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Spectral triples for finitely generated groups.

Sébastien Palcoux*

Abstract

Using Cayley graphs and Clifford algebras, we are able to give spectral triples for a large class of finitely generated groups (perhaps all), such that the phase of the Dirac operator is generically non-trivial.

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1 Basic definitions

Definition 1.1. A spectral triple (\mathcal{A}, H, D) is given by a unital \star -algebra \mathcal{A} represented on the Hilbert space H , and an unbounded operator D , called the Dirac operator, such that:

1. D is self-adjoint.
2. $(D^2 + I)^{-1}$ is compact.
3. $\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$ is dense in \mathcal{A} .

See the article [2] of G. Skandalis, dedicated to A. Connes and spectral triple.

Definition 1.2. A group Γ is finitely generated if it exists a finite generating set $S \subset \Gamma$. We always take S equals to S^{-1} and the identity element $e \notin S$. We can also defined a group by generators and relations: $\Gamma = \langle S \mid R \rangle$.

2 Geometric construction

Definition 2.1. Let $\ell : \Gamma \rightarrow \mathbb{N}$ be the word length related to S . Let \mathcal{G} be the Cayley graph with the distance $d(g, h) = \ell(gh^{-1})$, see [1].

Definition 2.2. Let $p : \Gamma \rightarrow \mathbb{N}$ with $p(g)$ the number of geodesic paths from g to the identity element e , on the Cayley graph \mathcal{G} .

Definition 2.3. Let $p_s : \Gamma \rightarrow \mathbb{N}$ with $p_s(g)$ the number of geodesic paths from g to e , starting by $s \in S$. Then $\sum_{s \in S} p_s = p$.

Remark 2.4. If $p_s(g) \neq 0$ then $p_s(g) = p(gs)$.

Definition 2.5. Let $\ell^2(\Gamma)$ be the canonical Hilbert space of base $(e_g)_{g \in \Gamma}$ and let ∂_s ($s \in S$) be the unbounded operator defined by:

$$\partial_s \cdot e_g = \frac{p_s(g)}{p(g)} \ell(g) e_g.$$

∂_s is a diagonal positive operator, so $\partial_s^* = \partial_s$ and $\partial_s \partial_{s'} = \partial_{s'} \partial_s$.

Example 2.6. Let $\Gamma = \mathbb{Z}^2$, $S = \{a, a^{-1}, b, b^{-1}\}$, $R = \{aba^{-1}b^{-1}\}$.
Let $g = a^n b^m = (n, m)$ with $n, m \in \mathbb{N}$, then $\ell(g) = n + m$, $p(g) = C_{n+m}^n$,
 $p_{a^{-1}}(g) = C_{n+m-1}^{n-1}$, $p_{b^{-1}}(g) = C_{n+m-1}^{m-1}$, and so $\frac{C_{n+m-1}^{n-1}}{C_{n+m}^n}(n + m) = n$.
Let $\partial_1 = \partial_{a^{-1}}$ and $\partial_2 = \partial_{b^{-1}}$, then:

$$\partial_1 e_{(n,m)} = n e_{(n,m)} \quad \text{and} \quad \partial_2 e_{(n,m)} = m e_{(n,m)},$$

as for the canonical derivations !

Example 2.7. Let $\Gamma = \mathbb{F}_2$, $S = \{a, a^{-1}, b, b^{-1}\}$, $R = \emptyset$. Let $g \in \mathbb{F}_2$, $g \neq e$,
then $p(g) = 1$, let $s \in S$ with $p_s(g) = 1$, then $\partial_s . e_g = \ell(g) . e_g$.

3 Clifford algebra

We quickly recall here the notion of Clifford algebra, for a more detailed exposition, see the course of A. Wassermann [3].

Definition 3.1. Let $\Lambda(\mathbb{R}^S)$ be the exterior algebra equals to $\bigoplus_{k=0}^{2d} \Lambda^k(\mathbb{R}^d)$,
with $2d = \text{card}(S)$ and $\Lambda^0(\mathbb{R}^d) = \mathbb{R}\Omega$. We called Ω the vacuum vector.
Recall that $v_1 \wedge v_2 = -v_2 \wedge v_1$ so that $v \wedge v = 0$.

Definition 3.2. Let a_v be the creation operator on $\Lambda(\mathbb{R}^S)$ defined by:

$$a_v(v_1 \wedge \dots \wedge v_r) = v \wedge v_1 \wedge \dots \wedge v_r \quad \text{and} \quad a_v(\Omega) = v$$

Reminder 3.3. The dual a_v^* is called the annihilation operator, then:

$$a_v^*(v_1 \wedge \dots \wedge v_r) = \sum_{i=0}^r (-1)^{i+1} (v, v_i) v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_r \quad \text{and} \quad a_v^*(\Omega) = 0$$

Reminder 3.4. Let $c_v = a_v + a_v^*$, then $c_v = c_v^*$ and $c_v c_w + c_w c_v = 2(v, w)I$.

Definition 3.5. The operators c_v generate the Clifford algebra $\text{Cliff}(\mathbb{R}^S)$.
Note that the operators c_v are bounded and that $\text{Cliff}(\mathbb{R}^S) . \Omega = \Lambda(\mathbb{R}^S)$.

