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ASYMPTOTIC-PRESERVING AND WELL-BALANCED SCHEME FOR THE 1D CATTANEO MODEL OF CHEMOTAXIS MOVEMENT IN BOTH HYPERBOLIC AND DIFFUSIVE REGIMES

LAURENT GOSSE*

Abstract. The original well-balanced (WB) framework [19, 14] relying on nonconservative (NC) products [34] is set up in order to efficiently treat the so-called Cattaneo model of chemotaxis in 1D [24]. It proceeds by concentrating the source terms onto Dirac masses: this allows to handle them by NC jump relations based on steady-state equations which can be integrated explicitly. A Riemann solver is deduced and the corresponding WB Godunov scheme completed with the standard Hoff-Smoller theory [24] for the diffusion-reaction equation ruling the evolution of the chemottractant concentration is studied in detail. Later, following former results [16, 17], a simple rewriting of the NC jump relations allows to generate another version of the same Godunov scheme which is well adapted to the parabolic scaling involving a small parameter ε . The standard BV framework is used to study the uniform stability of this Asymptotic-Preserving (AP) scheme with respect to ε allows to pass to the limit and derive a simple centered discretization of the Keller-Segel model. Finally, results by Filbet [10] permit to pass to the complementary limit when the space-step h is sent to zero. Numerical results are included to illustrate the feasibility and the efficiency of the method.

Key words. Chemotaxis modeling; discrete velocity kinetic model; non-conservative products; quasi-monotone source term; AP scheme; WB scheme.

AMS subject classifications. 65M06, 35L60.

1. Introduction.

1.1. Modeling of chemotaxis dynamics. This paper is concerned with the following semilinear model of chemotaxis movement, usually named after Cattaneo:

$$(1.1) \quad \begin{cases} \partial_t \rho + \partial_x J = 0 \\ \partial_t J + \lambda^2 \partial_x \rho = \rho \partial_x \varphi - J \\ \partial_t \varphi - D \partial_{xx} \varphi = \alpha \rho - \beta \varphi. \end{cases}$$

One can rewrite it in a more mathematically tractable way by introducing its diagonal variables (its Riemann invariants, in hyperbolic terminology),

$$w = \frac{1}{2} \left(\rho - \frac{J}{\lambda} \right), \quad z = \frac{1}{2} \left(\rho + \frac{J}{\lambda} \right),$$

which satisfy the following semilinear system of equations [18]:

$$(1.2) \quad \begin{cases} \partial_t w - \lambda \partial_x w = -\frac{1}{2\lambda} G(\partial_x \varphi; w, z) \\ \partial_t z + \lambda \partial_x z = \frac{1}{2\lambda} G(\partial_x \varphi; w, z) \\ \partial_t \varphi - D \partial_{xx} \varphi = \alpha \rho - \beta \varphi. \end{cases}$$

with the conventional choice of turning rates,

$$G(\partial_x \varphi; w, z) = (\partial_x \varphi + \lambda)w + (\partial_x \varphi - \lambda)z.$$

For stability reasons, we want the source term $G(\partial_x \varphi; \cdot, \cdot)$ to be quasi-monotone in the terminology of [38] and this leads to the well-known *subcharacteristic conditions*:

$$(1.3) \quad \partial_w G(\partial_x \varphi; w, z) \geq 0 \text{ and } \partial_z G(\partial_x \varphi; w, z) \leq 0 \Rightarrow |\partial_x \varphi| \leq \lambda.$$

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The system (1.2) can be seen as a 2-velocity discrete kinetic model with w, z being identified to f^\mp in the notation of [17], the density of particles moving with negative and positive speed, respectively. The coupling with φ is weak in the sense that the linear equation admits an explicit solution involving a convolution with a damped heat kernel and Duhamel's principle (as written in [26], proof of Lemma 4); thus $\partial_x \varphi$ is actually a function of $\varphi(t=0)$ and $\partial_x \rho$. We consider its parabolic scaling by introducing a small parameter $0 < \varepsilon \ll 1$, and imposing $\alpha^\varepsilon = \varepsilon^2 \alpha$, $\beta^\varepsilon = \varepsilon^2 \beta$, [9]

$$(1.4) \quad \begin{cases} \varepsilon^2 \partial_t w - \varepsilon \lambda \partial_x w = -\frac{1}{2\lambda} G(\partial_x \varphi; w, z) \\ \varepsilon^2 \partial_t z + \varepsilon \lambda \partial_x z = \frac{1}{2\lambda} G(\partial_x \varphi; w, z) \\ \varepsilon^2 (\partial_t \varphi - D \partial_{xx} \varphi) = \alpha^\varepsilon \rho - \beta^\varepsilon \varphi, \end{cases}$$

meaning that we substitute $t \rightarrow \varepsilon^2 t$, $x \rightarrow \varepsilon x$, and “we look at the system (1.2) from far away with a clock which turns very slowly”. It is expected that the behavior of (1.4) in the limit $\varepsilon \rightarrow 0$ approaches the one induced by the so-called Keller-Segel model which contains only a convection-diffusion equation for ρ coupled to φ :

$$(1.5) \quad \partial_t \rho + \partial_x (\partial_x \varphi \rho) = \lambda^2 \partial_{xx} \rho, \quad \partial_t \varphi - D \partial_{xx} \varphi = \alpha \rho - \beta \varphi.$$

The first equation of (1.5) is a special form of the classical linear Fokker-Planck (or conservative forward Kolmogorov) equation in divergence form; however, since φ satisfies a linear diffusion equation, $\partial_x \varphi$ isn't a confining potential since $\partial_{xx} \varphi$ surely changes its sign when $x \in \mathbb{R}$. Observe also that the equation on ρ doesn't admit BV-bounds except if $\partial_{xx} \varphi \equiv 0$, which is incompatible with the prescribed equation: this will create issues for the convergence of the Asymptotic-Preserving process. Imposing the “small reaction rates” $\alpha^\varepsilon, \beta^\varepsilon$ in (1.4) is somewhat necessary; in the opposite situation, we formally get $\varphi = \alpha \rho / \beta$, which leads to a nonlinear diffusion equation of the porous medium type, but endowed with the “wrong sign” (because $\alpha \geq 0, \beta \geq 0$):

$$\partial_t \rho + \frac{\alpha}{2\beta} \partial_{xx} (\rho^2) = \lambda^2 \partial_{xx} \rho.$$

From a mathematical point of view, this situation is related to the one ruling the behavior of the so-called Ruijgrok-Wu model of the Boltzmann equation, see [13, 36]. The derivation of efficient numerical schemes for this model has been carried out first in [30], and then in [17] where all the rigorous compactness estimates have been obtained together with numerical robustness by following the well-balanced canvas involving non-conservative products (see also [34, 14, 16]). The present text will exploit the same strategy, except that new (and quite substantial) difficulties arise from the coupling with the diffusion equation on φ .

1.2. WB and AP: two sides of the same coin. Asymptotic-Preserving and Well-Balanced schemes are two complementary methodologies which, besides having been introduced roughly at the same moment (see [29, 28, 30, 32] and [19, 14]), address similar numerical issues in different contexts. Well-balanced schemes were designed by Greenberg and LeRoux for scalar conservation laws with source terms with two main goals: **handling stiff source terms** (thus allowing for problems displaying 2 distinct characteristic time-scales) without any time-step restriction besides the usual convective CFL condition and **being fully consistent with a time-asymptotic behavior** ruled by steady-state equations. Thus, through an original concentration process of sources onto Dirac masses located at both interfaces of each computational cell, stiffness was disappearing because source terms were rendered by means of a

nonconservative jump relation induced by the atomic measures, that have been called later “zero waves” [1]. Clearly, if a source term can be discretized in a consistent way by means of a supplementary jump relation inside a Riemann solver, its size (hence its “reaction rate”) is not relevant when it comes to time-step restriction. Moreover, Riemann problems involving a source term concentrated on a Dirac measure present the advantage of still being self-similar (as opposed to so-called “generalized Riemann problems” [37]) hence they still can be exactly solved in many interesting cases. An exact resolution of these Riemann problems in a Godunov scheme allows to derive an exact weak solution at steady-state; indeed, steady-states for one-dimensional homogeneous systems of conservation laws (possibly non-conservative) consists in a succession of constant states separated by stationary discontinuities. Assuming we have derived the correct jump relation across all the zero-waves resulting from the numerical grid, thus ensuring consistency with the original problem, classical results yield that numerical viscosity effects completely disappear at steady states for the well-balanced Godunov scheme. Efforts have been made in [14, 17] (see §§3.3-4 and 2.1-2 respectively) to establish uniform BV-bounds in order to define correctly the nonconservative products following the general framework of weak limits [34].

It is at this point that it is easy to highlight the very close relation with the Asymptotic-Preserving methodology. There, the two main objectives are **handling the stiff parabolic scaling** of the type (1.4) for $\varepsilon \ll 1$ without heavy restrictions on the time-step and **being consistent with the time-dependent asymptotic behavior** given in our particular context by (1.5). This asymptotic behavior results from a delicate balance appearing between flux terms and the lower-order, but stiff, right-hand side as rigorous proofs clearly explain in *e.g.* [13, 36, 43]. Put this way, it comes with no surprise that well-balanced schemes, as soon as they can be efficiently stabilized in order to handle convective velocities blowing up in $\frac{1}{\varepsilon}$, furnish very reliable asymptotic-preserving discretizations. This is what has been shown in [16, 17] for the telegraph equations and 2×2 discrete kinetic models; here, we shall follow the same canvas in order to treat a more involved system arising from biological modelling. Obviously, there may exist AP schemes which don’t result from a WB Godunov scheme which convective step is treated implicitly in time and yield correct asymptotic limits as $\varepsilon \rightarrow 0$ [2, 5], however, the deep relation between these two numerical approaches shouldn’t be overlooked. For instance, the recent scheme for Fokker-Planck equations [46] is a well-balanced scheme which doesn’t tell his name.

This paper is organized as follows: §2 is devoted to recalling theoretical results shown mainly in [24, 26]. In §3, we study the well-balanced Godunov scheme for (1.2) in hyperbolic regime; in particular, smoothness estimates for the diffusive equation on φ are obtained from the Hoff-Smoller L^1 study of numerical schemes for the heat equation [25]. In §4, we consider the asymptotic-preserving rewriting of this Godunov scheme and show various stability estimates as $\varepsilon \rightarrow 0$. However, in contrast with former works [16, 17], it doesn’t seem possible to derive BV estimates which are uniform in both $\frac{1}{h}$, the maximum frequency allowed by the grid, and ε ; we think the reason is that Keller-Segel equations don’t have BV estimates at the continuous level. So it is necessary to pass first at the $\varepsilon \rightarrow 0$ limit while maintaining $h > 0$ fixed, and then later send $h \rightarrow 0$. Finally, §5 displays numerical results illustrating former stability estimates in both regimes and §6 gives concluding remarks.

