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The Brown-Golasiński model structure on strict ∞-groupoids revisited

Dimitri Ara and François Métayer

Abstract

We prove that the folk model structure on strict ∞-categories transfers to the category of strict ∞-groupoids (and more generally to the category of strict (∞, n)-categories), and that the resulting model structure on strict ∞-groupoids coincides with the one defined by Brown and Golasiński via crossed complexes.

Introduction

In [4], Brown and Golasiński build a model structure on the category of crossed complexes. In an earlier work [5], Brown and Higgins established an equivalence of categories between crossed complexes and strict ∞-groupoids, whence a model structure on the latter category.

On the other hand, there is a "folk" model structure on strict ∞-categories recently discovered by Lafont, Worytkiewicz and the second author [10], which extends previously known model structures on categories [8] and 2-categories [9]. Note that this model structure is also known as the "natural" or the "categorical" model structure by various authors.

This immediately raised the questions whether the folk model structure on ∞-categories may be transferred to ∞-groupoids by inclusion, and in that case whether the Brown-Golasiński model structure may be recovered this way.

The purpose of the present work is to show that both questions have affirmative answers.

Our paper is organized as follows. In Section 1, we recall the basic definitions of strict ∞-categories and strict ∞-groupoids. Then, in Section 2, we describe the Brown-Golasiński model structure on crossed complexes and ∞-groupoids. In Section 3, we prove the transfer theorem leading to the definition of the folk model structure on strict ∞-groupoids. Our proof uses neither crossed complexes nor the existence of the Brown-Golasiński model structure. Moreover, it also applies to the category of strict (∞, n)-categories for a fixed n (that is ∞-categories such that every m-arrow is invertible for

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$m > n$). Finally, in Section 4, we show that the two model structures previously defined on ∞-groupoids are in fact the same.

We thank Ronald Brown and Urs Schreiber, whose questions motivated us to go further in this work.

1 Strict ∞-groupoids

The purpose of this section is to introduce the definitions and notations about strict ∞-groupoids and their weak equivalences that we will use in the sequel of this paper. Our presentation is essentially the same as the one given in [2].

1.1 Globular sets

Let us denote by $\mathbf{O}$ the *globular category*, that is the category generated by the graph

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} \cdots \underset{\tau_{i-1}}{\overset{\sigma_{i-1}}{\rightrightarrows}} i \underset{\tau_i}{\overset{\sigma_i}{\rightrightarrows}} i+1 \underset{\tau_{i+1}}{\overset{\sigma_{i+1}}{\rightrightarrows}} \cdots$$

under the coglobular relations

$$\sigma_{i+1}\sigma_i = \tau_{i+1}\sigma_i \quad \text{and} \quad \sigma_{i+1}\tau_i = \tau_{i+1}\tau_i, \qquad i \geq 0.$$

A *globular set* or ∞*-graph* is a presheaf on $\mathbf{O}$. A globular set X amounts to a diagram of sets

$$\cdots \underset{t_{n+1}}{\overset{s_{n+1}}{\rightrightarrows}} X_{n+1} \underset{t_n}{\overset{s_n}{\rightrightarrows}} X_n \underset{t_{n-1}}{\overset{s_{n-1}}{\rightrightarrows}} \cdots \underset{t_1}{\overset{s_1}{\rightrightarrows}} X_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} X_0$$

satisfying the globular relations

$$s_i s_{i+1} = s_i t_{i+1} \quad \text{and} \quad t_i s_{i+1} = t_i t_{i+1}, \qquad i \geq 0.$$

For $i \geq j \geq 0$, we will denote by s^i_j and t^i_j the maps from X_i to X_j defined by

$$s^i_j = s_j \cdots s_{i-2} s_{i-1} \quad \text{and} \quad t^i_j = t_j \cdots t_{i-2} t_{i-1}.$$

If X is a globular set, we will call X_0 the set of *objects* of X and X_n for $n \geq 0$ the set of n*-arrows* or n*-cells*. The notation $u\colon x \to y$ will mean that u is an n-arrow for $n \geq 1$ whose source is an $(n-1)$-arrow x (that is $s_{n-1}(u) = x$) and whose target is an $(n-1)$-arrow y (that is $t_{n-1}(u) = y$). We will say that two n-arrows u and v are *parallel* if either $n = 0$, or $n \geq 1$ and u, v have same source and same target. For $i \geq j \geq 0$, if u is an i-arrow, we will often write $s_j(u)$ for $s^i_j(u)$ and similarly $t_j(u)$ for $t^i_j(u)$.

