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Abstract. In this article, we improve the Strichartz estimates obtained in [12] for the Schrödinger equation with white noise dispersion in one dimension. This allows us to prove global well posedness when a quintic critical nonlinearity is added to the equation. We finally show that the white noise dispersion is the limit of smooth random dispersion.

1. Introduction

The nonlinear Schrödinger equation with power nonlinearity is a common model in optics. It describes the propagation of waves in a nonlinear dispersive medium. It has been widely studied (see for instance [7], [26]). In the case of a focusing nonlinearity, it has the form

$$\begin{cases} i\dfrac{du}{dt} + \Delta u + |u|^{2\sigma} u = 0,\ x \in \mathbb{R},\ t > 0, \\ \\ u(0) = u_0,\ x \in \mathbb{R}^n. \end{cases}$$

It is well known that for subcritical nonlinearity, *i.e.* $\sigma < 2/n$, this equation is globally well posed in $L^2(\mathbb{R}^n)$ and in $H^1(\mathbb{R}^n)$ ([21], [22], [27]). Moreover, solitary waves are stable.

For critical, $\sigma = 2/n$, or supercritical, $\sigma > 2/n$, nonlinearity, the equation is locally well posed in $H^1(\mathbb{R}^n)$. It is known that there exists solutions which form singularities in finite time. On the contrary, initial data with small $H^1(\mathbb{R}^n)$ norm yield global solutions. Furthermore, solitary waves are unstable.

The effect of a noise on the behavior of the solutions has also been the object of several studies, both in the physical literature (see for instance [2], [5], [6], [17], [23], [28]) or in the mathematical literature (see for instance [8], [9], [10], [11], [14], [15],[19], [20]). Random effects may be taken into account at various places of the equation. A random forcing term or a random potential can be added. Also random diffraction index result as a random coefficient before the nonlinear term. Numerical and theoretical studies have shown that many interesting new behaviors may appear.

For instance, it has been shown that when a random potential which is white in time is added to the equation it may affect strongly the formation of singularities. If this random potential is smooth in space and the nonlinearity is supercritical, any initial data yields a solutions which blows up in finite time with positive probability. If the noise is additive, this is also true for critical nonlinearity. On the contrary, numerical experiments tend to show that,

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if the noise acts as a potential and is rough in space, the formation of singularities is prevented and the solution continue to propagate. The rigorous justification of such statement seems to be completely out of reach at present.

In this work, we consider a noisy dispersion. This is a natural model in dispersion managed optical fibers [1], [3], [4], [18], [24] (see also [29] for a deterministic periodic dispersion). The nonlinear Schrödinger equation with random dispersion has also been studied mathematically. In [24], the power law nonlinearity is replaced by a smooth bounded function and it is shown that, in a certain scaling, the solutions to the nonlinear Schrödinger equation converge to the solutions of the nonlinear Schrödinger equation with white noise dispersion. This result has been extended to the case of a subcritical nonlinearity in [12]. One of the main improvement in [12] is the use of Strichartz type estimates for white noise dispersion (see also [13] for the derivation of Strichartz estimates for a stochastic Nonlinear Schrödinger equation).

Note that Strichartz type estimates are not immediate for a white noise dispersion. We have an explicit formula of the fundamental solution for the linear equation as in the deterministic case:

$$u(t) = \frac{1}{\left(4i\pi\left(\beta(t)-\beta(s)\right)\right)^{d/2}} \int_{\mathbb{R}^d} \exp\left(i\frac{|x-y|^2}{4(\beta(t)-\beta(s))}\right) u_s(y)dy$$

is the solution of the linear equation with white noise dispersion with initial data u_s at time s (see Proposition 3.1).

Nevertheless, it is not obvious whether the Strichartz type estimate holds or not unlike the deterministic case. We have two difficulties to prove the Strichartz type estimate. One difficulty is that the dispersion coefficient is highly degenerate. In fact, for $\epsilon > s \geq 0$, the set $\{t \in (s,\epsilon) \ : \beta(t)-\beta(s) = 0\}$ has the cardinality of the continuum (see, e.g. [16], Example 4.1 in Section 7.4). Roughly speaking, in our problem, the dispersion coefficient has so many zeros that we can not expect that pathwise Strichartz estimates hold. Another difficulty is that the duality argument (or TT^* argument) does not work as well as in the deterministic case. "Duality" corresponds to solving the equation backwards. For stochastic equations, a backward equation has in general no solution unless the coefficient of the noise is considered as an unknown, which is not desirable in our situation.

In the present work, we show that in the one dimensional case it is possible to improve the Strichartz estimates obtained in [12] and as a result prove that the nonlinear Schrödinger equation with critical nonlinearity and white noise dispersion is globally well posed in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$. This confirms the fact that such a random dispersion has a strong stabilizing effect on the equation: in the quintic one dimensional case considered, it prevents the formation of singularities and yields global well posedness.

2. Preliminaries and main results

We consider the following stochastic nonlinear Schrödinger (NLS) equation with quintic nonlinearity on the real line

$$\left\{ \begin{array}{l} idu + \Delta u \circ d\beta + |u|^4 u \, dt = 0, \ x \in \mathbb{R}, \ t > 0, \\ u(0) = u_0, \ x \in \mathbb{R}. \end{array} \right. \tag{2.1}$$

The unknown u is a random process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ depending on $t > 0$ and $x \in \mathbb{R}$. The noise term is given by a brownian motion β associated to a stochastic basis

$(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. The product $\circ$ is a Stratonovich product. Classically, we transform this Stratonovitch equation into an Itô equation which is formally equivalent:

$$\begin{cases} idu + \frac{i}{2}\Delta^2 u\, dt + \Delta u\, d\beta + |u|^4 u\, dt = 0, \ x \in \mathbb{R}, \ t > 0, \\ u(0) = u_0. \end{cases} \tag{2.2}$$

It seems as if the principal part of (2.2) were the double Laplacian, which does not appear to be degenerate. But this is not true. Indeed, the explicit formula of the fundamental solution for the linear equation shows the high degeneracy of the principal part (see Proposition 3.1), as is already pointed out in Section 1.

We study this equation (2.2) in the framework of the $L^2(\mathbb{R})$ based Sobolev spaces denoted by $H^s(\mathbb{R})$, $s \geq 0$. We also use the spaces $L^p(\mathbb{R})$ to treat the nonlinear term thanks to the Strichartz estimates. Note that, in all the article, these are spaces of complex valued functions.

For time dependent functions on an interval $I \subset \mathbb{R}$ with values in a Banach space K, we use the spaces: $L^r(I; K)$, $r \geq 1$. Given a time dependent function f, we use two notations for its values at some time t depending on the context. We either write $f(t)$ or f_t.

The norm of a Banach space K is simply denoted by $\| \cdot \|_K$. When we consider random variables with values in a Banach space K, we use $L^p(\Omega; K)$, $p \geq 1$.

For spaces of predictible time dependent processes, we use the subscript $\mathcal{P}$. For instance $L^r_{\mathcal{P}}(\Omega; L^p(0, T; K))$ is the subspace of $L^r(\Omega; L^p(0, T; K))$ consisting of predictible processes.

Our main result is the following.

