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# The coordinate system $\mu \times \kappa$ on $L^2([0, 1]) \times L^2([0, 1])$ for the AKNS operator

L. Amour\*

## Abstract

A direct proof of the following result is given. The map  $\mu \times \kappa$  for the AKNS operator on the unit interval associated to the separated boundary conditions is one to one on  $L^2([0, 1]) \times L^2([0, 1])$ . The Schrödinger operators are considered in an appendix.

## 1 Introduction and statement of the result.

This short paper is concerned with the AKNS operator

$$\mathcal{H}(p, q) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} -q & p \\ p & q \end{pmatrix} \quad (1)$$

on  $[0, 1]$ . In (1) the potentials  $p$  and  $q$  are real-valued functions belonging to  $L^2([0, 1])$ .

Let us mention that  $\mathcal{H}(p, q)$  is the self-adjoint operator of the Lax pair associated to the one dimensional nonlinear cubic (defocusing) Schrödinger equation  $iu_t + u_{xx} - 2|u|^2u = 0$  (setting  $u = p + iq$ ). The operator  $\mathcal{H}(p, q)$  is also unitarily equivalent to the Zakharov-Shabat operator  $i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \varphi \\ \bar{\varphi} & 0 \end{pmatrix}$  (with  $\varphi = q - ip$ ). Moreover  $\mathcal{H}(p, q)$  is related to the first operator in the decomposition of the Dirac operator with a radial potential.

In this paper, the operator  $\mathcal{H}(p, q)$  is associated to the  $SL(\alpha, \beta)$  boundary conditions where  $(\alpha, \beta) \in [0, \pi]^2$ .

One says that  $F = \begin{pmatrix} Y \\ Z \end{pmatrix} \in H^1([0, 1]) \times H^1([0, 1])$  satisfies the  $SL(\alpha, \beta)$  boundary conditions if

$$\begin{cases} \cos \alpha Y(0) + \sin \alpha Z(0) = 0 \\ \cos \beta Y(1) + \sin \beta Z(1) = 0 \end{cases} \quad (2)$$

Fix  $(\alpha, \beta) \in [0, \pi]^2$ . The operator  $H(p, q)$  denotes the following self-adjoint operator

$$\begin{aligned} D(H(p, q)) &= \left\{ F = \begin{pmatrix} Y \\ Z \end{pmatrix} \in H^1([0, 1]) \times H^1([0, 1]) \text{ satisfying (2)} \right\} \\ H(p, q)F &= \mathcal{H}(p, q)F, \quad \forall F \in D(H(p, q)) \end{aligned}$$

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Let us recall the following points ([A]). For each  $(p, q) \in L^2([0, 1]) \times L^2([0, 1])$  the spectrum of  $H(p, q)$  denoted by  $\sigma(H(p, q))$  is a strictly increasing sequence of eigenvalues  $(\mu_k(p, q))_{k \in \mathbb{Z}}$ . Each eigenvalue is of multiplicity one. The asymptotic expansion is

$$(\mu_k(p, q) - k\pi - \alpha + \beta)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}). \quad (3)$$

For any complex number  $z$  and any  $(p, q) \in L^2([0, 1]) \times L^2([0, 1])$  the functions  $F_1(\cdot, z, p, q) = \begin{pmatrix} Y_1(\cdot, z, p, q) \\ Z_1(\cdot, z, p, q) \end{pmatrix}$  and  $F_2(\cdot, z, p, q) = \begin{pmatrix} Y_2(\cdot, z, p, q) \\ Z_2(\cdot, z, p, q) \end{pmatrix}$  denote the solutions to

$$\mathcal{H}(p, q)F = zF \quad (4)$$

on  $[0, 1]$  verifying the initial conditions

$$F_1(0, z, p, q) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad F_2(0, z, p, q) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In particular, any solution  $F = \begin{pmatrix} Y \\ Z \end{pmatrix}$  to (4) may be expressed as,

$$F(x) = Y(0)F_1(x, z, p, q) + Z(0)F_2(x, z, p, q), \quad \forall x \in [0, 1]. \quad (5)$$

