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b-chromatic number of cacti²

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Abstract

A b-colouring of a graph G is a proper colouring of G such that each colour contains a vertex that is adjacent to all other colours and the b-chromatic number $\chi_b(G)$ is the maximum number of colours used in a b-colouring of G . If $m(G)$ is the largest integer k such that G has at least k vertices with degree at least $k - 1$, then we know that $\chi_b(G) \leq m(G)$. Irving and Manlove [1] prove that, if T is a tree, then the b-chromatic number of T is at least $m(T) - 1$. In this paper, we prove that, if G is a connected cactus and $m(G) \geq 7$, then the b-chromatic number of G is at least $m(G) - 1$.

Keywords: Graph, complexity, b-colouring, cactus

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1 Introduction

Let G be a simple graph. A *proper coloring* of G is an assignment of colors to the vertices of G such that no two adjacent vertices have the same color. In a proper coloring, the color class of a color c is the set of vertices of G colored with color c . The *chromatic number* of G is the minimum integer $\chi(G)$ such that G has a proper coloring with $\chi(G)$ colors. Suppose that we have a proper coloring of G and a color class C such that every vertex in C is not adjacent to at least one other color class. We can change the color of the vertices in C , obtaining a proper coloring that uses less colors than before. However, as one can expect, we cannot apply this heuristic iteratively until we reach the chromatic number of G , since the coloring problem is \mathcal{NP} -hard. It was this idea that made Irving and Manlove introduce the notion of b-coloring in [1]. Intuitively, a b-coloring is a proper coloring that cannot be improved by the described heuristic and the b-chromatic number $\chi_b(G)$ of G measures the worst possible such coloring. More formally:

Definition 1.1 A vertex u in color class C is said to be a b-vertex if u has at least one neighbor in each color class other than C .

Definition 1.2 A b-coloring of a graph G is a proper coloring of G such that each color class contains at least one b-vertex.

Definition 1.3 The b-chromatic number of G is the largest integer k such that G has a b-coloring with k colors. We denote it by $\chi_b(G)$.

Naturally, we have that a proper coloring of G with $\chi(G)$ colors is a b-coloring of G , since it cannot be improved. So, $\chi(G) \leq \chi_b(G)$. For an upper bound, note that if G has a b-coloring with k colors, then G has at least k vertices with degree at least $k - 1$ (the b-vertices). So, if $m(G)$ is the largest integer such that G has at least $m(G)$ vertices with degree at least $m(G) - 1$, we know that G cannot have a b-coloring with more than $m(G)$ colors, i.e., $\chi_b(G) \leq m(G)$. This upper bound was introduced by Irving and Manlove in [1], where they also showed that the difference between $\chi_b(G)$ and $m(G)$ can be arbitrarily large for a general graph and that it is at most one for trees. In addition, the problem was proved to be \mathcal{NP} -hard [1], even when restricted to bipartite graphs [2].

We say that G is a cactus if G does not contain two cycles that share an edge. In this article, we prove that, if G is a connected cactus and $m(G) \geq 7$, then the difference between $\chi_b(G)$ and $m(G)$ is at most one and we can obtain $\chi_b(G)$ in polynomial time.

We say that G is *m-defective* if $m(G) = m$ and $\chi_b(G) < m$. Additionally, G is *minimal m-defective* if G is *m-defective* and any proper subgraph of G is not *m-defective*.

The general idea for our main result is as follows. We present the class of pivoted cacti and show that every pivoted cactus G is $m(G)$ -defective. Then, we show that $\chi_b(G) = m(G) - 1$, for every pivoted cactus G . Finally, we prove that a minimal $m(G)$ -defective cactus is pivoted. To do so, first we describe the structure of minimal $m(G)$ -defective cacti. We say that a subgraph H of G is a *b-kernel of G* if $m(H) = m(G)$ and $\chi_b(H) = m(G)$ implies $\chi_b(G) = m(G)$. If G is not a pivoted cactus, we find a special b-kernel H_G of G and we either show how to b-color H_G with $m(G)$ colors or show that H_G contains no minimal $m(G)$ -defective cactus as a subgraph. In any case, this implies that $\chi_b(H_G) = m(G)$ and, by the definition of a b-kernel, $\chi_b(G) = m(G)$. This proves that if G is not a pivoted cactus, then it is not $m(G)$ -defective. More details on this general idea are provided in the remainder of this paper.

