Inversion of a new V-line Radon transform and its numerical analysis

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Abstract. A new Radon transform defined on a discontinuous curve formed by a pair of half-lines forming a letter V is defined and studied. We establish its analytic inverse formula, its related filtered back-projection reconstruction procedure and its numerical analysis. These theoretical results allow the reconstruction of two-dimensional images of a radiating object from its Compton scattered rays measured on a one-dimensional collimated camera. Numerical simulations results illustrate the performance of the new imaging process.

1. Introduction

The Radon transform of a function $f(x,y)$ [1] has been defined as the integral of $f(x,y)$ on a family of curves in $\mathbb{R}^2$, such as straight lines, circles intersecting a fixed point, or more general classes of curves [2, 3]. It was R Basko who, in an attempt to model image formation in the so-called one-dimensional Compton camera, introduced a Radon transform on a pair of half-lines forming a V-letter [4]. The axis of his V-line swings around a point of the plane such that its vertex lies on a line with a variable opening angle between the two half-lines. Here we consider the class of V-line Radon transforms with fixed Oy-axis direction and vertex on the Ox-axis. Such Radon transforms may be of theoretical interest in integral geometry in the sense of Gel'fand [5]. An exact inverse formula will be established and its back-projection form derived. As a possible application we perform numerical simulations on the reconstruction of $f(x,y)$. This may be of use as a new imaging modality in nuclear medicine or in non-destructive testing [7].

2. The V-line Radon transform

2.1. Review of the standard Radon transform

The standard Radon transform maps a $L^1(\mathbb{R}^2)$-function $f(x,y)$ onto $\mathbb{R}f(p,\phi)$, according to

$$\mathbb{R}f(p,\phi) = \int_{\mathbb{R}^2} dx\,dy\, f(x,y)\,\delta(p - x\cos\phi - y\sin\phi), \tag{1}$$

in which (p,ϕ) are the parameters of the straight line: ϕ the angle of the unit normal vector $\mathbf{n}$ of the line with the axis Ox and p the distance from the polar coordinate system origin. There exists an explicit inverse formula, see *e.g.* [6]

$$f(x,y) = -\frac{1}{2\pi^2}\,\text{P.V.} \int_0^{\pi} d\phi \int_{\mathbb{R}} dp\, \frac{\mathbb{R}f(p,\phi)}{(p - x\cos\phi - y\sin\phi)^2}, \tag{2}$$

where P.V. denotes the Cauchy principal value of the p-integral. This inverse formula can be recast as a filtered back-projection formula, after reordering the integrations as follows

$$f(x,y) = \int_{\mathbb{R}^2} d\mathbf{k}\, |\mathbf{k}|\, e^{2i\pi\mathbf{k}\cdot\mathbf{r}} \left\{ \int_{\mathbb{R}^2} d\mathbf{r}'\, e^{-2i\pi(\mathbf{r}'\cdot\mathbf{k})} \int_0^{\pi} d\phi \int_{\mathbb{R}} dp\, \mathbb{R}f(p,\phi)\, \delta(p-(\mathbf{n}\cdot\mathbf{r}')) \right\}. \tag{3}$$

2.2. Definition of the V-line Radon transform

Consider a pair of half-lines meeting at a point **M** of abscissa ξ on the Ox-axis, each of which making an angle ω $(0<\omega<\pi/2)$ with the Oy-axis. Let $f(x,y)$ be a $L^1(\mathbb{R}^2)$ nonnegative continuous function with compact support in $\{\mathbb{R}^2|y>0\}$. We call

$$\mathbb{V}f(\zeta,\tau) = \int_0^{\infty} dr\, \big(f(\zeta + r\sin\omega, r\cos\omega) + f(\zeta - r\sin\omega, r\cos\omega) \big), \tag{4}$$

the V-line Radon transform of f, where $\tau = \tan\omega$.