If u and v are n-arrows, $X(u, v)$ will denote the globular set whose k-arrows are the $(n+k+1)$-arrows a of G such that $s_n(a) = u$ and $t_n(a) = v$. In particular, $X(u, v)_0$ is the set of $(n+1)$-arrows $a\colon u \to v$ in X.

1.2 Strict ∞-categories

An ∞-*precategory* is a globular set C endowed with maps

$$\begin{aligned} &*^i_j \colon (X_i, s^i_j) \times_{X_j} (t^i_j, X_i) \to X_i, \quad i > j \geq 0, \\ &k_i \colon X_{i-1} \to X_i, \quad i \geq 1, \end{aligned}$$

such that

1. for every (u, v) in $(X_i, s^i_j) \times_{X_j} (t^i_j, X_i)$ with $i > j \geq 0$, we have

$$s_{i-1}(u *^i_j v) = \begin{cases} s_{i-1}(v), & j = i-1 \\ s_{i-1}(u) *^{i-1}_j s_{i-1}(v), & j < i-1 \end{cases};$$

2. for every (u, v) in $(X_i, s^i_j) \times_{X_j} (t^i_j, X_i)$ with $i > j \geq 0$, we have

$$t_{i-1}(u *^i_j v) = \begin{cases} t_{i-1}(u), & j = i-1 \\ t_{i-1}(u) *^{i-1}_j t_{i-1}(v), & j < i-1 \end{cases};$$

3. for every u in X_i with $i \geq 0$, we have

$$s_i k_{i+1}(u) = u = t_i k_{i+1}(u).$$

For $i \geq j \geq 0$, we will denote by k^j_i the map from $X_j \to X_i$ defined by

$$k^j_i = k_i \cdots k_{j+2} k_{j+1}.$$

If u and v are n-arrows for $n \geq 1$ of an ∞-precategory, we will often write $u *_k v$ for $u *^n_k v$. If u is an n-arrow, we will often write 1_u for the iterated identity $k^n_m(u)$ in a dimension $m \geq n$ clear by the context.

Definition 1.1. *A* strict ∞-category *is an ∞-precategory X such that the following axioms are satisfied:*

1. *Associativity*
   for every (u, v, w) in $(X_i, s^i_j) \times_{X_j} (t^i_j, X_i, s^i_j) \times_{X_j} (t^i_j, X_i)$ with $i > j \geq 0$, we have

$$(u *_j v) *_j w = u *_j (v *_j w);$$

2. *Exchange law*
   for every (u, u', v, v') in

$$(X_i, s^i_j) \times_{X_j} (t^i_j, X_i, s^i_k) \times_{X_k} (t^i_k, X_i, s^i_j) \times_{X_j} (t^i_j, X_i),$$

   with $i > j > k \geq 0$, we have

$$(u *_j u') *_k (v *_j v') = (u *_k v) *_j (u' *_k v');$$

3. *Units*
   for every u in X_i with $i \geq 1$ and every j such that $i > j \geq 0$, we have

$$u *_j 1_{s_j(u)} = u = 1_{t_j(u)} *_j u;$$

4. *Functoriality of units*
   for every (u, v) in $(X_i, s_j^i) \times_{X_j} (t_j^i, X_i)$ with $i > j \geq 0$, we have the following equality between $(i+1)$-arrows:

$$1_{u *_j v} = 1_u *_j 1_v.$$

A morphism of strict ∞-categories *or* ∞-functor *is a morphism of globular sets compatible with the maps $*_j^i$ and k_i.*

We will denote by ∞-**Cat** the category of ∞-categories. This category is a full reflexive subcategory of the presheaf category of globular sets. Moreover, it is stable under filtered colimits. Hence, by Theorem 1.46 of [1], ∞-**Cat** is locally presentable.

Note that if u and v are two n-arrows of a strict ∞-category C, the globular set $C(u, v)$ inherits a structure of strict ∞-category.

1.3 Strict ∞-groupoids

Let C be a strict ∞-category and u an i-arrow for $i \geq 1$. For j such that $0 \leq j < i$, a $*_j^i$*-inverse* v of u is an i-arrow such that $s_j(v) = t_j(u)$ and $t_j(v) = s_j(u)$, satisfying

$$u *_j v = 1_{t_j(u)} \quad \text{and} \quad v *_j u = 1_{s_j(u)}.$$

It is easy to see that if it exists, such an inverse is unique. For $i > j \geq 0$, we will say that C *admits $*_j^i$-inverses* if every i-arrow of C admits a $*_j^i$-inverse.

