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ABSOLUTE ALGEBRA
II–IDEALS AND SPECTRA

PAUL LESCOT

Abstract. We study natural notions of ideals and spectra for algebras of characteristic one.

1. Introduction

Many different strands of thought converge in suggesting the existence of a "characteristic 1 analogue"of the usual algebra of fields. Among these let us mention tropical geometry ([2]), the geometry of algebraic groups([11]), their representations ([12]), and the Riemann hypothesis ([3],[4],[10]). Various such formalisms have been propounded, among which the most notable are Deitmar's theory of F_1–schemes ([6],[7]), Soulé's theory of F_1–objects([10]), and Zhu's characteristic one algebra ([13]).

In a previous work ([9]), we have developed Zhu's theory, and made clear its intimate connection with Deitmar's. By B_1 we shall denote (as in [9]) the set $\{0,1\}$ equipped with the usual operations of addition and multiplication, except that $1+1=1$. It is clear that this object satisfies all the axioms defining a field, except for the existence of symmetric elements for addition. In Castella's terminology ([1]), this is the smallest *characteristic* 1 *semifield* .

B_1–modules are defined in the obvious way, *i.e.* as commutative monoids with a zero element equipped with an external B_1–action satisfying the usual conditions.

Definition 1.1. ([9], Definition 2.3)A B_1-module is a commutative monoid M with zero element 0_M equipped with an external B_1-action (that is an application

$$(\lambda, x) \mapsto \lambda x$$

from $B_1 \times M$ to M), such that the following properties hold:

$$\forall(\lambda,\mu,x) \in B_1 \times B_1 \times M \quad (\lambda+\mu)x = \lambda x + \mu x \quad , \tag{1}$$

$$\forall(\lambda,x,y) \in B_1 \times M \times M \quad \lambda(x+y) = \lambda x + \lambda y \quad , \tag{2}$$

$$\forall x \in M \quad 1x = x \quad , \text{and} \tag{3}$$

$$\forall x \in M \quad 0x = 0_M \quad . \tag{4}$$

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We then define B_1–algebras in the natural way (slightly weaker than the one in [9],Definition 4.1) :

Definition 1.2. By a (commutative, unitary)B_1–algebra A we mean the data of a B_1–module A and of an associative and commutative multiplication on A that has a neutral element 1_A and is bilinear with respect to the operations of B_1–module.

If $1_A = 0_A$, then A has only one element ; in the other case, we may identify $B_1 = \{0, 1\}$ and the subalgebra $\{0_A, 1_A\}$ of A, and it turns out that A is a B_1–algebra in the sense of [9], Definition 4.1.

Conversely, any B_1–algebra in the sense of [9], Definition 4.1 is a B_1–algebra in the sense of the present paper.

Except when otherwise precised, we shall keep in force the definitions and notations of [9]. In particular, for E a set, $\mathcal{P}_f(E)$ will denote the set of its finite subsets, and

$$\begin{array}{rcl} j_E & : & E \to \mathcal{P}_f(E) \\ & & x \mapsto \{x\} \end{array}$$

the canonical injection.

In (commutative) ring theory, there is a bijection between congruences (in the sense of universal algebra) on a ring and ideals of the ring (cf. *e.g.*the proof of Corollary 2, p.68, in [8]). In the category of B_1–algebras, that correspondence breaks down. Thus we first consider ideals (§2), then congruences (§3) and we obtain a bijection between *saturated* ideals and *excellent* congruences. In §4 we discuss the connection between this theory and Deitmar's ideas ([6], [7]), in the line of [9], §5.

Castella([1],[2]) has developed a different theory that works more generally over an arbitrary *characteristic 1 semifield* (not necessarily B_1), and has some points of contact with ours. His notion of *idéal fermé* ([2], p.5) corresponds to our notion of *saturated ideal* . Nevertheless, his definition of quotient by an ideal is entirely different.

In a subsequent paper we shall investigate tensor products of B_1–algebras. Hopefully all these constructions will some day fit together within Connes and Consani's theory of hyperrings ([3],[4],[5]).

2. Ideals

Let A denote a B_1–algebra.

Definition 2.1. A subset I of A is termed an *ideal* of A if

- I is a B_1–submodule of A, *i.e.*
$$0 \in I \ ,$$
$$\forall (x,y) \in I^2 \ x+y \in I \ , \text{ and}$$
- $\forall x \in I \ \forall a \in A \ ax \in I$.

