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GROWTH OF A POPULATION OF BACTERIA IN A DYNAMICAL HOSTILE ENVIRONMENT

OLIVIER GARET AND RÉGINE MARCHAND

ABSTRACT. We study the growth of a population of bacteria in a dynamical hostile environment corresponding to the immune system of the colonised organism. The immune cells evolve as subcritical open clusters of oriented percolation and are perpetually reinforced by an immigration process, while the bacteria try to grow as a supercritical oriented percolation in the remaining empty space. For appropriate values of the parameters, we prove that the population of bacteria grows linearly. In this perspective, we build general tools to study dependent percolation models issued from renormalization processes.

1. A GROWTH MODEL IN DYNAMICAL HOSTILE ENVIRONMENT

We consider the following interacting particle system: at time $n = 0$, a particularly fertile bacterium (represented here by a type 1 particle) is submerged in a population of immune cells (type 2 particles) that are going to impede its development. The immune cells are not very fertile but benefit from a constant immigration process. Our aim is to find conditions that ensure, when the bacteria survive, that their growth is linear.

Thus, our system is described by a discrete time Markov chain taking its values in $\{0, 1, 2\}^{\mathbb{Z}^d}$. The transition between two states is in two steps. First, between time n and time $n+1/2$, each particle tries to colonize its neighbor sites: it succeeds with probability p if it is a type 1 particle, and with probability q if it is a type 2 particle. All events are independent, and in case of conflict, the type 2 particle wins. Next, between time $n+1/2$ and time $n+1$, the immigration of type 2 particles occurs: on each site, a type 2 particle appears with probability $\alpha > 0$, possibly taking the place of the particle previously occupying the site. Once again, all events are independent.

It is easy to see that, without type 2 particles, the bacteria would naturally grow as an oriented percolation with parameter p on the graph $\mathbb{Z}^d \times \mathbb{N}$. By classical arguments, there exists a critical probability $\bar{p}_c^{\text{alt}}(d+1)$ for the possibility to grow infinitely. Of course, we choose $p > \bar{p}_c^{\text{alt}}(d+1)$. On the contrary, we take $q < \bar{p}_c^{\text{alt}}(d+1)$, which corresponds to the poor virulence of type 2 particles. However, the constant immigration rate α guarantees that type 2 particles are always present in the organism.

Let us now describe the model more formally. We work, for $d \geq 1$, on the following graph:

- The set of sites is $\mathbb{V}^{d+1} = \{(z, n) \in \mathbb{Z}^d \times \mathbb{N}\}$.

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- We put an oriented edge from (z_1, n_1) to (z_2, n_2) if and only if $n_2 = n_1 + 1$ and $\|z_2 - z_1\|_1 \leq 1$; the set of these edges is denoted by $\vec{\mathbb{E}}_{\text{alt}}^{d+1}$.

The oriented edge from (z_1, n_1) to (z_2, n_2) can be identified with the couple $((z_1, z_2), n_2) \in \vec{\mathbb{E}}^d \times \mathbb{N}_*$, where $\vec{\mathbb{E}}^d$ is the set of oriented edges between nearest neighbors in \mathbb{Z}^d and of oriented loops: in $\vec{\mathbb{E}}^d$, there is an oriented edge between any two points z_1 and z_2 in \mathbb{Z}^d such that $\|z_1 - z_2\|_1 \leq 1$. Thus, we identify $\vec{\mathbb{E}}_{\text{alt}}^{d+1}$ and $\vec{\mathbb{E}}^d \times \mathbb{N}^*$.

We set $\tilde{\Omega} = \{0, 1\}^{\vec{\mathbb{E}}^d} \times \{0, 1\}^{\vec{\mathbb{E}}^d} \times \{0, 1\}^{\mathbb{Z}^d}$ and we endow the set $\Omega = \tilde{\Omega}^{\mathbb{N}^*}$ with its Borel σ -algebra and the probability $\mathbb{P} = \mathbb{P}_{p,q,\alpha} = \nu^{\otimes \mathbb{N}^*}$, where $\nu = \mathcal{B}(p)^{\otimes \vec{\mathbb{E}}^d} \otimes \mathcal{B}(q)^{\otimes \vec{\mathbb{E}}^d} \otimes \mathcal{B}(\alpha)^{\otimes \mathbb{Z}^d}$ and where $\mathcal{B}(p)$ stands for the Bernoulli law with parameter p .

Starting from the initial configuration $x \in \{0, 1, 2\}^{\mathbb{Z}^d}$, we define the Markov chain $(\eta_n^x)_{n \geq 0}$ taking its values in $\{0, 1, 2\}^{\mathbb{Z}^d}$ by

$$\eta_0^x = x \text{ and } \eta_{n+1}^x = f(\eta_n^x, \omega_{n+1})$$

where $f : \{0, 1, 2\}^{\mathbb{Z}^d} \times \tilde{\Omega} \rightarrow \{0, 1, 2\}^{\mathbb{Z}^d}$ is defined as follows:

$$f(x, ((\omega_1^e)_{e \in \vec{\mathbb{E}}^d}, (\omega_2^e)_{e \in \vec{\mathbb{E}}^d}, (\omega_3^k)_{k \in \mathbb{Z}^d})) = \left(\max_{k \in \mathbb{Z}^d} \left\{ \begin{array}{l} 2\omega_3^k, \\ 2 \max(\omega_2^{\{i,k\}} : \|i-k\|_1 \leq 1, x_i = 2), \\ \max(\omega_1^{\{i,k\}} : \|i-k\|_1 \leq 1, x_i = 1) \end{array} \right\} \right)_{k \in \mathbb{Z}^d}.$$

Note that type 2 particles do not see type 1 particles in their evolution, which explains why type 2 particles are assimilated to an environment. For two disjoint subsets E_1, E_2 of \mathbb{Z}^d , we also use the notation $\eta_n^{E_1, E_2} = \eta_n^{\mathbf{1}_{E_1} + 2\mathbf{1}_{E_2}}$.

We denote by $\eta_{1,n}^{E_1, E_2}$ (*resp.* $\eta_{2,n}^{E_2}$) the set of sites occupied by type 1 particles (*resp.* by type 2 particles) at time n , and we consider the evolution of the bacteria population $(\eta_{1,n}^{\{0\}, \emptyset})_{n \geq 0}$: can this process survive? Does it grow linearly when it survives? We naturally introduce the following extinction time and reaching times:

$$\begin{aligned} \tau_1^{E_1, E_2} &= \inf\{n \geq 0 : \eta_{1,n}^{E_1, E_2} = \emptyset\}; \\ \forall y \in \mathbb{Z}^d \quad t_1^{E_1, E_2}(y) &= \inf\{n \geq 0 : y \in \eta_{1,n}^{E_1, E_2}\}. \end{aligned}$$

Note that $\alpha \mapsto \mathbb{P}_{p,q,\alpha}(\tau_1^{E_1, E_2} = +\infty)$ is non-increasing and exhibits a phase transition. We first prove that this phase transition does not depend on the initial configuration of the environment:

Theorem 1.1. *For every $p > \vec{p}_c^{\text{alt}}(d+1)$ and every $q < \vec{p}_c^{\text{alt}}(d+1)$,*

$$\mathbb{P}_{p,q,\alpha}(\tau_1^{0, \mathbb{Z}^d \setminus \{0\}} = +\infty) > 0 \iff \mathbb{P}_{p,q,\alpha}(\tau_1^{0, \emptyset} = +\infty) > 0.$$

We thus define $\alpha_c(p, q) = \sup\{\alpha \geq 0 : \mathbb{P}_{p,q,\alpha}(\tau_1^{0, \emptyset} = +\infty) > 0\}$.

Our main result is the following:

Theorem 1.2. *For every $p > \vec{p}_c^{\text{alt}}(d+1)$ and every $q < \vec{p}_c^{\text{alt}}(d+1)$,*

$$0 < \alpha_c(p, q) < 1.$$

Moreover, for every $\alpha < \alpha_c(p, q)$, there exist positive constants A, B, C such that for every $E \subset \mathbb{Z}^d \setminus \{0\}$,

- (1) $\mathbb{P}_{p,q,\alpha}(\tau_1^{0,E} = +\infty) > 0$,
- (2) $\mathbb{P}_{p,q,\alpha}(\tau_1^{0,E} = +\infty, t^{0,E}(x) \geq C\|x\|_1 + t) \leq A \exp(-Bt)$,
- (3) $\mathbb{P}_{p,q,\alpha}(t < \tau_1^{0,E} < +\infty) \leq A \exp(-Bt)$.

We thus prove that if the immigration of type 2 particles is not too important, the bacteria population survive with positive probability, and, when it survives, grows linearly. We can also explain this model in terms of dependent oriented percolation: on the oriented graph $\mathbb{Z}^d \times \mathbb{N}$, we erase for each site $(z, n) \in \mathbb{Z}^d \times \mathbb{N}$, with probability α , the finite cluster of oriented percolation with parameter q starting from (z, n) . This random oriented graph is then given to the type 1 particle, which tries to develop as an oriented percolation with parameter p . Thus the growth of type 1 particles can be seen as a dependent oriented percolation model, with an unbounded but exponentially fast decreasing dependence. Our result ensures the linear growth of this oriented percolation when it percolates.

A natural question is then the existence of an asymptotic shape result:

Conjecture 1.3. *For every $p > \bar{p}_c^{alt}(d+1)$, for every $q < \bar{p}_c^{alt}(d+1)$ and every $\alpha \in (0, \alpha_c(p, q))$, there exists a norm μ on \mathbb{R}^d such that for any two disjoint subsets E_1 and E_2 in \mathbb{Z}^d with $E_1 \neq \emptyset$, we have for $\varepsilon > 0$: $\mathbb{P}_{p,q,\alpha}(\cdot | \tau_1^{E_1, E_2} = +\infty)$ almost surely, for every large enough t ,*

$$(1 - \varepsilon)B_\mu(0, 1) \subset \frac{1}{t}B_t \subset (1 + \varepsilon)B_\mu(0, 1),$$

where $B_t = \{x \in \mathbb{Z}^d : t_1^{E_1, E_2}(x) \leq t\} + [-1/2, 1/2]^d$.

We think that this result can be proved with subadditive methods similar to the ones we used in the case of the contact process in a random environment – see Garet–Marchand [11].