Theorem 2.1. *Let $u_0 \in L^2(\mathbb{R})$ a.s. be $\mathcal{F}_0$-measurable, then there exists a unique solution u to (2.2) with paths a.s. in $L^5_{loc}(0, \infty; L^{10}(\mathbb{R}))$; moreover, u has paths in $C(\mathbb{R}^+; L^2(\mathbb{R}))$, a.s. and*

$$\|u(t)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}, \quad a.s.$$

If in addition $u_0 \in H^1(\mathbb{R})$, then u has paths a.s. in $C(\mathbb{R}^+; H^1(\mathbb{R}))$.

As in [12], we use this result to justify rigorously the convergence of the solution of the following random equation

$$\begin{cases} i\frac{du}{dt} + \frac{1}{\varepsilon} m\left(\frac{t}{\varepsilon^2}\right) \partial_{xx} u + |u|^4 u = 0, \ x \in \mathbb{R}, \ t > 0, \\ u(0) = u_0, \, x \in \mathbb{R}, \end{cases} \tag{2.3}$$

to the solution of (2.2) provided that the real valued centered stationary random process $m(t)$ is continous and that for any $T > 0$, the process $t \mapsto \varepsilon \int_0^{t/\varepsilon^2} m(s) ds$ converges in distribution to a standard real valued Brownian motion in $C([0, T])$. This is a classical assumption and can be verified in many cases.

To our knowledge, Strichartz estimates are not available for equation (2.3). Hence we cannot get solutions in $L^2(\mathbb{R})$. Since the equation is set in space dimension 1, a local existence result can be easily proved in $H^1(\mathbb{R})$. For fixed ε, we do not expect to have global in time solutions, indeed with a quintic nonlinearity it is known that singularities appear for the deterministic nonlinear Schrödinger equation. In the following result, we prove that the lifetime of the solutions converges to infinity when ε goes to zero, and that solutions of (2.3) converge in distribution to the solutions of the white noise driven equation (2.2).

Theorem 2.2. *Suppose that m satisfies the above assumption. Then, for any $\varepsilon > 0$ and $u_0 \in H^1(\mathbb{R})$, there exists a unique solution u_ε of equation (2.3) with continuous paths in $H^1(\mathbb{R})$ which is defined on a random interval $[0, \tau_\varepsilon(u_0))$. Moreover, for any $T > 0$*

$$\lim_{\varepsilon \to 0} \mathbb{P}(\tau_\varepsilon(u_0) \leq T) = 0,$$

and the process $u_\varepsilon \mathbb{1}_{[\tau_\varepsilon > T]}$ converges in distribution to the solution u of (2.2) in $C([0,T]; H^1(\mathbb{R}))$.

Remark 2.3. *Note that there is a slight improvement compared to the result obtained in [12] where the convergence was not proved in the $H^1(\mathbb{R})$ topology. This result can be extended to initial data in $H^s(\mathbb{R})$ for $s \in (1/2, 1]$. In this case, the convergence holds in $C([0,T]; H^s(\mathbb{R}))$.*

3. The linear equation and Strichartz type estimates

The Strichartz estimates are crucial to study the deterministic equation. In [12], these have been generalized to a white noise dispersion. However, the result obtained there was not strong enough to treat the nonlinearity of the present article. We now show that in dimension 1, it is possible to get a better result.

We consider the following stochastic linear Schrödinger equation:

$$\begin{cases} idu + \dfrac{i}{2}\Delta^2 u\, dt + \Delta u\, d\beta = 0, \ t \geq s, \\ u(s) = u_s. \end{cases} \tag{3.1}$$

We have an explicit formula for the solutions of (3.1). We recall from [12], [24] the following result:

Proposition 3.1. *For any $s \leq T$ and $u_s \in \mathcal{S}'(\mathbb{R}^n)$, there exists a unique solution of (3.1) almost surely in $C([s,T]; \mathcal{S}'(\mathbb{R}^n))$ and adapted. Its Fourier transform in space is given by*

$$\hat{u}(t,\xi) = e^{-i|\xi|^2(\beta(t)-\beta(s))}\hat{u}_s(\xi), \ t \geq s, \ \xi \in \mathbb{R}^d.$$

Moreover, if $u_s \in H^\sigma(\mathbb{R})$ for some $\sigma \in \mathbb{R}$, then $u(\cdot) \in C([0,T]; H^\sigma(\mathbb{R}))$ a.s. and $\|u(t)\|_{H^\sigma} = \|u_s\|_{H^\sigma}$, a.s. for $t \geq s$.

If $u_s \in L^1(\mathbb{R})$, the solution u of (3.1) has the expression

$$u(t) = S(t,s)u_s := \frac{1}{(4i\pi\,(\beta(t)-\beta(s)))^{d/2}} \int_{\mathbb{R}^d} \exp\left(i\frac{|x-y|^2}{4(\beta(t)-\beta(s))}\right) u_s(y)dy, \ t \in [s,T]. \tag{3.2}$$

The idea is to obtain Strichartz estimate through smoothing effects of $S(t,s)$ as was done in the deterministic case in [25].

The first step is the following.

Proposition 3.2. *Let $f \in L^4_{\mathcal{P}}(\Omega; L^1(0,T; L^2(R)))$ then $t \mapsto D^{1/2}\left(\left|\int_0^t S(t,s)f(s)ds\right|^2\right)$ belongs to $L^2_{\mathcal{P}}(\Omega \times [0,T] \times \mathbb{R})$ and*

$$\mathbb{E}\int_0^T \left\| D^{1/2}\left(\left|\int_0^t S(t,s)f(s)ds\right|^2\right)\right\|^2_{L^2(\mathbb{R})} dt \leq 4\sqrt{2\pi}\, T^{1/2}\mathbb{E}\left(\|f\|^4_{L^1(0,T;L^2(\mathbb{R}))}\right).$$

Proof. By density, it is sufficient to prove that the inequality is valid for sufficiently smooth f. Set, for $\xi \in \mathbb{R}$,

$$A(\xi) = \left| \mathcal{F}\left[\left| \int_0^t S(t,s)f(s)ds \right|^2 \right](\xi) \right|^2 .$$

Then, by Plancherel identity,

$$\mathbb{E}\int_0^T \left\| D^{1/2}\left(\left| \int_0^t S(t,s)f(s)ds \right|^2 \right) \right\|^2_{L^2(\mathbb{R})} dt = \mathbb{E}\int_0^T \int_{\mathbb{R}} |\xi| A(\xi) d\xi dt.$$

We have, by Proposition 3.1 and easy computations,

$$\mathcal{F}\left[\left| \int_0^t S(t,s)f(s)ds \right|^2 \right](\xi) = \int_{\mathbb{R}}\int_0^t\int_0^t e^{-i(\beta_t-\beta_{s_1})(\xi-\xi_1)^2 + i(\beta_t-\beta_{s_2})\xi_1^2} \hat{f}_{s_1}(\xi-\xi_1)\bar{\hat{f}}_{s_2}(\xi_1) ds_1 ds_2 d\xi_1.$$

We deduce:

$$\begin{aligned} A(\xi) &= \iint_{\mathbb{R}^2} \iiiint_{[0,t]^4} e^{-i(\beta_t-\beta_{s_1})(\xi-\xi_1)^2+i(\beta_t-\beta_{s_2})\xi_1^2} e^{i(\beta_t-\beta_{s_3})(\xi-\xi_2)^2 - i(\beta_t-\beta_{s_4})\xi_2^2} \\ &\qquad \times \hat{f}_{s_1}(\xi-\xi_1)\bar{\hat{f}}_{s_2}(\xi_1)\bar{\hat{f}}_{s_3}(\xi-\xi_2)\bar{\bar{\hat{f}}}_{s_4}(\xi_2) ds_1 ds_2 ds_3 ds_4 d\xi_1 d\xi_2. \end{aligned}$$