$\{0\}$ and A are both ideals of A; by a *proper* ideal of A we mean an ideal different from A itself. We shall denote by $Id(A)$ the set of all ideals of A, and by $Max(A)$ the set of all maximal (proper) ideals of A.

Definition 2.2. An ideal I of A is *prime* if $I \neq A$ and,
$$\forall (x,y) \in A^2 \ \ [xy \in I \Longrightarrow x \in I \text{ or } y \in I] \ .$$

We shall denote by $Pr(A)$ the set of prime ideals of A.

Proposition 2.3.
$$Max(A) \subseteq Pr(A) \ .$$

Proof. The familiar ring–theoretic argument applies here : let $I \in Max(A)$, let us assume $xy \in I$ and $x \notin I$, and define
$$\begin{aligned} J &= I + Ax \\ &= {}_{def.}\{i + ax | i \in I, a \in A\} \ . \end{aligned}$$
Then one checks easily that J is an ideal of A, that $I \subseteq J$, and that $I \neq J$ (as $x = 0 + 1.x \in J$, and $x \notin I$); therefore, $J = A$. In particular, $1 \in J$; therefore one may find $i \in I$ and $a \in A$ such that $1 = i + ax$. But then one has
$$\begin{aligned} y &= 1.y \\ &= (i+ax)y \\ &= iy + a(xy) \\ &= yi + a(xy) \in I \ , \end{aligned}$$
as $i \in I$ and $xy \in I$. □

As for ordinary rings, the reciprocal inclusion need not hold : *e.g.*, for $A = B_1[x]$, $I = \{0\}$ is a prime ideal that is not maximal, as $I \subsetneq xA \subsetneq A$.

Theorem 2.4. *For S a subset of A, let*
$$W(S) := \{\mathcal{P} \in Pr(A) | S \subseteq \mathcal{P}\} \ .$$
Then the $(W(S))_{S \subseteq A}$ are the closed sets for a (Zariski) topology on $Pr(A)$.

Proof. The proof follows the usual lines, as
$$W(\emptyset) = Pr(A) \ ,$$
$$W(A) = \emptyset \ ,$$
$$\bigcap_{i \in I} W(S_i) = W(\bigcup_{i \in I} S_i) \ ,$$

and

$$W(S) \cup W(T) = W(ST) \ ,$$

where

$$ST := \{xy | x \in S, y \in T\} \ .$$

$\square$

3. Congruences

For the convenience of the reader, we shall repeat some of the definitions from [9],§4, with a slight change : we now allow the "trivial"congruence, *i.e.* the congruence such that $0 \simeq 1$.

Definition 3.1. We call *congruence* on the B_1–algebra A an equivalence relation $\sim$ on A such that

$$\forall (a, b, a^{'}, b^{'}) \in A^4 \;\; a \sim b \text{ and } a' \sim b' \Longrightarrow \;\; a + a' \sim b + b' \text{ and } aa' \sim bb' \; .$$

In our theory, congruences play the same role as equivalences modulo an ideal in commutative algebra ; in particular, for each congruence $\sim$ on A, the quotient set $A/\!\sim$ possesses a canonical structure of (possibly trivial) B_1–algebra.

Definition 3.2. On the set of congruences on the B_1–algebra A let us define an order $\geq$ by :

$$\sim_1 \geq \sim_2 \Longleftrightarrow \; \forall (a, b) \in A^2 \; a \sim_2 b \Longrightarrow a \sim_1 b \; .$$

The trivial congruence $\mathcal{C}_0(A) = A \times A$ is the greatest element for that order, and the equality relation $=_A$ on A the smallest. It is easy to see that, if $\sim_1 \geq \sim_2$, then there is a canonical surjective morphism

$$A/\!\sim_2 \twoheadrightarrow A/\!\sim_1 \; .$$

Definition 3.3. We shall denote by $MaxSpec(A)$ the set of all maximal nontrivial congruences on $\mathcal{A}$.

When A is the free B_1–algebra on n generators, $MaxSpec(A)$ consists of 2^n elements, and has been described in [9], Theorems 4.7 and 4.8. It is an easy consequence of Zorn's Lemma that any nontrivial congruence is contained in a maximal one.