Set

$$\begin{aligned} F_\alpha(x, z, p, q) &= \begin{pmatrix} Y_\alpha(x, z, p, q) \\ Z_\alpha(x, z, p, q) \end{pmatrix} \\ &= -\sin \alpha F_1(x, z, p, q) + \cos \alpha F_2(x, z, p, q). \end{aligned} \quad (6)$$

The  $\mu_k(p, q)$ 's are the simple zeros of the entire function

$$z \mapsto \cos \beta Y_\alpha(1, z, p, q) + \sin \beta Z_\alpha(1, z, p, q)$$

and  $F_\alpha(\cdot, \mu_k(p, q), p, q)$  is the eigenfunction (up to a non vanishing multiplicative factor) corresponding to  $\mu_k(p, q)$ . The sequence  $(\kappa_k(p, q))_{k \in \mathbb{Z}}$  is defined by

$$\kappa_k(p, q) = -\sin \beta Y_\alpha(1, z, p, q) + \cos \beta Z_\alpha(1, z, p, q), \quad \forall k \in \mathbb{Z}. \quad (7)$$

The maps  $\mu$  and  $\kappa$  are then defined on  $L^2([0, 1]) \times L^2([0, 1])$  by

$$\mu(p, q) = (\mu_k(p, q))_{k \in \mathbb{Z}}, \quad \kappa(p, q) = (\kappa_k(p, q))_{k \in \mathbb{Z}}. \quad (8)$$

It is known ([GG]) that  $\mu \times \kappa$  is one to one on  $H^1([0, 1]) \times H^1([0, 1])$ . The main result here is the following one.

**Theorem 1.1.** *The map  $\mu \times \kappa$  is one to one on  $L^2([0, 1]) \times L^2([0, 1])$ .*

According to Theorem 1.1 the map  $\mu \times \kappa$  may be viewed as a global coordinates system on  $L^2([0, 1]) \times L^2([0, 1])$ .

The goal here is to provide a quick and direct proof of Theorem 1.1. Note that this result certainly should be also proved starting from the Birkhoff coordinates ([GPT, Ch. III.8]) or from the action-angle coordinates. Also note that, although this result is stated in [GG], it is only proved in [GG] for potentials in  $H^1([0, 1]) \times H^1([0, 1])$  in order to follow the proof of [PT, Theorem 3.5] (see also [S] for singular AKNS operators with potentials in  $H^1([0, 1]) \times H^1([0, 1])$ ).

If one of the two parameters  $\alpha$  and  $\beta$  is not given then it is determined from (3). One may also replace the sequence  $(\kappa_k(p, q))_{k \in \mathbb{Z}}$  by the sequence of norming constants  $(\|F_\alpha(\cdot, \mu_k(p, q), p, q)\|_{L^2([0, 1]) \times L^2([0, 1])})_{k \in \mathbb{Z}}$ .

In order to derive Theorem 1.1 an entire function  $f$  is introduced in (12). This function is already used for the AKNS operator (see [DG]) in the proof of results like one spectrum together with the knowledge of the potentials on half of the interval uniquely determine the potentials on the whole interval (Hochstadt-Lieberman type's result). See also [L],[AR], ... in the case of the Schrödinger operator. The point of this paper is to remark that this function  $f$  still may be used to obtain Theorem 1.1.

More precisely, suppose that  $(p, q) \in L^2([0, 1]) \times L^2([0, 1])$ ,  $(\tilde{p}, \tilde{q}) \in L^2([0, 1]) \times L^2([0, 1])$  and define  $f$  by (12). The main step is to derive that  $f$  is entirely vanishing since the fact that  $f \equiv 0$  implies  $(p, q) = (\tilde{p}, \tilde{q})$  is provided by [DG]. To this end, the number of the zeros of  $f$  has to be sufficiently high. When  $\mu_k(p, q) = \mu_k(\tilde{p}, \tilde{q})$  for some  $k \in \mathbb{Z}$  then it is already known (see [DG]) that  $\mu_k(p, q)$  is a zero of  $f$ . The main fact is the following. If  $\mu_k(p, q) = \mu_k(\tilde{p}, \tilde{q})$  and  $\kappa_k(p, q) = \kappa_k(\tilde{p}, \tilde{q})$  for all  $k \in \mathbb{Z}$  then the  $\mu_k(p, q)$ 's becomes zeros of order at least *two* of  $f$  (Proposition 2.2).