Let us split $[0,t]^4 = \bigcup_{i=1,\dots,4} R_i$ with

$$R_i = \{(s_1,s_2,s_3,s_4) \in [0,t]^4;\ s_i = \max\{s_1,s_2,s_3,s_4\}\}$$

and split accordingly

$$A(\xi) = \sum_{i=1,\dots,4} I_i(\xi).$$

We then write, using $(\xi-\xi_1)^2 - \xi_1^2 - (\xi-\xi_2)^2 + \xi_2^2 = 2\xi(\xi_2-\xi_1)$,

$$\begin{aligned} \mathbb{E}\left(\int_{\mathbb{R}} |\xi| I_1(\xi) d\xi \right) &= \mathbb{E}\bigg(\int_{R_1}\iiint_{\mathbb{R}^3} |\xi| e^{-2i(\beta_t-\beta_{s_1})\xi(\xi_2-\xi_1) - i(\beta_{s_1}-\beta_{s_2})\xi_1^2 + i(\beta_{s_1}-\beta_{s_3})(\xi-\xi_2)^2 - i(\beta_{s_1}-\beta_{s_4})\xi_2^2} \\ &\qquad \times \hat{f}_{s_1}(\xi-\xi_1)\bar{\hat{f}}_{s_2}(\xi_1)\bar{\hat{f}}_{s_3}(\xi-\xi_2)\bar{\bar{\hat{f}}}_{s_4}(\xi_2) ds_1 ds_2 ds_3 ds_4 d\xi_1 d\xi_2 d\xi \bigg). \end{aligned}$$

Clearly $e^{-2i(\beta_t-\beta_{s_1})\xi(\xi_2-\xi_1)}$ is independent to the other factors. Moreover:

$$\mathbb{E}\left(e^{-2i(\beta_t-\beta_{s_1})\xi(\xi_2-\xi_1)} \right) = e^{-2(t-s_1)\xi^2(\xi_2-\xi_1)^2}.$$

We deduce

$$\begin{aligned} \mathbb{E}\left(\int_{\mathbb{R}} |\xi| I_1(\xi) d\xi \right) &\le \mathbb{E}\bigg(\int_{R_1}\iiint_{\mathbb{R}^3} |\xi| e^{-2(t-s_1)\xi^2(\xi_2-\xi_1)^2} |\hat{f}_{s_1}(\xi-\xi_1)||\bar{\hat{f}}_{s_2}(\xi_1)| \\ &\qquad \times |\hat{f}_{s_3}(\xi-\xi_2)||\bar{\hat{f}}_{s_4}(\xi_2)| ds_1 ds_2 ds_3 ds_4 d\xi_1 d\xi_2 d\xi \bigg). \end{aligned}$$

Note that

$$\begin{aligned}&\iiint_{\mathbb{R}^3}|\xi|e^{-2(t-s_1)\xi^2(\xi_2-\xi_1)^2}|\hat{f}_{s_1}(\xi-\xi_1)||\hat{\bar{f}}_{s_2}(\xi_1)||\hat{f}_{s_3}(\xi-\xi_2)||\hat{\bar{f}}_{s_4}(\xi_2)|d\xi_1 d\xi_2 d\xi\\&=\int_{\mathbb{R}}|\xi|\left(\int_{\mathbb{R}}|\hat{f}_{s_1}(\xi-\xi_1)||\hat{\bar{f}}_{s_2}(\xi_1)|\left(\int_{\mathbb{R}}e^{-2(t-s_1)\xi^2(\xi_2-\xi_1)^2}|\hat{f}_{s_3}(\xi-\xi_2)||\hat{\bar{f}}_{s_4}(\xi_2)|d\xi_2\right)d\xi_1\right)d\xi.\end{aligned}$$

Since $\int_{\mathbb{R}}e^{-2(t-s_1)\xi^2\eta^2}d\eta=\frac{\sqrt{\pi}}{|\xi|\big(2(t-s_1)\big)^{1/2}}$, we deduce by Young's and Schwarz's inequalities:

$$\begin{aligned}&\iiint_{\mathbb{R}^3}|\xi|e^{-2(t-s_1)\xi^2(\xi_2-\xi_1)^2}|\hat{f}_{s_1}(\xi-\xi_1)||\hat{\bar{f}}_{s_2}(\xi_1)||\hat{f}_{s_3}(\xi-\xi_2)||\hat{\bar{f}}_{s_4}(\xi_2)|d\xi_1 d\xi_2 d\xi\\
&\le\frac{\sqrt{\pi}}{\big(2(t-s_1)\big)^{1/2}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\hat{f}_{s_1}(\xi-\xi_1)|^2|\hat{\bar{f}}_{s_2}(\xi_1)|^2d\xi_1\right)^{1/2}\left(\int_{\mathbb{R}}|\hat{f}_{s_3}(\xi-\xi_2)|^2|\hat{\bar{f}}_{s_4}(\xi_2)|^2d\xi_2\right)^{1/2}d\xi\\
&\le\frac{\sqrt{\pi}}{\big(2(t-s_1)\big)^{1/2}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\hat{f}_{s_1}(\xi-\xi_1)|^2|\hat{\bar{f}}_{s_2}(\xi_1)|^2d\xi_1d\xi\right)^{1/2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\hat{f}_{s_3}(\xi-\xi_2)|^2|\hat{\bar{f}}_{s_4}(\xi_2)|^2d\xi_2d\xi\right)^{1/2}\\
&=\frac{\sqrt{\pi}}{\big(2(t-s_1)\big)^{1/2}}\|f_{s_1}\|_{L^2(\mathbb{R})}\|f_{s_2}\|_{L^2(\mathbb{R})}\|f_{s_3}\|_{L^2(\mathbb{R})}\|f_{s_4}\|_{L^2(\mathbb{R})}.\end{aligned}$$

It follows

$$\mathbb{E}\left(\int_{\mathbb{R}}|\xi|I_1(\xi)d\xi\right)\le\mathbb{E}\int_{R_1}\frac{\sqrt{\pi}}{\big(2(t-s_1)\big)^{1/2}}\|f_{s_1}\|_{L^2(\mathbb{R})}\|f_{s_2}\|_{L^2(\mathbb{R})}\|f_{s_3}\|_{L^2(\mathbb{R})}\|f_{s_4}\|_{L^2(\mathbb{R})}ds_1ds_2ds_3ds_4$$

and

$$\mathbb{E}\int_0^T\int_{\mathbb{R}}|\xi|I_1(\xi)d\xi dt\le\sqrt{2\pi}T^{1/2}\mathbb{E}\left(\left(\int_0^T\|f_s\|_{L^2(\mathbb{R})}ds\right)^4\right).$$