Definition 3.4. A congruence $\sim$ on A is said to be prime if $\sim \neq \mathcal{C}_0(A)$ and

$$ab \sim 0 \Longrightarrow a \sim 0 \text{ or } b \sim 0 \; ;$$

we shall denote the set of prime congruences on A by $Spec(A)$.

One has

Theorem 3.5.

$$MaxSpec(A) \subseteq Spec(A) \; .$$

Proof. We shall repeat an argument already used at the beginning of the proof of Theorem 4.8 in [9]. Let $\sim \in Maxspec(A)$, and let $(u, v) \in A^2$ be such that $uv \sim 0$, and $u \nsim 0$.

Define the relation $\mathcal{R}_u$ on A by :

$$x \, \mathcal{R}_u \, y \equiv \exists (a, b) \in A^2 \;\; x + ua \sim y + ub \; .$$

It is very easy to see that $\mathcal{R}_u$ is compatible with addition and multiplication, and that $x \sim y$ implies $x \, \mathcal{R}_u \, y$. Furthermore $0 \, \mathcal{R}_u \, u$, and $0 \nsim u$, therefore

$$\sim \neq \mathcal{R}_u \; .$$

It follows that $\mathcal{R}_u$ is a congruence, and that $\mathcal{R}_u > \sim$, whence $\mathcal{R}_u = \mathcal{C}_0(A)$. In particular $0 \; \mathcal{R}_u \; 1$, therefore one may find $(a, b) \in A^2$ such that :

$$0 + ua = 1 + ub \; ,$$

i.e.

$$ua = 1 + ub \ .$$

But then

$$(uv)a = v(ua) = v(1 + ub) = v + uvb$$

and from $uv \sim 0$ follows :

$$0 = 0a \sim (uv)a = v + uvb \sim v + 0b = v \ ,$$

that is $v \sim 0$. Therefore $\sim$ is prime. □

Definition 3.6. For $\mathcal{R}$ a congruence, let us set

$$I(\mathcal{R}) := \{x \in A | x \ \mathcal{R} \ 0\} \ ;$$

obviously, $I(\mathcal{R})$ is an ideal of A.

Conversely, we have

Theorem 3.7. *Let J be an ideal of A, then there is a unique smallest congruence (denoted by $\mathcal{R}_J$) such that*

$$(\forall x \in J) \ \ x \ \mathcal{R}_J \ 0 \ .$$

One has

$$\forall (x, y) \in A^2 \ \ x \ \mathcal{R}_J \ y \Longleftrightarrow \ (\exists z \in J) \ x + z = y + z \ .$$

Furthermore $\overline{J} := I(\mathcal{R}_J)$ is an ideal of A (in fact

$$\overline{J} = \{x \in A | \ (\exists z \in J) \ x + z = z\})$$

and the mapping $J \mapsto \overline{J}$ is a closure operator (i.e. $J \subseteq \overline{J}$ and $\overline{\overline{J}} = \overline{J}$) on $Id(A)$.

Proof. Let us define a relation $\mathcal{R}_J$ on A by

$$\forall (x, y) \in A^2 \ \ x \ \mathcal{R}_J \ y \equiv (\exists z \in J) \ x + z = y + z \ . \tag{3.1}$$

Then from $a\mathcal{R}_J b$ and $a^{'}\mathcal{R}_J b^{'}$ follows the existence of $(c, c^{'}) \in J^2$ with

$$a + c = b + c$$

and $a^{'} + c^{'} = b^{'} + c^{'}$. Then $c + c^{'} \in J$ and

$$\begin{aligned} (a + a^{'}) + (c + c^{'}) &= (a + c) + (a^{'} + c^{'}) \\ &= (b + c) + (b^{'} + c^{'}) \\ &= (b + b^{'}) + (c + c^{'}) \ , \end{aligned}$$

whence $a + a^{'}\mathcal{R}_J b + b^{'}$.

Let now $x = a^{'}c + bc^{'} \in J$; then

$$\begin{aligned} aa^{'} + x &= aa^{'} + a^{'}c + bc^{'} \\ &= a^{'}(a + c) + bc^{'} \\ &= a^{'}(b + c) + bc^{'} \\ &= b(a^{'} + c^{'}) + a^{'}c \\ &= b(b^{'} + c^{'}) + a^{'}c \\ &= bb^{'} + x \ , \end{aligned}$$

whence $aa^{'}\mathcal{R}_J bb^{'}$. We have shown $\mathcal{R}_J$ to be a congruence on A. Obviously, for $x \in J$, one has $x + x = x = 0 + x$, whence $x \ \mathcal{R}_J \ 0$.