In the case of Hochstadt-Lieberman type of results,  $(p, q) = (\tilde{p}, \tilde{q})$  on  $[\frac{1}{2}, 1]$  implies that  $f$  is of type 1 (at most) and the  $\mu_k(p, q)$ 's are zeros of  $f$ . In the case of Theorem 1.1,  $f$  is of type 2 (at most) and the  $\mu_k(p, q)$ 's are zeros of order (at least) *two* of  $f$ . That is to say, in the second case, the type and the number of zeros are multiplied by two. Actually one has to pay attention not only to the type of  $f$  but also to polynomial factors. Precisely, one has  $f(z) = O\left(\frac{e^{2|\Im z|}}{|\Im z|^{\frac{1}{2}}}\right)$  and the power  $\frac{1}{|\Im z|^{\frac{1}{2}}}$  is used in the proof.

It is proved in Proposition 2.3 that the  $\mu_k(p, q)$ 's being zeros of order 2 of  $f$  lead to a first estimate on  $N_f$  ( $N_f$  is a function related to the number of zeros of  $f$ , see (17)) and the type, the order and the power  $\frac{1}{|\Im z|^{\frac{1}{2}}}$  give rise in Proposition 2.4 using Jensen's Theorem to a second estimate on  $N_f$  when  $f \not\equiv 0$ . These two estimates on  $N_f$  are not compatible. Therefore  $f$  is entirely vanishing.

An appendix is concerned with Schrödinger operators associated with Dirichlet boundary conditions. It is noticed that the proof of Theorem 1.1 may be adapted to obtain a new proof of the fact that  $\mu \times \kappa$  is one to one on  $L^2([0, 1])$  ([PT]). In particular this proof does not rely on the existence of an involution (see the proof of [PT]) and may be also extended to  $L^1([0, 1])$ .

The next section is devoted to the proof of Theorem 1.1.

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## 2 Proof of Theorem 1.1

Let us begin the proof with the following remark.

**Remark 2.1.** *It is sufficient to prove Theorem 1.1 in the particular case  $(\alpha, \beta) = (0, 0)$ .*

Indeed, defining

$$G_{ab}(x) = \begin{pmatrix} \cos(ax + b) & \sin(ax + b) \\ -\sin(ax + b) & \cos(ax + b) \end{pmatrix}, \quad \forall x \in [0, 1]$$

and

$$g_{ab} : \begin{cases} L^2([0, 1]) \times L^2([0, 1]) & \longrightarrow & L^2([0, 1]) \times L^2([0, 1]) \\ \begin{pmatrix} u \\ v \end{pmatrix} & \longmapsto & G_{ab} \begin{pmatrix} u \\ v \end{pmatrix} \end{cases} \quad (9)$$

for  $(a, b) \in \mathbb{R}^2$ , it is checked by direct computations that

$$(i) \quad \mathcal{H}(p, q)F = zF \iff \mathcal{H}\left(\left(g_{2a, 2b} \begin{pmatrix} p \\ q \end{pmatrix}\right)^T\right)g_{ab}F = (z + a)g_{ab}F$$

(ii)  $F$  satisfies the  $SL(0, 0)$  boundary conditions if and only if  $g_{\alpha-\beta, -\alpha}F$  verifies the  $SL(\alpha, \beta)$  boundary conditions.

Suppose that  $(\alpha, \beta) \in [0, \pi]^2$  and  $(p, q) \in L^2([0, 1]) \times L^2([0, 1])$  are given and let us temporarily emphasize on the dependance on  $(\alpha, \beta)$  by writing  $\mu_k(\alpha, \beta, p, q)$  and  $\kappa_k(\alpha, \beta, p, q)$  instead of  $\mu_k(p, q)$  and  $\kappa_k(p, q)$  respectively. Set  $\begin{pmatrix} p^\sharp \\ q^\sharp \end{pmatrix} = g_{-2(\alpha-\beta), 2\alpha} \begin{pmatrix} p \\ q \end{pmatrix}$ . The above points (i)(ii) yield that  $\mu_k(\alpha, \beta, p, q) = \mu_k(0, 0, p^\sharp, q^\sharp) + \alpha - \beta$  and  $F_\alpha(\cdot, \mu_k(\alpha, \beta, p, q), p, q) = g_{\alpha-\beta, -\alpha}F_\alpha(\cdot, \mu_k(0, 0, p^\sharp, q^\sharp), p^\sharp, q^\sharp)$  since the two sides of the last equality are eigenfunction for the same eigenvalue and the same initial condition. Then (7) shows that  $\kappa_k(\alpha, \beta, p, q) = \kappa_k(0, 0, p^\sharp, q^\sharp)$ . Moreover,  $\|(p, q)\|_{L^2([0, 1]) \times L^2([0, 1])} = \|(p^\sharp, q^\sharp)\|_{L^2([0, 1]) \times L^2([0, 1])}$ . This explains Remark 2.1.