Definition 1.2. *A* strict ∞-groupoid *is a strict ∞-category which admits $*_j^i$-inverses for every $i > j \geq 0$. We will denote by ∞-**Grp** the full subcategory of ∞-**Cat** whose objects are strict ∞-groupoids.*

Let $n \geq 0$. A strict (∞, n)-category *is a strict ∞-category which admits $*_j^i$-inverses for every $i > j \geq n$. We will denote by (∞, n)-**Cat** the full subcategory of ∞-**Cat** whose objects are strict (∞, n)-categories. Note that for $n = 0$ we recover the category of strict ∞-groupoids.*

The same argument as for ∞-**Cat** shows that ∞-**Grp** is a locally presentable category.

If G is a strict ∞-groupoid and u is an i-arrow of G for $i \geq 1$, we will denote by $w_j^i(u)$ or simply by $w_j(u)$ the $*_j^i$-inverse of u and by u^{-1} the $*_{i-1}^i$-inverse. Note that if u and v are two n-arrows of a strict ∞-groupoid G, the strict ∞-category $G(u, v)$ is a strict ∞-groupoid.

Proposition 1.3. *Let C be a strict ∞-category. Then the following assertions are equivalent:*

1. *C is a strict ∞-groupoid;*

2. *C admits $*^i_{i-1}$-inverses for every $i \geq 1$;*

3. *C admits $*^i_0$-inverses for every $i \geq 1$;*

4. *for all $i \geq 1$, there exists j satisfying $0 \leq j < i$ such that C admits $*^i_j$-inverses.*

Proof. By induction, it suffices to show that for every $i > j > k \geq 0$, if C admits $*^j_k$-inverses, then C admits $*^i_k$-inverses if and only if it admits $*^i_j$-inverses. By using the fact that the 2-graph

$$C_i \underset{t^i_j}{\overset{s^i_j}{\rightrightarrows}} C_j \underset{t^j_k}{\overset{s^j_k}{\rightrightarrows}} C_k$$

has a natural structure of 2-category, one can assume that $k = 0$, $j = 1$ and $i = 2$. The result is thus a consequence of the following lemma. □

Lemma 1.4. *Let C be a 2-category whose 1-arrows are invertible. Then a 2-arrow is invertible for horizontal composition (i.e., $*^2_0$) if and only if it is invertible for vertical composition (i.e., $*^2_1$).*

Proof. Let $a\colon u \to v$ be a 2-arrow. Suppose a admits a horizontal inverse a^*. Then $v *_0 a^* *_0 u$ is a vertical inverse. Conversely, suppose a admits a vertical inverse a^{-1}. Then $v^{-1} *_0 a^{-1} *_0 u^{-1}$ is a horizontal inverse. □

1.4 Weak equivalences of strict ∞-groupoids

Let G be a strict ∞-groupoid. An n-arrow u of G is *homotopic* to another n-arrow v if there exists an $(n+1)$-arrow from u to v. This implies that the arrows u and v are parallel. If u is homotopic to v, we will write $u \sim v$. The relation $\sim$ is an equivalence relation on G_n: the properties with respect to source and target of the maps k_{n+1}, w^n_{n-1} and $*^n_{n-1}$ imply respectively that $\sim$ is reflexive, symmetric and transitive.

Let us denote by $\overline{G_n}$ the quotient of G_n by $\sim$. The composition

$$*^n_{n-1}\colon G_n \times_{G_{n-1}} G_n \to G_n$$

induces a map

$$*^n_{n-1}\colon \overline{G_n} \times_{G_{n-1}} \overline{G_n} \to \overline{G_n},$$

thanks to the properties with respect to source and target of the composition $*^{n+1}_{n-1}$. For $n \geq 1$, we can thus define a groupoid $\varpi_n(G)$ whose objects are $(n-1)$-arrows of G and whose morphisms are elements of $\overline{G_n}$. It is clear that ϖ_n defines a functor from the category of strict ∞-groupoids to the category of groupoids.