2. Theoretical results for the 1D Cattaneo model. In all the sequel, we shall tacitly assume all the restrictions which are necessary to have existence and uniqueness of solutions for both (1.2) and (1.5). Here, we give a quick review of these

results from [24, 26]; let us consider the Cauchy problem for (1.2), equivalently for (1.1), on the complete real line (boundary conditions **(bc1)** in [24])

$$w(t=0, \cdot) = w^0, \quad z(t=0, \cdot) = z^0, \quad \varphi(t=0, \cdot) = \varphi^0 \text{ for } x \in \mathbb{R},$$

which implies that $\rho(t=0, \cdot) = w^0 + z^0$ and $J(t=0, \cdot) = \lambda(z^0 - w^0)$. For technical reasons, besides $\alpha \geq 0$ and $\beta \geq 0$, it is customary to assume:

1. nonnegative initial values with compact support: $w^0 \geq 0, z^0 \geq 0, \varphi^0 \geq 0$; moreover, some smoothness is required $w^0, z^0 \in L^\infty(\mathbb{R}), \varphi^0 \in W^{1,\infty}(\mathbb{R})$.
2. nonnegative turning rates: this is equivalent to the quasi-monotonicity of (1.2), which is ensured by the subcharacteristic condition. It is also postulated that turning rates should be symmetric when $\partial_x \varphi$ changes its sign.
3. local Lipschitz continuity and boundedness of turning rates: in our case, they inherit all the smoothness of $\partial_x \varphi$. This assumption allows to enforce the quasi-monotonicity property if turning rates are taken as $\pm \max(0, \lambda - |\partial_x \varphi|)$.

Under all these assumptions, Hillen and Stevens prove in [24] that:

THEOREM 2.1. *There exists a maximal time T_{max} and a unique solution to (1.2),*

$$(w, z, \varphi) \in L^\infty([0, T_{max}[; L^\infty(\mathbb{R})^2 \times W^{1,\infty}(\mathbb{R})),$$

with T_{max} possibly infinite. If $T_{max} < +\infty$, then $\lim_{t \rightarrow T_{max}} \|\varphi(t, \cdot)\|_{W^{1,\infty}(\mathbb{R})} \rightarrow +\infty$. Some further results have been published in [26], including the parabolic limit of (1.4):

THEOREM 2.2. *Assume all the former hypotheses and moreover, that $w^0, z^0, \varphi^0 \in C^\infty(\mathbb{R})$ with compact support and steady-state initial distribution of $\varphi(t=0, \cdot)$:*

$$\beta \varphi^0 = D \partial_{xx} \varphi^0 + \alpha(w^0 + z^0).$$

There exists a unique solution to (1.2) for any $T > 0$, $(w, z, \varphi) \in C([0, T[, W^{1,\infty}(\mathbb{R})^3)$. Passing to the limit with $\varepsilon \rightarrow 0$ in (1.2) in order to derive (1.5) is also studied in [26]; however, we prefer to refer to [9]. The framework of this paper corresponds to their “third scaling” called “small reaction rates”. An interesting estimate obtained in [26] concerns the evolution in time of the norm of φ in $W^{1,\infty}$:

LEMMA 2.3. *If $\rho \in L^\infty(\mathbb{R}^+, L^2 \cap L^\infty(\mathbb{R}))$, then for any $t > 0$, φ satisfies:*

$$\begin{aligned} \|\varphi(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq C \|\rho^0\|_{L^1(\mathbb{R})}, \\ \|\partial_x \varphi(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq C \left(1 + \|\rho^0\|_{L^1(\mathbb{R})} \left[1 + \max(0, \log t) + \left| \log \left(\sup_{\tau \leq t} \|\rho(\tau, \cdot)\|_{L^2(\mathbb{R})} \right) \right| \right] \right), \end{aligned}$$

with C depending only on $\alpha, \beta \in (\mathbb{R}^+)^2$.

3. Space localization and Godunov scheme in hyperbolic regime. The hyperbolic regime corresponds to a value of the relaxation parameter $\varepsilon \simeq 1$; for simplicity, we shall assume $\varepsilon = 1$ in this section and drop the corresponding indexes. In all the sequel, we shall work with a uniform numerical grid where the space step is denoted by h , the time step, Δt ; both are linked through the classical CFL condition

$$\lambda \Delta t \leq h.$$

3.1. Sources on a Dirac comb: jump relations across zero-waves. We now carefully follow the ideas already presented in [16, 17], that is to say, we pass from (1.2) to the non-linear and non-conservative system:

$$(3.1) \quad \begin{cases} \partial_t w - \lambda \partial_x w = -\frac{1}{2\lambda} \sum_{j \in \mathbb{Z}} h G(\partial_x \varphi; w, z) \delta(x - (j - \frac{1}{2})h) \\ \partial_t z + \lambda \partial_x z = \frac{1}{2\lambda} \sum_{j \in \mathbb{Z}} h G(\partial_x \varphi; w, z) \delta(x - (j - \frac{1}{2})h) \\ \partial_t \varphi - D \partial_{xx} \varphi = \alpha \rho - \beta \varphi, \end{cases}$$

where $\delta(\cdot)$ stands for the Dirac mass in $x = 0$. Clearly, in the context of discontinuous w, z , this formulation looks like being unstable because of the products “Heaviside \times Dirac” appearing on the right-hand side. However, it has been rigorously shown in [17] that these non-conservative products can be rigorously defined as weak limits in the framework of [34] thanks to the uniform BV estimates which come from the linear convection in (1.2) (similar estimates for scalar balance laws are given in [14]).

The presence of the Dirac masses induces new discontinuities on the locations $(j - \frac{1}{2})h$, $j \in \mathbb{Z}$ which are called the “zero waves”; in order to solve the Riemann problem for (3.1), we must derive appropriate jump relations. Let us denote by \bar{w}, \bar{z} the microscopic profiles which are shrunk inside the non-conservative products: they are to satisfy the stationary equations of (1.2) which read, for $x \in [0, h]$,

$$(3.2) \quad 2\lambda^2 \partial_x \begin{pmatrix} \bar{w} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \partial_x \varphi + \lambda & \partial_x \varphi - \lambda \\ \partial_x \varphi + \lambda & \partial_x \varphi - \lambda \end{pmatrix} \begin{pmatrix} \bar{w} \\ \bar{z} \end{pmatrix}.$$

The solution can be written explicitly, with obvious notation:

$$\bar{J}(h) \equiv \bar{J}(0), \quad \bar{\rho}(h) = \left[\bar{\rho}(0) - \frac{h\bar{J}(0)}{\lambda^2} \right] \exp(h\partial_x \varphi / \lambda^2).$$

At this microscopic scale, the quantity $\partial_x \varphi$ is a constant. Hence we have the result:

LEMMA 3.1. *For any $h > 0$, the stationary equations of (1.2) yield the following jump relations across the zero-waves of (3.1) located in $(j - \frac{1}{2})h$, $j \in \mathbb{Z}$:*

$$(3.3) \quad \bar{w}(0) = \frac{2}{1+B} \bar{w}(h) + \frac{1-A}{1+B} \bar{z}(0), \quad \bar{z}(h) = -\frac{1-B}{1+B} \bar{w}(h) + \frac{A+B}{1+B} \bar{z}(0),$$

with the notation:

$$A = \left(1 - \frac{h}{\lambda}\right) \exp(h\partial_x \varphi / \lambda^2), \quad B = \left(1 + \frac{h}{\lambda}\right) \exp(h\partial_x \varphi / \lambda^2).$$

In particular, the following important relation holds: (flux conservation)

$$(3.4) \quad \bar{w}(0) - \bar{w}(h) = \frac{1-B}{1+B} \bar{w}(h) + \frac{1-A}{1+B} \bar{z}(0) = \bar{z}(0) - \bar{z}(h).$$

Proof. We rewrite $\bar{\rho}(h)$ with the diagonal variables \bar{w}, \bar{z} :

$$\bar{w}(h) + \bar{z}(h) = \left(\bar{w}(0) + \bar{z}(0) + h \frac{\bar{w}(0) - \bar{z}(0)}{\lambda} \right) \exp\left(\frac{h\partial_x \varphi}{\lambda^2}\right),$$

and we also have that $\bar{w}(h) - \bar{w}(0) = \bar{z}(0) - \bar{z}(h)$. This leads to the following system:

$$\begin{pmatrix} (1 + \frac{h}{\lambda}) \exp(h \frac{\partial_x \varphi}{\lambda^2}) & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{w}(0) \\ \bar{z}(h) \end{pmatrix} = \begin{pmatrix} 1 & -(1 - \frac{h}{\lambda}) \exp(h \frac{\partial_x \varphi}{\lambda^2}) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{w}(h) \\ \bar{z}(0) \end{pmatrix}.$$

The matrix on the left is always invertible because its determinant $1 + (1 + \frac{h}{\lambda}) \exp(h \frac{\partial_x \varphi}{\lambda^2})$ is strictly positive. The coefficients A and B are taken from these matrices; inverting and multiplying them yields the jump relations (3.4). \square

It is interesting to observe that a simple linearization of exponentials gives:

$$\begin{aligned} 1 &\geq \frac{1-A}{1+B} = \frac{\exp(-h\partial_x \varphi / \lambda^2) - 1 + h/\lambda}{\exp(-h\partial_x \varphi / \lambda^2) + 1 + h/\lambda} \simeq \frac{h}{2\lambda} \left(\frac{\lambda - \partial_x \varphi}{\lambda - (\partial_x \varphi - \lambda)h/2\lambda} \right) \geq 0, \\ 0 &\geq \frac{1-B}{1+B} = \frac{\exp(-h\partial_x \varphi / \lambda^2) - 1 - h/\lambda}{\exp(-h\partial_x \varphi / \lambda^2) + 1 + h/\lambda} \simeq -\frac{h}{2\lambda} \left(\frac{\partial_x \varphi + \lambda}{\lambda - (\partial_x \varphi - \lambda)h/2\lambda} \right) \geq -1. \end{aligned}$$

These linearizations will be useful for establishing consistency as $h \rightarrow 0$ since $|\partial_x \varphi|$ remains bounded for $D, \beta \geq 0$ are big enough. In practice, a first order divided difference computed at each interface $x = (j - \frac{1}{2})h$ and $t = n\Delta t$ will appear in place of $\partial_x \varphi$; there are no zero-waves involved in the discretization of φ , obviously.

3.2. Hoff-Smoller theory for the equation on φ . It is well-known that strictly parabolic equations induce an instantaneous regularizing effect which, in our context, makes $\varphi(t, \cdot)$ a $W^{2,p}(\mathbb{R})$ function for $t > 0$. This feature can't exactly be reproduced by a simple finite-difference scheme, however, Hoff and Smoller showed that a good deal of it still holds for conventional centered discretizations. In order to present part of their results, let us first present the 1D heat equation:

$$(3.5) \quad \partial_t v = D \partial_{xx} v, \quad v_j^{n+1} = v_j^n + \frac{D\Delta t}{h^2} (v_{j+1}^n - 2v_j^n + v_{j-1}^n).$$

Hoff and Smoller [25] introduce next the concept of “numerical fundamental solution”:

DEFINITION 1. *For any $n \in \mathbb{N}$, the numerical fundamental solution to (3.5) is the sequence $(\mathcal{E}_j^n)_{j \in \mathbb{Z}} \in \ell^1 \cap \ell^\infty(\mathbb{Z})$ satisfying:*

$$\mathcal{E}_j^0 = \frac{1}{h} \delta_{j=0}, \quad \mathcal{E}_j^{n+1} = \mathcal{E}_j^n \left(1 - \frac{2D\Delta t}{h^2} \right) + \frac{D\Delta t}{h^2} (\mathcal{E}_{j+1}^n + \mathcal{E}_{j-1}^n).$$

It is nonnegative if $2D\Delta t \leq h^2$.