2.3. Inversion formula

To obtain the analytic inversion formula, we show that the relation between the one-variable Fourier transforms of the function and of its V-line Radon transform is, after appropriate change of variables, a simple cosine-Fourier transform, hence readily invertible. Let $(\widetilde{f}(p,y), \text{resp.} \widetilde{\mathbb{V}f(p,\tau)})$ be the $(x, \text{resp.}\zeta)$-Fourier transform of $(f(x,y), \text{resp.}\mathbb{V}f(\zeta,\tau))$

$$f(x,y) = \int_{-\infty}^{\infty} dp\, e^{2i\pi px}\, \widetilde{f}(p,y) \quad \text{resp.} \quad \mathbb{V}f(\zeta,\tau) = \int_{-\infty}^{\infty} d\zeta\, e^{2i\pi p\zeta}\, \widetilde{\mathbb{V}f(p,\tau)}. \tag{5}$$

Let $z = r\cos\omega$ be a new integration variable. Since $0<\omega<\pi/2$, $\cos\omega = \frac{1}{\sqrt{1+\tau^2}} > 0$ and $z>0$. Then eq. (4) reads

$$\frac{\widetilde{\mathbb{V}f(p,\tau)}}{2\sqrt{1+\tau^2}} = \int_0^{\infty} dz\, \widetilde{f}(p,z)\, \cos(2\pi p\tau z). \tag{6}$$

This is precisely a cosine-Fourier transform. Its inverse can be computed straightforwardly under the form

$$\widetilde{f}(p,|v/p|) = 2|p| \int_0^{\infty} d\tau\, \cos(2\pi\tau v) \int_{\mathbb{R}} d\zeta\, e^{-2i\pi\zeta p} \left(\frac{\mathbb{V}f(\zeta,\tau)}{\sqrt{1+\tau^2}} \right). \tag{7}$$

Putting $y = |v/p| > 0$, $f(x,y)$ is reconstructed by inverse Fourier transform in p and is expressed as a triple integral on the V-line Radon data

$$f(x,y) = \int_{-\infty}^{\infty} |p|\, dp\, e^{2i\pi px} \int_0^{\infty} d\tau\, \frac{2\cos(2\pi p\tau y)}{\sqrt{1+\tau^2}} \int_{-\infty}^{\infty} d\zeta\, e^{-2i\pi\zeta p}\, \mathbb{V}f(\zeta,\tau), \tag{8}$$

where we have made use of $\cos(2\pi\tau v) = \cos(2\pi\tau|v|) = \cos(2\pi\tau y|p|) = \cos(2\pi\tau yp)$. For $y>0$, explicit intermediate integrations yield the final reconstruction formula :

$$f(x,y) = -\frac{1}{2\pi^2} \int_0^{\infty} \frac{d\tau}{\sqrt{1+\tau^2}}\, \text{P.V.} \left(\int_{\mathbb{R}} d\zeta \left(\frac{\mathbb{V}f(\zeta,\tau)}{(x+\tau y-\zeta)^2} + \frac{\mathbb{V}f(\zeta,\tau)}{(x-\tau y-\zeta)^2} \right) \right). \tag{9}$$

2.4. Filtered back-projection form of inversion

Equation (9) can be put also under the form of a filtered back-projection as in the case of the standard Radon transform

$$f(x,y) = \int_{\mathbb{R}^2} d\mathbf{k}\, |\mathbf{k}|\, e^{2i\pi \mathbf{k}\cdot\mathbf{r}} \int_{\mathbb{R}^2} d\mathbf{r}'\, e^{-2i\pi(\mathbf{r}'\cdot\mathbf{k})}$$

$$\left\{ \int_0^{\pi/2} d\omega \int_{\mathbb{R}} du\, \mathbb{V}f\left(\frac{u}{\cos\omega}, \frac{\sin\omega}{\cos\omega}\right) \left[\delta(u-(\mathbf{n}\cdot\mathbf{r}')) + \delta(u-(\mathbf{n}'\cdot\mathbf{r}'))\right] \right\}. \quad (10)$$

$\mathbf{n}$ and $\mathbf{n}'$ are unit vectors normal to the two branches of the V-line. The term in curly brackets is just the Barrett summation image of the V-line Radon transform but over a half angular range only with the same filtering as in Radon transform.