We can find a certain number of similar competition mechanisms in the literature under the name of hierarchical competition (see Durrett–Møller [5]), of contact process (or oriented percolation) in a dynamical random environment (see Broman [3], Luo [16], Remenik [17], Steif–Warfheimer [18]), or without any specific denomination (see Durrett–Swindle [10], Durrett–Schinazi [7]). The common characteristic of these models is that one type of particles (here type 2 particles) evolves in a Markovian way, and that the second type evolves as a contact process or an oriented percolation in the remaining empty space.

In our paper, we are going to use renormalization techniques. This is not surprising: the efficiency of such techniques in the study of particle systems has been known for long, see for instance Bramson–Durrett [2], or Durrett [4, 9]. But if the use of renormalization is usual to prove that survival occurs with positive probability, it is not so easy to implement to study a system conditioned to survive. Indeed, the renormalization procedures tend to destroy the independence properties given by the Markovianity and the tried and tested restart arguments described in Durrett [6] must be adapted with some care. While the general idea remains simple, the implementation is quite technical and, for the moment, there are no ready-made tools for this kind of situation. In the perspective of future works, we build tools in the spirit of the theorem of Liggett–Schonmann–Stacey [14] but in the context

of dependent oriented percolations resulting from renormalization procedures – see Theorem 2.5.

2. COMPARISON AND COUPLING RESULTS

While the setting of static renormalization can be defined quite formally, there are other types of renormalization that are harder to classify: they all have in common to consider local events that cannot be defined in an absolute way, but rather depend on a local component and also on the past of the renormalization process. This past can be associated to a time line as in Bezuidenhout–Grimmett [1] and Durrett [8], or to a sequence of spatial boxes as in Grimmett–Marstrand [13].

After renormalization, we are led to study a dependent oriented percolation process. The fact that this process survives with positive probability can be proved quite directly from the comparison result of Liggett–Schonmann–Stacey [14]. However, when one wants to study the oriented percolation process conditioned to survive, things are more intricate: for instance, we want to give a sense to the assertion “conditioned on its survival, the oriented percolation process on $\mathbb{Z}^d \times \mathbb{N}$ built from the renormalization process stochastically dominates an independent oriented percolation process with parameter as large as we want”. The aim is of course to transfer the properties of the supercritical independent percolation process to the dependent percolation process.

We work on the graph $\mathbb{Z}^d \times \mathbb{N}$, as defined in the introduction. We consider $\Omega = \{0, 1\}^{\vec{\mathbb{E}}^{d+1}}$ endowed with its Borel σ -algebra and the probability $\mathbb{P}_p = \mathcal{B}(p)^{\otimes \vec{\mathbb{E}}^{d+1}}$: the edges such that $\omega_e = 1$ are said to be open, the other ones are closed. For two sites v, w in $\mathbb{Z}^d \times \mathbb{N}$, we denote by $v \rightarrow w$ the existence of an open oriented path from v to w . The critical probability is denoted by $\vec{p}_c^{\text{alt}}(d+1)$. The time translations θ_n on Ω are defined by $\theta_n((\omega_{(e,k)})_{e \in \vec{\mathbb{L}}^d, k \geq 1}) = (\omega_{(e,k+n)})_{e \in \vec{\mathbb{L}}^d, k \geq 1}$. We set, for $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,

$$\begin{aligned} \xi_n^x &= \{y \in \mathbb{Z}^d : (x, 0) \rightarrow (y, n)\}, \\ \tau^x &= \min\{n \in \mathbb{N} : \xi_n^x = \emptyset\}, \\ H_n^x &= \bigcup_{0 \leq k \leq n} \xi_k^x, \\ K_n^x &= \{y \in \mathbb{Z}^d : \xi_n^x(y) = \xi_n^{\mathbb{Z}^d}(y)\}. \end{aligned}$$

As for the contact process, $(H_n^x)_{n \geq 0}$ and $(K_n^x)_{n \geq 0}$ grow linearly in case of survival:

Lemma 2.1. *We consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$. For every $p > \vec{p}_c^{\text{alt}}(d+1)$, there exist strictly positive constants A, B, C such that for every $x \in \mathbb{Z}^d$, for every $L, n > 0$:*

$$\begin{aligned} \mathbb{P}_p(\tau^x = +\infty, [-L, L]^d \not\subset K_{C(L+n)}^x) &\leq Ae^{-Bn} \\ \mathbb{P}_p(\tau^x = +\infty, [-L, L]^d \not\subset H_{C(L+n)}^x) &\leq Ae^{-Bn}. \end{aligned}$$

Proof. For the contact process, Durrett [8] showed how to deduce an analogous result from the construction of Bezuidenhout–Grimmett [1]. \square

We now recall the comparison theorem of Liggett–Schonmann–Stacey [14]. In the following, for two edges e and f in $\vec{\mathbb{E}}^d$, we denote by $d(e, f)$ the distance from $\|\cdot\|_1$ between the middles of the edges e and f .

Proposition 2.2. *Let $d \geq 1$ be fixed. For every $M \geq 1$, there exists a function g_M from $[0, 1]$ to $[0, 1]$ with $\lim_{q \rightarrow 1} g_M(q) = 1$ and such that if μ is a probability measure on $\Omega = \{0, 1\}^{\vec{\mathbb{L}}^d}$ satisfying: for every $e \in \vec{\mathbb{L}}^d$, $\mu(\omega_e = 1 | \sigma(\omega_f, d(e, f) \geq M)) \geq q$, then μ stochastically dominates a product of Bernoulli law with parameter $g_M(q)$:*

$$\mu \succeq \mathcal{B}(g_M(q))^{\otimes \vec{\mathbb{L}}^d}.$$

Relying on this theorem, we are going to prove analogous results for a certain class of dependent oriented percolations:

Definition 2.3. *Let $d \geq 1$ be fixed. Let M be a positive integer and $q \in (0, 1)$.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{G}_n)_{n \geq 0}$. We assume that, on this probability space, a random field $(W_e^n)_{e \in \vec{\mathbb{L}}^d, n \geq 1}$ taking its values in $\{0, 1\}$ is defined. This field gives the states – open or closed – of the edges in $\vec{\mathbb{E}}_{alt}^{d+1}$. We say that the law of the field $(W_e^n)_{e \in \vec{\mathbb{L}}^d, n \geq 1}$ is in $\mathcal{C}_d(M, q)$ if it satisfies the two following conditions.

- $\forall n \geq 1, \forall e \in \vec{\mathbb{E}}^d \quad W_e^n \in \mathcal{G}_n;$
- $\forall n \geq 0, \forall e \in \vec{\mathbb{E}}^d \quad \mathbb{P}[W_e^{n+1} = 1 | \mathcal{G}_n \vee \sigma(W_f^{n+1}, d(e, f) \geq M)] \geq q.$

First, we give a stochastic comparison between fields in $\mathcal{C}_d(M, q)$ and Bernoulli product measures:

Lemma 2.4. *Let $d, M \geq 1$ be positive integers and $q \in (0, 1)$.*

If the distribution of $(W_e^n)_{e \in \vec{\mathbb{L}}^d, n \geq 1}$ belongs to $\mathcal{C}_d(M, q)$, then for each n , the distribution of the field $(W_e^{n+k})_{e \in \vec{\mathbb{L}}^d, k \geq 1}$ conditioned by \mathcal{G}_n stochastically dominates $\mathcal{B}(g_M(q))^{\otimes \vec{\mathbb{E}}_{alt}^{d+1}}$.

Precisely, for each $n \geq 0$, for each $A \in \mathcal{G}_n$, and each non-decreasing bounded function f , we have

$$\mathbb{E}_W[\mathbb{1}_A(f \circ \theta_n)] \geq \mathbb{P}(A) \int_{\{0,1\}^{\vec{\mathbb{E}}_{alt}^{d+1}}} f \, d\mathcal{B}(g_M(q))^{\otimes \vec{\mathbb{E}}_{alt}^{d+1}}.$$

Proof. Let $E = \{0, 1\}^{\vec{\mathbb{L}}^d}$, $q' = g_M(q)$ and fix $n \geq 1$. We will show that for each non-negative integer k , for every non-decreasing bounded function f that only depends on the k first time coordinates, we have

$$\mathbb{E}[\mathbb{1}_A f(W^{n+1}, W^{n+2}, \dots, W^{n+k})] \geq \mathbb{P}(A) \int f \, d\mathcal{B}(q')^{\otimes \vec{\mathbb{E}}_{alt}^{d+1}}.$$

When $k = 0$, f is constant and the result is obvious.

Suppose the result holds for k and let us prove it for $k + 1$.

Let h be a non-decreasing bounded function on E^{k+1} and consider $A \in \mathcal{G}_n$. Since we work on a Polish space, we can disintegrate \mathbb{P} with respect to the σ -field \mathcal{G}_{n+k} : then, we have

$$\begin{aligned} & \mathbb{E}[\mathbb{1}_A h(W^{n+1}, \dots, W^{n+k+1})] \\ &= \mathbb{E}[\mathbb{1}_A \mathbb{E}[h(W^{n+1}, \dots, W^{n+k+1}) | \mathcal{G}_{n+k}]] \\ &= \int_A \int_{\Omega} h(W^{n+1}(\omega'), \dots, W^{n+k}(\omega'), W^{n+k+1}(\omega')) \, d\mathbb{P}^{\mathcal{G}_{n+k}, \omega'}(\omega') \, d\mathbb{P}(\omega) \\ &= \int_A \int_{\Omega} h(W^{n+1}(\omega), \dots, W^{n+k}(\omega), W_{n+k+1}(\omega')) \, d\mathbb{P}^{\mathcal{G}_{n+k}, \omega'}(\omega') \, d\mathbb{P}(\omega). \end{aligned}$$