We denote by “*” the discrete convolution product:

$$\forall a, b \in \ell^1 \cap \ell^\infty(\mathbb{Z}), \quad (a * b)_j := \sum_{\ell \in \mathbb{Z}} h a_{\ell-j} b_\ell,$$

which satisfies (besides commutativity):

$$\sum_{j \in \mathbb{Z}} h |(a * b)_j| \leq \left(\sum_{j \in \mathbb{Z}} h |a_j| \right) \left(\sum_{j \in \mathbb{Z}} h |b_j| \right), \quad \sup_{j \in \mathbb{Z}} |(a * b)_j| \leq \sup_{j \in \mathbb{Z}} |a_j| \left(\sum_{j \in \mathbb{Z}} h |b_j| \right).$$

We propose therefore to define a piecewise constant approximation $\tilde{\varphi}^h(n\Delta t, \cdot)$ for any $n \in \mathbb{N}$ as usual, $\varphi^h(t, x) := \varphi_j^n$ for $t, x \in [n\Delta t, (n+1)\Delta t] \times [(j - \frac{1}{2})h, (j + \frac{1}{2})h[$; a consequence of Theorem 2.1 in [25] is the explicit form of the following scheme,

$$\varphi_j^{n+1} (1 + \Delta t \beta) = \varphi_j^n + \alpha \Delta t \rho_j^n + \frac{D\Delta t}{h^2} (\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n),$$

which involves a discrete Duhamel's principle:

$$(3.6) \quad \varphi_j^n = (1 + \Delta t \beta)^{-n} (\mathcal{E}^n * \varphi^0) + \frac{\alpha \Delta t}{1 + \beta \Delta t} \sum_{k=0}^{n-1} (1 + \Delta t \beta)^{k-(n-1)} (\mathcal{E}^{(n-1)-k} * \rho^k).$$

This notation is used hereafter:

$$\forall j, n \in \mathbb{Z} \times \mathbb{N}, \quad (\partial_x \varphi)_{j+\frac{1}{2}}^n := \frac{1}{h} (\varphi_{j+1}^n - \varphi_j^n).$$

The main interest in choosing this scheme is the numerical analogue of the regularizing effect which holds at the continuous level as stated in Theorem 2.2 of [25]:

THEOREM 3.2. Assume $2\Delta t D \leq h^2$, then:

$$\forall n \in \mathbb{N}, \quad \sum_{j \in \mathbb{Z}} \mathcal{E}_j^n = 1, \quad 0 \leq \mathcal{E}_j^n \leq \min \left(\frac{1}{h}, \frac{C_0}{\sqrt{n\Delta t}} \right),$$

and moreover,

$$\sum_{j \in \mathbb{Z}} |\mathcal{E}_{j+1}^n - \mathcal{E}_j^n| \leq \frac{C_1}{\sqrt{n\Delta t}}, \quad \frac{1}{h} \sum_{j \in \mathbb{Z}} |\mathcal{E}_{j+1}^n - 2\mathcal{E}_j^n + \mathcal{E}_{j-1}^n| \leq \frac{C_2}{n\Delta t}.$$

The constants C_0, C_1, C_2 depend only on $D \geq 0$.

It is now possible to study the time propagation of the subcharacteristic condition:

LEMMA 3.3. Let $2\Delta t D = O(h^2)$ and assume that $\varphi^0 \in L^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $\rho^h(t, \cdot) \in L^1 \cap BV(\mathbb{R})$ for any $t \geq 0$. Then, for $\Delta t > 0$, the subcharacteristic condition holds for any $n \in \mathbb{N}$ if one of the two following requirements is met:

1.

$$\alpha \sup_n TV(\rho^h(n\Delta t, \cdot)) \leq \beta \left(\lambda - (1 + \beta\Delta t)^{-n} \sup_j |(\partial_x \varphi)_{j+\frac{1}{2}}^0| \right),$$

2.

$$\lambda - (1 + \beta\Delta t)^{-n} \sup_j |(\partial_x \varphi)_{j+\frac{1}{2}}^0| \geq S_n \|\rho^h\|_{L^\infty} + \frac{2\alpha\Delta t}{h(1 + \beta\Delta t)} \|\rho^h(n\Delta t, \cdot)\|_{L^1(\mathbb{R})},$$

with S_n depends on n, α, β and C_1 . Moreover, there also holds:

$$TV((\partial_x \varphi)(n\Delta t, \cdot)) := \sum_{j \in \mathbb{Z}} |(\partial_x \varphi)_{j+\frac{1}{2}}^n - (\partial_x \varphi)_{j-\frac{1}{2}}^n| \leq (1 + \beta\Delta t)^{-n} \frac{C_2}{n\Delta t} \|\varphi^0\|_{L^1(\mathbb{R})} + \tilde{S}_n \|\rho^h\|_{L^\infty} + \frac{4\alpha\Delta t}{h^2(1 + \beta\Delta t)} \|\rho^h(n\Delta t, \cdot)\|_{L^1(\mathbb{R})}.$$

Proof. Thanks to the first estimate of Theorem 3.2, everything starts with

$$(\partial_x \varphi)_{j+\frac{1}{2}}^n - \lambda = \mathcal{E}^n * \left(\frac{(\partial_x \varphi)_{j+\frac{1}{2}}^0}{(1 + \beta\Delta t)^n} - \lambda \right) + \frac{\alpha\Delta t}{1 + \beta\Delta t} \sum_{k=0}^{n-1} \frac{\delta[\mathcal{E}^{(n-1)-k} * \rho^k]}{h(1 + \beta\Delta t)^{(n-1)-k}},$$

which should be negative. The first term is handled by taking advantage of the resulting convex combination and taking the supremum on $j \in \mathbb{Z}$. The second term is delicate as we must decide on which part we want to apply the divided difference operator denoted here $\delta[\cdot]/h$ for easiness in reading. The first solution, which leads to Point 1 in Lemma 3.3 is to apply it on ρ^h and suppose this function has bounded total variation in the space variable. In this case, it remains only to observe that:

$$\frac{\alpha\Delta t}{1 + \beta\Delta t} \sum_{k=0}^{n-1} (1 + \beta\Delta t)^{k-(n-1)} \leq \frac{\alpha}{\beta}.$$

To prove Point 2, we proceed by applying the divided difference on the fundamental solution. However, it can't be directly applied for $k = n - 1$, but in this case, we see that $TV(\mathcal{E}^0) = 2/h$. The rest of the summation can be controlled by means of:

$$\|\rho^h\|_{L^\infty} \left\{ \sum_{k=0}^{n-2} (1 + \beta\Delta t)^{k-n} \frac{C_1 \alpha \Delta t}{\sqrt{((n-1)-k)\Delta t}} \right\} =: \|\rho^h\|_{L^\infty} S_n.$$

Now, in order to prove the estimate on the total variation of the first-order divided difference, we observe that $TV(\delta\mathcal{E}^0/h) = \|\delta^2\mathcal{E}^0/h^2\|_{\ell^1(\mathbb{Z})} = 4/h^2$ and we compute:

$$(\partial_x \varphi)_{j+\frac{1}{2}}^n - (\partial_x \varphi)_{j-\frac{1}{2}}^n = \frac{\varphi^0 * [\delta^2 \mathcal{E}^n]}{h^2(1 + \beta \Delta t)^n} + \frac{\alpha \Delta t}{1 + \beta \Delta t} \sum_{k=0}^{n-1} \frac{\delta^2[\mathcal{E}^{(n-1)-k} * \rho^k]}{h^2(1 + \beta \Delta t)^{(n-1)-k}}.$$

We proceed by directly applying the estimates of Theorem 3.2 to the first term. The new constant \tilde{S}_n is given by the following summation:

$$\tilde{S}_n := \sum_{k=0}^{n-2} \frac{C_2 \alpha \Delta t}{((n-1)-k)\Delta t} (1 + \beta \Delta t)^{k-n}.$$

Finally, the last term corresponding to $k = n - 1$ is bounded by:

$$\frac{\alpha \Delta t}{1 + \beta \Delta t} TV(\delta\mathcal{E}^0/h) \|\rho^h(n\Delta t, \cdot)\|_{L^1(\mathbb{R})} \leq \frac{4\alpha \Delta t}{h^2(1 + \beta \Delta t)} \|\rho^h(n\Delta t, \cdot)\|_{L^1(\mathbb{R})}.$$

A ℓ^∞ bound on the second order divided difference of φ^h can be obtained the same way by replacing $\|\varphi^0\|_{L^1(\mathbb{R})}$ and $\|\rho^h(n\Delta t, \cdot)\|_{L^1(\mathbb{R})}$ by $\|\varphi^0\|_{L^\infty(\mathbb{R})}$ and $\|\rho^h(n\Delta t, \cdot)\|_{L^\infty(\mathbb{R})}$, respectively. \square

In the sequel (see Lemma 3.4), we shall see that the propagation in time of the subcharacteristic condition ensures that the $L^p(\mathbb{R})$ norms of $\rho(t, \cdot)$ remain bounded as a consequence of the quasi-monotonicity of the source term G **only** which is a consequence of the subcharacteristic condition. Establishing a BV-bound for $\rho^h(t, \cdot)$ asks for a ℓ^∞ bound on the second order divided difference of $\varphi^h(t, \cdot)$. Concerning the choice of having studied an explicit time discretization on the diffusion equation of φ , there is a simple way to circumvent the restrictive parabolic CFL condition: it suffices to choose the time-step for the diffusion equation as $\lambda h/D$ times the one for the hyperbolic system and to keep ρ constant during the $D/(\lambda h)$ sub-iterations. This adjustment will disappear when setting up the Asymptotic-Preserving process.

3.3. Riemann solver and a Godunov scheme in hyperbolic regime.

Lemma 3.1 is the key to produce a Riemann solver for the non-conservative system (3.1) and thus a Godunov scheme. Let's denote $w_j^n \simeq w(n\Delta t, jh)$ and $z_j^n \simeq w(n\Delta t, jh)$ for any $j \in \mathbb{Z}$ and $n \in \mathbb{N}$; these numerical approximations induce piecewise constant functions $w^h(t, x)$ and $z^h(t, x)$ such that:

$$w^h(t, x) := w_j^n, \quad z^h(t, x) := z_j^n \text{ for } t, x \in [n\Delta t, (n+1)\Delta t[\times \left[(j - \frac{1}{2})h, (j + \frac{1}{2})h \right[.$$

The Godunov scheme proceeds in defining a control cell $](j - \frac{1}{2})h, (j + \frac{1}{2})h[\times]n\Delta t, (n+1)\Delta t[$ around each point $x_j = jh$, solving a Riemann problem on both interfaces $(j \pm \frac{1}{2})h$ and averaging: see Fig. 3.3. Within the notation of this figure, it holds:

$$w_j^{n+1} = w_j^n + \frac{\lambda \Delta t}{h} \left(w_{j+\frac{1}{2}}^* - w_j^n \right), \quad z_j^{n+1} = z_j^n - \frac{\lambda \Delta t}{h} \left(z_j^n - z_{j-\frac{1}{2}}^* \right).$$

A first consequence is the conservation property on the $\rho^h := w^h + z^h$ variable:

$$(3.7) \quad \rho^h((n+1)\Delta t, jh) := \rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{h} \left(\underbrace{\lambda(z_j^n - w_{j+\frac{1}{2}}^*)}_{(z_{j+\frac{1}{2}}^* - w_{j+1}^n)} - \lambda(z_{j-\frac{1}{2}}^* - w_j^n) \right).$$