Let us mention that in the case  $(\alpha, \beta) = (0, 0)$  the  $\mu_k(p, q)$ 's are the zeros of the entire function  $z \mapsto Y_2(1, z, p, q)$ ,  $F_2(\cdot, \mu_k(p, q), p, q)$  is the eigenfunction corresponding to the eigenvalue  $\mu_k(p, q)$  (up to a non zero multiplicative factor) and  $\kappa_k(p, q) = Z_2(1, \mu_k(p, q), p, q)$  for all  $k \in \mathbb{Z}$ . The asymptotic expansions of the quantities used in the sequel are (see [GKP, Prop. I.3])

$$\begin{aligned} Y_2(x, z, p, q) &= \sin zx + o(e^{|\Im z|x}) \\ Z_2(x, z, p, q) &= \cos zx + o(e^{|\Im z|x}) \end{aligned} \quad (10)$$

as  $|z| \rightarrow +\infty$  uniformly in  $x \in [0, 1]$  for fixed  $(p, q)$  and (see (3))

$$(\mu_k(p, q) - k\pi)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}). \quad (11)$$

Fix  $(p, q) \in L^2([0, 1]) \times L^2([0, 1])$  and  $(\tilde{p}, \tilde{q}) \in L^2([0, 1]) \times L^2([0, 1])$ . According to Remark 2.1 it is supposed from now on that  $(\alpha, \beta) = (0, 0)$ . It shall be proved that if  $\mu_k(p, q) = \mu_k(\tilde{p}, \tilde{q})$  and  $\kappa_k(p, q) = \kappa_k(\tilde{p}, \tilde{q})$  for all  $k \in \mathbb{Z}$  then  $(p, q) = (\tilde{p}, \tilde{q})$ .

Define the entire function (see [DG]),

$$f(z) = \int_0^1 \{Y_2(x, z, p, q)Z_2(x, z, \tilde{p}, \tilde{q}) + Z_2(x, z, p, q)Y_2(x, z, \tilde{p}, \tilde{q})\} (p(x) - \tilde{p}(x)) + \{Z_2(x, z, p, q)Z_2(x, z, \tilde{p}, \tilde{q}) - Y_2(x, z, p, q)Y_2(x, z, \tilde{p}, \tilde{q})\} (q(x) - \tilde{q}(x)) dx \quad (12)$$

for all  $z \in \mathbb{C}$ .

**Proposition 2.2.** (i) Suppose that  $\mu_k(p, q) = \mu_k(\tilde{p}, \tilde{q})$  for all  $k \in \mathbb{Z}$ . Then the  $\mu_k(p, q)$ 's are zeros of the function  $f$ .

(ii) If in addition  $\kappa_k(p, q) = \kappa_k(\tilde{p}, \tilde{q})$  for all  $k \in \mathbb{Z}$  then the  $\mu_k(p, q)$ 's are zeros of order at least two of  $f$ .

It is actually proved that if  $\mu_k(p, q) = \mu_k(\tilde{p}, \tilde{q})$  for all  $k \in \mathbb{Z}$  and if  $\kappa_\ell(p, q) = \kappa_\ell(\tilde{p}, \tilde{q})$  for some  $\ell \in \mathbb{Z}$  then  $\mu_\ell(p, q)$  is a zero of order at least two of  $f$ .