Since we supposed that the distribution of $(W_e^n)_{e \in \vec{\mathbb{L}}^d, n \geq 1}$ belongs to $\mathcal{C}_d(M, q)$, the distribution of $(W_e^{n+k+1})_{e \in \vec{\mathbb{L}}^d}$ under $\mathbb{P}^{\mathcal{G}_{n+k}, \omega}$ satisfies, for every fixed ω , the assumptions of the Liggett–Schonmann–Stacey comparison Theorem (Theorem 2.2). Thus, it stochastically dominates $\mathcal{B}(q')^{\otimes \vec{\mathbb{L}}^d}$, which gives

$$\begin{aligned} & \int_{\Omega} h(W^{n+1}(\omega), \dots, W^{n+k}(\omega), W^{n+k+1}(\omega')) d\mathbb{P}^{\mathcal{G}_{n+k}, \omega}(\omega') \\ & \geq \int_E h(W^{n+1}(\omega), \dots, W^{n+k}(\omega), x) d\mathcal{B}(q')^{\otimes \vec{\mathbb{L}}^d}(x) = f(W^{n+1}(\omega), \dots, W^{n+k}(\omega)), \end{aligned}$$

where f is defined by

$$(4) \quad f(y_1, \dots, y_k) = \int_E h(y_1, \dots, y_k, x) d\mathcal{B}(q')^{\otimes \vec{\mathbb{L}}^d}(x).$$

Thus we obtain

$$\mathbb{E}[\mathbb{1}_A h(W^{n+1}, \dots, W^{n+k})] \geq \int_A f(W^{n+1}, \dots, W^{n+k}) d\mathbb{P}.$$

But by the induction assumption,

$$\int_A f(W^{n+1}, \dots, W^{n+k}) d\mathbb{P} \geq \mathbb{P}(A) \int_{E^k} f(y_1, \dots, y_k) d(\mathcal{B}(q')^{\otimes \vec{\mathbb{L}}^d})^{\otimes k},$$

which, from Definition (4), gives the desired result. \square

Then, we associate to every $\{0, 1\}$ -valued random field $(W_e^n)_{e \in \vec{\mathbb{L}}^d, n \geq 1}$ an oriented percolation process $(\xi_n^0(W))_{n \geq 1} = (\xi_n^0)_{n \geq 1}$ starting from $(0, 0)$ and defined in the usual way:

$$\begin{cases} \xi_0^0 = \{0\} \\ \xi_{n+1}^0 = \{x \in \mathbb{Z}^d : \exists y \in \xi_n^0 \quad W_{(y,x)}^{n+1} = 1\}. \end{cases}$$

For simplicity, we will often say “oriented percolation in $\mathcal{C}_d(M, q)$ ” instead of “oriented percolation associated to a field $\chi \in \mathcal{C}_d(M, q)$ ”.

We define the extinction time of the oriented percolation associated to W and starting from $(0, 0)$:

$$\tau^0(W) = \tau^0 = \inf\{n \geq 1 : \xi_n^0 = \emptyset\}.$$

The following result allows a coupling between surviving dependent percolation in $\mathcal{C}_d(M, q)$ and supercritical Bernoulli percolation:

Theorem 2.5. *Let $d, M \geq 1$ be fixed positive integers and let $q \in (0, 1)$ be such that $g_M(q) > \vec{p}_c^{\text{alt}}(d+1)$.*

There exist positive constants β, γ such that for each field $\chi \in \mathcal{C}_d(M, q)$, we can find a probability space where live a field $W = (W_e^n)_{e \in \vec{\mathbb{L}}^d, n \geq 1}$, a field $(W_e'^n)_{e \in \vec{\mathbb{L}}^d, n \geq 1}$, a \mathbb{N} -valued random variable T and a \mathbb{Z}^d -valued random variable D such that

- $\|D\|_1 \leq T$ and $\mathbb{E}[\exp(\beta T)] \leq \gamma$;
- The field $(W_e^n)_{e \in \vec{\mathbb{L}}^d, n \geq 1}$ follows the distribution χ and $\mathbb{P}(\tau^0(W) = +\infty) > 0$;
- $T = \tau^0(W)$ on the event $\{\tau^0(W) < +\infty\}$;
- Conditioning by $\{\tau^0(W) = +\infty\}$, the field $(W_e'^n)_{e \in \vec{\mathbb{L}}^d, n \geq 1}$ has the same distribution as independent oriented percolation with parameter $g_M(q)$ starting

from $(0, 0)$ and conditioned to survival; moreover, on $\{\tau^0(W) = +\infty\}$, we have

$$\forall n \geq 0 \quad \xi_{T+n}^0(W) \supset D + \xi_n^0(W').$$

Proof. Define $q' = g_M(q)$.

Let E_1, \dots, E_n be finite subsets of \mathbb{Z}^d . We define $E = (E_1, \dots, E_n)$ and $|E| = n$. The event

$$A_E = \bigcap_{i=1}^{|E|} \{\xi_i^0 = E_i\}$$

is in \mathcal{G}_n ; on this event, the history of the directed percolation process starting from $(0, 0)$ up to time n is characterized by E . We call that E an history.

From now on, we only consider histories satisfying $\chi(A_E) > 0$; for such an history, we define a probability measure m_E on $\{0, 1\}^{\vec{\mathbb{L}}^d \times \mathbb{N}^*}$ by

$$m_E(B) = \chi((W^{|E|+k})_{k \geq 1} \in B | A_E);$$

we call it the law of the dependent oriented percolation with history E . Thanks to Lemma 2.4, the probability measure m_E stochastically dominates $\mathcal{B}(q')^{\otimes \vec{\mathbb{L}}^d \times \mathbb{N}^*}$.

Thus, Strassen's Theorem ([19], see also Lindvall [15]) allows to build a law ν_E on $(\{0, 1\}^{\vec{\mathbb{L}}^d \times \mathbb{N}^*})^2$ with marginals m_E and $\mathcal{B}(q')^{\otimes \vec{\mathbb{L}}^d \times \mathbb{N}^*}$ and is concentrated on $\{x \geq y\}$. For every history E , the law ν_E allows to make a coupling between the state of the bonds of dependent and of independent oriented percolations with common history E . Now, we can construct on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a family of $(\{0, 1\}^{\vec{\mathbb{L}}^d \times \mathbb{N}^*})^2$ -valued independent processes $({}^E\eta, {}^E\eta')_E$, that are indexed by the collection of all histories E , in such a way that for every history E ,

$$({}^E\eta_e^n, {}^E\eta'_e{}^n)_{e \in \vec{\mathbb{L}}^d, n \geq 1} \stackrel{\text{law}}{=} \nu_E.$$

We denote by ${}^E\tau^x$ the time where the directed percolation related to ${}^E\eta'$ and starting from x dies. To simplify the notation, we also write $\xi_n({}^E\eta)$ to denote the state at time n of the percolation process with history E ; for instance, $\xi_0({}^E\eta) = E_{|E|}$. We also denote by ${}^E N^x = (\xi_1({}^E\eta), \dots, \xi_{{}^E\tau^x}({}^E\eta))$ the sequence of the configurations occupied up to time ${}^E\tau^x$ by the percolation process associated to η with history E , and denote by ${}^E L^x$ its terminal configuration. Then, we put $\varepsilon_0 = \{0\}$, $t_0 = 0$, $x_0 = 0$ and

- If $t_i < +\infty$ and $\varepsilon_i L^{x_i} \neq \emptyset$, then

$$t_{i+1} = t_i + \varepsilon_i \tau^{x_i}.$$

If $t_{i+1} < +\infty$, we set $x_{i+1} = \min \varepsilon_n L^{x_n}$ and $\varepsilon_{i+1} = (\varepsilon_i, \varepsilon_i N^{x_i})$.

If $t_{i+1} = +\infty$, we set $x_{i+1} = 0$.

- Otherwise, we set $t_{i+1} = +\infty$ et $x_{i+1} = 0$.

Then, we define

$$K = \min\{k \geq 1 : t_{k+1} = +\infty\}, \quad T = t_K, \quad \text{and} \quad D = x_K.$$

For $i \leq K$ and $e \in \vec{\mathbb{E}}^d$, we put $W_e^n = \varepsilon_i \eta_e^{n-t_i}$ for $n \in [t_i, t_{i+1}[$.

Finally, for each $n \geq 1$ and each $e \in \vec{\mathbb{E}}^d$, we define $W_e'^n = \varepsilon_K \eta_{e-x_K}'^n$.

This procedure, close to the classical so-called "restart argument" can be described as follows: starting from 0, we exhibit with $\{^0\}\nu$ a coupling between dependent and independent percolations up to time $t_1 = \{^0\}\tau^0$ when independent percolation dies. Then, we record the history of the dependent percolation in ε_1 ,

and pick some point x_1 occupied by the dependent percolation process in the terminal configuration. We then construct another coupling $\varepsilon_1 \nu$ between the dependent percolation and some new independent percolation process starting from x_1 , following this coupling until time t_2 when the new independent percolation also dies. We can complement the history of the dependent percolation and get ε_2 , then choose x_2 occupied by the dependent percolation process in the terminal configuration, and so on. This procedure stops either because we find a time t_K when our K th independent percolation process survives, or because the dependent percolation process died together with the independent one.

Let us denote by \mathcal{T}_n the σ -field generated by the $({}^E\eta, {}^E\eta')_{|E| \leq n}$. We have, for $\alpha > 0$,

$$\begin{aligned} & \mathbb{E}[\exp(\alpha^{\varepsilon_n} \tau^{x_n}) \mathbb{1}_{\{K > n\}} | \mathcal{T}_n] \\ &= \mathbb{E}[\exp(\alpha^{\varepsilon_n} \tau^{x_n}) \mathbb{1}_{\{t_{n+1} < +\infty\}} | \mathcal{T}_n] \\ &= \mathbb{E}[\exp(\alpha^{\varepsilon_n} \tau^{x_n}) \mathbb{1}_{\{t_n < +\infty, \varepsilon_{n-1} L^{x_{n-1}} \neq \emptyset, \varepsilon_n \tau^{x_n} < +\infty\}} | \mathcal{T}_n] \\ &\leq \mathbb{1}_{\{K > n-1\}} \int \mathbb{1}_{\{\tau^0 < +\infty\}} \exp(\alpha \tau^0) d\mathcal{B}(q')^{\otimes \vec{\mathbb{E}}^d \times \mathbb{N}^*}. \end{aligned}$$

Thus, since $q' > \vec{p}_c^{\text{alt}}(d+1)$, if we put $r = \int \mathbb{1}_{\{\tau^0 < +\infty\}} \exp(\alpha \tau^0) d\mathcal{B}(q')^{\otimes \vec{\mathbb{E}}^d \times \mathbb{N}^*}$, we can choose $\alpha > 0$ small enough to have $r < 1$; then

$$\begin{aligned} \mathbb{E}[\exp(\alpha t_{n+1}) \mathbb{1}_{\{K = n+1\}}] &\leq \mathbb{E}[\exp(\alpha(\varepsilon_0 \tau^{x_0} + \dots + \varepsilon_n \tau^{x_n})) \mathbb{1}_{\{K > n\}}] \\ &\leq r \mathbb{E}[\exp(\alpha t_n) \mathbb{1}_{\{K > n-1\}}] \leq r^{n+1}, \\ \text{then } \mathbb{E}[\exp(\alpha T)] &\leq \sum_{i=0}^{+\infty} r^{i+1} = \frac{r}{1-r}. \end{aligned}$$

Particularly, since $K \leq T$, we get the existence of exponential moments for K , and the fact that K is almost surely finite.