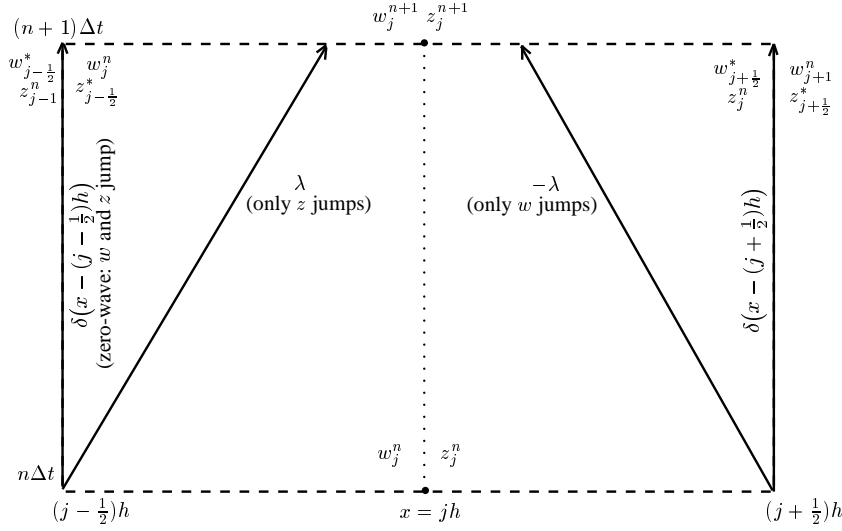


FIG. 3.1. Illustration of the control cell for the non-conservative Godunov scheme

Clearly, the “interface values” which result from the inclusion of the zero-waves yield:

$$(3.8) \quad \begin{cases} w_j^{n+1} &= w_j^n + \frac{\lambda \Delta t}{h} (w_{j+1}^n - w_j^n) \\ &\quad + \frac{\lambda \Delta t}{h(1+B_{j+\frac{1}{2}}^n)} \left((1-B_{j+\frac{1}{2}}^n)w_{j+1}^n + (1-A_{j+\frac{1}{2}}^n)z_j^n \right), \\ z_j^{n+1} &= z_j^n - \frac{\lambda \Delta t}{h} (z_j^n - z_{j-1}^n) \\ &\quad - \frac{\lambda \Delta t}{h(1+B_{j-\frac{1}{2}}^n)} \left((1-B_{j-\frac{1}{2}}^n)w_j^n + (1-A_{j-\frac{1}{2}}^n)z_{j-1}^n \right). \end{cases}$$

We stress that since A and B are computed at the borders of each control cell, the term $\partial_x \varphi$ is very well defined because odd derivatives “live on the staggered grid”.

PROPOSITION 3.4. *Let w^0 and z^0 belong to $L^p(\mathbb{R})$, $1 \leq p \leq \infty$; under both the CFL condition $\lambda \Delta t \leq h$ and the subcharacteristic restriction $\lambda \geq |\partial_x \varphi|$, there holds:*

$$(3.9) \quad \forall t \in \mathbb{R}^+, \quad \|w^h(t, \cdot)\|_{L^p(\mathbb{R})} + \|z^h(t, \cdot)\|_{L^p(\mathbb{R})} \leq \|w^0\|_{L^p(\mathbb{R})} + \|z^0\|_{L^p(\mathbb{R})}.$$

Proof. One checks that the linear well-balanced scheme is a convex combination:

$$|w_j^{n+1}| \leq |w_j^n| \left(1 - \frac{\lambda \Delta t}{h} \right) + \frac{\lambda \Delta t}{h} |w_{j+1}^n| \left(1 + \frac{1-B_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} \right) + \frac{\lambda \Delta t (1-A_{j+\frac{1}{2}}^n)}{h(1+B_{j+\frac{1}{2}}^n)} |z_j^n|,$$

and our assumptions ensure that $-1 \leq \frac{1-B_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} \leq 0$ and $0 \leq \frac{1-A_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} \leq 1$, hence

$$|z_j^{n+1}| \leq |z_j^n| \left(1 - \frac{\lambda \Delta t}{h} \right) + \frac{\lambda \Delta t}{h} |z_{j-1}^n| \left(1 - \frac{1-A_{j-\frac{1}{2}}^n}{1+B_{j-\frac{1}{2}}^n} \right) - \frac{\lambda \Delta t (1-B_{j-\frac{1}{2}}^n)}{h(1+B_{j-\frac{1}{2}}^n)} |w_j^n|.$$

It remains to sum up or to take the supremum on $j \in \mathbb{Z}$ in order to conclude. \square

The bound (3.9) is **crucial**; however, since the subsystem of (3.1) ruling only w and z is **not** translation-invariant because of $\partial_x \varphi$, it doesn't lead to a BV-bound as

directly as one could hope for at first glance. However, this bound completes nicely Lemma 3.3 because if we choose $p = 1$ in (3.9), we see that the hypothesis made on $\|\rho^h(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|w^h(t, \cdot)\|_{L^1(\mathbb{R})} + \|z^h(t, \cdot)\|_{L^1(\mathbb{R})}$ is satisfied for instance if we assume that $w^0 \geq 0$ and $z^0 \geq 0$, which amounts to asking for a small momentum initially.

3.4. Compactness for the Well-Balanced Godunov scheme. To establish strong convergence of w^h and z^h toward the unique solution of (1.2), we need a bound on the total variation of $w^h(t, \cdot)$ and $z^h(t, \cdot)$.

LEMMA 3.5. *Let $\beta \geq 0$, $\varphi^0 \in W^{1,\infty}(\mathbb{R})$ and $w^0, z^0 \in L^1 \cap BV(\mathbb{R})$; under both the CFL condition $\lambda \Delta t \leq h$ and the subcharacteristic restriction $\lambda \geq \sup_{j,n} |(\partial_x \varphi)_{j+\frac{1}{2}}^n|$, the following BV-bounds hold for any $n \in \mathbb{N}$:*

$$(3.10) \quad TV(w^h(n\Delta t, \cdot)) + TV(z^h(n\Delta t, \cdot)) \leq TV(w^0) + TV(z^0) + \frac{L}{\lambda} n\Delta t (\|w^0\|_{L^1(\mathbb{R})} + \|z^0\|_{L^1(\mathbb{R})}),$$

where L is the Lipschitz constant of $\left(\frac{1-A}{1+B}, \frac{1-B}{1+B}\right)$ depending on the values of $(\partial_x \varphi)_{j+\frac{1}{2}}^n$.

Proof. One proceeds by computing the differentiating the well-balanced scheme:

$$\begin{aligned} |w_{j+1}^{n+1} - w_j^{n+1}| &\leq \left(1 - \frac{\lambda \Delta t}{h}\right) |w_{j+1}^n - w_j^n| + \frac{\lambda \Delta t}{h} \left(1 + \frac{1-B_{j+\frac{3}{2}}^n}{1+B_{j+\frac{3}{2}}^n}\right) |w_{j+2}^n - w_{j+1}^n| + \\ &\frac{\lambda \Delta t}{h} \left|\frac{1-B_{j+\frac{3}{2}}^n}{1+B_{j+\frac{3}{2}}^n} - \frac{1-B_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n}\right| |w_{j+1}^n| + \frac{\lambda \Delta t}{h} \frac{1-A_{j+\frac{3}{2}}^n}{1+B_{j+\frac{3}{2}}^n} |z_{j+1}^n - z_j^n| + \frac{\lambda \Delta t}{h} \left|\frac{1-A_{j+\frac{3}{2}}^n}{1+B_{j+\frac{3}{2}}^n} - \frac{1-A_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n}\right| |z_j^n|, \end{aligned}$$

where the same properties on $\frac{1-A}{1+B} \geq 0$, $\frac{1-B}{1+B} \leq 0$ are used, and then

$$\begin{aligned} |z_{j+1}^{n+1} - z_j^{n+1}| &\leq \left(1 - \frac{\lambda \Delta t}{h}\right) |z_{j+1}^n - z_j^n| + \frac{\lambda \Delta t}{h} \left(1 - \frac{1-A_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n}\right) |z_j^n - z_{j-1}^n| + \\ &\frac{\lambda \Delta t}{h} \left|\frac{1-A_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} - \frac{1-A_{j-\frac{1}{2}}^n}{1+B_{j-\frac{1}{2}}^n}\right| |z_{j-1}^n| - \frac{\lambda \Delta t}{h} \frac{1-B_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} |w_{j+1}^n - w_j^n| + \frac{\lambda \Delta t}{h} \left|\frac{1-B_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} - \frac{1-B_{j-\frac{1}{2}}^n}{1+B_{j-\frac{1}{2}}^n}\right| |w_j^n|. \end{aligned}$$

Summing on $j \in \mathbb{Z}$, we have cancellations except for the terms rendering from the oscillations of $\frac{1-A}{1+B}$ and $\frac{1-B}{1+B}$ which depend on x through $\partial_x \varphi$. They read:

$$O := 2 \frac{\lambda \Delta t}{h} \sum_{j \in \mathbb{Z}} \left| \frac{1-B_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} - \frac{1-B_{j-\frac{1}{2}}^n}{1+B_{j-\frac{1}{2}}^n} \right| |w_j^n| + \left| \frac{1-A_{j+\frac{3}{2}}^n}{1+B_{j+\frac{3}{2}}^n} - \frac{1-A_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} \right| |z_j^n|.$$

Now, we know that these two quantities are perturbations of $h(\partial_x \varphi \mp \lambda)/2\lambda^2$. Moreover, from the discrete regularizing effect of [25] and Lemma 3.3, we deduce their Lipschitz regularity (recall that the denominator never vanishes), so there exists a Lipschitz constant L depending on the initial data and the grid parameters such that

$$O \leq 2L \frac{\lambda \Delta t}{h} \frac{h}{2\lambda^2} \sum_{j \in \mathbb{Z}} |w_j^n| + |z_j^n| \leq \frac{L \Delta t}{\lambda} (\|w^0\|_{L^1(\mathbb{R})} + \|z^0\|_{L^1(\mathbb{R})}),$$

the last inequality coming from Lemma 3.4. \square

REMARK 1. *One sees here the big difficulty in treating the system (1.2): the BV-bound (3.10) needs both second order divided differences for $\varphi^h(t, \cdot)$ to be either in $\ell^1(\mathbb{Z})$, or in $\ell^\infty(\mathbb{Z})$ for claiming that L is uniformly bounded and also the subcharacteristic condition in order to keep the source term G quasi-monotone and ensure that the L^p bounds (3.9) hold through time.*

We switch to time-equicontinuity:

LEMMA 3.6. *Under the hypotheses of Lemma 3.5, the piecewise constant approximations w^h, z^h are endowed with a L^1 -modulus of time equicontinuity:*

$$\|w^h(t + \Delta t, \cdot) - w^h(t, \cdot)\|_{L^1(\mathbb{R})} + \|z^h(t + \Delta t, \cdot) - z^h(t, \cdot)\|_{L^1(\mathbb{R})} \leq \Delta t [TV(w^h(t, \cdot)) + TV(z^h(t, \cdot)) + C (\|w^0\|_{L^1(\mathbb{R})} + \|z^0\|_{L^1(\mathbb{R})})],$$

for any $t \in \mathbb{R}^+$ and $C := \sup_{j,n} \left(\left| \frac{1-A_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} \right|, \left| \frac{1-B_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} \right| \right)$.