*Proof of Proposition 2.2:* For  $F = \begin{pmatrix} Y \\ Z \end{pmatrix}$  and  $\tilde{F} = \begin{pmatrix} \tilde{Y} \\ \tilde{Z} \end{pmatrix}$  in  $L^2([0, 1]) \times L^2([0, 1])$  set  $\text{wronskian}(F, \tilde{F}) := Y\tilde{Z} - Z\tilde{Y}$ . Part (i) of Proposition 2.2 is already observed in [DG] and follows from

$$z \in \sigma(H(p, q)) \cap \sigma(H(\tilde{p}, \tilde{q})) \implies f(z) = -\text{wronskian}(F_2(x, z, p, q), F_2(x, z, \tilde{p}, \tilde{q})) \Big|_{x=0}^{x=1} = 0. \quad (13)$$

(ii) The symbol  $\dot{\cdot}$  denotes the abbreviated notation  $\frac{\partial}{\partial z}$ . By direct computations,

$$\dot{f}(z) = \dot{Z}_2(x, z, p, q)Y_2(x, z, \tilde{p}, \tilde{q}) + Z_2(x, z, p, q)\dot{Y}_2(x, z, \tilde{p}, \tilde{q}) - \dot{Y}_2(x, z, p, q)Z_2(x, z, \tilde{p}, \tilde{q}) - Y_2(x, z, p, q)\dot{Z}_2(x, z, \tilde{p}, \tilde{q}) \Big|_{x=0}^{x=1}, \quad (14)$$

then

$$\dot{f}(\mu_\ell(p, q)) = Z_2(1, \mu_\ell(p, q), p, q)\dot{Y}_2(1, \mu_\ell(p, q), \tilde{p}, \tilde{q}) - \dot{Y}_2(1, \mu_\ell(p, q), p, q)Z_2(1, \mu_\ell(p, q), \tilde{p}, \tilde{q}), \quad (15)$$

for all  $\ell \in \mathbb{Z}$ . Equality (15) follows from  $Y_2(0, z, p, q) = 0$  for all  $z \in \mathbb{C}$  implying  $\dot{Y}_2(0, z, p, q) = 0$  for all  $z \in \mathbb{C}$  and from  $Y_2(1, \mu_k(p, q), p, q) = 0$  for all  $k \in \mathbb{Z}$ .

Furthermore, using infinite products expansions

$$Y_2(1, z, p, q) = (z - \mu_0(p, q)) \prod_{k \in \mathbb{Z}} \frac{\mu_k(p, q) - z}{k\pi}, \quad (16)$$

for all  $z \in \mathbb{C}$ . In (16),  $\prod_{k \in \mathbb{Z}}$  means  $\lim_{n \rightarrow +\infty} \prod_{|k| \leq n}$ . Equality (16) comes from the estimates  $(z - \mu_0(p, q)) \prod_{k \in \mathbb{Z}} \frac{\mu_k(p, q) - z}{k\pi} = \sin z(1 + o(1))$  uniformly on  $\Gamma_n = \{z \in \mathbb{C} \mid |z| = (n + \frac{1}{2})\pi\}$  as  $n \rightarrow +\infty$  ([DG, Lemma 5], see also [GKP]) together with  $Y_2(1, z, p, q) = \sin z + o(e^{|\Im z|})$  as  $|z| \rightarrow +\infty$  (see (10)),  $|\sin z| > \frac{e^{|\Im z|}}{4}$  on  $\Gamma_n$  and the maximum modulus principle.

In particular (16) yields that for each  $z \in \mathbb{C}$ , the sequence  $(\mu_k(p, q))_{k \in \mathbb{Z}}$  determines  $Y_2(1, z, p, q)$  and also  $\dot{Y}_2(1, z, p, q)$ . Thus, according to (15), if  $\mu_k(p, q) = \mu_k(\tilde{p}, \tilde{q})$  for all  $k \in \mathbb{Z}$  and if  $\kappa_\ell(p, q) = \kappa_\ell(\tilde{p}, \tilde{q})$  for some  $\ell \in \mathbb{Z}$  then  $\dot{f}(\mu_\ell(p, q)) = 0$ . This completes the proof of (ii).  $\square$

For any  $t > 0$  let  $n_f(t)$  denotes the number of zeros counted with their multiplicity of the function  $f$  inside the closed ball of radius  $t$  and centered at the origin.