Stacking the conditional laws up, we can check that the field W has the desired distribution.

Assume that $\tau^0(W) < +\infty$ and $K = k$. Then $t_k < +\infty$ and $t_{k+1} = +\infty$. For each $n \in [t_k, +\infty[$, we have by construction $(W_e^n)_{e \in \vec{\mathbb{E}}^d} = (\varepsilon_k \eta_e^{n-t_k})_{e \in \vec{\mathbb{E}}^d}$. If $\varepsilon_k L^{x_k} \neq \emptyset$, then $t_{k+1} = t_k + \varepsilon_k \tau^{x_k} = +\infty$, which implies that $\varepsilon_k \tau^{x_k} = +\infty$, which can not happen because $\tau^0(W) < +\infty$. Thus $\varepsilon_k L^{x_k} = \emptyset$, so $\tau^0(W) \leq t_k = T$. The inequality $\tau^0(W) \geq t_k$ directly follows from the inclusion between independent and dependent percolations. Finally, if $\tau^0(W) < +\infty$, then $T = \tau^0(W)$.

On the event $\{\tau^0(W) = +\infty\}$, we have by construction $D \in \xi_T^0(W)$, so the inclusion property gives $\forall n \geq 0 \quad \xi_{T+n}^0(W) \supset D + \xi_n^0(W')$. Finally, noting that $\{\tau^0(W) = +\infty, K = n, \varepsilon_n = E, x_n = x\} \subset \{{}^E\tau^x = +\infty\}$, we get by independence that for every Borel set B in $\{0, 1\}^{\vec{\mathbb{E}}^{d+1}}$,

$$\begin{aligned} & \mathbb{P}(\tau^0(W) = +\infty, K = n, \varepsilon_n = E, x_n = x, W' \in B) \\ &= \mathbb{P}(\tau^0(W) = +\infty, K = n, \varepsilon_n = E, x_n = x, {}^E\eta' \in (-x).B) \\ &= \mathbb{P}(\tau^0(W) = +\infty, K \geq n, \varepsilon_n = E, x_n = x, {}^E\eta' \in (-x).B, {}^E\tau^x = +\infty) \\ &= \mathbb{P}(\tau^0(W) = +\infty, K \geq n, \varepsilon_n = E, x_n = x) \mathbb{P}({}^E\tau^x = +\infty, {}^E\eta' \in (-x).B) \\ &= \mathbb{P}(\tau^0(W) = +\infty, K \geq n, \varepsilon_n = E, x_n = x) \mathbb{P}_{q'}(\tau^0 = +\infty, B) \end{aligned}$$

Along those lines, $x.B = \{(\eta_{c+x}^n)_{e \in \vec{\mathbb{E}}^d, n \geq 1} : \eta \in B\}$.

Summing on all possible values for E, n, x , we obtain

$$\mathbb{P}(\tau^0(W) = +\infty, W' \in B) = c\mathbb{P}_{q'}(\tau^0 = +\infty, B),$$

and further $\mathbb{P}(W' \in B | \tau^0(W) = +\infty) = \mathbb{P}_{q'}(B | \tau^0 = +\infty)$. \square

3. SOME PROPERTIES OF DEPENDENT ORIENTED PERCOLATION

The coupling Theorem 2.5 permits to transfer some properties from supercritical independent oriented percolations to dependent oriented percolations in $\mathcal{C}_d(M, q)$ for q close to 1. Practically, those processes often arise after the use of a dynamical renormalization scheme.

As a by-product of the proof of Theorem 2.5, we can get information on the exponential moments for extinction times. For oriented Bernoulli percolation, a Peierls-like argument shows that

$$(5) \quad \liminf_{p \rightarrow 1} \int_{\beta > 0} \mathbb{1}_{\{\tau^0 < +\infty\}} \exp(\beta\tau^0) d\mathbb{P}_p = 0,$$

which can be transposed to the dependent fields of $\mathcal{C}_d(M, q)$ as follows:

Corollary 3.1. *Let $\varepsilon > 0$ and $M > 1$. There exist $\beta > 0$ and $q < 1$ such that for each $\chi \in \mathcal{C}_d(M, q)$,*

$$\mathbb{E}_\chi[\mathbb{1}_{\{\tau^0 < +\infty\}} \exp(\beta\tau^0)] \leq \varepsilon.$$

Proof. We observed in the proof of Theorem 2.5 that $T = \tau^0$ when $\{\tau^0 < +\infty\}$. We also have the bound

$$\mathbb{E}_\chi[\mathbb{1}_{\{\tau^0 < +\infty\}} \exp(\beta\tau^0)] \leq \mathbb{E}_\chi[e^{\beta T}] \leq \frac{r}{1-r},$$

with $r = \int \mathbb{1}_{\{\tau^0 < +\infty\}} \exp(\beta\tau^0) d\mathbb{P}_{g_M(q)}$; the result then follows from (5). \square

As a direct application of the coupling Theorem 2.5, the linear growth of the set H_n of points reached before time n , given in Lemma 2.1 for independent directed percolation, can be transposed to any dependent percolation in $\mathcal{C}_d(M, q)$:

Corollary 3.2. *Let $d, M \geq 1$ be fixed positive integers, and let $q \in (0, 1)$ be such that $g_M(q) > \vec{p}_c^{alt}(d+1)$. There exist positive constants β, D_1, D_2 and random variables $(S^y)_{y \in \mathbb{Z}^d}$ such that*

$$\forall y \in \mathbb{Z}^d \quad \mathbb{E}[e^{\beta S^y}] \leq D_2,$$

and such that for each field $\chi \in \mathcal{C}_d(M, q)$, the directed percolation associated to χ satisfies: on the event $\{\tau^y = +\infty\}$,

$$\forall n \in \mathbb{N} \quad y + [-D_1 n, D_1 n]^d \subset H_{S^y+n}^y.$$

Having in mind an accurate study of certain particle systems, it could be interesting to have estimates on the density of bi-infinite points in the dependent oriented percolation. Thus, we define

$$\begin{aligned} G(x, y) &= \{k \in \mathbb{N} \mid (x, 0) \rightarrow (y, k) \rightarrow \infty\} \\ \gamma(\theta, x, y) &= \inf\{n \in \mathbb{N} : \forall k \geq n \mid |\{0, \dots, k\} \cap G(x, y)| \geq \theta k\} \end{aligned}$$

Corollary 3.3. *Let $M > 1$. There exist $q_0 < 1$ and positive constants A, B, θ, α such that for each $\chi \in \mathcal{C}_d(M, q_0)$, we have*

$$\forall x, y \in \mathbb{Z}^d \quad \forall n \geq 0 \quad \mathbb{P}(+\infty > \gamma(\theta, x, y) > \alpha \|x - y\|_1 + n) \leq Ae^{-Bn}.$$

For instance, those estimates allow to study the large deviations of the asymptotic shape of the contact process [12]. Considering Theorem 2.5, Lemma 3.3 will easily follow from the independent case.

Lemma 3.4. *We consider independent directed percolation on $\mathbb{Z}^d \times \mathbb{N}$. There exist positive constants A, B, θ, α and $p < 1$ such that for every $x, y \in \mathbb{Z}^d$,*

$$(6) \quad \forall n \in \mathbb{N} \quad \mathbb{P}_p(\tau^x = +\infty, \gamma(\theta, x, y) \geq \alpha \|y - x\|_\infty + n) \leq A \exp(-Bn).$$

Proof. We actually prove the following simpler result: there exists p close to 1, positive constants A, B, C', θ such that $\forall x \in \mathbb{Z}^d \quad \forall n \in \mathbb{N}$

$$(7) \quad \mathbb{P}_p \left(\left| \{k \in \{C'\|x\|_\infty, \dots, C'\|x\|_\infty + n : (0, 0) \rightarrow (x, k) \rightarrow \infty\} \right| \leq \theta n \right) \leq A \exp(-Bn).$$

Let us show that (7) implies (6). We note that $\gamma(\theta, x, y)$ has the same distribution as $\gamma(\theta, 0, y - x)$, that $\theta < 1$ and use (7):

$$\begin{aligned} & \mathbb{P}_p \left(\tau^0 = +\infty, \gamma(\theta, 0, x) \geq \frac{C'}{\theta} \|x\|_\infty + n \right) \\ &= \mathbb{P}_p \left(\left| \{l \in \{0..k\} : (0, 0) \rightarrow (x, l) \rightarrow +\infty\} \right| \leq \theta k \right) \\ &\leq \mathbb{P}_p \left(\left| \{l \in \{C'\|x\|_\infty, \dots, C'\|x\|_\infty + k\} : (0, 0) \rightarrow (x, l) \rightarrow +\infty\} \right| \leq \theta k \right) \\ &\leq \sum_{k \geq n} \mathbb{P}_p(\tau^0 = +\infty, \left| \{l \in \{C'\|x\|_\infty, \dots, C'\|x\|_\infty + k\} : (0, 0) \rightarrow (x, l) \rightarrow +\infty\} \right| \leq \theta k) \\ &\leq \sum_{k \geq n} A \exp(-Bk). \end{aligned}$$

Taking $\alpha = C'/\theta$, this proves (6).