The three other terms are treated similarly and the result follows. □

Proposition 3.3. *There exists a constant $\kappa>0$ such that for any $s\in\mathbb{R}$, $T\ge 0$ and $f\in L^4_{\mathcal{P}}(\Omega;L^1(s,s+T;L^2(\mathbb{R})))$, the mapping $t\mapsto\int_s^t S(t,\sigma)f(\sigma)d\sigma$ belongs to $L^4_{\mathcal{P}}(\Omega;L^5(s,s+T;L^{10}(\mathbb{R})))$ and*

$$\left\|\int_s^\cdot S(\cdot,\sigma)f(\sigma)d\sigma\right\|_{L^4(\Omega;L^5(s,s+T;L^{10}(\mathbb{R})))}\le\kappa T^{1/10}\|f\|_{L^4(\Omega;L^1(s,s+T;L^2(\mathbb{R})))}$$

Remark 3.4. *This result is very similar to the classical Strichartz estimates in the case of dimension 1 considered here. Indeed $(5,10)$ and $(\infty,2)$ are admissible pairs. However, it is more powerful. Indeed, we have the extra factor $T^{1/10}$. This is a major difference and allows us to construct solution for the quintic nonlinearity. Recall that in the deterministic case, it is known that there are singular solutions for this equation. The proof below extends easily to the same result with $(5,10)$ replaced by any admissible pair (r,p), i.e. satisfying $\frac{2}{r}=\frac{1}{2}-\frac{1}{p}$. Of course, the power of T changes in this case; but it remains positive.*

Proof: We treat only the case $s = 0$. The generalization is easy. Also, it is sufficient to prove that the inequality holds for sufficiently smooth f.

We use the following Lemma. Its proof is given below for the reader's convenience.

Lemma 3.5. *Let $g \in L^1(\mathbb{R})$ such that $D^{1/2}g \in L^2(\mathbb{R})$, then $g \in L^5(\mathbb{R})$ and*

$$\|g\|_{L^5(\mathbb{R})} \le C\|g\|_{L^1(\mathbb{R})}^{1/5}\|D^{1/2}g\|_{L^2(\mathbb{R})}^{4/5}.$$

Let us write

$$\left\|\int_0^\cdot S(\cdot,\sigma)f(\sigma)d\sigma\right\|_{L^4(\Omega;L^5(0,T;L^{10}(\mathbb{R})))}^4 = \left\|\left|\int_0^\cdot S(\cdot,\sigma)f(\sigma)d\sigma\right|^2\right\|_{L^2(\Omega;L^{5/2}(0,T;L^5(\mathbb{R})))}^2.$$

Therefore, by Lemma 3.5, Hölder inequality and Proposition 3.2,

$$\begin{aligned}
&\left\|\int_0^\cdot S(\cdot,\sigma)f(\sigma)d\sigma\right\|_{L^4(\Omega;L^5(0,T;L^{10}(\mathbb{R})))}^4 \\
&\le c\mathbb{E}\left(\left(\int_0^T \left\|\left|\int_0^t S(t;\sigma)f_\sigma d\sigma\right|^2\right\|_{L^1(\mathbb{R})}^{1/2} \left\|D^{1/2}\left|\int_0^t S(t;\sigma)f_\sigma d\sigma\right|^2\right\|_{L^2(\mathbb{R})}^2 dt\right)^{4/5}\right) \\
&\le c\mathbb{E}\left(\left\|\left|\int_0^\cdot S(\cdot;\sigma)f_\sigma d\sigma\right|^2\right\|_{L^\infty(0,T;L^1(\mathbb{R}))}^{2/5} \left\|D^{1/2}\left|\int_0^\cdot S(\cdot;\sigma)f_\sigma d\sigma\right|^2\right\|_{L^2(0,T;L^2(\mathbb{R}))}^{8/5}\right) \\
&\le c\mathbb{E}\left(\left\|\left|\int_0^\cdot S(\cdot;\sigma)f_\sigma d\sigma\right|^2\right\|_{L^\infty(0,T;L^1(\mathbb{R}))}^2\right)^{1/5} \mathbb{E}\left(\left\|D^{1/2}\left|\int_0^\cdot S(\cdot;\sigma)f_\sigma d\sigma\right|^2\right\|_{L^2(0,T;L^2(\mathbb{R}))}^2\right)^{4/5} \\
&\le T^{2/5}\mathbb{E}\left(\|f\|_{L^1(0,T;L^2(\mathbb{R}))}^4\right).
\end{aligned}$$

□

Proof of Lemma 3.5: By Gagliardo-Nirenberg inequality, we have:

$$\|g\|_{L^5(\mathbb{R})} \le c\|D^{1/2}g\|_{L^2(\mathbb{R})}^{3/5}\|g\|_{L^2(\mathbb{R})}^{2/5}. \tag{3.3}$$

Moreover

$$\begin{aligned}
\|g\|_{L^2(\mathbb{R})}^2 = \|\hat g\|_{L^2(\mathbb{R})}^2 &= \int_{|\xi|\ge R} |\hat g(\xi)|^2 d\xi + \int_{|\xi|\le R} |\hat g(\xi)|^2 d\xi \\
&\le \int_{|\xi|\ge R} \frac{|\xi|}{R}|\hat g(\xi)|^2 d\xi + 2R\|\hat g\|_{L^\infty(\mathbb{R})}^2 \\
&\le \frac{1}{R}\|D^{1/2}g\|_{L^2(\mathbb{R})}^2 + 2R\|g\|_{L^1(\mathbb{R})}^2
\end{aligned}$$

It suffices to take $R = \|D^{1/2}g\|_{L^2(\mathbb{R})}\|g\|_{L^1(\mathbb{R})}^{-1}$ and to insert the result in (3.3) to conclude. □

We also need to have estimates on the action of $S(t,s)$ on an initial data.

Proposition 3.6. *Let $s \geq 0$ and $u_s \in L^4(\Omega; L^2(\mathbb{R}))$ be $\mathcal{F}_s$ measurable, then $t \mapsto S(t,s)u_s$ belongs to $L^4_{\mathcal{P}}(\Omega; L^5(s, s+T; L^{10}(\mathbb{R})))$ and*

$$\|S(\cdot, s)u_s\|_{L^4(\Omega;L^5(s,s+T;L^{10}(\mathbb{R})))} \leq cT^{1/10}\|u_s\|_{L^4(\Omega;L^2(\mathbb{R}))}.$$

Proof: The proof is similar. Again, we only treat the case $s = 0$. We first write:

$$\left|\mathcal{F}\left(|S(t,0)u_0|^2\right)\right|^2 = \iint_{\mathbb{R}^2} e^{-2i\beta_t\xi(\xi_2-\xi_1)}\hat{u}_0(\xi-\xi_1)\hat{\bar{u}}_0(\xi_1)\bar{\hat{u}}_0(\xi-\xi_2)\bar{\hat{\bar{u}}}_0(\xi_2)d\xi_1 d\xi_2$$

and

$$\begin{aligned}
&\mathbb{E}\left(\left\|D^{1/2}\,|S(t,0)u_0|^2\right\|^2_{L^2(0,T;L^2(\mathbb{R}))}\right)\\
&= \mathbb{E}\int_0^T \iiint_{\mathbb{R}^3} |\xi| e^{-2t\xi^2(\xi_2-\xi_1)^2}\hat{u}_0(\xi-\xi_1)\hat{\bar{u}}_0(\xi_1)\bar{\hat{u}}_0(\xi-\xi_2)\bar{\hat{\bar{u}}}(\xi_2)d\xi_1 d\xi_2 d\xi dt\\
&\leq \mathbb{E}\int_0^T \int_{\mathbb{R}} |\xi| \left(\int_{\mathbb{R}} \hat{u}_0(\xi-\xi_1)\hat{\bar{u}}_0(\xi_1)\left(\int_{\mathbb{R}} e^{-2t\xi^2(\xi_2-\xi_1)^2}\bar{\hat{u}}_0(\xi-\xi_2)\bar{\hat{\bar{u}}}(\xi_2)d\xi_2\right)d\xi_1\right)d\xi dt.
\end{aligned}$$