Proof. It suffices to add up the schemes on $w_j^{n+1} - w_j^n$ and $z_j^{n+1} - z_j^n$, to take the modulus, multiply by h and sum on $j \in \mathbb{Z}$:

$$h \sum_{j \in \mathbb{Z}} |w_j^{n+1} - w_j^n| + |z_j^{n+1} - z_j^n| \leq \Delta t \sum_{j \in \mathbb{Z}} |w_{j+1}^n - w_j^n| + |z_{j+1}^n - z_j^n| + 2\Delta t \lambda \sup_j \left(\left| \frac{1-A_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} \right|, \left| \frac{1-B_{j+\frac{1}{2}}^n}{1+B_{j+\frac{1}{2}}^n} \right| \right) h \sum_{j \in \mathbb{Z}} |w_{j+1}^n| + |z_j^n|.$$

Applying the BV-bound (3.10) and the L^p -bound (3.9) together with the properties of $\varphi^h(t, \cdot)$ yields the conclusion. \square

Finally we are in position to prove the convergence of our well-balanced scheme toward the unique solution of (1.2):

THEOREM 3.7. *Let $\beta \geq 0$, $\varphi^0 \in L^1 \cap W^{1,\infty}(\mathbb{R})$ and $w^0, z^0 \in L^1 \cap BV(\mathbb{R})$; under both the CFL and the subcharacteristic restrictions, $\lambda \Delta t \leq h$, $\lambda \geq \text{Lip}(\varphi^h(t, \cdot))$, the sequence w^h, z^h, φ^h converges strongly as $h \rightarrow 0$ toward the unique solution of (1.2).*

Proof. The proof consists in checking both the Lax requirements: stability and consistency. Lemmas 3.9, 3.5 and 3.6 ensure that one can extract a subsequence indexed by $h_k \rightarrow 0$ which converges strongly in $L^1_{loc}(\mathbb{R}_*^+ \times \mathbb{R})$. The results from [24] yield the strong convergence of φ^h generated by the finite differences scheme (3.6). Concerning the consistency, the diagonal convective part is very classical. The consistency for the source term G can be obtained from the linearization of the exponentials for small h and $\lambda > 0$: (these quantities appear always multiplied by Δt)

$$\begin{aligned} \frac{\lambda}{h} \left(\frac{1-A}{1+B} \right) &\simeq \frac{\lambda}{h} \frac{h}{2\lambda} \left(\frac{\lambda - \partial_x \varphi^h}{\lambda - (\partial_x \varphi^h - \lambda)h/2\lambda} \right) \rightarrow \frac{1}{2\lambda} (\lambda - \partial_x \varphi), \\ \frac{\lambda}{h} \left(\frac{1-B}{1+B} \right) &\simeq -\frac{\lambda}{h} \frac{h}{2\lambda} \left(\frac{\partial_x \varphi^h + \lambda}{\lambda - (\partial_x \varphi^h - \lambda)h/2\lambda} \right) \rightarrow -\frac{1}{2\lambda} (\lambda + \partial_x \varphi). \end{aligned}$$

By uniqueness of the limit proved in [24, 26], the whole sequence converges. \square

REMARK 2. *Concerning the preservation of steady-states, the Godunov scheme on w_j^n and z_j^n preserves **all** the steady-states of the 2×2 system. The issue comes from the centered scheme (3.6) on φ : clearly, even if we furnish an initial datum φ^0 which satisfies $\partial_t \varphi^0 = 0$ as in [24, 26], it is likely to be perturbed and a spurious dynamic may be ignited. However, we aren't aware of any well-balanced scheme for the diffusion equation; a remedy may be to approximate the equation on φ by a system of "hyperbolic heat equations" (like in [16]), build a traditional well-balanced scheme on this approximation, and then plug it inside the present framework. This would constitute a complex system of 2 Cattaneo models (weakly) coupled by φ and ρ .*

4. Diffusive scaling through a modification of NC jump relations. In this section, we adopt the diffusive scaling (1.4) and, following [16, 17], we investigate how the preceding well-balanced scheme can handle the limit $\varepsilon \rightarrow 0$ under the simple parabolic CFL restriction $\Delta t = O(h^2)$. Clearly, the convective part is to be treated implicitly in time, but this isn't costly as it is linear.

4.1. Rewriting jump relations makes the Maxwellian appear. The first thing to observe is that the parabolic scaling can be handled in the steady-state equations (3.2) by simply changing $h \rightarrow h/\varepsilon$. Hence it makes sense to introduce new coefficients defined as follows with $\lambda > 0$ and $\varepsilon > 0$:

$$A_\varepsilon = \left(1 - \frac{h}{\varepsilon\lambda}\right) \exp(h\partial_x\varphi/\varepsilon\lambda^2), \quad B_\varepsilon = \left(1 + \frac{h}{\varepsilon\lambda}\right) \exp(h\partial_x\varphi/\varepsilon\lambda^2).$$

We stress that when we rescale $x \rightarrow \varepsilon x$, the quantity $\partial_x\varphi$ appearing in the jump relations becomes $\partial_x\varphi/\varepsilon$ **which remains bounded**. In the sequel and in the set of rescaled variables, we shall continue to work with the numerical approximation:

$$\forall j, n \in \mathbb{Z} \times \mathbb{N}, \quad (\partial_x\varphi)_{j+\frac{1}{2}}^n := \frac{1}{h} (\varphi_{j+1}^n - \varphi_j^n).$$

Starting from here, we shall use the **convention** of writing $\partial_x\varphi$ as the space derivative of φ with respect to the rescaled variable εx , so the ε in the denominator drops. Following [16], we rewrite the jump relations (3.3) occurring through the zero-waves:

$$\begin{aligned} \bar{w}(0) &= \frac{2}{1+B_\varepsilon} \bar{w}(h) + \frac{1-A_\varepsilon}{1+B_\varepsilon} \bar{z}(0) \\ &= \bar{z}(0) - \frac{A_\varepsilon+B_\varepsilon}{1+B_\varepsilon} \bar{z}(0) + \frac{2}{1+B_\varepsilon} \bar{w}(h) \\ &= \bar{z}(0) + \frac{A_\varepsilon+B_\varepsilon}{1+B_\varepsilon} (\bar{w}(h) - \bar{z}(0)) + \frac{2-(A_\varepsilon+B_\varepsilon)}{1+B_\varepsilon} \bar{w}(h), \end{aligned}$$

and,

$$\begin{aligned} \bar{z}(h) &= \bar{z}(0) - \frac{1-B_\varepsilon}{1+B_\varepsilon} \bar{w}(h) - \frac{1-A_\varepsilon}{1+B_\varepsilon} \bar{z}(0) \\ &= \bar{w}(h) - \frac{2}{1+B_\varepsilon} \bar{w}(h) + \frac{A_\varepsilon+B_\varepsilon}{1+B_\varepsilon} \bar{z}(0) \\ &= \bar{w}(h) + \frac{A_\varepsilon+B_\varepsilon}{1+B_\varepsilon} (\bar{z}(0) - \bar{w}(h)) - \frac{2-(A_\varepsilon+B_\varepsilon)}{1+B_\varepsilon} \bar{w}(h). \end{aligned}$$

Observe also that for any $\varepsilon > 0$, $A_\varepsilon + B_\varepsilon = 2 \exp(h\partial_x\varphi/\lambda^2)$. We rewrite the well-balanced Godunov scheme of the former section with these jump relations and treating part of the convective term implicitly; denoting $C_\varepsilon := 2 - (A_\varepsilon + B_\varepsilon)$, it comes:

$$(4.1) \left\{ \begin{aligned} w_j^{n+1} &= w_j^n + \frac{\lambda\Delta t}{\varepsilon h} (z_j^{n+1} - w_j^{n+1}) \\ &\quad + \frac{\lambda\Delta t \left(A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n \right)}{\varepsilon h \left(1 + B_{\varepsilon,j+\frac{1}{2}}^n \right)} (w_{j+1}^n - z_j^n) + \frac{\lambda\Delta t C_{\varepsilon,j+\frac{1}{2}}^n}{\varepsilon h \left(1 + B_{\varepsilon,j+\frac{1}{2}}^n \right)} w_{j+1}^n, \\ z_j^{n+1} &= z_j^n - \frac{\lambda\Delta t}{\varepsilon h} (z_j^{n+1} - w_j^{n+1}) \\ &\quad - \frac{\lambda\Delta t \left(A_{\varepsilon,j-\frac{1}{2}}^n + B_{\varepsilon,j-\frac{1}{2}}^n \right)}{\varepsilon h \left(1 + B_{\varepsilon,j-\frac{1}{2}}^n \right)} (w_j^n - z_{j-1}^n) - \frac{\lambda\Delta t C_{\varepsilon,j-\frac{1}{2}}^n}{\varepsilon h \left(1 + B_{\varepsilon,j-\frac{1}{2}}^n \right)} w_j^n. \end{aligned} \right.$$

Let us pause here in order to distinguish between the various terms appearing in this new rewriting of **the same** Godunov scheme:

- the implicit term $z_j^{n+1} - w_j^{n+1}$ is penalized by ε ; it is a Maxwellian term which is meant to enforce $w^h = z^h = \frac{\rho^h}{2}$ in the limit $\varepsilon \rightarrow 0$.
- the coefficient $\frac{\lambda\Delta t}{\varepsilon h} \frac{A_\varepsilon+B_\varepsilon}{1+B_\varepsilon} = \frac{2\lambda\Delta t \exp(h\partial_x\varphi/\lambda^2)}{\varepsilon h (1+(1+h/\varepsilon\lambda)) \exp(h\partial_x\varphi/\lambda^2)} \rightarrow \frac{2\lambda^2\Delta t}{h^2}$ as $\varepsilon \rightarrow 0$; it is therefore meant to generate the centered discretization of the diffusion term $\lambda^2\partial_{xx}\rho$ when both the equations are added. No spurious term in h remains, this is one part of the AP property.

- finally, the coefficient $\frac{\lambda\Delta t C_\varepsilon}{\varepsilon h(1+B_\varepsilon)}$ asks for more involved computations:

$$\begin{aligned} \frac{2\lambda\Delta t(1 - \exp(\frac{h\partial_x\varphi}{\lambda^2}))}{\varepsilon h(1 + (1 + h/\lambda)\exp(\frac{h\partial_x\varphi}{\lambda^2}))} &= \frac{2\lambda^2\Delta t(1 - \exp(\frac{h\partial_x\varphi}{\lambda^2}))}{\varepsilon h\lambda + (\varepsilon h\lambda + h^2)\exp(\frac{h\partial_x\varphi}{\lambda^2})} \\ &\simeq \frac{2\lambda^2\Delta t}{h^2} \left(\exp(-\frac{h\partial_x\varphi}{\varepsilon\lambda^2}) - 1 \right) \\ &\simeq \frac{2\Delta t\partial_x\varphi}{h}, \quad \varepsilon \rightarrow 0. \end{aligned}$$

The space derivative of φ inside the zero-wave has been rescaled according to the convention previously discussed. The AP property is complete.