Let now $\mathcal{R}$ be a congruence on A such that

$$\forall x \in J \;\; x \; \mathcal{R} \; 0 \; ,$$

and let $(x, y) \in A^2$ with $x \; \mathcal{R}_J \; y$; then there is $z \in J$ such that $x + z = y + z$. But then $z \; \mathcal{R}_J \; 0$, whence

$$x = x + 0 \;\; \mathcal{R} \; x + z = y + z \; \mathcal{R} \; y + 0 = y \; ,$$

whence $x\mathcal{R}y$, and $\mathcal{R}_J \subseteq \mathcal{R}$.

Now, let $\overline{J} =_{def} I(\mathcal{R}_J)$; as seen above, $\overline{J}$ is an ideal of A, and $J \subseteq \overline{J}$ by definition of $\mathcal{R}_J$. The description of $\overline{J}$ follows from (3.1). Furthermore, by definition of $\overline{J}$, $\mathcal{R}_J$ is a congruence on A such that

$$\forall x \in \overline{J} \;\; x \; \mathcal{R}_J \; 0 \; ,$$

whence $\mathcal{R}_{\overline{J}} \leq \mathcal{R}_J$ and

$$\begin{aligned} \overline{\overline{J}} &= I(\mathcal{R}_{\overline{J}}) \\ &\subseteq I(\mathcal{R}_J) \\ &= \overline{J} \\ &\subseteq \overline{\overline{J}} \; , \text{ and} \end{aligned}$$

$$\overline{J} = \overline{\overline{J}}.$$

□

An ideal J of A will be termed *saturated* if $J = \overline{J}$.

Congruences of the type $\mathcal{R}_J$(for J an ideal of A) will be termed excellent. It follows from the next Theorem that $\mathcal{R}$ is excellent if and only if $\mathcal{R} = \mathcal{R}_{I(\mathcal{R})}$.

Theorem 3.8. *For an arbitrary congruence $\mathcal{R}$, $\mathcal{R}_{I(\mathcal{R})}$ is the largest excellent congruence that is $\leq \mathcal{R}$. Consequently, for $J \in Id(A)$, one has $J = \overline{J}$ if and only if there exists a congruence $\mathcal{R}$ on A such that $I(\mathcal{R}) = J$; if this is the case, then $\mathcal{R}_J$ is the smallest such congruence.*

Proof. By definition, $\mathcal{R}_{I(\mathcal{R})}$ is excellent ; if $x \; \mathcal{R}_{I(\mathcal{R})} \; y$, there is $z \in I(\mathcal{R})$ with

$$x + z = y + z.$$

But then $z \; \mathcal{R} \; 0$ and

$$x = x + 0 \; \mathcal{R} \; x + z = y + z \; \mathcal{R} \; y + 0 = y \; ,$$

whence $x\mathcal{R}y$, and $\mathcal{R}_{I(\mathcal{R})} \leq \mathcal{R}$.

If $\mathcal{R}_J \leq \mathcal{R}$, then

$$J \subseteq I(\mathcal{R}_J) \subseteq I(\mathcal{R}) \; ,$$

whence $J \subseteq I(\mathcal{R})$ and $\mathcal{R}_J \leq \mathcal{R}_{I(\mathcal{R})}$: the first assertion follows.

Let us assume $J = I(\mathcal{R})$ for some $\mathcal{R}$; then

$$\begin{aligned} J &\subseteq \overline{J} \\ &= I(\mathcal{R}_J) \\ &= I(\mathcal{R}_{I(\mathcal{R})}) \\ &\subseteq I(\mathcal{R}) \text{ (as } \mathcal{R}_{I(\mathcal{R})} \leq \mathcal{R}) \\ &= J \ , \end{aligned}$$

whence $J = \overline{J}$.