Therefore by Young's and Schwarz's inequalities:

$$\begin{aligned}
\mathbb{E}\left(\left\|D^{1/2}\,|S(t,0)u_0|^2\right\|^2_{L^2(0,T;L^2(\mathbb{R}))}\right) &\leq \mathbb{E}\int_0^T \sqrt{\pi}t^{-1/2}\mathbb{E}\left(\|u_0\|^4_{L^2(\mathbb{R})}\right)dt\\
&\leq 2\sqrt{\pi}T^{1/2}\mathbb{E}\left(\|u_0\|^4_{L^2(\mathbb{R})}\right).
\end{aligned}$$

We then use Lemma 3.5 and Hölder inequality:

$$\begin{aligned}
&\|S(\cdot,0)u_0\|_{L^4(\Omega;L^5(0,T;L^{10}(\mathbb{R})))}\\
&\leq c\left\||S(\cdot,0,u_0)|^2\right\|^{1/10}_{L^2(\Omega;L^\infty(0,T;L^1(\mathbb{R})))}\left\|D^{1/2}\,|S(\cdot,0,u_0)|^2\right\|^{4/10}_{L^2(\Omega;L^2(0,T;L^2(\mathbb{R})))}\\
&\leq cT^{1/10}\mathbb{E}\left(\|u_0\|^4_{L^2(\mathbb{R})}\right).
\end{aligned}$$

□

4. Proof of Theorem 2.1

As is classical, we first construct a local solution of equation (2.2) thanks to a cut-off of the nonlinearity. Proceeding as in in [8], [9], [12], we take $\theta \in C_0^\infty(\mathbb{R})$ be such that $\theta = 1$ on $[0,1]$, $\theta = 0$ on $[2,\infty)$ and for $s \in \mathbb{R}$, $u \in L^5_{loc}(s,\infty; L^{10}(\mathbb{R}))$, $R \geq 1$ and $t \geq 0$, we set

$$\theta_R^s(u)(t) = \theta\left(\frac{\|u\|_{L^5(s,s+t;L^{10}(\mathbb{R}))}}{R}\right).$$

For $s = 0$, we set $\theta_R^0 = \theta_R$.

The truncated form of equation (2.2) is given by

$$\left\{\begin{array}{l} idu^R + \dfrac{i}{2}\Delta^2 u^R\,dt + \Delta u^R d\beta + \theta_R(u^R)|u^R|^4 u^R\,dt = 0,\\ u^R(0) = u_0. \end{array}\right. \tag{4.1}$$

We interpret it in the mild sense

$$u^R(t) = S(t,0)u_0 + i\int_0^t S(t,s)\theta_R(u^R)(s)|u^R(s)|^4u^R(s)ds. \tag{4.2}$$

Proposition 4.1. *For any $\mathcal{F}_0$-measurable $u_0 \in L^4(\Omega; L^2(\mathbb{R}))$, there exists a unique solution of (4.2) u^R in $L^4_{\mathcal{P}}(\Omega; L^5(0,T; L^{10}(\mathbb{R}))))$ for any $T > 0$. Moreover u^R is a weak solution of (4.1) in the sense that for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ and any $t \geq 0$,*

$$\begin{aligned} & i(u^R(t) - u_0, \varphi)_{L^2(\mathbb{R})} \\ = \; & -\frac{i}{2}\int_0^t (u^R, \Delta^2\varphi)_{L^2(\mathbb{R})}ds - \int_0^t \theta_R(u^R)(|u^R|^4u^R, \varphi)_{L^2(\mathbb{R})}ds - \int_0^t (u^R, \Delta\varphi)_{L^2(\mathbb{R})}d\beta(s), \; a.s. \end{aligned}$$

Finally, the $L^2(\mathbb{R})$ norm is conserved:

$$\|u^R(t)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}, \; t \geq 0, \; a.s.$$

and $u \in C([0,T]; L^2(\mathbb{R}))$ a.s.

Proof. In order to lighten the notations we omit the R dependence in this proof. By Proposition 3.6, we know that $S(\cdot,0)u_0 \in L^4_{\mathcal{P}}(\Omega; L^5(0,T; L^{10}(\mathbb{R}))))$. Then, by Proposition 3.3, for $u, v \in L^4_{\mathcal{P}}(\Omega; L^5(0,T; L^{10}(\mathbb{R}))))$,

$$\begin{aligned} & \left\| \int_0^t S(t,s)\left(\theta(u)(s)|u(s)|^4u(s) - \theta(v)(s)|v(s)|^4v(s)\right)ds \right\|_{L^4(\Omega; L^5(0,T; L^{10}(\mathbb{R}))))} \\ & \leq cT^{1/10}\left\|\theta(u)|u|^4u - \theta(v)|v|^4v\right\|_{L^4(\Omega; L^1(0,T; L^2(\mathbb{R})))} \\ & \leq cT^{1/10}R^4\|u-v\|_{L^4(\Omega; L^5(0,T; L^{10}(\mathbb{R}))))}. \end{aligned}$$

It follows that

$$\mathcal{T}^R : u \mapsto S(t,0)u_0 + i\int_0^t S(t,s)\theta(u(s))|u(s)|^4u(s)ds \tag{4.3}$$

defines a strict contraction on $L^4_{\mathcal{P}}(\Omega; L^5(0,T; L^{10}(\mathbb{R}))))$ provided $T \leq T_0$ where T_0 depends only on R. Iterating this construction, one easily ends the proof of the first statement. The proof that u is in fact a weak solution is classical.

Let $M \geq 0$ and $u_M = P_M u$ be a regularization of the solution u defined by a truncation in Fourier space: $\hat{u}_M(t,\xi) = \theta\left(\frac{|\xi|}{M}\right)\hat{u}(t,\xi)$. We deduce from the weak form of the equation that

$$idu_M + \frac{i}{2}\Delta^2 u_M\,dt + \Delta u_M d\beta + P_M\left(\theta(u)|u|^4u\right)\,dt = 0.$$

We apply Itô formula to $\|u_M\|^2_{L^2(\mathbb{R})}$ and obtain

$$\|u_M(t)\|^2_{L^2(\mathbb{R})} = \|u_0\|^2_{L^2(\mathbb{R})} + Re\left(i\int_0^t \left(\theta(u)|u|^4u, u_M\right)ds\right), \; t \in [0,T].$$

We know that $u \in L^5(0,T;L^{10}(\mathbb{R}))$ a.s. By the integral equation,

$$\begin{aligned}\|u(t)\|_{L^2(\mathbb{R})} &\le \|S(t,0)u_0\|_{L^2(\mathbb{R})} + \int_0^t \|S(t,s)\theta(u(s))|u(s)|^4 u(s)\|_{L^2(\mathbb{R})} ds \\ &\le \|u_0\|_{L^2(\mathbb{R})} + \int_0^t \|u(s)\|^5_{L^{10}(\mathbb{R})} ds.\end{aligned}$$