Set

$$N_f(R) = \int_0^R \frac{n_f(t)}{t} dt, \quad (17)$$

for any  $R > 0$ .

On one side it is derived that

**Proposition 2.3.** *Let  $R_j = \min(-\mu_{-j}(p, q), \mu_j(p, q))$ ,  $\forall j \in \mathbb{N}$ . Then the sequence  $(N_f(R_j) - \frac{4}{\pi}R_j)_{j \in \mathbb{N}}$  is bounded from below.*

*Proof of Proposition 2.3:* Define  $r_j = \max(-\mu_{-j}(p, q), \mu_j(p, q))$ ,  $\forall j \in \mathbb{N}$ . Let  $k_0 \in \mathbb{N}$  be such that (see (11))  $R_k > 0$  for all  $k \geq k_0$ . According to Proposition 2.2

$$\begin{aligned} n_f(t) &\geq 4k, & \forall t \in [R_k, r_k[ \\ n_f(t) &\geq 4k + 2, & \forall t \in [r_k, R_{k+1}[, \quad \forall k \geq k_0 \end{aligned} \quad (18)$$

in such a way that

$$\begin{aligned} N_f(R_j) &\geq \sum_{k=k_0}^{j-1} \int_{R_k}^{r_k} \frac{4k}{t} dt + \sum_{k=k_0}^{j-1} \int_{r_k}^{R_{k+1}} \frac{4k+2}{t} dt + O(1) \\ &\geq - \sum_{k=k_0}^{j-1} \int_{R_k}^{r_k} \frac{2}{t} dt + \sum_{k=k_0}^{j-1} \int_{R_k}^{R_{k+1}} \frac{4k+2}{t} dt + O(1), \end{aligned} \quad (19)$$

as  $j \rightarrow +\infty$ . From (11),  $(r_k - k\pi) \in \ell^2(\mathbb{N})$ ,  $(R_k - k\pi) \in \ell^2(\mathbb{N})$  and therefore

$$\sum_{k=k_0}^{j-1} \ln \frac{r_k}{R_k} = O(1), \text{ as } j \rightarrow +\infty. \quad (20)$$

Writing,

$$(4k+2)(\ln R_{k+1} - \ln R_k) = (4(k+1)-2) \ln R_{k+1} - (4k-2) \ln R_k - 4 \ln R_k$$

it is clear that

$$\sum_{k=k_0}^{j-1} (4k+2)(\ln R_{k+1} - \ln R_k) = (4j-2) \ln R_j - 4 \sum_{k=k_0}^{j-1} \ln R_k + O(1), \text{ as } j \rightarrow +\infty. \quad (21)$$

That is to say, (19)(20)(21) shows

$$N_f(R_j) \geq (4j-2) \ln R_j - 4 \sum_{k=k_0}^{j-1} \ln R_k + O(1), \text{ as } j \rightarrow +\infty. \quad (22)$$

From (11),  $R_j = j\pi + s_j$  for all  $j \in \mathbb{N}$  with  $(s_j)_{j \in \mathbb{N}} \in \ell^2(\mathbb{N})$ .

First, using  $\ln j! = (j + \frac{1}{2}) \ln j - j + O(1)$  and  $\sum_{k=k_0}^{j-1} \ln(1 + \frac{s_k}{k\pi}) = O(1)$ , it is deduced that

$$\sum_{k=k_0}^{j-1} \ln R_k = j \ln \pi + (j - \frac{1}{2}) \ln j - j + O(1). \quad (23)$$

Next,

$$\ln R_j = \ln j + O(1), \quad j \ln R_j = j \ln j + j \ln \pi + O(1), \text{ as } j \rightarrow +\infty. \quad (24)$$

Consequently, (22)(23)(24) yield

$$\begin{aligned} N_f(R_j) &\geq 4j + O(1) \\ &\geq \frac{4}{\pi} R_j + O(1), \text{ as } j \rightarrow +\infty \end{aligned}$$

since the terms behaving like  $\ln j$  and  $j \ln j$  disappear. This proves Proposition 2.3.  $\square$

On the other side, it is proved that

**Proposition 2.4.** *If the function  $f$  is non vanishing then  $\lim_{R \rightarrow +\infty} N_f(R) - \frac{4}{\pi} R = -\infty$ .*