Conversely, if $J = \overline{J}$ then $I(\mathcal{R}_J) = \overline{J} = J$, and from $I(\mathcal{R}) = J$ follows $\mathcal{R}_J = \mathcal{R}_{I(\mathcal{R})} \leq \mathcal{R}$. □

Remark 3.9.
- $\mathcal{R}_J$ is prime if and only if $\overline{J}$ is prime.
- $\mathcal{R}_J$ is maximal if and only if $\overline{J}$ is maximal among proper saturated ideals.

Let us note that the last condition does not imply the maximality of J : for example, let $A = B_1[x]$ and $J = xA$; then $\mathcal{R}_J$ is a maximal congruence on A (see [9], Theorem 4.7), but $J = \overline{J}$ is not a maximal ideal in A, as $J \subsetneq xA+(1+x)A \subsetneq A$.

Definition 3.10. The ideal I is **absolutely prime** if $I \neq A$ and

$$ab \ \mathcal{R}_I \ ac \Longrightarrow (a \in \overline{I}) \text{ or } b \ \mathcal{R}_I \ c \ .$$

Theorem 3.11. *An absolutely prime saturated ideal is prime.*

Proof. Let I be absolutely prime and saturated, and let us assume $ab \in I$; then $ab \ \mathcal{R}_I \ 0 = a.0$, whence, by hypothesis, $a \in \overline{I} = I$ or $b \ \mathcal{R}_I \ 0$; but, in the second case, $b \in \overline{I} = I$: I is prime. □

Remark 3.12. Clearly, if I is absolutely prime, then $\overline{I}$ either equals A or is prime.

Theorem 3.13. *A maximal ideal is prime and absolutely prime.*

Proof. Let I be maximal ; then I is prime according to Proposition 2.3.

Let us assume

$$ab \ \mathcal{R}_I \ ac \text{ and } (a \notin \overline{I});$$

then there is a $x \in I$ such that

$$ab + x = ac + x \ .$$

As $a \notin \overline{I}$, $\overline{I} \neq A$, whence, due to the maximality of I, $\overline{I} = I$: I is saturated.

Let now

$$J := \{y \in A | (\exists z \in I) yb + z = yc + z\};$$

then J is an ideal of A : it is clear that it is a submonoid, and from $y \in J$ and $y^{'} \in A$ follow $yb + z = yc + z$ for some $z \in I$, whence :

$$(y^{'}y)b + y^{'}z = y^{'}(yb + z) = y^{'}(yc + z) = (y^{'}y)c + y^{'}z \ ,$$

therefore $y^{'}y \in J$, as $y^{'}z \in I$.

Furthermore, for $i \in I$, let $z := ib + ic \in I$; then

$$ib + z = ib + ib + ic = ib + ic = ic + ib = ic + ic + ib = ic + z$$

whence $i \in J$: $I \subseteq J$. As $a \in J$ and $a \notin I$, one has $I \neq J$, whence, from the maximality of I, $I = A$. But then $1 \in I$, *i.e.* there is $z \in I$ such that $b + z = c + z$, that is $b \ \mathcal{R}_I \ c$. □

Remark 3.14. Here are some relevant examples :

- $A = B_1[x]$ and $I = x^2B_1[x]$: I is saturated, but is neither prime nor absolutely prime.
- $A = B_1[x]$ and $I = xA + (1+x)A = \{a = \sum_{n=0}^{+\infty} a_n x^n | a_0 = 0 \text{ or } a_1 = 1\}$. As $ab \in I$ implies $a_0b_0 = 0$ or $a_0b_1 + a_1b_0 = 1$, I is prime ; as $\overline{I} = A$, it is also absolutely prime ; but , for the same reason, it is not saturated.
- Let $G =< \tau >= Z_2$ denote a group of order 2, $A = \mathcal{F}(G)$, and $I = \{0\}$; then I is prime and saturated, but not absolutely prime, as, setting
$$u = j_G(\tau) \ ,$$
one has
$$(1+u)u = u + 1 = (1+u)(1+u)$$
but $1 + u \notin \overline{I} = \{0\}$ and $u \not\mathcal{R}_I 1 + u$.
- In $A = B_1[x]$, the ideal $I = (1+x^2)A$ is absolutely prime (as $\overline{I} = A$), but not prime as $(1+x)(1+x+x^2) = (1+x)(1+x^2) \in I$, but $1 + x \notin I$ and $1 + x + x^2 \notin I$.