We deduce that $u \in L^\infty(0,T;L^2(\mathbb{R}))$ a.s. and

$$\lim_{M\to\infty} u_M = u \text{ in } L^\infty(0,T;L^2(\mathbb{R})), \text{ a.s.}$$

we may let M go to infinity in the above equality and obtain

$$\lim_{M\to\infty} \|u_M(t)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}, \ t \in [0,T], \text{ a.s.}$$

This implies $u(t) \in L^2(\mathbb{R})$ for any $t \in [0,T]$ and $\|u(t)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$. As easily seen from the weak form of the equation, u is almost surely continuous with values in $H^{-4}(\mathbb{R})$. It follows that u is weakly continuous with values in $L^2(\mathbb{R})$. Finally the continuity of $t \mapsto \|u(t)\|_{L^2(\mathbb{R})}$ implies $u \in C([0,T];L^2(\mathbb{R}))$. □

The construction of a global solution and the end of the proof of Theorem 2.1 are now very similar to what was done in [12]. We briefly recall the ideas for the reader's convenience.

There is no loss of generality in assuming that $u_0 \in L^2(\mathbb{R})$ is deterministic. Uniqueness is clear since two solutions are solutions of the truncated equation on a random interval. We fix T_0 and construct a solution on $[0,T_0]$.

We define

$$\tau_R = \inf\{t \in [0,T], \ \|u^R\|_{L^5(0,t;L^{10}(\mathbb{R}))} \ge R\}$$

so that u^R is a solution of (2.2) on $[0,\tau_R]$.

Lemma 4.2. *There exist constants c_1, c_2 such that if*

$$T^{2/5} \le c_1 R^{-16}$$

then

$$\mathbb{P}(\tau_R \le T) \le \frac{c_2\|u_0\|^4_{L^2(\mathbb{R})}}{R^4}$$

Proof. We write

$$u^R(t)\mathbb{1}_{[0,\tau_R]}(t) = S(t,0)u_0\mathbb{1}_{[0,\tau_R]}(t) + i\int_0^t S(t,s)|u^R|^4 u^R \mathbb{1}_{[0,\tau_R]}(s) ds \mathbb{1}_{[0,\tau_R]}(t). \tag{4.4}$$

Thus for $T \le T_0$

$$\begin{aligned}\|u^R\mathbb{1}_{[0,\tau_R]}\|_{L^5(0,T;L^{10}(\mathbb{R}))} &\le \|S(\cdot,0)u_0\mathbb{1}_{[0,\tau_R]}\|_{L^5(0,T;L^{10}(\mathbb{R}))} \\ &+\|\int_0^t S(t,s)|u^R|^4 u^R\mathbb{1}_{[0,\tau_R]}(s)ds\|_{L^5(0,T;L^{10}(\mathbb{R}))}.\end{aligned}$$

Proposition 3.3 and Proposition 3.6 yield

$$\begin{aligned}\mathbb{E}\left(\|u^R \mathbb{1}_{[0,\tau_R]}\|^4_{L^5(0,T;L^{10}(\mathbb{R}))}\right) &\le c(T_0)\|u_0\|^4_{L^2(\mathbb{R})} + c\,T^{2/5}\mathbb{E}\left(\left\|\left|u^R\right|^5 \mathbb{1}_{[0,\tau_R)}\right\|^4_{L^1(0,T;L^2(\mathbb{R}))}\right)\\ &\le c(T_0)\|u_0\|^4_{L^2(\mathbb{R})} + c\,T^{2/5}\mathbb{E}\left(\left\|u^R \mathbb{1}_{[0,\tau_R)}\right\|^{20}_{L^5(0,T;L^{10}(\mathbb{R}))}\right)\\ &\le c(T_0)\|u_0\|^4_{L^2(\mathbb{R})} + c\,T^{2/5}R^{16}\mathbb{E}\left(\left\|u^R \mathbb{1}_{[0,\tau_R)}\right\|^{4}_{L^5(0,T;L^{10}(\mathbb{R}))}\right)\end{aligned}$$

Hence, if $c\,T^{2/5}R^{16} \le \frac{1}{2}$,

$$\mathbb{E}\left(\|u^R \mathbb{1}_{[0,\tau_R]}\|^4_{L^5(0,T;L^{10}(\mathbb{R}))}\right) \le 2c(T_0)\|u_0\|^4_{L^2(\mathbb{R})}$$

and by Markov inequality

$$\mathbb{P}(\tau_R \le T) \le \frac{2c(T_0)\|u_0\|^4_{L^2(\mathbb{R})}}{R^4}.$$

□

In order to construct a solution to (2.2) on $[0, T_0]$, we iterate the local construction. We fix $R > 0$ and have a local solution on $[0, \tau_R]$. We set $\tau_R^0 = \tau_R$. We then consider recursively the equation for u. For $n \ge 0$, we set $T_R^n = \sum_{k=0}^{n} \tau_R^n$ and define :

$$u(t + T_R^n) = S(t + T_R^n, T_R^n)u(T_R^n) + \int_0^t S(t + T_R^n, s + T_R^n)\theta_R^{T_R^n}(u)(s)|u(s + T_R^n)|^{2\sigma}u(s + T_R^n)ds.$$

The local construction can be reproduced and we obtain a unique global solution of this equation on $[T_R^n, T_R^n + \tau_R^{n+1}]$ where

$$\tau_R^{n+1} = \inf\{t \in [0, T],\ |u|_{L^5(T_R^n, t+T_R^n; L^{10}(\mathbb{R}))} \ge R\}.$$

We thus obtain a solution of the non truncated equation on $\left[0, \sum_{n=0}^{\infty} \tau_R^n\right]$. By Lemma 4.2, the strong Markov property and the conservation of the $L^2(\mathbb{R})$ norm

$$\mathbb{P}(\tau_R^{n+1} \le T|\mathcal{F}_{T_R^n}) = \mathbb{P}(\tau_R^{n+1} \le T|u(T_R^n)) \le \frac{c_2|u(T_R^n)|^4_{L^2(\mathbb{R})}}{R^4} = \frac{c_2|u_0|^4_{L^2(\mathbb{R})}}{R^4},\ a.s.,$$

provided $T^{2/5} \le c_1\, R^{-16}$. Note that

$$\mathbb{P}\left(\lim_{n\to+\infty} \tau_R^n = 0\right) = \lim_{\varepsilon\to 0}\lim_{N\to+\infty} \mathbb{P}(\tau_R^n \le \varepsilon,\ n \ge N).$$

Finally we choose R large enough and $\varepsilon^{2/5} \le c_1\, R^{-16}$ so that, for all $n \in \mathbb{N}$,

$$\mathbb{P}(\tau_R^{n+1} \le \varepsilon|\mathcal{F}_{T_R^n}) \le \frac{1}{2},\ a.s.$$