*Proof of Proposition 2.4:* According to  $Y_2(x, z, p, q) = \sin zx + O(e^{2|\Im z|x})$  and  $Z_2(x, z, p, q) = \cos zx + O(e^{2|\Im z|x})$ ,

$$f(z) = \int_0^1 (\sin 2zx + O(e^{2|\Im z|x})) (p(x) - \tilde{p}(x)) + (\cos 2zx + O(e^{2|\Im z|x})) (q(x) - \tilde{q}(x)) dx. \quad (25)$$

In particular, one deduces using Cauchy-Schwarz's inequality that

$$f(z) = O\left(\frac{e^{2|\Im z|}}{|\Im z|^{\frac{1}{2}}}\|(p - \tilde{p}, q - \tilde{q})\|_{L^2([0,1]) \times L^2([0,1])}\right). \quad (26)$$

First suppose that  $f(0) \neq 0$ . Then Jensen's Theorem for the function  $f$  may be written as

$$N_f(R) = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(Re^{i\theta})| d\theta. \quad (27)$$



Therefore (27) with (26) imply (see [L] for a similar computation) that the following equality is satisfied

$$N_f(R) = \frac{4}{\pi}R - \frac{1}{2} \ln R + O(1), \quad \text{as } R \rightarrow +\infty. \quad (28)$$

Next assume that  $f$  has a zero of order  $m$  at the origin. For  $t > 0$  define  $n_f^*(t)$  as the number of zeros of  $f$  in  $\{z \in \mathbb{C} \mid 0 < |z| \leq t\}$  and for  $R > 0$  set  $N_f^*(R) = \int_0^R \frac{n_f^*(t)}{t} dt$ . In that case, Jensen's Theorem reads as

$$N_f^*(R) = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(Re^{i\theta})| d\theta - m \ln R - \ln \frac{f^{(m)}(0)}{m}$$

implying

$$N_f^*(R) = \frac{4}{\pi}R - \left(\frac{1}{2} + m\right) \ln R + O(1), \quad \text{as } R \rightarrow +\infty. \quad (29)$$

Since  $n_f(t) = n_f^*(t) + m$  then  $N_f(t) = N_f^*(t) + m \ln R + O(1)$  and (28) stays valid in that case. The proof of Proposition 2.4 is finished.  $\square$

Proposition 2.3 and Proposition 2.4 imply that  $f$  is entirely vanishing on  $\mathbb{C}$ . This yields using the same argument as [DG] that  $(p, q) = (\tilde{p}, \tilde{q})$ . The proof of Theorem 1.1 is complete.

### 3 Appendix

Let  $H(q) = -\frac{d^2}{dx^2} + q$  on  $[0, 1]$  associated with the Dirichlet boundary conditions  $y(0) = y(1) = 0$  with  $q \in L^1([0, 1])$ . Let us recall the following facts ([LG],[PT], ...). Set  $y_2(\cdot, z, q)$  the solution to  $H(q)y = zy$  satisfying  $y(0) = 0$  and  $y'(0) = 1$ . It is known that  $y_2(x, z, q) = \frac{\sin \sqrt{z}}{\sqrt{z}} + O\left(\frac{e^{|\Im \sqrt{z}|x}}{z}\right)$ . The spectrum of  $H(q)$  is a sequence of strictly increasing simple eigenvalues  $(\mu_n(q))_{n \geq 1}$  satisfying  $\mu_n(q) = n^2\pi^2 + \int_0^1 q(x)dx + o(1)$  as  $n \rightarrow +\infty$ . The  $\mu_n(q)$ 's are the zeros of the entire function  $y_2(1, \cdot, q)$  and for each  $n \geq 1$ ,  $y_2(\cdot, \mu_n(q), q)$  is an eigenfunction corresponding to  $\mu_n(q)$ . Let  $\kappa_n(q) = y_2'(1, \mu_n(q), q)$ ,  $\forall n \geq 1$  and define  $\mu(q) = (\mu_n(q))_{n \geq 1}$ ,  $\kappa(q) = (\kappa_n(q))_{n \geq 1}$ .

The purpose of this appendix is to provide another quick proof (similar to the one of Theorem 1.1 for the AKNS operator) of the following result. In particular, our proof do not rely on the existence of an involution.