Let us now prove (7). We define

$$\tilde{I}_\infty = \{(x, n) \in \mathbb{V}^{d+1} : \mathbb{Z}^d \times \{0\} \rightarrow (x, n) \rightarrow \infty\}.$$

One notes that if $x \in K_k$ and $(x, k) \in \tilde{I}_\infty$, then by the definition of the coupled region, $(0, 0) \rightarrow (x, k) \rightarrow \infty$. We take $C' = \lceil 1/C \rceil$, where C comes from Lemma 2.1: we choose any θ with $0 < \theta < 1/4$ then

$$\begin{aligned} & \mathbb{P}_p(\tau^0 = +\infty, \left| \{k \in \{C'\|x\|_\infty, \dots, C'\|x\|_\infty + n\} : (0, 0) \rightarrow (x, k) \rightarrow \infty\} \right| \leq \theta n) \\ &\leq \mathbb{P}_p(\tau^0 = +\infty, \exists k \geq C'\|x\|_\infty + \frac{n}{2}, K_k \not\supset [-CC'\|x\|_\infty, CC'\|x\|_\infty]^d) \\ &\quad + \mathbb{P}_p(\left| \{k \in \{C'\|x\|_\infty + n/2, \dots, C'\|x\|_\infty + n\} : (x, k) \in \tilde{I}_\infty\} \right| \leq \theta n) \end{aligned}$$

For the first term, we use Lemma 2.1:

$$\begin{aligned} & \mathbb{P}_p(\tau^0 = +\infty, \exists k \geq C'\|x\|_\infty + \frac{n}{2}, K_k \not\supset [-CC'\|x\|_\infty, CC'\|x\|_\infty]^d) \\ &\leq \sum_{k \geq n/2} \mathbb{P}_p(\tau^0 = +\infty, K_{C'\|x\|_\infty + k} \not\supset [-CC'\|x\|_\infty, CC'\|x\|_\infty]^d) \leq \sum_{k \geq n/2} A \exp(-Bk). \end{aligned}$$

To control the second term, we use the following lemma:

Lemma 3.5. *Consider independent directed percolation on $\mathbb{Z}^d \times \mathbb{N}$. For each $\rho \in (0, 1)$, there exists $p_0(\rho) < 1$ such that for each $p > p_0(\rho)$,*

$$\forall \text{ finite } A \subset \{0\} \times \mathbb{N} \quad \mathbb{P}_p(A \cap \tilde{I}_\infty = \emptyset) \leq 16\rho^{|A|-2}.$$

We omit the proof, which is classical. Basically, it is based on contour arguments – see for example Durrett [6].

With the notations of Lemma 3.5, we then choose $0 < \rho < 1$ such that $2\rho^{1/2} < 1$ and obtain, for $p \geq p_0(\rho)$,

$$\begin{aligned} & \mathbb{P}_p \left(\left| k \in \left\{ C' \|x\|_\infty + \frac{n}{2}, \dots, C' \|x\|_\infty + n \right\} : (x, k) \in \tilde{I}_\infty \right| \leq \theta n \right) \\ & \leq 2^{n/2+1} 16 \rho^{n/2 - \theta n - 3}. \end{aligned}$$

This concludes the proof of (7), and therefore of the Lemma. \square

4. APPLICATION TO THE MODEL

4.1. First remarks. We begin by proving that the positivity of the probability of survival for the bacteria does not depend on the initial condition of the environment. We can note that Steif and Warfheimer [18] have proved a similar result for the model introduced by Broman [3].

Proof of Theorem 1.1. Let $p, q, \alpha > 0$ be such that $\mathbb{P}_{p,q,\alpha}(\tau^{0,\emptyset} = +\infty) > 0$. We want to show that $\mathbb{P}_{p,q,\alpha}(\tau^{0,\mathbb{Z}^d \setminus \{0\}} = +\infty) > 0$. Let us denote by C_n the event : “there exists $x \in [-n, n]^d$ such that (x, n) is linked to $\mathbb{Z}^d \times \{0\}$ by open bonds of directed oriented percolation with parameter q ”: by a time reversal argument, we get

$$\mathbb{P}_{p,q,\alpha}(C_n) \leq (2n+1)^d \mathbb{P}_q(T > n) \leq A \exp(-Bn),$$

where T is the extinction time of some subcritical oriented percolation process with parameter q . We conclude that, if $A_N = \bigcap_{k \geq N} C_k^c$,

$$\lim_{N \rightarrow +\infty} \mathbb{P}_{p,q,\alpha}(A_{N-1} \circ \theta_1) = \lim_{N \rightarrow +\infty} \mathbb{P}_{p,q,\alpha}(A_{N-1}) = 1,$$

$$\text{whence } \lim_{N \rightarrow +\infty} \mathbb{P}_{p,q,\alpha}(\tau^{0,\emptyset} = +\infty, A_{N-1} \circ \theta_1) = \mathbb{P}_{p,q,\alpha}(\tau^{0,\emptyset} = +\infty).$$

In particular, there exists N such that $\mathbb{P}_{p,q,\alpha}(\tau^{0,\emptyset} = +\infty, A_{N-1} \circ \theta_1) > 0$. Let us denote by B the event: “all the oriented edges issued from $[-3N, 3N]^d \times \{0\}$ are closed for the percolation with parameter q ”. By independence, one has

$$\mathbb{P}_{p,q,\alpha}(\tau^{0,\emptyset} = +\infty, A_{N-1} \circ \theta_1, B) = \mathbb{P}_{p,q,\alpha}(\tau^{0,\emptyset} = +\infty, A_{N-1} \circ \theta_1) \mathbb{P}_{p,q,\alpha}(B) > 0.$$

It remains to prove that $\tau^{0,\mathbb{Z}^d \setminus \{0\}} = +\infty$ holds on this event. It is sufficient to prove that the processes $(\eta_{1,n}^{0,\mathbb{Z}^d \setminus \{0\}})_{n \geq 0}$ and $(\eta_{1,n}^{0,\emptyset})_{n \geq 0}$ coincide on this event; but because of the definition of the dynamics, it is sufficient to note that on the event $(A_{N-1} \circ \theta_1) \cap B$, we have

$$\forall n \geq 1 \quad \eta_{2,n}^\emptyset \cap [-n, n]^d = \eta_{2,n}^{\mathbb{Z}^d \setminus \{0\}} \cap [-n, n]^d,$$

which ends the proof. \square

The end of this section is devoted to the proof of Theorem 1.2.

The idea of the proof is to define a local block event with probability close to 1, that expresses the fact that if the bacterium occupies a sufficiently large area at a given place, it will presumably extend itself a bit further. If the associated block process percolates, then the linear growth is ensured by Theorem 2.5. With a restart argument, we will find a point of the space-time, not too far from the origin,

where the bacterium occupies a sufficiently large area and where the associated block process percolates, which will give the desired result.

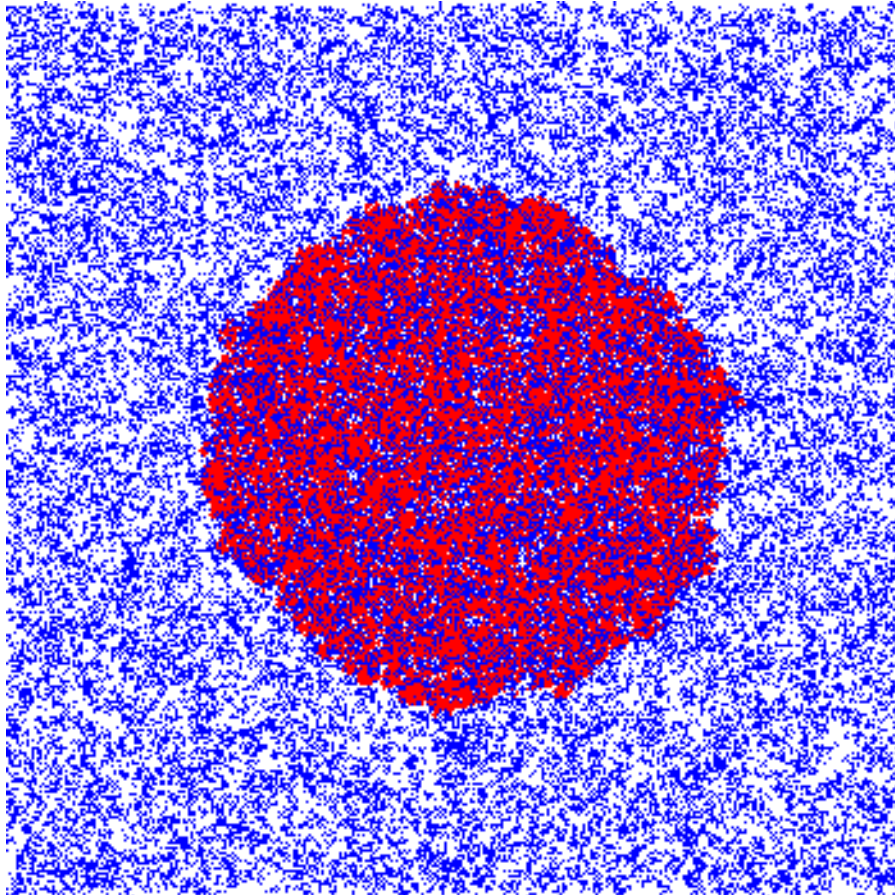


FIGURE 1. A simulation with $p = 0.7$ and $q = 0.25$.

The statement in Theorem 1.2 actually contains two results that must be proved separately. On one side, there is the fact that $\alpha_c > 0$, on the other side the fact that for $\alpha < \alpha_c$, the process, when surviving, linearly grows. We can find in the literature many examples of block events similar to the ones we will use. Most of these papers take inspiration from the Bezuidenhout–Grimmett article [1]: *The critical contact process dies out*. We think that for this kind of dynamical renormalization schemes, the existence of a coupling between the dependent oriented percolation of blocks and a Bernoulli oriented percolation conditioned to survive is barely explained in the literature. This led us to write Theorem 2.5. We chose to focus here on the case where α is small and the renormalization event simpler. We thus provide a complete proof of the fact that $\alpha_c > 0$ and of the linear growth in case of survival for α small. The construction for $\alpha < \alpha_c$ is technically more subtle, but also widely used, so to avoid the repetition of similar proofs, we just describe the considered events, referring the reader to Bezuidenhout–Grimmett [1] and Steif–Warfheimer [18] for a complete description of the steps of the proof.