Remark 3.6. \mathbb{R}^S admits the orthonormal basis $(v_s)_{s \in S}$.
We will write c_s instead of c_{v_s} , so that $[c_s, c_{s'}]_+ = 2\delta_{s,s'}I$.

4 Dirac operator and class \mathcal{C}

Definition 4.1. Let H be the Hilbert space $\Lambda(\mathbb{R}^S) \overline{\otimes} \ell^2(\Gamma)$, we define a self-adjoint operator D on a dense domain of H by:

$$D = \sum_{s \in S} c_s \otimes \partial_s.$$

Lemma 4.2. $D^2 = I \otimes (\sum_{s \in S} \partial_s^2)$.

Proof. $D^2 = \sum_{s, s' \in S} c_s c_{s'} \otimes \partial_s \partial_{s'} = \frac{1}{2} \sum (c_s c_{s'} + c_{s'} c_s) \otimes \partial_s \partial_{s'} = I \otimes (\sum \partial_s^2)$ \square

Lemma 4.3. $(D^2 + I)^{-1}$ is compact.

Proof. First of all $\dim(\Lambda(\mathbb{R}^S)) = 2^{2d} < \infty$.

Next $\sum_{s \in S} \frac{p_s(g)}{p(g)} = 1$, then it exists $s_o \in S$ such that $\frac{p_{s_o}(g)}{p(g)} \geq \frac{1}{2d}$.

Now $\sum_{s \in S} [\ell(g) \frac{p_s(g)}{p(g)}]^2 \geq [\ell(g) \frac{p_{s_o}(g)}{p(g)}]^2 \geq [\frac{\ell(g)}{2d}]^2$. \square

Definition 4.4. Let \mathcal{C} be the class of finitely generated group Γ such that it exists a finite generating set $S \subset \Gamma$ (with $S = S^{-1}$ and $e \notin S$) such that $\forall s \in S$ and $\forall g, h \in \Gamma$ (with $h \neq e$):

$$\left| \frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)} \right| \leq \frac{K_g}{\ell(h)}$$

with $K_g \in \mathbb{R}_+$ depending only on g .

Examples 4.5. The class \mathcal{C} is stable by direct or free product, it contains \mathbb{Z}^n , \mathbb{F}_n , the finite groups, and probably every amenable or hyperbolic groups.

Question 4.6. Does \mathcal{C} contains every finitely generated groups ?

From now, the group Γ is in the class \mathcal{C} .

Definition 4.7. Let $\mathcal{A} = C_r^* \Gamma$, acting on H by $I \otimes u_g$, $g \in \Gamma$.

Proposition 4.8. $\{a \in \mathcal{A} \mid [D, a] \in B(H)\}$ is dense in \mathcal{A} .

Proof. $[D, u_g](\Omega \otimes e_h) = \dots = (\sum_{s \in S} [\ell(gh) \frac{p_s(gh)}{p(gh)} - \ell(h) \frac{p_s(h)}{p(h)}] v_s) \otimes e_{gh}$.

Now $|\ell(gh) \frac{p_s(gh)}{p(gh)} - \ell(h) \frac{p_s(h)}{p(h)}| \leq \ell(gh) \left| \frac{p_s(gh)}{p(gh)} - \frac{p_s(h)}{p(h)} \right| + |\ell(gh) - \ell(h)| \frac{p_s(h)}{p(h)}$
 $\leq K_g \frac{\ell(gh)}{\ell(h)} + \ell(g) \leq K_g(\ell(g) + 1) + \ell(g)$. It's bounded in h . \square

Theorem 4.9. (\mathcal{A}, H, D) is a spectral triple.

The phase of D is generically non-trivial.

5 The class $\tilde{\mathcal{C}}$

Definition 5.1. Let $\alpha \geq 1$, let $p(g, \alpha)$ be the number of paths of length $\leq \alpha \ell(g)$, from g to e , and let $p_s(g, \alpha)$ be the number of such paths starting by $s \in S$.

Definition 5.2. Let $\tilde{\mathcal{C}}$ be the class of finitely generated group Γ such that it exists a finite generating set $S \subset \Gamma$ (with $S = S^{-1}$ and $e \notin S$), $\exists \alpha \geq 1$, such that $\forall s \in S$ and $\forall g, h \in \Gamma$ (with $h \neq e$):

$$\left| \frac{p_s(gh, \alpha)}{p(gh, \alpha)} - \frac{p_s(h, \alpha)}{p(h, \alpha)} \right| \leq \frac{K_g}{\ell(h)}$$

with $K_g \in \mathbb{R}_+$ depending only on g .

Theorem 5.3. Let Γ be a group on the class $\tilde{\mathcal{C}}$ for a fixed $\alpha \geq 1$, and let D_α defined as previously using $\partial_{s, \alpha} \cdot e_g := \frac{p_s(g, \alpha)}{p(g, \alpha)} \ell(g) e_g$, then $(\mathcal{A}, H, D_\alpha)$ is a spectral triple and the phase of D_α is generically non-trivial.

Remark 5.4. The class $\tilde{\mathcal{C}}$ contains the class \mathcal{C} , but it's not clear that it contains anything else. The same question is asked: does $\tilde{\mathcal{C}}$ contains the amenable groups, the hyperbolic groups or every finitely generated groups ?

References

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