**Theorem 3.1.** ([PT]) *The map  $\mu \times \kappa$  is one to one on  $L^2([0, 1])$ .*

**Remark 3.2.** *Actually we can prove here  $\mu \times \kappa$  is one to one on  $L^1([0, 1])$ .*

*Proof :* First the proof in the  $L^2([0, 1])$  is given since it is similar to the case of AKNS operators. Let  $q \in L^2([0, 1])$ ,  $\tilde{q} \in L^2([0, 1])$  and suppose that the  $\mu_n(q)$ 's and the  $\mu_n(\tilde{q})$ 's are strictly positive (since the map  $q \mapsto q + C$  translates the whole spectrum). As in (12) define

$$f(z) = \int_0^1 y_2(x, z^2, q)y_2(x, z^2, q)(q(x) - \tilde{q}(x))dx, \quad \forall z \in \mathbb{C}.$$

Suppose that  $(\mu \times \kappa)(q) = (\mu \times \kappa)(\tilde{q})$ . Similarly to Proposition 2.2, it is obtained that the  $\pm\sqrt{\mu_n(q)}$ 's are zeros of order at least two of the entire function  $f$ , implying that  $n_f(t) \geq 4k$  for all  $t \in [\sqrt{\mu_k(q)}, \sqrt{\mu_{k+1}(q)}[$  ( $2k$  since  $\sqrt{\mu_n(q)}$  and  $-\sqrt{\mu_n(q)}$  are zeros, and  $2k$  since these zeros are order of at least 2). Set  $R_j = \sqrt{\mu_j(q)}$ ,  $\forall j \geq 1$ . It is then deduced analogously to Proposition 2.3 that  $N_f(R_j) \geq \sum_{k=1}^j \int_{R_k}^{R_{k+1}} \frac{4k}{t} dt$  implying

$$\left( N_f(R_j) - \frac{4}{\pi} R_j + 2 \ln R_j \right)_{j \geq 1} \text{ is bounded from below.} \quad (30)$$

Besides, the counterpart to Proposition 2.3 is written as follows. One has  $y_2(x, z^2, q)y_2(x, z^2, \tilde{q}) = O\left(\frac{e^{2|\Im z|x}}{|z|^2}\right)$  and the Cauchy-Schwarz inequality provides an additional  $|z|^{\frac{1}{2}}$  in the estimates for  $f$ . Namely,

$$f(z) = O\left(\frac{e^{2|\Im z|x}}{|z|^{\frac{5}{2}}}\|q - \tilde{q}\|_{L^2([0,1])}\right), \quad (31)$$

implying (Jensen's Theorem), if  $f \not\equiv 0$ , that

$$N_f(R) - \frac{4}{\pi}R + \frac{5}{2} \ln R = O(1), \text{ as } R \rightarrow +\infty. \quad (32)$$

Therefore (30)(32) yields  $f \equiv 0$  implying (see [L] or [AR]) that  $q = \tilde{q}$ . The proof of Theorem 3.1 is then complete in the  $L^2([0, 1])$  case.

In the case  $q \in L^1([0, 1])$  and  $\tilde{q} \in L^1([0, 1])$  the proof is modified as follows. Equality (31) becomes (instead of applying Cauchy-Schwarz inequality an argument is borrowed to [L])

$$|f(z)| \leq C \frac{e^{2|\Im z|x}}{|z|^2} (e^{-\varepsilon|\Im z|} + \delta_\varepsilon), \quad (33)$$

for all  $\varepsilon > 0$ , where  $\delta_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and for some  $C > 0$ . Note that that  $C$  and  $\delta_\varepsilon$  depends only on  $\|q - \tilde{q}\|_{L^1([0,1])}$ . The estimate (33) gives rise to (Jensen's Theorem, see [L])

$$\lim_{R \rightarrow +\infty} N_f(R) - \frac{4}{\pi}R + 2 \ln R = -\infty \quad (34)$$

if  $f \neq 0$ . Since (30) is still valid for  $L^1$  potentials then (30)(34) prove that  $f \equiv 0$  and  $q = \tilde{q}$  as before. The proofs of Theorem 3.1 and Remark 3.2 are finished.  $\square$

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