2 Pivoted Cacti

In this section, we define the class of pivoted cacti and show that if G is a pivoted cactus, then $\chi_b(G) = m(G) - 1$. This implies that every pivoted cactus G is $m(G)$ -defective.

Let $G = (V, E)$ be a connected cactus. We say that a vertex v in V is a *dense vertex* if $d(v) \geq m(G) - 1$. Let $M(G)$ denote the set of dense vertices of G .

Let V' be any subset of $m(G)$ vertices of $M(G)$. Let $u \in V \setminus V'$ and $v \in V'$. If u and v are adjacent or have a common neighbor w in V' with $d(w) = m(G) - 1$, then we say that v is *reachable from u within V'* . Note that if $u \in V \setminus V'$ reaches every vertex of V' , then there is no b-colouring of G with V' as the set of b-vertices. Observe that it also happens if we have a pair of vertices $u, v \in V \setminus V'$ that reaches every vertex of V' but one, say w : we could colour u or v with the colour of w , but the other one cannot be coloured without repeating some colour in the neighborhood of a vertex of V' with degree $m(G) - 1$. So, we say that V' *encircles vertex u in $V \setminus V'$* if every vertex v in V' is reachable from u within V' . Below, we describe the situations where we have a pair that prevent V' from being a set of b-vertices of some b-colouring of G . We say that V' *encircles the pair $x, y \in V \setminus V'$* if it does not encircle x or y and if one of the following occurs:

- (E1) There are $V'' \subset V'$ and $u, v \in V''$ such that $|V''| = m(G) - 1$, $\langle x, u, y, v \rangle$

is a cycle and:

- (a) $d(u) = d(v) = m(G) - 1$ and every $w \in V'' \setminus \{u, v\}$ is adjacent to u or v ; or
 - (b) $d(u) = m(G) - 1$ and every $w \in V'' \setminus \{u, v\}$ is adjacent to u ; or
 - (c) $d(u) = m(G)$, $d(v) = m(G) - 1$ and every $w \in V' \setminus \{u, v\}$ is adjacent to u or v ; or
 - (d) $d(u) = m(G)$ and every $w \in V' \setminus \{u, v\}$ is adjacent to u .
- (E2) There are $V'' \subseteq V'$ and $u, v, w \in V''$ such that $|V''| \geq m(G) - 1$, $\langle x, u, v, y, w \rangle$ is a cycle, $d(u) = d(v) = m(G) - 1$, every $w' \in V'' \setminus \{u, v, w\}$ is adjacent to w , and
- (a) $V'' = V'$ and $d(w) = m(G)$; or
 - (b) $V'' \subset V'$ and $d(w) = m(G) - 1$.

Let V' be a subset of $m(G)$ vertices of $M(G)$. We say that V' is a *good set* if it does not encircle any vertex or pair of vertices and every $u \in V \setminus V'$ with degree at least $m(G)$ is either adjacent to some vertex in V' with degree $m(G) - 1$ or is within a path between two vertices of V' of length at most three, whose internal vertices are not in V' . If G does not have a good set, we say that G is a *pivoted cactus*.

The following lemma shows the possible number of encircled vertices or encircled pairs.

Lemma 2.1 *Let V' be any set of $m(G)$ dense vertices. If $m(G) \geq 7$, then V' encircles at most two vertices, or at most one pair of vertices.*

By using the structural properties presented in Lemma 2.1, we can prove that if $|M(G)| = m(G)$ and situation E1 or E2 occurs (in which case, we know that $\chi_b(G) < m(G)$), then we can b-color G with $m(G) - 1$ colors. However, the following lemma shows us that there is a situation where G has more than $m(G)$ dense vertices and still cannot be b-colored with $m(G)$ colors.