Then, since $\mathbb{P}(\tau_R^M \leq \varepsilon | \mathcal{F}_{T_R^{M-1}}) = \mathbb{E}\left(\mathbb{1}_{\tau_R^M \leq \varepsilon} | \mathcal{F}_{T_R^{M-1}}\right)$, we have for $0 \leq N \leq M$:

$$\begin{aligned}\mathbb{P}(\tau_R^n \leq \varepsilon,\ M \geq n \geq N) &= \mathbb{E}\left(\prod_{M \geq n \geq N} \mathbb{1}_{\tau_R^n \leq \varepsilon}\right)\\ &= \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}_{\tau_R^M \leq \varepsilon} | \mathcal{F}_{T_R^{M-1}}\right) \prod_{M-1 \geq n \geq N} \mathbb{1}_{\tau_R^n \leq \varepsilon}\right)\\ &\leq \frac{1}{2}\mathbb{E}\left(\prod_{M-1 \geq n \geq N} \mathbb{1}_{\tau_R^n \leq \varepsilon}\right).\end{aligned}$$

Repeating the last inequality, we deduce

$$\mathbb{P}(\tau_R^n \leq \varepsilon,\ M \geq n \geq N) \leq \frac{1}{2^{M-N}}$$

and

$$\mathbb{P}(\tau_R^n \leq \varepsilon,\ n \geq N) \leq \lim_{M \to \infty} \mathbb{P}(\tau_R^n \leq \varepsilon,\ M \geq n \geq N) \leq \lim_{M \to \infty} \frac{1}{2^{M-N}} = 0.$$

Hence, $\mathbb{P}(\lim_{n \to +\infty} \tau_R^n = 0) = 0$ so that $\tau_R^0 + \cdots + \tau_R^n$ goes to infinity a.s. and we have constructed a global solution.

The conservation of the L^2-norm and the fact that $u \in C(\mathbb{R}^+; L^2(\mathbb{R}))$ a.s. was proved in Theorem 4.1.

Finally, assume that $u_0 \in H^1(\mathbb{R})$. Then going back to $\mathcal{T}^R$ defined in (4.3), and applying the same estimates as in the proof of Lemma 4.2, after having taken first order space derivatives, lead to

$$\begin{aligned}&\|\mathcal{T}^R u\|_{L^4(\Omega; L^5(0,T; W^{1,10}(\mathbb{R}))}\\ &\leq C T_0^{1/10} \|u_0\|_{H^1(\mathbb{R})} + C' T^{1/10} R^{16} \|u\|_{L^4(\Omega; L^5(0,T; W^{1,10}(\mathbb{R}))}\end{aligned}$$

This proves that $B = B(0, R_0)$, the ball of radius R_0 in $L^4(\Omega; L^5(0,T; W^{1,10}(\mathbb{R}))$ is invariant by $\mathcal{T}^R$ provided $T \leq \tilde{T}_0$, where $\tilde{T}_0$ depends only on R and not on R_0. Since closed balls of $L^4(\Omega; L^5(0,T; W^{1,10}(\mathbb{R}))$ are closed in $L^4(\Omega; L^5(0,T; W^{1,10}(\mathbb{R}))$, this implies that the fixed point of $\mathcal{T}^R$, which is the solution u^R of (4.2), is in $L^4(\Omega; L^5(0,T; W^{1,10}(\mathbb{R}))$.

We deduce that u has paths in $L^5(0, T_0; W^{1,10}(\mathbb{R})$ and $|u|^4 u$ in $L^1(0, T_0; H^1(\mathbb{R}))$.

It is easily proved that $t \mapsto \int_0^t S(t,s) f(s) ds$ is in $L^p(\Omega; C([0,T]; H^1(\mathbb{R}))$ provided $f \in L^p(\Omega; L^1(0,T; H^1(\mathbb{R})))$ and that $t \mapsto S(t,0)u_0$ is in $L^p(\Omega; C([0,T]; H^1(\mathbb{R}))$ for $u_0 \in L^p(\Omega; H^1(\mathbb{R}))$.

By a localization argument, we conclude that u is continuous with values in $H^1(R)$ for $u_0 \in H^1(\mathbb{R})$ □

5. Equation (2.1) as limit of NLS equation with random dispersion

The proof of Theorem 2.2 uses similar arguments as in [12], however there are some modifications which enable us to get a stronger result. We fix $T \geq 0$.

Consider the following nonlinear Schrödinger equation written in the mild form:

$$u_n(t) = S_n(t)u_0 + i\int_0^t S_n(t,\sigma)F(|u(\sigma)|^2)u(\sigma)d\sigma,$$

where F is a smooth function with compact support, n is a real valued function and we have denoted by $S_n(t,\sigma) = \mathcal{F}^{-1}e^{-i(n(t)-n(\sigma))\xi^2/2}\mathcal{F}$, the evolution operator associated to the linear equation

$$i\frac{dv}{dt} + \dot{n}(t)\partial_{xx}v = 0,\ x\in\mathbb{R},\ t>0.$$

Since $S_n(t,\sigma)$ is an isometry on $H^1(\mathbb{R})$, it is easily shown that for $u_0 \in H^1(\mathbb{R})$ there exists a unique u_n in $C([0,T];H^1(\mathbb{R}))$, provided that n is a continuous function of t.

Let (n_k) be a sequence in $C([0,T];\mathbb{R})$ which converges to $n \in C([0,T];\mathbb{R})$ uniformly on $[0,T]$. Then, for $u_0 \in H^1(\mathbb{R})$, we have

$$\begin{aligned}
\|u_{n_k}(t) - u_n(t)\|_{H^1(\mathbb{R})} &\le \|(S_{n_k}(t,0) - S_n(t,0))\,u_0\|_{H^1(\mathbb{R})} \\
&+ \int_0^t \left\|(S_{n_k}(t,\sigma) - S_n(t,\sigma))\,F(|u_n(\sigma)|^2)u_n(\sigma)\right\|_{H^1(\mathbb{R})} d\sigma \\
&+ \int_0^t \left\|S_{n_k}(t,\sigma)\left(F(|u_n(\sigma)|^2)u_n(\sigma) - F(|u_{n_k}(\sigma)|^2)u_{n_k}(\sigma)\right)\right\|_{H^1(\mathbb{R})} d\sigma
\end{aligned}$$

Since F is smooth and has compact support, there exists M_F such that

$$\begin{aligned}
\|F(|u|^2)u - F(|v|^2)v\|_{H^1(\mathbb{R})} &\le M_F\left(\|u-v\|_{H^1(\mathbb{R})} + \|u\|_{H^1(\mathbb{R})}\|u-v\|_{L^\infty(\mathbb{R})}\right) \\
&\le M_F\left(\|u-v\|_{H^1(\mathbb{R})} + \|u\|_{H^1(\mathbb{R})}\|u-v\|_{H^1(\mathbb{R})}\right).
\end{aligned}$$

Since $S_{n_k}(t,\sigma)$ is an isometry, we deduce

$$\begin{aligned}
\int_0^t &\left\|S_{n_k}(t,\sigma)\left(F(|u_n(\sigma)|^2)u_n(\sigma) - F(|u_{n_k}(\sigma)|^2)u_{n_k}(\sigma)\right)\right\|_{H^1(\mathbb{R})} d\sigma \\
&\le C\int_0^t \|u_n(\sigma) - u_{n_k}(\sigma)\|_{H^1(\mathbb{R})} d\sigma
\end{aligned}$$

with $C = M_F\left(1 + \sup_{t\in[0,T]}\|u_n(t)\|_{H^1(\mathbb{R})}\right)$. It is easily checked that

$$\|(S_{n_k}(t,0) - S_n(t,0))\,u_0\|_{H^1(\mathbb{R})} \to 0 \tag{5.1}$$

as $k\to\infty$. Finally, note that $\{u_n(\sigma);\ \sigma\in[0,T]\}$ is compact in $H^1(\mathbb{R})$. By continuity of $u \mapsto F(|u|^2)u$ on $H^1(\mathbb{R})$, we deduce that $\{F(|u_n(\sigma)|^2)u_n(\sigma);\ \sigma\in[0,T]\}$ is also compact in $H^1(\mathbb{R})$. It follows that for any δ, we can find an R_δ such that

$$\sup_{\sigma\in[0,T]}\left\||\xi|\mathcal{F}\left(F(|u_n(\sigma)|^2)u_n(\sigma)\right)1_{|\xi|\ge R_\delta}\right\|_{L^2(\mathbb{R})} \le \delta.$$