As the implicit convection terms are linear, it is possible to invert it explicitly; following [16, 17], we introduce the notations: $a = 1 + \frac{\lambda\Delta t}{\varepsilon h}$, $b = \frac{\lambda\Delta t}{\varepsilon h} \geq 0$. Inverting the matrix appearing in (4.1) gives the following scheme:

$$(4.2) \quad \begin{cases} w_j^{n+1} = \frac{a}{a+b} \left(w_j^n + \delta W_{j+\frac{1}{2}}^n \right) + \frac{b}{a+b} \left(z_j^n - \delta Z_{j-\frac{1}{2}}^n \right), \\ z_j^{n+1} = \frac{b}{a+b} \left(w_j^n + \delta W_{j+\frac{1}{2}}^n \right) + \frac{a}{a+b} \left(z_j^n - \delta Z_{j-\frac{1}{2}}^n \right), \end{cases}$$

where the new quantities read,

$$\delta W_{j+\frac{1}{2}}^n = \frac{\lambda\Delta t}{\varepsilon h} \left[\left(\frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1 + B_{\varepsilon,j+\frac{1}{2}}^n} + \frac{C_{\varepsilon,j+\frac{1}{2}}^n}{1 + B_{\varepsilon,j+\frac{1}{2}}^n} \right) w_{j+1}^n - \frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1 + B_{\varepsilon,j+\frac{1}{2}}^n} z_j^n \right],$$

and $\delta Z_{j-\frac{1}{2}}^n = \delta W_{j-\frac{1}{2}}^n$. For any value of $\varepsilon \geq 0$, there holds for $j \in \mathbb{Z}$: $A_\varepsilon + B_\varepsilon \geq 0$, $A_\varepsilon + B_\varepsilon + C_\varepsilon = 2$. Moreover, we shall hereafter impose the “parabolic CFL restriction”:

$$(4.3) \quad 1 \geq \frac{\lambda\Delta t}{\varepsilon h} \left(\frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n + C_{\varepsilon,j-\frac{1}{2}}^n}{1 + B_{\varepsilon,j+\frac{1}{2}}^n} \right) \simeq \frac{2\Delta t}{h^2} (|\varphi_{j+1}^n - \varphi_j^n| + \lambda^2).$$

We are now in position to study L^p bounds and BV-bounds by seeking to rewrite (4.1) as a convex combination of the neighboring cells. As in the previous section, we define the following functions for any value of $\varepsilon > 0$,

$$\forall (j, n) \in \mathbb{Z} \times \mathbb{N}, \quad w^h(n\Delta t, jh) := w_j^n, \quad z^h(n\Delta t, jh) := z_j^n,$$

with the numerical values w_j^n and z_j^n being generated by (4.1). We still treat the diffusion equation on φ by means of the explicit centered scheme (3.6) which is stable under the parabolic CFL condition $2D\Delta t \leq h^2$.

LEMMA 4.1. *Assume that $\varepsilon \leq \frac{\lambda\Delta t}{h}$ and the CFL condition (4.3) holds, then:*

$$(4.4) \quad \forall t \in \mathbb{R}^+, \quad \|w^h(t, \cdot)\|_{L^p(\mathbb{R})} + \|z^h(t, \cdot)\|_{L^p(\mathbb{R})} \leq \|w^0\|_{L^p(\mathbb{R})} + \|z^0\|_{L^p(\mathbb{R})}.$$

Proof. The proof is computationally tedious but consists only in checking the nonnegativity of certain quantities which are moreover asked to equal 1 when they

are summed up. Let us rewrite the equation on w_j^{n+1} first:

$$\begin{aligned} w_j^{n+1} &= w_{j+1}^n \frac{ab}{a+b} \left(\frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} + \frac{C_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \right) \\ &\quad + z_j^n \frac{b}{a+b} \left(1 - a \frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \right) \\ &\quad + \frac{w_j^n}{a+b} \left[a - b^2 \left(\frac{A_{\varepsilon,j-\frac{1}{2}}^n + B_{\varepsilon,j-\frac{1}{2}}^n}{1+B_{\varepsilon,j-\frac{1}{2}}^n} + \frac{C_{\varepsilon,j-\frac{1}{2}}^n}{1+B_{\varepsilon,j-\frac{1}{2}}^n} \right) \right] \\ &\quad + z_{j-1}^n \frac{b^2}{a+b} \left(\frac{A_{\varepsilon,j-\frac{1}{2}}^n + B_{\varepsilon,j-\frac{1}{2}}^n}{1+B_{\varepsilon,j-\frac{1}{2}}^n} \right). \end{aligned}$$

At this point, the only thing to notice is that, since $a = 1 + b$,

$$a - b^2 \frac{A + B + C}{1 + B} = 1 + b \underbrace{\left[1 - b \left(\frac{A + B + C}{1 + B} \right) \right]}_{\geq 0 \text{ by (4.3)}} \geq 0,$$

and (4.3) also implies that the coefficient on z_j^n is nonnegative. Let's pass to z_j^{n+1} :

$$\begin{aligned} z_j^{n+1} &= w_{j+1}^n \frac{b^2}{a+b} \left(\frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} + \frac{C_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \right) \\ &\quad + \frac{z_j^n}{a+b} \left(a - b^2 \frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \right) \\ &\quad + w_j^n \frac{b}{a+b} \left[1 - a \left(\frac{A_{\varepsilon,j-\frac{1}{2}}^n + B_{\varepsilon,j-\frac{1}{2}}^n}{1+B_{\varepsilon,j-\frac{1}{2}}^n} + \frac{C_{\varepsilon,j-\frac{1}{2}}^n}{1+B_{\varepsilon,j-\frac{1}{2}}^n} \right) \right] \\ &\quad + z_{j-1}^n \frac{ab}{a+b} \left(\frac{A_{\varepsilon,j-\frac{1}{2}}^n + B_{\varepsilon,j-\frac{1}{2}}^n}{1+B_{\varepsilon,j-\frac{1}{2}}^n} \right). \end{aligned}$$

The coefficient on w_j^n is nonnegative if $1 - \frac{1}{b} \geq 0$ and this is ensured by $\varepsilon \leq \frac{\lambda \Delta t}{h} \leq 1$. One can now take the moduli in both equations and sum up; it comes that,

$$\sum_{j \in \mathbb{Z}} h (|w_j^{n+1}| + |z_j^{n+1}|) \leq \sum_{j \in \mathbb{Z}} h (|w_j^n| + |z_j^n|).$$

□

The BV-bound will be obtained the same way, except that the space dependence of the coefficients A, B, C through $\partial_x \varphi$ will make the computations even more intricate.

PROPOSITION 4.2. *Let $\varepsilon \leq \frac{\lambda \Delta t}{h}$; assume that for any $n \in \mathbb{N}$, $(\partial_x \varphi)_{j+\frac{1}{2}}^n - (\partial_x \varphi)_{j-\frac{1}{2}}^n$ changes sign only at a finite number N of locations $j_1^n, j_2^n, \dots, j_N^n \in \mathbb{Z}$, that the CFL condition (4.3) holds and that $\text{Lip}(\varphi^h) \simeq Lh$ for some $L \in \mathbb{R}^+$, then for any $t \in \mathbb{R}^+$:*

$$(4.5) \quad \begin{aligned} TV(w^h(t, \cdot)) + TV(z^h(t, \cdot)) &\leq \exp(2Lt) (TV(w^0) + TV(z^0)) + O(\varepsilon) + \\ &\quad N(\|w^0\|_{L^\infty(\mathbb{R})} + \|z^0\|_{L^\infty(\mathbb{R})})(\exp(2tL) - 1). \end{aligned}$$

In the special case where $(\partial_x \varphi)_{j+\frac{1}{2}}^n \equiv (\partial_x \varphi)_{j-\frac{1}{2}}^n$ for all $j \in \mathbb{Z}$, the TVD property holds:

$$\forall t \in \mathbb{R}^+, \quad TV(w^h(t, \cdot)) + TV(z^h(t, \cdot)) \leq TV(w^0) + TV(z^0) \text{ with } L = 0.$$

Proof. We proceed as in the proof of Lemma 4.1, with a first set of terms acting on differences like $w_{j+1}^n - w_j^n$ and $z_{j+1}^n - z_j^n$; we don't repeat the computations as

they are very similar. We concentrate on the new terms arising from the lack of translation-invariance and without loss of generality, we assume that $N = 2$ as the situation for higher N can be handled the same way (moreover, the asymptotic profile of the heat equation is the Gaussian function which has only 2 inflexion points in \mathbb{R}).

$$\begin{aligned} & w_{j+1}^n \frac{ab}{a+b} \left[\frac{A_{\varepsilon,j+\frac{3}{2}}^n + B_{\varepsilon,j+\frac{3}{2}}^n + C_{\varepsilon,j+\frac{3}{2}}^n}{1+B_{\varepsilon,j+\frac{3}{2}}^n} - \frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n + C_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \right] \\ & - z_j^n \frac{ab}{a+b} \left[\frac{A_{\varepsilon,j+\frac{3}{2}}^n + B_{\varepsilon,j+\frac{3}{2}}^n}{1+B_{\varepsilon,j+\frac{3}{2}}^n} - \frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \right] \\ & - w_j^n \frac{b^2}{a+b} \left[\frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n + C_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} - \frac{A_{\varepsilon,j-\frac{1}{2}}^n + B_{\varepsilon,j-\frac{1}{2}}^n + C_{\varepsilon,j-\frac{1}{2}}^n}{1+B_{\varepsilon,j-\frac{1}{2}}^n} \right] \\ & + z_{j-1}^n \frac{b^2}{a+b} \left[\frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} - \frac{A_{\varepsilon,j-\frac{1}{2}}^n + B_{\varepsilon,j-\frac{1}{2}}^n}{1+B_{\varepsilon,j-\frac{1}{2}}^n} \right], \end{aligned}$$

for the equation on w_j^n , and

$$\begin{aligned} & w_{j+1}^n \frac{b^2}{a+b} \left[\frac{A_{\varepsilon,j+\frac{3}{2}}^n + B_{\varepsilon,j+\frac{3}{2}}^n + C_{\varepsilon,j+\frac{3}{2}}^n}{1+B_{\varepsilon,j+\frac{3}{2}}^n} - \frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n + C_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \right] \\ & - z_j^n \frac{b^2}{a+b} \left[\frac{A_{\varepsilon,j+\frac{3}{2}}^n + B_{\varepsilon,j+\frac{3}{2}}^n}{1+B_{\varepsilon,j+\frac{3}{2}}^n} - \frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \right] \\ & - w_j^n \frac{a}{a+b} \left[\frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n + C_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} - \frac{A_{\varepsilon,j-\frac{1}{2}}^n + B_{\varepsilon,j-\frac{1}{2}}^n + C_{\varepsilon,j-\frac{1}{2}}^n}{1+B_{\varepsilon,j-\frac{1}{2}}^n} \right] \\ & + z_{j-1}^n \frac{ab}{a+b} \left[\frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} - \frac{A_{\varepsilon,j-\frac{1}{2}}^n + B_{\varepsilon,j-\frac{1}{2}}^n}{1+B_{\varepsilon,j-\frac{1}{2}}^n} \right], \end{aligned}$$

for the equation on z_j^n . Summing moduli on $j \in \mathbb{Z}$ yields on the one hand:

$$2b \sum_{j \in \mathbb{Z}} |z_j^n| \left| \frac{A_{\varepsilon,j+\frac{3}{2}}^n + B_{\varepsilon,j+\frac{3}{2}}^n}{1+B_{\varepsilon,j+\frac{3}{2}}^n} - \frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \right| = O(\varepsilon),$$

and this quantity is of the order of ε since $b \frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \rightarrow 2\lambda^2 \Delta t / h^2$ (independent of j) as $\varepsilon \rightarrow 0$. On the other hand, let us introduce $j_1^n \leq j_2^n$ which are the points where the second order divided difference of $\varphi^h(n\Delta t, \cdot)$ changes its sign; for $j \in [j_1^n, j_2^n]$, $(\partial_x \varphi)_{j+\frac{1}{2}}^n - (\partial_x \varphi)_{j-\frac{1}{2}}^n \leq 0$, and it is nonnegative elsewhere. Taking moduli in both the former equalities and linearizing the exponentials yields, up to $O(\varepsilon)$,