We shall denote by $Pr_s(A)$ the set of saturated prime ideals of A, and by $Max_s(A)$ the set of saturated maximal ideals of A ; these sets are naturally equipped with a topology induced by the topology on $Pr(A)$ described in Theorem 2.4.

Proposition 3.15. *For $S \subseteq A$, let*
$$V(S) = \{\mathcal{R} \in Spec(A) | S \subseteq I(\mathcal{R})\} \ .$$
Then the $V(S)_{S \subseteq A}$ are the closed sets for a topology on $Spec(A)$ (the "Zariski topology"on $Spec(A)$). If A is integral (in the sense that $A \neq \{0\}$ and
$$ab = 0 \Longrightarrow a = 0 \ \ or \ b = 0$$
in A), then $=_A \in Spec(A)$ and $\overline{\{=_A\}} = Spec(A)$.

Proof. The reasoning is the same as in the proof of Theorem 2.4, remarking that
$$V(\emptyset) = Spec(A) \ ,$$
$$V(A) = \emptyset \ ,$$
$$V(\cup_{i \in I} S_i) = \bigcap_{i \in I} V(S_i) \ ,$$
and
$$V(S) \cup V(T) = V(ST) \ .$$
One may also remark that $V(S) = I^{-1}(W(S))$, whence the topology in question is the initial topology induced on $Spec(A)$ by the mapping
$$I : Spec(A) \to Pr(A)$$
and the Zariski topology on $Pr(A)$. The last assertion is easy to check. □

4. Links with Deitmar's theory.

Let $\mathcal{D}$ denote Deitmar's category of F_1–rings, i.e. the category of commutative monoids.

For $M \in \mathcal{D}$, let $Spec_{\mathcal{D}}(M)$ denote the set of prime ideals in M (including $\emptyset$), and, for $\mathcal{P} \in Spec_{\mathcal{D}}(M)$, let $\tilde{\mathcal{P}}$ denote the ideal of $\mathcal{F}(M) = B_1[M]$ generated by $j_M(\mathcal{P})$. For $S \subseteq \mathcal{F}(M)$, S_0 will denote the set of $m \in M$ such that $j_M(m)(= \{m\} \in \mathcal{F}(M)$: cf. [9],Theorem 3.1) be a component of some $s \in S$, *i.e.*:

$$S_0 = \{m \in M | (\exists s \in S) j_M(m) + s = s\}.$$

Proposition 4.1. *$\tilde{\mathcal{P}}$ is a saturated prime ideal of $\mathcal{F}(M)$ and one has an isomorphism*

$$\frac{\mathcal{F}(M)}{\mathcal{R}_{\tilde{\mathcal{P}}}} \simeq \mathcal{F}(M \setminus \mathcal{P})$$

(note that, as $\mathcal{P}$ is a prime ideal of M, $M \setminus \mathcal{P}$ is a monoid for the induced operation).

Proof. As $\mathcal{P}$ is an ideal of M, $\tilde{\mathcal{P}}$ equals the B_1–submodule of $\mathcal{F}(M)$ generated by the $(j_M(p))_{p \in \mathcal{P}}$; in particular it is generated by a subset of the set of minimal (for the order relation associated to the B_1–module structure of $\mathcal{F}(M)$—see [**?**],Theorem 2.5) elements of $\mathcal{F}(M)$, whence it is saturated. It is also clear that

$$\mathcal{F}(M) = G \oplus \tilde{\mathcal{P}}$$

where G denotes the (free) B_1–submodule of $\mathcal{F}(M)$ generated by the

$$j_M(x)_{x \in M \setminus \mathcal{P}} \ ;$$

but the mapping

$$\vartheta : M \setminus \mathcal{P} \to G$$
$$x \mapsto j_M(x)$$

turns G into the free B_1–algebra on $M \setminus \mathcal{P}$, whence

$$G \simeq \mathcal{F}(M \setminus \mathcal{P}) \ .$$

It now follows that, for $(a, a^{'}) \in G^2$ and $(x, x^{'}) \in \tilde{\mathcal{P}}^2$, $a + x \ \mathcal{R}_{\tilde{\mathcal{P}}} \ a^{'} + x^{'}$ if and only if $a = a^{'}$. Whence

$$\frac{\mathcal{F}(M)}{\mathcal{R}_{\tilde{\mathcal{P}}}} \simeq G \simeq \mathcal{F}(M \setminus \mathcal{P}) \ .$$

