quad a_v^*(\Omega) = 0$$

**Reminder 4.4.** Let  $c_v = a_v + a_v^*$ , then  $c_v = c_v^*$  and  $c_v c_w + c_w c_v = 2(v, w)I$ .

**Definition 4.5.** The operators  $c_v$  generate the Clifford algebra  $\text{Cliff}(\mathbb{R}^S)$ . Note that the operators  $c_v$  are bounded and that  $\text{Cliff}(\mathbb{R}^S).\Omega = \Lambda(\mathbb{R}^S)$ .

**Remark 4.6.**  $\mathbb{R}^S$  admits the orthonormal basis  $(v_s)_{s \in S}$ . We will write  $c_s$  instead of  $c_{v_s}$ , so that  $[c_s, c_{s'}]_+ = 2\delta_{s,s'}I$ .

## 5 Class $\mathcal{C}$ and Dirac operator

**Definition 5.1.** Let  $H$  be the Hilbert space  $\Lambda(\mathbb{R}^S) \overline{\otimes} \ell^2(\Gamma)$ , we define a self-adjoint operator  $D$  on a dense domain of  $H$  by:

$$D = \sum_{s \in S} c_s \otimes \partial_s.$$

**Lemma 5.2.**  $D^2 = I \otimes (\sum_{s \in S} \partial_s^2)$ .

*Proof.*  $D^2 = \sum_{s, s' \in S} c_s c_{s'} \otimes \partial_s \partial_{s'} = \frac{1}{2} \sum (c_s c_{s'} + c_{s'} c_s) \otimes \partial_s \partial_{s'} = I \otimes (\sum \partial_s^2)$   $\square$

**Lemma 5.3.**  $(D^2 + I)^{-1}$  is compact.

*Proof.* First of all  $\dim(\Lambda(\mathbb{R}^S)) = 2^{2d} < \infty$ .

Next  $\sum_{s \in S} \frac{p_s(g)}{p(g)} = 1$ , then it exists  $s_o \in S$  such that  $\frac{p_{s_o}(g)}{p(g)} \geq \frac{1}{2d}$ .

Now  $\sum_{s \in S} [\ell(g) \frac{p_s(g)}{p(g)}]^2 \geq [\ell(g) \frac{p_{s_o}(g)}{p(g)}]^2 \geq [\frac{\ell(g)}{2d}]^2$ .  $\square$

**Definition 5.4.** Let  $\mathcal{C}$  be the class of finitely generated group  $\Gamma$  such that it exists a finite generating set  $S \subset \Gamma$  (with  $S = S^{-1}$  and  $e \notin S$ ) such that  $\forall s \in S$  and  $\forall g, h \in \Gamma$  (with  $h \neq e$ ):

$$\left| \frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)} \right| \leq \frac{K_g}{\ell(h)}$$

with  $K_g \in \mathbb{R}_+$  depending only on  $g$ .

**Examples 5.5.** The class  $\mathcal{C}$  is stable by direct or free product, it contains  $\mathbb{Z}^n$ ,  $\mathbb{F}_n$ , the finite groups, and probably every amenable or hyperbolic groups.

**Warning 5.6.** From now, the group  $\Gamma$  is in the class  $\mathcal{C}$ .

**Definition 5.7.** Let  $\mathcal{A} = C_r^* \Gamma$ , acting on  $H$  by  $I \otimes u_g$ ,  $g \in \Gamma$ .

**Proposition 5.8.**  $\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$  is dense in  $\mathcal{A}$ .

*Proof.*  $[D, u_g](\Omega \otimes e_h) = \dots = (\sum_{s \in S} [\ell(gh) \frac{p_s(gh)}{p(gh)} - \ell(h) \frac{p_s(h)}{p(h)}] v_s) \otimes e_{gh}$ .

Now  $|\ell(gh) \frac{p_s(gh)}{p(gh)} - \ell(h) \frac{p_s(h)}{p(h)}| \leq \ell(gh) \left| \frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)} \right| + |\ell(gh) - \ell(h)| \frac{p_s(h)}{p(h)}$   
 $\leq K_g \frac{\ell(gh)}{\ell(h)} + \ell(g) \leq K_g(\ell(g) + 1) + \ell(g)$ . It's bounded in  $h$ .

Now let  $v \in \Lambda(\mathbb{R}^S)$  then it exists  $X \in \text{Cliff}(\mathbb{R}^S)$  with  $v = X\Omega$ . By linearity

we can restrict to  $X = c_{s_1} \dots c_{s_r}$  with  $s_i \in S$  and  $i \neq j$  implies  $s_i \neq s_j$ .