$$\begin{aligned} 2b \sum_{j \in [j_1^n, j_2^n]} |w_j^n| \left| \frac{1}{1+B_{\varepsilon,j+\frac{1}{2}}^n} - \frac{1}{1+B_{\varepsilon,j-\frac{1}{2}}^n} \right| & \simeq 2b \sum_{j \in [j_1^n, j_2^n]} |w_j^n| \left(\frac{1}{1+B_{\varepsilon,j+\frac{1}{2}}^n} - \frac{1}{1+B_{\varepsilon,j-\frac{1}{2}}^n} \right) \\ & \simeq \sum_{j \in [j_1^n, j_2^n]} \frac{bC_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} (|w_j^n| - |w_{j+1}^n|) \\ & \quad + \frac{bC_{\varepsilon,j_1^n+\frac{1}{2}}^n}{1+B_{\varepsilon,j_1^n+\frac{1}{2}}^n} |w_{j_1^n}^n| - \frac{bC_{\varepsilon,j_2^n+\frac{1}{2}}^n}{1+B_{\varepsilon,j_2^n+\frac{1}{2}}^n} |w_{j_2^n}^n| \\ & \leq \sum_{j \in [j_1^n, j_2^n]} \left| \frac{bC_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \right| |w_{j+1}^n - w_j^n| \\ & \quad + \left| \frac{bC_{\varepsilon,j_1^n+\frac{1}{2}}^n}{1+B_{\varepsilon,j_1^n+\frac{1}{2}}^n} \right| |w_{j_1^n}^n| + \left| \frac{bC_{\varepsilon,j_2^n+\frac{1}{2}}^n}{1+B_{\varepsilon,j_2^n+\frac{1}{2}}^n} \right| |w_{j_2^n}^n| \end{aligned}$$

where we exploited first the concavity of $\varphi^h(n\Delta t, \cdot)$ (which gives that $\frac{C_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n}$ is increasing with respect to j), and then made a summation by parts including the

boundary terms. Form the strong assumption made on the Lipschitz constant of φ^h ,

$$\left| \frac{bC_{\varepsilon, j+\frac{1}{2}}^n}{1+B_{\varepsilon, j+\frac{1}{2}}^n} \right| \leq 2 \frac{\Delta t}{h} \left(\sup_j |(\partial_x \varphi)_{j+\frac{1}{2}}^n| + O(\varepsilon) \right) \leq 2\Delta t (L + O(\varepsilon)),$$

and this allows to include nonnegative $O(\Delta t)$ terms. Finally, we obtain:

$$\begin{aligned} \sum_{j \in \mathbb{Z}} |w_{j+1}^{n+1} - w_j^{n+1}| + |z_{j+1}^{n+1} - z_j^{n+1}| &\leq (1 + 2\Delta t L) \sum_{j \in \mathbb{Z}} |w_{j+1}^n - w_j^n| + |z_{j+1}^n - z_j^n| \\ &\quad + O(\varepsilon) + 4\Delta t L \sup_j |w_j^n|. \end{aligned}$$

From (4.4), we get the decay in time of the L^∞ norms and since the number of inflexion points is supposed finite, the last term is bounded. Now, this quantity can be summed up to $n = 0$ thanks to the Δt which appear in the all the terms responsible for an increase of the total variation in space. More precisely,

$$\begin{aligned} TV(w^h(n\Delta t, \cdot)) + TV(z^h(n\Delta t, \cdot)) &\leq (1 + 2\Delta t L)^n (TV(w^0) + TV(z^0)) + O(\varepsilon) \\ &\quad + 4Lip(\varphi^h) \Delta t (\|w^0\|_{L^\infty(\mathbb{R})} + \|z^0\|_{L^\infty(\mathbb{R})}) \frac{1 - (1 + 2\Delta t L)^n}{1 - (1 + 2\Delta t L)} \\ &\leq \exp(2n\Delta t L) (TV(w^0) + TV(z^0)) + O(\varepsilon) \\ &\quad + 2(\|w^0\|_{L^\infty(\mathbb{R})} + \|z^0\|_{L^\infty(\mathbb{R})}) (\exp(2n\Delta t L) - 1), \end{aligned}$$

where we have used (4.4) and the formula for the summation of a geometric sequence. The more general case where more than 2 inflexion points appear can be treated the same way at the price of more intricate computations. \square

4.2. Maxwellian control and stability of the diffusive scheme. The situation as presented in Proposition 4.2 is very delicate: it illustrates the fact that the approximation process, which generates stable BV numerical solutions hopefully stable uniformly in $\varepsilon \ll 1$ (see Lemma below) is incompatible with the Keller-Segel system (1.5) which appears to be the limit equation. Indeed, such a linear Fokker-Planck equation endowed with a potential which is neither divergence-free (in the 1D context, this reduces to $\partial_{xx}\varphi = 0$) nor confining (meaning $\partial_{xx}\varphi \leq 0$) doesn't admit generally solutions which are total-variation bounded. Hence the bound (4.5) exists for any $h > 0$, but blows up and become useless in the limit $h \rightarrow 0$ because there exists no BV-theory for the continuous 1D Keller-Segel system set on the real line.

The bound (4.5) allows to control the deviation from the Maxwellian equilibrium:

LEMMA 4.3. *Under the assumptions of Proposition 4.2, if $\|w^0 - z^0\|_{L^1(\mathbb{R})} = O(\varepsilon)$ (well-prepared initial data) and for ε small enough, there holds for any $t > 0$:*

$$(4.6) \quad \|w^h(t, \cdot) - z^h(t, \cdot)\|_{L^1(\mathbb{R})} = O(\varepsilon).$$

Proof. We subtract the equations appearing in the semi-implicit scheme (4.1):

$$\begin{aligned} (1 + 2b) (w_j^{n+1} - z_j^{n+1}) &= w_j^n - z_j^n + \frac{\lambda \Delta t \left(A_{\varepsilon, j+\frac{1}{2}}^n + B_{\varepsilon, j+\frac{1}{2}}^n \right)}{\varepsilon h \left(1 + B_{\varepsilon, j+\frac{1}{2}}^n \right)} (w_{j+1}^n - z_j^n) \\ &\quad + \frac{\lambda \Delta t \left(A_{\varepsilon, j-\frac{1}{2}}^n + B_{\varepsilon, j-\frac{1}{2}}^n \right)}{\varepsilon h \left(1 + B_{\varepsilon, j-\frac{1}{2}}^n \right)} (w_j^n - z_{j-1}^n) + \frac{\lambda \Delta t C_{\varepsilon, j+\frac{1}{2}}^n}{\varepsilon h \left(1 + B_{\varepsilon, j+\frac{1}{2}}^n \right)} w_{j+1}^n + \frac{\lambda \Delta t C_{\varepsilon, j-\frac{1}{2}}^n}{\varepsilon h \left(1 + B_{\varepsilon, j-\frac{1}{2}}^n \right)} w_j^n. \end{aligned}$$

We add and subtract the following 2 terms,

$$\frac{\lambda \Delta t (A_{\varepsilon, j+\frac{1}{2}}^n + B_{\varepsilon, j+\frac{1}{2}}^n)}{\varepsilon h (1 + B_{\varepsilon, j+\frac{1}{2}}^n)} w_j^n, \quad \text{and} \quad - \frac{\lambda \Delta t (A_{\varepsilon, j-\frac{1}{2}}^n + B_{\varepsilon, j-\frac{1}{2}}^n)}{\varepsilon h (1 + B_{\varepsilon, j-\frac{1}{2}}^n)} z_j^n,$$

in the preceding equation in order to make appear all the available Maxwellian terms together with other terms which can be controlled by the L^p bounds and the BV-bound. Thus, we take the modulus and sum on $j \in \mathbb{Z}$ to obtain first

$$\begin{aligned} (1+2b)(w_j^{n+1} - z_j^{n+1}) &= (w_j^n - z_j^n) \left(1 + b \frac{A_{\varepsilon, j+\frac{1}{2}}^n + B_{\varepsilon, j+\frac{1}{2}}^n}{1+B_{\varepsilon, j+\frac{1}{2}}^n} + b \frac{A_{\varepsilon, j-\frac{1}{2}}^n + B_{\varepsilon, j-\frac{1}{2}}^n}{1+B_{\varepsilon, j-\frac{1}{2}}^n} \right) \\ &\quad + b \frac{A_{\varepsilon, j+\frac{1}{2}}^n + B_{\varepsilon, j+\frac{1}{2}}^n}{1+B_{\varepsilon, j+\frac{1}{2}}^n} (w_{j+1}^n - w_j^n) - b \frac{A_{\varepsilon, j-\frac{1}{2}}^n + B_{\varepsilon, j-\frac{1}{2}}^n}{1+B_{\varepsilon, j-\frac{1}{2}}^n} (z_j^n - z_{j-1}^n) \\ &\quad + b \frac{C_{\varepsilon, j+\frac{1}{2}}^n}{1+B_{\varepsilon, j+\frac{1}{2}}^n} w_{j+1}^n + b \frac{C_{\varepsilon, j-\frac{1}{2}}^n}{1+B_{\varepsilon, j-\frac{1}{2}}^n} w_j^n, \end{aligned}$$

and then, as $\frac{1+2b \frac{A+B}{1+B}}{1+2b} \simeq \frac{h^2 + 4\lambda \Delta t}{h^2(1+2\Delta t \lambda / \varepsilon h)} \simeq \frac{\varepsilon}{h}$,

$$\begin{aligned} \sum_{j \in \mathbb{Z}} h |w_j^{n+1} - z_j^{n+1}| &\leq \frac{\varepsilon}{h} \sum_{j \in \mathbb{Z}} h |w_j^n - z_j^n| + \frac{2b}{1+2b} \sum_{j \in \mathbb{Z}} \left| \frac{C_{\varepsilon, j-\frac{1}{2}}^n}{1+B_{\varepsilon, j-\frac{1}{2}}^n} \right| h |w_j^n| \\ &\quad + \frac{b}{1+2b} \sum_{j \in \mathbb{Z}} h \left| \frac{A_{\varepsilon, j+\frac{1}{2}}^n + B_{\varepsilon, j+\frac{1}{2}}^n}{1+B_{\varepsilon, j+\frac{1}{2}}^n} \right| (|w_{j+1}^n - w_j^n| + |z_{j+1}^n - z_j^n|). \end{aligned}$$

At this point, one notices that, up to an error of the order of $\varepsilon \ll 1$,

$$\frac{A_{\varepsilon, j+\frac{1}{2}}^n + B_{\varepsilon, j+\frac{1}{2}}^n}{1+B_{\varepsilon, j+\frac{1}{2}}^n} \simeq \frac{2\lambda \varepsilon}{h}, \quad \frac{C_{\varepsilon, j-\frac{1}{2}}^n}{1+B_{\varepsilon, j-\frac{1}{2}}^n} \simeq \frac{2\varepsilon (\partial_x \varphi)_{j+\frac{1}{2}}^n}{\lambda}, \quad \frac{2b}{1+2b} \simeq 1,$$

so, taking advantage of (4.4), it comes for any $t \in \mathbb{R}^+$:

$$\begin{aligned} \|w^h(t + \Delta t, \cdot) - z^h(t + \Delta t, \cdot)\|_{L^1(\mathbb{R})} &\leq \frac{\varepsilon}{h} \|w^h(t, \cdot) - z^h(t, \cdot)\|_{L^1(\mathbb{R})} \\ &\quad + \lambda \varepsilon (TV(w^h(t, \cdot)) + TV(z^h(t, \cdot))) + 2\varepsilon (\|w^0\|_{L^1(\mathbb{R})} + \|z^0\|_{L^1(\mathbb{R})}). \end{aligned}$$

□

4.3. Compactness of the Asymptotic-Preserving scheme. With all the estimates (4.4), (4.5) and (4.6), it is routine to establish the L^1 time equicontinuity property, so we omit the proof of the following lemma:

LEMMA 4.4. *Under all the assumptions of Lemma 4.3, one has for any $t > 0$:*

$$\|w^h(t + \Delta t, \cdot) - w^h(t, \cdot)\|_{L^1(\mathbb{R})} + \|z^h(t + \Delta t, \cdot) - z^h(t, \cdot)\|_{L^1(\mathbb{R})} \leq O(\Delta t).$$

At this point, since the BV-bound (4.5) blows up as $h \rightarrow 0$, we cannot state results as strong as those of [16, 17]; the strategy will be first, to establish strong convergence of $\rho^h = w^h + z^h$ for $\varepsilon \rightarrow 0$ **only**, keeping $h > 0$ in order to take advantage of (4.5), toward a piecewise constant function satisfying a numerical scheme which is a perturbation of a centered discretization of the Keller-Segel model (1.5). We stress that in this limiting process, nothing happens concerning the numerical treatment of the discrete equation which rules the time evolution of φ^h as it never deals with w^h and z^h , but only with ρ^h . And second, to derive the convergence of ρ^h, φ^h as $h \rightarrow 0$ because it is a drastic simplification of the 2D finite volume scheme studied by Filbet [10].