□

Theorem 4.2. *The mapping*

$$\psi : \mathcal{P} \mapsto \tilde{\mathcal{P}}$$

defines a bijection between the set of prime ideals of M and the set of saturated prime ideals of $\mathcal{F}(M)$. Consequently, the mapping

$$\varphi_M : Spec_{\mathcal{D}}(M) \to Spec(\mathcal{F}(M))$$
$$\mathcal{P} \mapsto \mathcal{R}_{\tilde{\mathcal{P}}}$$

defines a bijection between the set $Spec_{\mathcal{D}}(M)$ and the set of excellent congruences on $Spec(\mathcal{F}(M))$ corresponding to saturated prime ideals. Its image is dense in $Spec(\mathcal{F}(M))$. Furthermore φ_M is an homeomorphism on its image.

Proof. It follows from the proof of Proposition 4.1 that

$$\mathcal{P} = \{x \in M | j_M(x) \in \tilde{\mathcal{P}}\} \ ,$$

whence the injectivity of ψ. Let I denote a saturated prime ideal in $\mathcal{F}(M)$, and let $\mathcal{P} := j_M^{-1}(I)$; it is clear that $\mathcal{P}$ is a prime ideal in M. Let $y \in I$; then, for each $x \in M$ such that $j_M(x) \leq y$, one has $j_M(x) + y = y$, whence, as I is saturated, $j_M(x) \in I$ and $x \in \mathcal{P}$. As

$$y = \sum_{x \in M; j_M(x) \leq y} j_M(x) \ ,$$

one has $y \in Vect_{B_1}(j_M(\mathcal{P}))$. Therefore $I = Vect_{B_1}(j_M(\mathcal{P})) = \tilde{\mathcal{P}}$, thus ψ is a bijection, the inverse of which is given by (for I a saturated ideal of $\mathcal{F}(M)$)

$$\psi^{-1}(I) = {j_M}^{-1}(I) \ .$$

But $\varphi_M = \alpha \circ \psi$ where

$$\begin{aligned} \alpha \quad : \quad & Id_s(\mathcal{F}(M)) \to Spec(\mathcal{F}(M)) \\ & I \mapsto \mathcal{R}_I \ . \end{aligned}$$

According to Theorems 3.7 and 3.8, α is an injection, whence $\varphi_M = \alpha \circ \psi$ is a bijection on its image $\alpha(\psi(Spec_{\mathcal{D}}(M)))$. That image contains the congruence

$$\alpha(\psi(\emptyset)) = \alpha(\{0\}) = \mathcal{R}_{\{0\}} \ ,$$

i.e. the congruence $=_{\mathcal{F}(M)}$, the equality on $\mathcal{F}(M)$. But $\{=_{\mathcal{F}(M)}\}$ is dense in $Spec(\mathcal{F}(M))$ (see Proposition 3.15), whence so is $\varphi_M(Spec_{\mathcal{D}}(M))$.

Let now $F = V(S)(S \subseteq \mathcal{F}(M))$; then, for $\mathcal{P}$ a prime ideal of M,

$$\begin{aligned} \varphi_M(\mathcal{P}) \in V(S) &\Longleftrightarrow \tilde{\mathcal{P}} \in V(S) \\ &\Longleftrightarrow S \subseteq I(\mathcal{R}_{\tilde{\mathcal{P}}}) \\ &\Longleftrightarrow S \subseteq \overline{\tilde{\mathcal{P}}} = \tilde{\mathcal{P}} \\ &\Longleftrightarrow S_0 \subseteq \mathcal{P} \ , \end{aligned}$$

whence

$$\varphi_M^{-1}(V(S)) = V_{\mathcal{D}}(S_0)$$

and the continuity of φ_M. Similarly, for $N \subseteq M$, let

$$V_{\mathcal{D}}(N) = \{\mathcal{P} \in Spec_{\mathcal{D}}(M) | N \subseteq \mathcal{P}\}$$

be a closed subset of $Spec_{\mathcal{D}}(M)$ in Deitmar's sense ([6],p.89). Then it is clear from the above considerations that

$$\varphi_M(V_{\mathcal{D}}(N)) = Im(\varphi_M) \cap V(j_M(N))$$

whence φ_M is closed on its image, and the result. □

Example 4.3. If C_n is the free monoid on $n \geq 1$ generators $x_1, ..., x_n$, then $Spec_{\mathcal{D}}(C_n)$ has 2^n elements (the $(\mathcal{P}_J)_{J \subseteq \{1,...,n\}}$: $\mathcal{P}_J := \bigcup_{j \in J} x_j C_n$), whence there are exactly 2^n excellent prime congruences on $\mathcal{F}(C_n)$.