3. Numerical simulations

The V-line Radon transform arises from modeling two-dimensional emission imaging process by Compton scattered gamma rays. The image reconstruction is performed using the filtered back-projection inversion formula (10) of the V-line Radon transform. In figures (1,2,3) we show an original thyroid medical phantom, its V-line Radon transform data and its reconstruction from simulated data. Back-projection on V-lines generates more artifacts than back-projection on straight lines in standard two-dimensional Radon Transform, due to the existence of more spurious line intersections. In order to reduce these artifacts, the Hann filter may be used. Despite these limitations, the small structures in the object are clearly reconstructed. These results illustrate undoubtedly the feasibility of the new imaging modality.

4. Conclusion

In this paper, a novel class of Radon transform defined on a discontinuous line having the shape of a V letter is shown to be analytically invertible and the corresponding filtered back-projection inversion procedure is very much akin with that of the standard Radon transform. This formula is then used for numerical simulations of a medical thyroid phantom. The results show clearly the feasibility of reconstruction and suggest applications in many fields such as nuclear medicine or non-destructive material testing.

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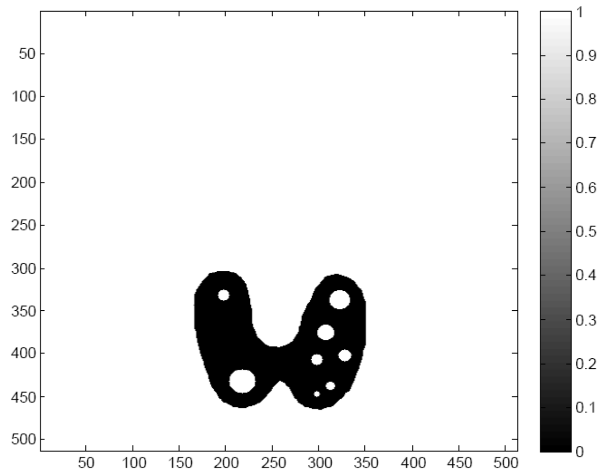

Figure 1. Original thyroid phantom.

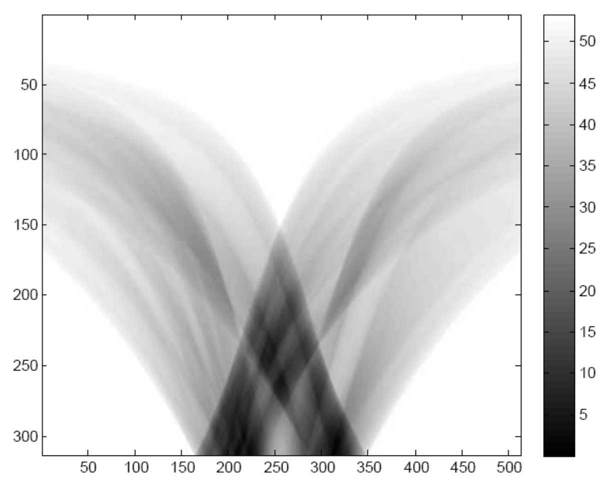

Figure 2. The V-line Radon transform of the thyroid with $d\omega = 0.005$ rad.

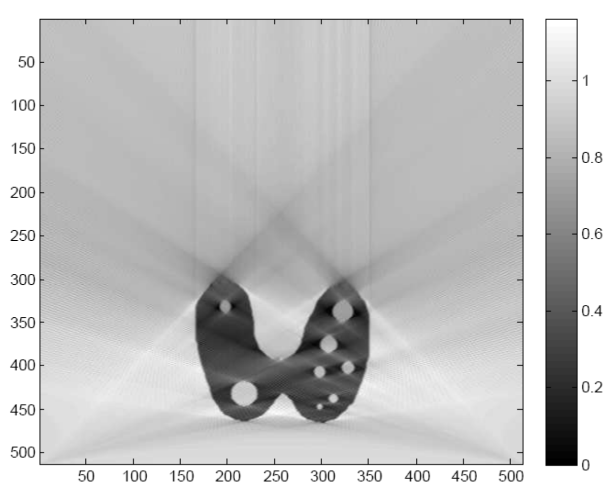

Figure 3. Filtered back-projection reconstruction with ($d\omega = 0.005$ rad).