THEOREM 4.5. *Under the assumptions of Lemma 4.3, for any $n \in \mathbb{N}$:*

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{(\partial_x \varphi)_{j+\frac{1}{2}}^n \rho_{j+1}^n - (\partial_x \varphi)_{j-\frac{1}{2}}^n \rho_j^n}{h} = \lambda^2 \frac{\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n}{h^2} + O(\varepsilon).$$

In particular, the sequences w^h, z^h are relatively compact in $L_{loc}^1(\mathbb{R}_+^ \times \mathbb{R})$ as $\varepsilon \rightarrow 0$ with $h > 0$ fixed and the remaining term in $O(\varepsilon)$ converges to zero in L^1 .*

Proof. One proceeds simply by adding up both equations appearing in (4.1): the Maxwellian terms treated implicitly cancel each other. Then, taking advantage of the facts that both $b \frac{A_{\varepsilon,j+\frac{1}{2}}^n + B_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \rightarrow 2\lambda^2 \Delta t / h^2$ and $b \frac{C_{\varepsilon,j+\frac{1}{2}}^n}{1+B_{\varepsilon,j+\frac{1}{2}}^n} \rightarrow 2(\partial_x \varphi)_{j+\frac{1}{2}}^n \Delta t / h$ and $w_j^n, z_j^n \rightarrow \rho_j^n / 2$ when $\varepsilon \rightarrow 0$ with $h > 0$ gives the aforementioned scheme on ρ^h . \square

In the limit $\varepsilon \rightarrow 0$, the conventional centered discretization acting on the Keller-Segel model (1.5), for small enough initial data, allows to apply the convergence result in [10] to pass later to the complementary limit $h \rightarrow 0$ with a completely different functional framework (in particular, no BV-bound is necessary and the convergence is weak). There is no change on the scheme on φ^h (3.6) thanks to the use of the rescaled parameters α^ε and β^ε which are called the “small reaction rates” in [9].

5. Numerical results. The convergence results obtained in the preceding section are qualitatively different from the ones of [16, 17]; indeed, in these former works, it was possible to consider for instance passing to the limit simultaneously in $\varepsilon \rightarrow 0$, $h \rightarrow 0$ with $\varepsilon = h^\gamma$, $\gamma > 1$. Here, since (4.5) blows up as $h \rightarrow 0$, one must pass first to the limit $\varepsilon \rightarrow 0$, $h > 0$, and later $h \rightarrow 0$ as a distinct process.

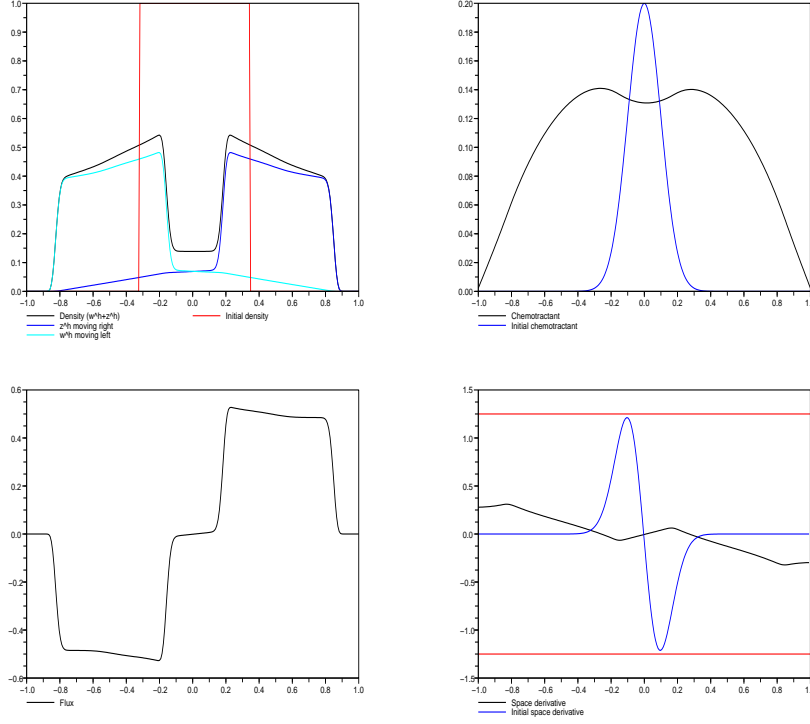
5.1. Hyperbolic regime. We consider the simple test-case of the propagation of Riemann initial data in the computational domain $x \in [-1, 1]$:

$$(5.1) \quad w^0 = z^0 = \frac{1}{2} \chi_{[-\frac{1}{3}, \frac{1}{3}]}, \quad \varphi^0(x) = \frac{1}{5} \exp(-50x^2), \quad \alpha = 15, \quad \beta = 35, \quad D = 5,$$

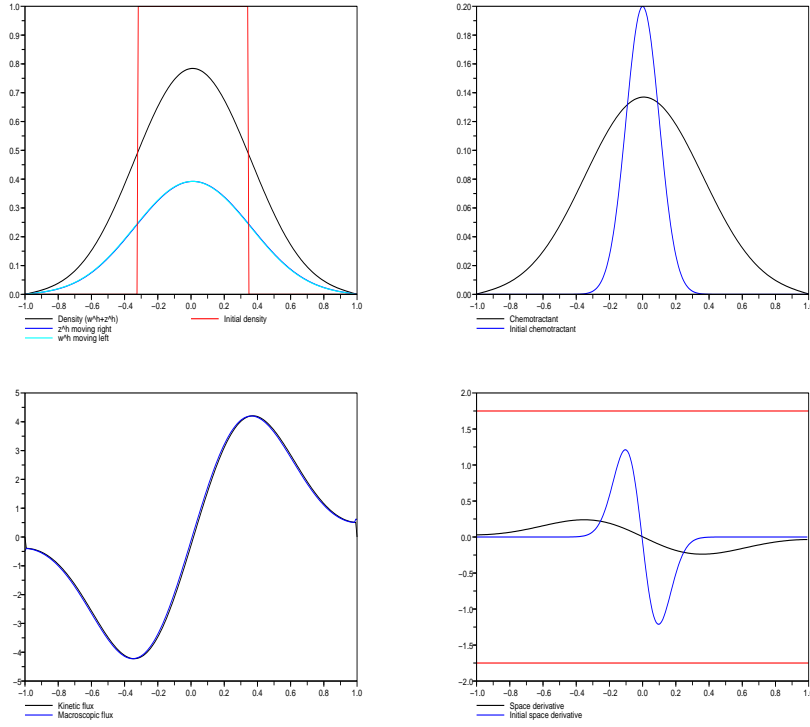
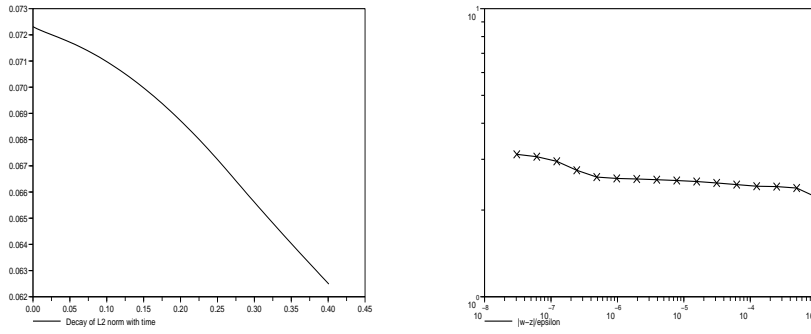
with 255 grid points, which gives $h = 0.0078$. The CFL number is chosen so as to get $\lambda \Delta t = 0.9h$ with $\lambda = 1.25$ and the results at time $t = 0.4$ are shown in Fig. 5.1. The (initially Maxwellian) kinetic densities split symmetrically between the ones moving in positive and negative direction. The space derivative $\partial_x \varphi$ remains always below the red lines which correspond to the maximal values $\pm \lambda$ thus ensures quasi-monotonicity and consequently the L^p and BV-bounds (3.9) and (3.10).

5.2. Diffusive regime. We kept exactly the same parameters (except for $\lambda = 1.75$) for checking the ability of the numerical scheme (4.2) which comes from the inversion of the implicit terms in (4.1). Clearly, the time-step has to be modified according to the parabolic CFL restriction (4.3); we used $\lambda^2 \Delta t = 0.3h^2$ and iterated up to $t = 0.02$ to produce the results of Fig. 5.2 with the choice $\varepsilon = 0.001 \leq h$.

5.3. Numerical decay properties. In Fig. 5.3, we display on the 2 preceding concrete examples some theoretical properties shown in the former sections. For the hyperbolic test-case, we show the realization of the estimate (3.9) in the particular case $p = 2$: the decay in time is very neat. For the parabolic test-case, we display the L^1 norm of the Maxwellian term divided by 16 values of ε (the L^1 norm of the flux J/λ) for the Riemann data (5.1) at time $t = 0.01$, thus illustrating the estimate (4.6).

FIG. 5.1. *Hyperbolic test-case with data (5.1) at time $t = 0.4$.*

6. Conclusion and outlook. The present work can be extended in various directions: for instance, one may think about implementing the quasi-linear hyperbolic models which parabolic limits have been considered in [9]. The main obstacle on this road is the handling of the non-linear resonance phenomenon, [1, 27], which occurs when characteristic wave speeds vanish thus deeply complicating the structure of the well-balanced scheme. A theoretically simpler extension but perhaps more asking computationally could be the development of a 2D approach with a more sophisticated method than simple dimensional splitting. From the point of view of both Asymptotic-Preserving and Well-Balanced methodologies, the linear diffusion equation on φ is not the most well-suited because of its infinite speed of propagation and its asymptotic profiles endowed with several inflexion points. Perhaps a better model could be the classical porous medium equation, which shares the advantage of a very smooth solution inside the interfaces [45], but which propagates at a finite speed and possesses concavity properties [4] which should reveal themselves useful in the derivation of BV-bounds similar to (4.5). Lastly, there is an interesting connection between such a nonlinear Keller-Segel model and the asymptotic system emerging from WKB expansions for linear wave propagation as the relation between the eikonal and the porous medium equations is a well-known fact, [3, 35]. Hence, if the exponent of the nonlinear diffusion equation is close to one, we may expect a behaviour somewhat similar to the one reported in [15] except that no concentrations should occur thanks to the linear diffusion term appearing in the continuity equation on ρ .

FIG. 5.2. *Parabolic test-case with data (5.1) at time $t = 0.02$.*FIG. 5.3. *Evolution of the L^2 norms (left) and Maxwellian divided by ε in L^1 (right).*

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