Example 4.4. If G is a group, then $\{0\}$ is the only saturated prime ideal in $\mathcal{F}(G)$. In fact A and $\{0\}$ are the only saturated ideals : this applies to the two examples 5.5 and 5.6 in [9].

As seen in Theorems 3.8 and 4.2, saturated prime ideals play the crucial role in the theory of B_1–algebras of the form $\mathcal{F}(M)$; as a matter of mere curiosity, we shall now determine all saturated absolutely prime ideals in such algebras.

Lemma 4.5. *If $\mathcal{F}(M)$ is simplifiable (in the sense that $ab = ac$ and $a \neq 0$ implies $b = c$), then M is trivial (i.e. has only one element).*

Proof. Let $x \in M$, and $y = j_M(x)$; then, using the idempotence of $+$, we get

$$(1+y)(1+y+y^2) = 1+y+y^2+y^3 = (1+y)(1+y^2)$$

whence (as $1+y \neq 0$), $1+y+y^2 = 1+y^2$. As $\mathcal{F}(M)$ is a free B_1–module on $j_M(M)$, it follows that $y = 1$ or $y = y^2$; in the second case

$$(1+y)(1+y) = 1+y+y^2 = 1+y = (1+y).1$$

whence $1+y = 1$ and $y = 1$ (by the same argument). Therefore $j_M(x) = 1_{\mathcal{F}(M)}$ for each $x \in M$; the injectivity of j_M now yields $x = 1_M$, whence M has only one element. □

It follows from this Lemma that saturated absolutely prime ideals are quite rare in B_1–algebras of monoids ; in fact one has

Proposition 4.6. *If I is a saturated absolutely prime ideal in $\mathcal{F}(M)$, then the only invertible element of M is 1 and $I = \mathcal{F}(M \setminus \{1\})$.*

Remark 4.7. The converse is clear.

Proof. Let I be a saturated absolutely prime ideal in $\mathcal{F}(M)$; according to Theorem 4.2 there is a (unique) prime ideal $\mathcal{P}$ of M such that $I = \tilde{\mathcal{P}}$. By definition of an absolutely prime ideal, $\dfrac{\mathcal{F}(M)}{\mathcal{R}_{\tilde{\mathcal{P}}}} = \dfrac{\mathcal{F}(M)}{\mathcal{R}_I}$ is simplifiable ; but

$$\frac{\mathcal{F}(M)}{\mathcal{R}_{\tilde{\mathcal{P}}}} \simeq \mathcal{F}(M \setminus \mathcal{P})$$

according to Proposition 4.1, hence $\mathcal{F}(M \setminus \mathcal{P})$ is simplifiable. Now Lemma 4.5 yields that $M \setminus \mathcal{P} = \{1_M\}$, *i.e.* $\mathcal{P} = M \setminus \{1_M\}$; $\mathcal{P}$ being an ideal of M, no nonidentity element of M is invertible. □

The following fact was also observed by Castella (see [2], p.3) :

Proposition 4.8. *If K is a finite nontrivial B_1–algebra embeddable in a B_1–field, then $K = \{1\}$.*

Proof. Let $K \subseteq L$, L denoting a B_1–field ; then L is simplifiable, hence so is K. In particular, for each $a \in K \setminus \{0\}$, the mapping

$$\begin{aligned} m_a : K &\to K \\ x &\mapsto ax \end{aligned}$$

is an injection, hence a bijection as K is finite. Let $y = \sum_{x \in K} x$; as $y = y + 1$, $y \neq 0$. But

$$\begin{aligned} ay &= \sum_{x \in K} ax \\ &= \sum_{x \in K} x \\ &= y \\ &= 1.y \ , \end{aligned}$$

whence $a = 1$ and

$$K = \{0, 1\} \simeq B_1 \ .$$

□

5. Bibliography

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