4.2. The block event. Let $I, L \in \mathbb{N}^*$ with $I < L$. Recall that the constant C is given in Lemma 2.1. We let

$$T = 6CL \text{ and } J = 2(L + T).$$

For $\bar{k} \in \mathbb{Z}^d, x \in [-L, L]^d$ and $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$, we define the following event:

$$A(\bar{k}, x, u) = \left\{ \begin{array}{l} \exists s \in [-L, L]^d, \\ 2L(\bar{k} + u) + s + [-I, I]^d \subset \eta_{1,T}^{2L\bar{k}+x+[-I,I]^d, \mathbb{Z}^d \setminus (2L\bar{k}+[-J,J]^d)}, \\ \eta_{2,T}^{2L\bar{k}+x+[-I,I]^d, \mathbb{Z}^d \setminus (2L\bar{k}+[-J,J]^d)} \cap (2L(\bar{k} + u) + [-J, J]^d) = \emptyset, \\ 2L\bar{k} + [-L, L]^d \subset \bigcup_{0 \leq t \leq T} \eta_{1,t}^{2L\bar{k}+x+[-I,I]^d, \mathbb{Z}^d \setminus (2L\bar{k}+[-J,J]^d)}. \end{array} \right\}.$$

If $A(\bar{k}, x, u)$ holds, we denote by $s(\bar{k}, x, u)$ an element s satisfying the condition above.

Let us briefly explain the signification of the event $A(\bar{k}, x, u)$: obviously, $\eta_{1,T}^{A,B}$ is non-decreasing with respect to A , non-increasing with respect to B , whereas $\eta_{2,T}^B$ is non-decreasing with respect to B . Thus, if $A(\bar{k}, x, u)$ holds and if one knows that at time 0, the block $2L\bar{k} + x + [-I, I]^d$ is full of “1” and the block $2L\bar{k} + [-J, J]^d$ contains no “2”, then one knows that analogous conditions will be fulfilled around $2L(\bar{k} + u)$ at time T . Of course, the idea is to follow a chain of such events in an oriented percolation and to draw a path ensuring the development of the bacteria.

Lemma 4.1. *For each $p > \bar{p}_c^{alt}(d+1)$, each $q < \bar{p}_c^{alt}(d+1)$, and each $\varepsilon > 0$, we can find integers $I < L$ large enough and $\alpha > 0$ small enough such that for every $\bar{k} \in \mathbb{Z}^d, x \in [-L, L]^d$ and $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$,*

$$(8) \quad \mathbb{P}_{p,q,\alpha}(A(\bar{k}, x, u)) \geq 1 - \varepsilon.$$

Moreover, as soon as $\|\bar{k} - \bar{l}\|_1 > 4 + 18C$, for every $x, y \in [-L, L]^d$, every $u, v \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$ and $\|v\|_1 \leq 1$, the events $A(\bar{k}, x, u)$ and $A(\bar{l}, y, v)$ are independent.

Proof. First note that $\mathbb{P}_{p,q,\alpha}(A(\bar{k}, x, u)) = \mathbb{P}_{p,q,\alpha}(A(\bar{0}, x, u))$, which allows to consider only the case $\bar{n} = \bar{0}$.

Under $\mathbb{P}_{p,q,\alpha}$, the collection of random variables $\omega_1 = (\omega_{1,n}^e)_{e \in \mathbb{E}^d, n \in \mathbb{N}^*}$ – resp. $\omega_2 = (\omega_{2,n}^e)_{e \in \mathbb{E}^d, n \in \mathbb{N}^*}$ – has the law of the bonds of an independent directed percolation with parameter p – resp. q . We realize these percolation structures on Ω , keeping the notation of the introduction: thus, $(\xi_n^A(\omega_1))_{n \geq 0}$ is under $\mathbb{P}_{p,q,\alpha}$ a directed Bernoulli percolation process with parameter p starting from the set A , and $(\tau^x(\omega_2))_{n \geq 0}$ is under $\mathbb{P}_{p,q,\alpha}$ the extinction time for a directed Bernoulli percolation process with parameter q starting from x .

Let $\varepsilon > 0$. We choose two integers I, L with $I < L$ – their values will be fixed later. Define

$$B(x, u) = \left\{ \begin{array}{l} \exists s \in [-L, L]^d \quad 2Lu + s + [-I, I]^d \subset \xi_T^{x+[-I,I]^d}(\omega_1), \\ \forall (y, n) \in [-(4L+2T), (4L+2T)]^d \times \{1, \dots, T\} \\ \quad \omega_3^{y,n} = 0 \text{ and } \tau^y \circ \theta_n(\omega_2) \leq T/2, \\ [-L, L]^d \subset \bigcup_{0 \leq t \leq T} \xi_t^{x+[-I,I]^d}(\omega_1). \end{array} \right\}.$$

We will show that $B(x, u) \subset A(\bar{0}, x, u)$ and also that one can choose I and L in such a way that $\mathbb{P}_{p,q,\alpha}(B(x, u)) \geq 1 - \varepsilon$, which will give the desired result. The

advantage of using B is that it does not deal with the competition process, using only the directed percolation and the immigration processes. Thus, it is easier to estimate its probability.

Step 1. Let us show that $B(x, u) \subset A(\bar{0}, x, u)$.

The existence of a convenient s for the condition of $A(\bar{0}, x, u)$ is given by $B(x, u)$ for the oriented percolation with parameter p embedded in the model. We have now to verify that our event ensures that the type 2 particles can not disturb the progress of type 1 particles.

Note $A = x + [-I, I]^d$ and $B = \mathbb{Z}^d \setminus [-J, J]^d$. At time 0, the smaller distance between points in $\eta_{1,0}^{A,B}$ and $\eta_{2,0}^B$ is at least $2L + 2T - (L + I) > 2T$. In the zone $[-J, J]^d$, there is no immigration between time 0 and time T , so $\eta_{1,t}^{A,B}$ and $\eta_{2,t}^B$ get closer at a speed that does not exceed 2 per time unit; thus at time T , the type 2 particles could not disturb the move of type 1 particles yet.

It remains to see that $\eta_{2,T}$ can not reach $2Lu + [-J, J]^d$. Remember that there is no immigration between time 0 and time T in the area $[-(4L + 2T), (4L + 2T)]^d$. Moreover, type 2 particles that are outside $[-(4L + 2T), (4L + 2T)]^d$ at time 0 do not have enough time to reach $2Lu + [-J, J]^d$ at time T , so only type 2 particles that were already inside $[-(4L + 2T), (4L + 2T)]^d$ at time 0 must be considered. But these ones are all dead at time $T/2$. This completes the proof of the inclusion.

Step 2. Bounding the probability of $B(x, u)$ from below. Remember that $\mathbb{P} = \mathbb{P}_{p,q,\alpha}$.

We first choose an integer I large enough to have

$$(9) \quad \forall x \in \mathbb{Z}^d \quad \mathbb{P}(\tau^{x+[-I, I]^d}(\omega_1) = +\infty) \geq 1 - \varepsilon/12.$$

By the FKG inequality, $\mathbb{P}(\forall y \in [-I, I]^d, \tau^y(\omega_1) = +\infty) > 0$. Translation invariance and ergodicity of \mathbb{P} then give

$$\lim_{L \rightarrow +\infty} \mathbb{P}(\exists n \in [0, L] : \forall y \in nu + [-I, I]^d, \tau^y(\omega_1) = +\infty) = 1.$$

Then, let $L_1 > I$ be such that for each $L \geq L_1$,

$$\mathbb{P}(\exists n \in [0, L] : \forall y \in nu + [-I, I]^d, \tau^y(\omega_1) = +\infty) > 1 - \frac{\varepsilon}{12}.$$

Let $L \geq L_1$. By a time reversal argument, we have for each $t > 0$,

$$(10) \quad \mathbb{P} \left(\begin{array}{c} \exists n \in [0, L] : \\ nu + [-I, I]^d \subset \xi_t^{\mathbb{Z}^d}(\omega_1) \end{array} \right) = \mathbb{P} \left(\begin{array}{c} \exists n \in [0, L] : \\ \forall y \in nu + [-I, I]^d, \tau^y(\omega_1) \geq n \end{array} \right) \geq 1 - \frac{\varepsilon}{12}.$$

Now, Lemma 2.1 gives the existence of some $L_2 \geq L_1$ such that for each $L \geq L_2$, we have simultaneously

$$(11) \quad \begin{aligned} & \mathbb{P}(\exists y \in [-2L, 2L]^d : \tau^y(\omega_1) = +\infty, Lu + [-2L, 2L]^d \not\subset K_{6CL}^y(\omega_1)) \\ & \leq (4L + 1)^d \mathbb{P}(\tau^0(\omega_1) = +\infty, [-5L, 5L]^d \not\subset K_{6CL}^0(\omega_1)) \leq \varepsilon/12, \end{aligned}$$

$$\text{and} \quad \mathbb{P}(\exists y \in [-2L, 2L]^d : \tau^y(\omega_1) = +\infty, [-L, L]^d \not\subset H_{6CL}^y(\omega_1))$$

$$(12) \quad \leq (4L + 1)^d \mathbb{P}(\tau^0(\omega_1) = +\infty, [-3L, 3L]^d \not\subset H_{6CL}^0(\omega_1)) \leq \varepsilon/12.$$

With (10) and (11), we get

$$\begin{aligned}
& \mathbb{P}(\tau^{x+[-I, I]^d}(\omega_1) = +\infty, \forall n \in [0, L] \quad Lu + nu + [-I, I]^d \not\subset \xi_T^{x+[-I, I]^d}(\omega_1)) \\
& \leq \mathbb{P}(\exists y \in x + [-I, I]^d : \tau^y(\omega_1) = +\infty, Lu + [-2L, 2L]^d \not\subset K_T^y(\omega_1)) \\
& \quad + \mathbb{P}(\forall n \in [0, L] \quad Lu + nu + [-I, I]^d \not\subset \xi_T^{\mathbb{Z}^d}(\omega_1)) \\
& \leq \frac{\varepsilon}{6}.
\end{aligned}$$

With (9) and (12), we conclude that for each $x \in [-L, L]^d$

$$(13) \quad \mathbb{P} \left(\begin{array}{c} H_T^{x+[-I, I]^d}(\omega_1) \supset [-L, L]^d, \\ \exists n \in [0, L] \quad (L+n)u + [-I, I]^d \subset \xi_T^{x+[-I, I]^d}(\omega_1) \end{array} \right) \geq 1 - \frac{\varepsilon}{3}.$$

Since $q < \vec{p}_c^{\text{alt}}(d+1)$, there exist positive constants A, B such that for each L , $\mathbb{P}(\exists y \in [-(4L+2T), (4L+2T)]^d : \tau^y(\omega_2) > T/2) \leq (8L+4T+1)^d A \exp(-BT/2)$.