Moreover, there exists $N_\delta \in \mathbb{N}$ such that, for $k \ge N_\delta$,

$$\sup_{0\le\sigma\le t\le T}\left\||\xi|\left(e^{-i(n(t)-n(s))\xi^2/2} - e^{-i(n_k(t)-n_k(s))\xi^2/2}\right)\mathcal{F}\left(F(|u_n(\sigma)|^2)u_n(\sigma)\right)1_{|\xi|\le R_\delta}\right\|_{L^2(\mathbb{R})} \le \delta.$$

We deduce

$$\int_0^t \left\|(S_{n_k}(t,\sigma)-S_n(t,\sigma))\,F(|u_n(\sigma)|^2)u_n(\sigma)\right\|_{H^1(\mathbb{R})}d\sigma \le 3T\delta$$

for $k\ge N_\delta$. By (5.1), we may assume that

$$\|(S_{n_k}(t,0)-S_n(t,0))\,u_0\|_{H^1(\mathbb{R})}\le\delta$$

for $k\ge N_\delta$. By Gronwall Lemma, we finally prove

$$\sup_{t\in[0,T]}\|u_{n_k}(t)-u_n(t)\|_{H^1(\mathbb{R})}\le(3T+1)e^{CT}\delta.$$

This proves that the map $n\to u_n$ is continuous form $C([0,T])$ into $C([0,T];H^1(\mathbb{R}))$.

Under our assumption, the process $t\mapsto\int_0^t\frac{1}{\varepsilon}m(\frac{s}{\varepsilon^2})ds$ converges in distribution in $C([0,T])$ to a brownian motion, and so we deduce that the solution of

$$\left\{\begin{array}{l} i\dfrac{du}{dt}+\dfrac{1}{\varepsilon}m(\dfrac{t}{\varepsilon^2})\partial_{xx}u+F(|u|^2)u=0,\ x\in\mathbb{R},\ t>0,\\ \\ u(0)=u_0,\ x\in\mathbb{R}, \end{array}\right. \tag{5.2}$$

converges in distribution in $C([0,T];H^1(\mathbb{R}))$ to the solution of

$$\left\{\begin{array}{l} idu+\Delta u\circ d\beta+F(|u|^2)u\,dt=0,\ x\in\mathbb{R},\ t>0,\\ u(0)=u_0,\ x\in\mathbb{R}. \end{array}\right.$$

We now want to extend this result to the original power nonlinear term. Let us introduce the truncated equations, where θ is as in section 4,

$$\left\{\begin{array}{l} i\dfrac{du}{dt}+\dfrac{1}{\varepsilon}m(\dfrac{t}{\varepsilon^2})\partial_{xx}u+\theta\left(\dfrac{|u|^2}{M}\right)|u|^4u=0,\ x\in\mathbb{R},\ t>0,\\ u(0)=u_0,\ x\in\mathbb{R}, \end{array}\right. \tag{5.3}$$

and

$$\left\{\begin{array}{l} idu+\Delta u\circ d\beta+\theta\left(\dfrac{|u|^2}{M}\right)|u|^4u\,dt=0,\ x\in\mathbb{R},\ t>0,\\ u(0)=u_0,\ x\in\mathbb{R}. \end{array}\right. \tag{5.4}$$

We denote by u_ε^M and u^M their respective solutions. By the previous arguments, these solutions exist and are unique in $C([0,T];H^1(\mathbb{R}))$. Note that setting

$$\widetilde{\tau}_\varepsilon^M=\inf\{t\ge0:\ \|u_\varepsilon^M(t)\|_{L^\infty(\mathbb{R})}\ge M\}$$

and $u_\varepsilon=u_\varepsilon^M$ on $[0,\widetilde{\tau}_\varepsilon^M]$, defines a unique local solution u_ε of equation (2.3) on $[0,\tau_\varepsilon)$ with $\tau_\varepsilon=\lim_{M\to\infty}\widetilde{\tau}_\varepsilon^M$.

We also set

$$\widetilde{\tau}^M=\inf\{t\ge0:\ \|u^M(t)\|_{L^\infty(\mathbb{R})}\ge M\}.$$

By the above result, for each M, u_ε^M converges to u^M in distribution in $C([0,T];H^1(\mathbb{R}))$. By Skorohod Theorem, after a change of probability space, we can assume that for each M the convergence of u_ε^M to u^M holds almost surely in $C([0,T];H^1(\mathbb{R}))$. To conclude, let us notice that for $0<\delta\le1$, if

$$\widetilde{\tau}^{M-1}\ge T \text{ and } \|u_\varepsilon^M-u^M\|_{C([0,T];H^1(\mathbb{R}))}\le\delta$$

then $u^M = u$, the solution of (2.2), on $[0,T]$. Moreover, by the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, we have

$$\|u_\varepsilon^M - u^M\|_{C([0,T];L^\infty(\mathbb{R}))} \leq c\delta$$

for some $c > 0$. We deduce $|u_\varepsilon^M|_{C([0,T];L^\infty(\mathbb{R}))} \leq M$ provided δ is small enough. Therefore

$$\tau_\varepsilon > \widetilde{\tau}_\varepsilon^M \geq T \text{ and } u_\varepsilon^M = u_\varepsilon \text{ on } [0,T].$$

It follows that for $\delta > 0$ small enough,

$$\begin{aligned} & \mathbb{P}(\tau_\varepsilon(u_0) \leq T) + \mathbb{P}(\tau_\varepsilon(u_0) > T \text{ and } \|u_\varepsilon - u\|_{C([0,T];H^1(\mathbb{R}))} > \delta) \\ \leq \;& \mathbb{P}(\|u_\varepsilon^M - u^M\|_{C([0,T];H^1(\mathbb{R}))} > \delta) + \mathbb{P}(\tilde{\tau}^{M-1} < T). \end{aligned}$$

Since $u_0 \in H^1(\mathbb{R})$, we know that u is almost surely in $C(\mathbb{R}^+; H^1(\mathbb{R}))$; we deduce

$$\lim_{M\to\infty} \mathbb{P}(\tilde{\tau}^{M-1} < T) = 0.$$

Choosing first M large and then ε small we obtain

$$\lim_{\varepsilon\to 0} \mathbb{P}(\tau_\varepsilon(u_0) \leq T) = 0$$

and

$$\lim_{\varepsilon\to 0} \mathbb{P}(\tau_\varepsilon(u_0) > T \text{ and } \|u_\varepsilon - u\|_{C([0,T];H^1(\mathbb{R}))} > \delta) = 0$$

The result follows. □

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