Lemma 2.2 *Let G be a connected cactus with $|M(G)| > m(G)$, $m(G) \geq 7$, and let V' be a set of $m(G) + 1$ dense vertices of G containing all vertices with degree greater than $m(G) - 1$. Then, G does not have a good set if and only if $V' = M(G)$ and there are vertices $u, v \in V'$ and $w \in V \setminus V'$ such that $d(u) = d(v) = m(G) - 1$, $\{u, v, w\}$ forms a triangle in G and every vertex in V' is adjacent to u or to v .*

Again, we can use the structural properties presented in Lemma 2.2 to prove the following lemma by giving a b-coloring of G with $m(G) - 1$ colors.

Lemma 2.3 *If G is a pivoted connected cactus, then $\chi_b(G) = m(G) - 1$.*

3 Non pivoted Cacti

In this section, we prove that if a cactus G has a good set, then it can be b-coloured with $m(G)$ colours, giving us the desired result. To do so, we first describe the structure of minimal m -defective cacti.

Theorem 3.1 *If G is a minimal $m(G)$ -defective cactus and $m(G) \geq 7$, then $|M(G)| = m(G)$ and $d(v) = m(G) - 1$, for all $v \in M(G)$. Furthermore, if $u \in V(G) \setminus M(G)$, then any neighbor of u is a dense vertex of G .*

Theorem 3.2 *If G is a minimal $m(G)$ -defective cactus with $m(G) \geq 7$, then either $M(G)$ encircles a vertex or $M(G)$ encircles a pair of vertices as in [E1.a](#) or [E2.b](#).*

To link minimal m -defective cacti with non-pivoted cacti, note that if G has no minimal $m(G)$ -defective subgraph, then G is not defective. This implies that $\chi_b(G) = m(G)$. We also use the following lemma.

Lemma 3.3 *If G is a non-pivoted cactus and V' is a good set of G , then $G[V' \cup N(V')]$ is a b-kernel of G .*

Now, let G be a non-pivoted cactus and let V' be a good set of G . Consider the b-kernel $H_G = G[V' \cup N(V')]$ of G . If H_G contains no minimal $m(G)$ -defective subgraph, then H_G is not $m(G)$ -defective itself and we have $\chi_b(H) = m(G)$. Since H_G is a b-kernel, then $\chi_b(G) = m(G)$. On the other hand, if H_G has a minimal $m(G)$ -defective subgraph, then using the structural properties given by [Theorem 3.2](#) and the fact that V' does not encircle a vertex or pair of vertices in H_G , it is possible to construct a b-colouring of G with $m(G)$ colours. So, the following theorem is true:

Theorem 3.4 *If G is a non-pivoted cactus with $m(G) \geq 7$, then $\chi_b(G) = m(G)$.*

4 Closing Comments

Our main result is the following theorem:

Theorem 4.1 *If G is a connected cactus with $m(G) \geq 7$, then $m(G) - 1 \leq \chi_b(G) \leq m(G)$.*

In addition, we found an algorithm that gives an optimum b-coloring of a pivoted cactus and an algorithm to find a good set of a non-pivoted cactus. We also have an algorithm to extend a maximum b-coloring of the b-kernel H_G

into a maximum b-coloring of G . However, we could not find an algorithm that optimally colors H_G and the proof that $\chi_b(H_G) = m(G)$ is not constructive. Thus, we can decide in polynomial time the value of $\chi_b(G)$, but we cannot always provide such coloring.

References

- [1] Irving, R.W. and Manlove, D.F., The b-chromatic number of a graph, *Discrete Applied Mathematics*, 91(1999), pages 127-141.
- [2] Kratochvil, J. and Tuza, Z. and VOIGT, M. On the b-chromatic number of graphs, *Lecture Notes In Computer Science*, Vol. 2573, pag. 310-320, 2002.