One deduces that there exists some integer $L_3 \geq L_2$ such that for each $L \geq L_3$,

$$(14) \quad \mathbb{P}(\exists y \in [-(4L+2T), (4L+2T)]^d : \tau^y(\omega_2) > T/2) \leq \varepsilon/3.$$

Now fix $L \geq L_3$ and choose $\alpha > 0$ small enough to have

$$(15) \quad \mathbb{P}(\exists (y, n) \in [-(4L+2T), (4L+2T)]^d \times \{1, \dots, T\} \quad \omega_{3,n}^y = 1) \leq \varepsilon/3.$$

We conclude by putting (13), (14) and (15) together. \square

4.3. Block events percolation. Let $I < L$ be fixed integers. First, for each $x \in \mathbb{Z}^d$, we will build a field $({}^x W_{(z,u)}^n)_{n \geq 1, z \in \mathbb{Z}^d, \|u\|_1 \leq 1}$ from the events defined above. The random variable $W_{(z,u)}^{n+1}$ will give the state of the oriented bond between the macroscopic sites (z, n) and $(z+u, n+1)$; those sites correspond to the coordinates of the boxes $(2Lz, nT) + [-L, L]^d \times [1, T]$ and $(2L(z+u), (n+1)T) + [-L, L]^d \times [1, T]$. The field $({}^x W_{(z,u)}^n)_{n \geq 1, z \in \mathbb{Z}^d, \|u\|_1 \leq 1}$ then defines a macroscopic dynamical dependent oriented percolation.

For $x \in \mathbb{Z}^d$, we denote by $[x]_{2L} \in \mathbb{Z}^d$ the unique integer such that

$$x \in 2L[x]_{2L} + [-L, L]^d \text{ and } \{x\}_{2L} = x - 2L[x]_{2L} \in [-L, L]^d.$$

We set $d_0^x([x]_{2L}) = \{x\}_{2L}$ and also $d_0^x(\bar{k}) = +\infty$ for every $\bar{k} \in \mathbb{Z}^d$ that is not equal to $[x]_{2L}$. Then, for each $\bar{k} \in \mathbb{Z}^d$, each $u \in \mathbb{Z}^d$ with $\|u\|_1 \leq 1$ and each $n \geq 1$, we recursively define:

- If $d_n^x(\bar{k}) = +\infty$, ${}^x W_{(\bar{k}, u)}^{n+1} = 1$.
- Otherwise,

$${}^x W_{(\bar{k}, u)}^{n+1} = \mathbb{1}_{A(\bar{k}, d_n^x(\bar{k}), u)} \circ \theta_{nT},$$

$$d_{n+1}^x(\bar{k}) = \min\{s(\bar{k} - u, d_n^x(\bar{k} - u), u) \circ \theta_{nT} : \|u\|_1 \leq 1, d_n^x(\bar{k} - u) \neq +\infty\}.$$

Let $\mathcal{G}_n = \sigma(\omega_1^{e,k}, \omega_2^{e,k}, \omega_3^{x,k}, e \in \vec{\mathbb{E}}^d, x \in \mathbb{Z}^d, k \leq nT)$. Note that conditionally to \mathcal{G}_n , the random variables ${}^x W_{(\bar{k}, u)}^{n+1}$ et ${}^x W_{(\bar{l}, v)}^{n+1}$ are independent as soon as $\|\bar{k} - \bar{l}\|_1 > 4 + 18C$. Then we take $M = 5 + 18C$, and prove the following lemma:

Lemma 4.2. *For each $p > \vec{p}_c^{\text{alt}}(d+1)$, $q < \vec{p}_c^{\text{alt}}(d+1)$, and $q_0 < 1$, we can find some integers $I < L$ and a parameter $\alpha > 0$ such that for each $x \in \mathbb{Z}^d$,*

$$\text{the law of } ({}^x W_e^n)_{n \geq 0, e \in \vec{\mathbb{E}}^d} \text{ under } \mathbb{P}_{p,q,\alpha} \text{ belongs to } \mathcal{C}(M, q_0).$$

Proof. Note that for every $x, \bar{k} \in \mathbb{Z}^d$, for each $n \geq 1$, the variable $d_n^x(\bar{k})$ is \mathcal{G}_n -measurable, and so does ${}^x W_{(\bar{k}, u)}^n$.

Let us now consider $x, \bar{k} \in \mathbb{Z}^d$, $n \geq 0$ and $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$: Lemma 4.1 ensures that

$$\begin{aligned} & \mathbb{E}_{p,q,\alpha}[{}^x W_{(\bar{k}, u)}^{n+1} | \mathcal{G}_n \vee \sigma({}^x W_{(\bar{l}, v)}^{n+1}, \|v\|_1 \leq 1, \|\bar{l} - \bar{k}\|_1 \geq M)] \\ &= \mathbb{E}_{p,q,\alpha}[{}^x W_{(\bar{k}, u)}^{n+1} | \mathcal{G}_n] \\ &= \mathbb{1}_{\{d_n^x(\bar{k}) = +\infty\}} + \mathbb{1}_{\{d_n^x(\bar{k}) < +\infty\}} \mathbb{P}_{p,q,\alpha}[{}^x W_{(\bar{k}, u)}^{n+1} = 1 | d_n^x(\bar{k}) < +\infty] \\ &= \mathbb{1}_{\{d_n^x(\bar{k}) = +\infty\}} + \mathbb{1}_{\{d_n^x(\bar{k}) < +\infty\}} \mathbb{P}_{p,q,\alpha}[A(\bar{k}, d_n^x(\bar{k}), u)]. \end{aligned}$$

With Lemma 4.1, one can find some integers $I < L$ and a parameter $\alpha > 0$ in such a way that

$$\mathbb{E}_{p,q,\alpha}[{}^x W_{(\bar{k}, u)}^{n+1} | \mathcal{G}_n \vee \sigma({}^x W_{(\bar{l}, v)}^{n+1}, \|v\|_1 \leq 1, \|\bar{l} - \bar{k}\|_1 \geq M)] \geq q_0.$$

This completes the proof of the Lemma. \square

4.4. An abstract restart procedure. We formalize here the restart procedure for Markov chains.

Let E be the state space where our Markov chains $(X_n^x)_{n \geq 0}$ evolve, where $x \in E$ denotes the starting point of the chain. We suppose that we have on our disposal a set $\tilde{\Omega}$, an update function $f : E \times \tilde{\Omega} \rightarrow E$, and a probability measure ν on $\tilde{\Omega}$ in such a way that on the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (\tilde{\Omega}^{\mathbb{N}^*}, \mathcal{B}(\tilde{\Omega}^{\mathbb{N}^*}), \nu^{\otimes \mathbb{N}^*})$, endowed with the natural filtering $(\mathcal{F}_n)_{n \geq 0}$ given by $\mathcal{F}_n = \sigma(\omega \mapsto \omega_k : k \leq n)$, the chains $(X_n^x)_{n \geq 0}$ starting from the different states enjoy the following representation:

$$\begin{cases} X_0^x(\omega) = x \\ X_{n+1}^x(\omega) = f(X_n^x(\omega), \omega_{n+1}). \end{cases}$$

As usual, we define $\theta : \Omega \rightarrow \Omega$ which maps $\omega = (\omega_n)_{n \geq 1}$ to $\theta\omega = (\omega_{n+1})_{n \geq 1}$. We assume that for each $x \in E$, we have defined a $(\mathcal{F}_n)_{n \geq 0}$ -adapted stopping time T^x , a \mathcal{F}_{T^x} -measurable function G^x and a \mathcal{F} -measurable function F^x . Now, we are interested in the following quantities:

$$\begin{aligned} T_0^x = 0 \text{ and } T_{k+1}^x &= \begin{cases} +\infty & \text{if } T_k^x = +\infty \\ T_k^x + T^{x_k}(\theta_{T_k^x}) & \text{with } x_k = X_{\theta_{T_k^x}}^x \text{ otherwise;} \end{cases} \\ K^x &= \inf\{k \geq 0 : T_{k+1}^x = +\infty\}; \\ M^x &= \sum_{k=0}^{K^x-1} G^{x_k}(\theta_{T_k^x}) + F^{X^{x_K}}(\theta_{T_K^x}). \end{aligned}$$

We wish to control the exponential moments of the M^x 's with the help of exponential bounds for G^x and F^x . In numerous applications to directed percolation or to the contact process, T^x is the extinction time of the process (or of some embedded process) starting from the smallest point (in lexicographic order) in the configuration x .

Lemma 4.3. *We suppose that there exist real numbers $A > 0$, $c < 1$, $p > 0$, $\beta > 0$, and that the real-valued functions $(G^x)_{x \in E}, (F^x)_{x \in E}$ defined above are such that*

$$\forall x \in E \quad \begin{cases} \mathbf{G}(x) = \mathbb{E}[\exp(\beta G^x) \mathbf{1}_{\{T^x < +\infty\}}] \leq c; \\ \mathbf{F}(x) = \mathbb{E}[\mathbf{1}_{\{T^x = +\infty\}} \exp(\beta F^x)] \leq A; \\ \mathbf{T}(x) = \mathbb{P}(T^x = +\infty) \geq p. \end{cases}$$

Then, for each $x \in E$, K^x is \mathbb{P} -almost surely finite and

$$\mathbb{E}[\exp(\beta M^x)] \leq \frac{A}{1-c} < +\infty.$$

Before the proof, we note that we could give a statement about Markov chains avoiding the use of an update function, by working directly with the trajectory space of the Markov chain rather than with the generic underlying space: in that way, $\mathbb{P}(T^x = +\infty)$ would be replaced by $\mathbb{P}^x(T = +\infty)$ and a lot of formulas would be simpler. However, the processes we plan to apply this lemma to are often built from a graphical construction (here, the Ω where the growth model lives) and the functions G, H we plan to apply the lemma to are defined from the graphical representation, and not from the Markov chain.

Proof. We can assume without loss of generality that $\beta = 1$. Let $x \in E$ be fixed. At first, we have for each $n \geq 0$

$$\begin{aligned} \mathbb{P}[K^x > n | \mathcal{F}_{T_n^x}] &= \mathbb{P}(T_{n+1}^x < +\infty | \mathcal{F}_{T_n^x}) = \mathbb{P}(T_n^x < +\infty, T^{x_n}(\theta_{T_n^x}) < +\infty | \mathcal{F}_{T_n^x}) \\ &= \mathbf{1}_{\{T_n^x < +\infty\}} (1 - \mathbf{T}(x_n)) \\ &\leq (1-p) \mathbf{1}_{\{T_n^x < +\infty\}} = \mathbf{1}_{\{K^x > n-1\}} (1-p), \end{aligned}$$

then $\mathbb{P}(K^x > n) \leq (1-p)\mathbb{P}(K^x > n-1)$, which ensures that K^x is \mathbb{P} -almost surely finite.