Warning, the following commutant  $[\cdot, X]$  is a graded commutant:

$$[D, u_g](v \otimes e_h) = ([D, u_g], X) + (-1)^r X [D, u_g](\Omega \otimes e_h)$$

$$[[D, u_g], X] = [[D, X], u_g]$$

$$[D, X] = [\sum_{s \in S} c_s \otimes \partial_s, X] = \sum_{s \in S} [c_s, X] \otimes \partial_s$$

$$[c_s, X] = 2 \sum (-1)^{i+1} \delta_{s, s_i} c_{s_1} \dots c_{s_{i-1}} c_{s_{i+1}} \dots c_{s_r}, \quad s_i \neq s_j \text{ if } i \neq j, \text{ then:}$$

$$[D, X] = \sum X_i \otimes \partial_{s_i} \text{ with } X_i = (-1)^{i+1} 2 c_{s_1} \dots c_{s_{i-1}} c_{s_{i+1}} \dots c_{s_r}$$

$$[[D, X], u_g](\Omega \otimes e_h) = \sum [\ell(gh) \frac{p_{s_i}(gh)}{p(gh)} - \ell(h) \frac{p_{s_i}(h)}{p(h)}] X_i \Omega \otimes e_{gh}.$$

Now  $X$  and  $X_i$  are bounded; the result follows.  $\square$

**Theorem 5.9.** *( $\mathcal{A}, H, D$ ) is a spectral triple; the phase of  $D$  is generically non-trivial.*

*Proof.* We use lemma 5.3 and proposition 5.8. The phase is generically non-trivial, because it's non-trivial for  $\Gamma = \mathbb{Z}^2$  or  $\mathbb{F}_2$ .  $\square$

## 6 Homogenization and general case

The class  $\mathcal{C}$  doesn't contain every finitely generated group, we have the following counterexample:

**Counterexample 6.1.** *Baumslag-Solitar group  $B(2, 1)$ ,  $S = \{a, a^{-1}, b, b^{-1}\}$ ,  $R = \{a^2 b a^{-1} b^{-1}\}$ ,  $a^{2^n} = b^{n-1} a^2 b^{1-n}$ ,  $\ell(a^{2^n}) = 2n$  with  $n > 1$ ,*

$$\left| \frac{p_b(a \cdot a^{2^n})}{p(a \cdot a^{2^n})} - \frac{p_b(a^{2^n})}{p(a^{2^n})} \right| = \left| \frac{2}{4} - \frac{2}{2} \right| = \frac{1}{2}$$

*It should rest to prove that such a failure is independant of the presentation...*

Then, to obtain a construction in the general case, we need to operate a little homogenization. Let  $\Gamma = \langle S \mid R \rangle$  be a finitely generated group.

**Definition 6.2.** *Let  $\mathbb{B}_\Gamma^n = \{g \in \Gamma \mid \ell(g) \leq n\}$  the ball of radius  $n$ .*

**Definition 6.3.** *Let  $\mathbb{S}_\Gamma^n = \{g \in \Gamma \mid \ell(g) = n\}$  the sphere of radius  $n$ .*

**Definition 6.4.** *Let  $\mu$  be the probability measure on  $\Gamma$  defined by:*

$$\mu(g) = (\#\mathbb{S}_\Gamma^{\ell(g)}) 2^{\ell(g)+1}^{-1}.$$

**Definition 6.5.** Let  $E_{\Gamma,S}$  be the set of smooth functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  with:

- (1)  $f(0) = 0$ ,  $f \geq 0$ ,  $f' \geq 0$  and  $f'' \leq 0$ .
- (2)  $\lambda_f := \lim_{x \rightarrow \infty} (f'(x)) \leq 1$ .
- (3)  $\forall s \in S$  and  $\forall g, h \in \Gamma$  (with  $h \neq e$ ):

$$\left| \sum_{\gamma \in \mathbb{B}_{\Gamma}^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \frac{\mu(\gamma gh)}{\mu(\mathbb{B}_{\Gamma}^{f(\ell(gh))}.gh)} - \sum_{\gamma \in \mathbb{B}_{\Gamma}^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \frac{\mu(\gamma h)}{\mu(\mathbb{B}_{\Gamma}^{f(\ell(h))}.h)} \right| \leq \frac{K_g}{\ell(h)}$$

with  $K_g \in \mathbb{R}_+$  depending only on  $g$ .

**Remark 6.6.** The condition (2) is well-defined because  $f'$  is decreasing and minorated by 0, so convergent at infinity.

**Remark 6.7.** Let  $f \in E_{\Gamma,S}$ , if  $\exists \alpha > 0$  with  $f(\alpha) = 0$  then  $f = 0$ .

**Lemma 6.8.**  $E_{\Gamma,S}$  is non empty.

*Proof.* We will show that  $f(x) = x + \log_2(x+1)$  defines a function in  $E_{\Gamma,S}$ . First of all  $\mathbb{B}_{\Gamma}^{\log_2(\ell(g))} \subset (\mathbb{B}_{\Gamma}^{f(\ell(g))}.g)$ , then  $\mu(\mathbb{B}_{\Gamma}^{f(\ell(g))}.g) \geq \mu(\mathbb{B}_{\Gamma}^{\log_2(\ell(g))}) \geq 1 - \frac{1}{\ell(g)}$ .