Let $S_{-1}^x = 1$ and, for $k \geq 0$, put

$$S_k^x = \exp\left(\sum_{i=0}^k G^{x_i}(\theta_{T_i^x})\right) \mathbf{1}_{\{T_{k+1}^x < +\infty\}}.$$

We note that S_k^x is $\mathcal{F}_{T_{k+1}^x}$ -measurable. For $k \geq 0$, one has

$$\exp(M^x) \mathbf{1}_{\{K^x = k\}} = S_{k-1}^x \mathbf{1}_{\{T^{x_k} \circ \theta_{T_k^x} = +\infty\}} \exp(F^{x_k}),$$

hence by the strong Markov property $\mathbb{E}[\exp(M^x) \mathbf{1}_{\{K^x = k\}} | \mathcal{F}_{T_k^x}] = S_{k-1}^x \mathbf{F}(x_k)$, then

$$\mathbb{E}[\exp(M^x) \mathbf{1}_{\{K^x = k\}}] \leq A \mathbb{E}[S_{k-1}^x].$$

For $k \geq 1$, the strong Markov property gives again

$$\mathbb{E}[S_{k+1}^x | \mathcal{F}_{T_{k+1}^x}] = S_k^x \times \mathbf{G}(x_{k+1}),$$

then $\mathbb{E}[S_{k+1}^x] \leq c \mathbb{E}[S_k^x]$, and $\mathbb{E}[\exp(M^x) \mathbf{1}_{\{K^x = k\}}] \leq A c^k$. We conclude the proof by summing on k . \square

4.5. Proof of Theorem 1.2 for small α .

Proof. The inequality $\alpha_c \leq 1 - \frac{1}{2d+1}$ easily follows from a counting argument. Let $p > \vec{p}_c^{\text{alt}}(d+1)$, $q < \vec{p}_c^{\text{alt}}(d+1)$, and take $M = 5 + 18C$ as previously. By

Lemma 3.1, we can find $q_0 < 1$ with $g_M(q_0) > \bar{p}_c^{\text{alt}}(d+1)$ and $\beta_0 > 0$ such that for each field $\chi \in \mathcal{C}_d(M, q_0)$:

$$(16) \quad \mathbb{E}_\chi[\mathbb{1}_{\{\tau^0 < +\infty\}} \exp(\beta_0 \tau^0)] \leq \frac{1}{2}.$$

We choose I, L, α as determined by Lemma 4.2. We will prove that for this α , the survival of the bacteria is possible, as well as the other announced estimates. Let $x \in \{0, 1, 2\}^{\mathbb{Z}^d}$ be some configuration; we denote by $E_1(x)$ the set of sites occupied by type 1 particles in configuration x . If $E_1(x) \neq \emptyset$, we denote by $j(x)$ the smallest point in $E_1(x)$ (in lexicographic order). Note that there exists $c > 0$ such that

$$(17) \quad \forall x \in \{0, 1, 2\}^{\mathbb{Z}^d} \quad \mathbb{P}_{p,q,\alpha}(\eta_{1,4dT}^x \supset j(x) + [-4T, 4T]^d) \geq c.$$

Indeed, it is sufficient to open for ω_1 every bond in $B = (j(x), 0) + [-4dT - 1, 4dT + 1]^d \times [0, 4T]$, to close for ω_2 every bond in B , and to forbid in ω_3 every birth of type 2 in B : all of this corresponds to fixing a finite number of coordinates in ω , which can be done with a positive probability.

If the event in (17) happens, we have at time $4dT$ a large box $j(x) + [-4T, 4T]^d$ occupied by type 1 particles. From this box, we can start the macroscopic percolation by building the random field ${}^x W = (j(x)W_e^n \circ \theta_{4dT})_{e \in \vec{\mathbb{E}}^d, n \geq 0}$. The choice we made for I, L, α and Lemma 4.2 ensure that ${}^x W$ belongs to $\mathcal{C}_d(M, q_0)$. Since $g_M(q_0) > \bar{p}_c^{\text{alt}}(d+1)$, (17) gives

$$\mathbb{P}_{p,q,\alpha}(\tau_1^x = +\infty) \geq \mathbb{P}_{p,q,\alpha}(\eta_{1,4dT}^x \supset j(x) + [-4T, 4T]^d) \mathbb{P}_{g_M(q_0)}(\tau^0 = +\infty) > 0,$$

which proves (1).

To show the exponential estimates, we will apply Lemma 4.3. If $E_1(x) = \emptyset$, we let $T^x = +\infty$; otherwise, let

$$T^x = \begin{cases} 4dT & \text{if the event in (17) does not occur,} \\ 4dT + T \times \tau^{[j(x)]_{2L}} \circ \theta_{4dT} & \text{otherwise,} \end{cases}$$

where $\tau^{[j(x)]_{2L}}$ represents the extinction time in the percolation ${}^x W$ starting from the macroscopic site $[j(x)]_{2L}$ containing $j(x)$.

For each $x \in \{0, 1, 2\}^{\mathbb{Z}^d}$ such that $E_1(x) \neq \emptyset$, we have

$$\mathbb{P}_{p,q,\alpha}(T^x = +\infty) \geq c \mathbb{P}_{g_M(q_0)}(\tau^0 = +\infty).$$

We take $G^x = T^x$; for $0 < \beta_1 < \beta_0$, Inequality (16) gives:

$$\begin{aligned} \mathbb{E}_{p,q,\alpha}[e^{\beta_1 T^x} \mathbb{1}_{\{T^x < +\infty\}}] &\leq e^{\beta_1 4dT} \sup_{\chi \in \mathcal{C}(M, q_0)} \mathbb{E}_\chi[\mathbb{1}_{\{\tau^0 < +\infty\}} \exp(\beta_0 \tau^0)] \\ &\leq e^{\beta_1 4dT} / 2 \leq 2/3 \end{aligned}$$

provided that β_1 is small enough. We take $F^\emptyset = 0$ and for $x \neq \emptyset$, $F^x = T \times S^{[j(x)]_{2L}} \circ \theta_{4dT}$ where S has been defined in Corollary 3.2. This corollary moreover gives the existence of exponential moments for S . Thus, the restart lemma ensures that the variable

$$M^x = T_K^x + F^{\eta_{T_K^x}^x} \circ \theta_{T_K^x}$$

admits exponential moments.

Let us begin to work on the event $\{\tau_1^x = +\infty\}$. In that case, $\eta_{1, T_K^x}^x$ is non-empty and, at time $T_K^x + 4dT$, the bacteria occupy a large box $j(\eta_{T_K^x + 4dT}^x) + [-4T, 4T]^d$, from which the macroscopic percolation lives forever; moreover,

$$M^x = T_K^x + T \times S^{[j(\eta_{T_K^x + 4dT}^x)]_{2L}} \circ \theta_{T_K^x + 4dT}.$$

By the definition of the macroscopic percolation, if the bond ${}^{j(\eta_{T_K^x+4dT}^x)}W_{\bar{k},u}^n \circ \theta_{T_K^x+4dT}$ is open, then every point in the box $2L\bar{k} + [-L, L]^d$ is visited by the bacteria between time $T_K^x + 4dT + nT$ and time $T_K^x + 4dT + (n+1)T$. In particular, using Corollary 3.2, it comes that

$$\forall n \in \mathbb{N} \quad 2L[j(\eta_{T_K^x+4dT}^x)]_{2L} + [-2nD_1L, 2nD_1L]^d \subset \bigcup_{0 \leq m \leq n+M^x+4dT} \eta_{1,m}^x;$$

we then deduce Estimate (2) and the existence of exponential moments for M^x and T_K^x .

Finally, since $\{\tau_1^x < +\infty\} \subset \{\tau_1^x \leq M^x\}$, Estimate (3) follows from the bound for the exponential moments of M^x given by Lemma 4.3; this completes the proof of Theorem 1.2 for small α . \square

4.6. The case $\alpha < \alpha_c(p, q)$: the Bezuidenhout–Grimmett way. We fix p, q, α such that

$$\mathbb{P}_{p,q,\alpha}(\tau^{0, \mathbb{Z}^d \setminus \{0\}} = +\infty) = \mathbb{P}(\tau^{0, \mathbb{Z}^d \setminus \{0\}} = +\infty) > 0,$$

or in other words such that $\alpha < \alpha_c(p, q)$. The proof for the linear growth of the bacteria as soon as it survives is quite close from the one we just completed in the case of a small α .

However, the block events differ because we can not use here the linear growth of directed percolation. We reproduce the strategy developed by Bezuidenhout–Grimmett [1] for the contact process on \mathbb{Z}^d , which is also the one followed by Steif–Warfheimer [18] in the case of a contact process where the death rate depends on a dynamical environment.

Our block event is as follows, denoting by V the set of $e \in \mathbb{Z}^d$ with $\|e\|_1 \leq 1$:

Lemma 4.4. *Let $\varepsilon > 0$ and $k \geq 1$. There exist $n, a, b \in \mathbb{N}$ with $n < a$ such that for every $(x, t) \in [-a, a]^d \times [0, b]$, each $v \in V$, with $\mathbb{P}_{p,q,\alpha}$ -probability at least $1 - \varepsilon$, there exists $(y, s) \in (2kav + [-a, a]^d) \times [5kb, (5k+1)b]$ such that, starting from configuration $\mathbb{1}_{x+[-n,n]^d} + 2 \cdot \mathbb{1}_{\mathbb{Z}^d \setminus \{x+[-n,n]^d\}}$ at time t , every point in the target $(y, s) + [-n, n]^d \times \{0\}$ is reached by type 1 particles, thanks to an infection path which stays in the region:*

$$\mathcal{A} = \bigcup_{j=0}^{k-1} (2jav + [-5a, 5a]^d) \times (5jb + [0, 6b]).$$

For the construction of the directed percolation associated to those block events, the σ -field \mathcal{G}_n appearing in Theorem 2.5 must take into account the fact that the temporal position of the target zone is not deterministic. This part is well detailed in Steif–Warfheimer [18].

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