$$\begin{aligned} & \left| \sum_{\gamma \in \mathbb{B}_{\Gamma}^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \frac{\mu(\gamma gh)}{\mu(\mathbb{B}_{\Gamma}^{f(\ell(gh))}.gh)} - \sum_{\gamma \in \mathbb{B}_{\Gamma}^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \frac{\mu(\gamma h)}{\mu(\mathbb{B}_{\Gamma}^{f(\ell(h))}.h)} \right| \leq \\ & \quad |(1/\mu(\mathbb{B}_{\Gamma}^{f(\ell(gh))}.gh) - 1) \sum_{\gamma \in \mathbb{B}_{\Gamma}^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \mu(\gamma gh)| + \\ & \quad |(1/\mu(\mathbb{B}_{\Gamma}^{f(\ell(h))}.h) - 1) \sum_{\gamma \in \mathbb{B}_{\Gamma}^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \frac{\mu(\gamma h)}{\mu(\mathbb{B}_{\Gamma}^{f(\ell(h))}.h)}| \\ & \quad + \left| \sum_{\gamma \in \mathbb{B}_{\Gamma}^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \mu(\gamma gh) - \sum_{\gamma \in \mathbb{B}_{\Gamma}^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \mu(\gamma h) \right| \leq \\ & 1/\ell(gh) + 1/\ell(h) + \left| \sum_{\gamma \in \Gamma \setminus \mathbb{B}_{\Gamma}^{f(\ell(gh))}} \frac{p_s(\gamma gh)}{p(\gamma gh)} \mu(\gamma gh) - \sum_{\gamma \in \Gamma \setminus \mathbb{B}_{\Gamma}^{f(\ell(h))}} \frac{p_s(\gamma h)}{p(\gamma h)} \mu(\gamma h) \right| + \\ & \quad \left| \sum_{\gamma \in \Gamma} \frac{p_s(\gamma gh)}{p(\gamma gh)} \mu(\gamma gh) - \sum_{\gamma \in \Gamma} \frac{p_s(\gamma h)}{p(\gamma h)} \mu(\gamma h) \right| \end{aligned}$$

$$\leq 2/\ell(gh) + 2/\ell(h) \leq 2/(\ell(g)\ell(h) + \ell(h)) + 2/\ell(h) = \frac{1/(\ell(g) + 1)}{\ell(h)}$$

The others conditions for  $f$  to be in  $E_{\Gamma,S}$  are evidents.  $\square$

**Remark 6.9.** *The class  $\mathcal{C}$  is the class of group  $\Gamma$  for which  $\exists S$  with  $0 \in E_{\Gamma,S}$ .*

We are led to a classification of finitely generated groups into three types:

**Definition 6.10.** *Let  $\lambda_{\Gamma} := \min_{S \subset \Gamma} \min_{f \in E_{\Gamma,S}} \lambda_f$ , with  $S$  and  $\lambda_f$  as previously. By definition  $\lambda_{\Gamma} \in [0, 1]$  then:*

**A<sub>0</sub>:**  $\Gamma$  is a type  $A_0$  group if  $\lambda_{\Gamma} = 0$  and  $\exists f$  with  $\lambda_f = 0$ .

**A <sub>$\lambda$</sub> :**  $\Gamma$  is a type  $A_{\lambda}$  group if it is not  $A_0$  and  $0 \leq \lambda_{\Gamma} < 1$ .

**A<sub>1</sub>:**  $\Gamma$  is a type  $A_1$  group if  $\lambda_{\Gamma} = 1$ .

**Definition 6.11.** *Let  $A_0^+$  be the subclass of type  $A_{\lambda}$  groups with  $\lambda_{\Gamma} = 0$ .*

**Remark 6.12.** *Every groups of the class  $\mathcal{C}$  (as  $\mathbb{Z}^n, \mathbb{F}^n$  etc...) are of type  $A_0$ .*

**Conjecture 6.13.** *Every Baumslag-Solitar groups  $B(n, m)$  are of type  $A_0$ .*

**Problem 6.14.** *Existence of type  $A_0^+$ ,  $A_{\lambda}$  or  $A_1$  groups.*

**Definition 6.15.** *Let  $\Gamma = \langle S \mid R \rangle$  be a finitely generated group, let  $f \in E_{\Gamma,S}$  be a function with (almost) minimal growth, we define  $\tilde{D}$  as  $D$  using:*

$$\tilde{\partial}_s \cdot e_g = \left[ \sum_{\gamma \in \mathbb{B}_{f(\ell(g))}^{\Gamma}} \frac{p_s(\gamma g)}{p(\gamma g)} \frac{\mu(\gamma g)}{\mu(\mathbb{B}_{f(\ell(g))}^{\Gamma} \cdot g)} \right] \ell(g) e_g.$$

**Remark 6.16.** *For the class  $\mathcal{C}$ , we can take  $f = 0$  and so  $\tilde{D} = D$ .*

**Theorem 6.17.**  *$(\mathcal{A}, H, \tilde{D})$  is a spectral triple; the phase of  $\tilde{D}$  is generically non-trivial.*

*Proof.* The proof runs exactly as for theorem 5.9.  $\square$

**Definition 6.18.** *Let  $\mathcal{C}^{\perp}$  be the class of groups  $\Gamma$  such that  $\forall S, \forall f \in E_{\Gamma,S}$ , the phase of  $\tilde{D}$  is always trivial.*

**Problem 6.19.** *Existence of groups of class  $\mathcal{C}^{\perp}$ .*



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