Definition 1.5. *The set of* connected components *of G is*

$$\pi_0(G) = \pi_0(\varpi_1(G)) = \overline{G_0}.$$

For $n \geq 1$ and x an object of G, the n-th homotopy group of G at x *is*

$$\pi_n(G,x) = \pi_1(\varpi_n(G), 1_x) = \mathsf{Aut}_{\varpi_n(G)}(1_x).$$

By functoriality of the ϖ_n's, π_0 induces a functor from the category of strict ∞-groupoids to the category of sets, and π_n, for $n \geq 1$, induces a functor from the category of pointed strict ∞-groupoids to the category of groups. By the Eckmann-Hilton argument, the groups $\pi_n(G,x)$ are abelian for $n \geq 2$. More generally, if u and v are two $(n-1)$-arrows for $n \geq 1$ we set

$$\pi_n(G,u,v) = \mathsf{Hom}_{\varpi_n(G)}(u,v) \quad \text{and} \quad \pi_n(G,u) = \pi_n(G,u,u).$$

Definition 1.6. *A morphism $f\colon G \to H$ of strict ∞-groupoids is a* weak equivalence of strict ∞-groupoids *if the map $\pi_0(f)\colon \pi_0(G) \to \pi_0(H)$ is a bijection and if for all $n \geq 1$ and all object x of G, the morphism $\pi_n(f,x)\colon \pi_n(G,x) \to \pi_n(H,f(x))$ is a group isomorphism. We will denote by $\mathcal{W}_{\mathrm{gr}}$ the class of such weak equivalences.*

Proposition 1.7. *Let $f\colon G \to H$ be a morphism of strict ∞-groupoids. The following conditions are equivalent:*

1. *f is a weak equivalence of strict ∞-groupoids;*
2. *$\pi_0(f)\colon \pi_0(G) \to \pi_0(H)$ is a bijection and for all $n \geq 1$ and every $(n-1)$-arrow u of G, f induces a bijection*
$$\pi_n(G,u) \to \pi_n(H,f(u));$$
3. *$\varpi_1(f)\colon \varpi_1(G) \to \varpi_1(H)$ is an equivalence of categories and for all $n \geq 2$ and every pair (u,v) of parallel $(n-1)$-arrows of G, f induces a bijection*
$$\pi_n(G,u,v) \to \pi_n(H,f(u),f(v));$$
4. *$\varpi_1(f)\colon \varpi_1(G) \to \varpi_1(H)$ is full and essentially surjective, and for all $n \geq 2$ and every pair (u,v) of parallel $(n-1)$-arrows of G, f induces a surjection*
$$\pi_n(G,u,v) \to \pi_n(H,f(u),f(v)).$$

Proof. $1 \Rightarrow 2)$ The case $n = 1$ is obvious. Let $n \geq 2$ and let u be an $(n-1)$-arrow of G. Set $x = s_0(u)$. The map

$$\pi_n(G,x) \to \pi_n(G,u)$$

which sends an n-arrow $a\colon 1_x \to 1_x$ to the n-arrow $1_u *_0 a\colon u \to u$, is an isomorphism. Moreover f commutes with this isomorphism, that is the square

$$\begin{tikzcd}
\pi_n(G,x) \arrow[r] \arrow[d] & \pi_n(G,u) \arrow[d] \\
\pi_n(H,f(x)) \arrow[r] & \pi_n(H,f(u))
\end{tikzcd}$$

is commutative. The map $\pi_n(G,u) \to \pi_n(H,f(u))$ is thus a bijection for $n \geq 2$.

$2 \Rightarrow 3)$ Let $n \geq 1$ and let u, v be two parallel $(n-1)$-arrows of G. Suppose there exists an n-arrow $a\colon u \to v$ in G. The map

$$\pi_n(G,u) \to \pi_n(G,u,v)$$

which sends an n-arrow $b\colon u \to u$ to the n-arrow $a *_{n-1} b\colon u \to v$, is a bijection. Moreover f commutes with this bijection, that is the square

$$\begin{tikzcd}
\pi_n(G,u) \arrow[r] \arrow[d] & \pi_n(G,u,v) \arrow[d] \\
\pi_n(H,f(u)) \arrow[r] & \pi_n(H,f(u),f(v))
\end{tikzcd}$$

is commutative.

Thus to conclude it suffices to show that if there exists an n-arrow $b\colon f(u) \to f(v)$ in H, then there exists an n-arrow $a\colon u \to v$ in G. It is clear when $n = 1$ by injectivity of $\pi_0(f)$. Let $n \geq 2$ and let $b\colon f(u) \to f(v)$ be an n-arrow of H. Set $x = s_{n-2}(u)$. The arrow $1_{f(u)^{-1}} *_{n-2} b$ is an n-arrow of H from $1_{f(x)}\colon f(x) \to f(x)$ to $f(u)^{-1} *_{n-2} f(v)\colon f(x) \to f(x)$. Since the map

$$\pi_{n-1}(G,x) \to \pi_{n-1}(H,f(x))$$

is injective, there exists an n-arrow a' of G from 1_x to $u^{-1} *_{n-2} v$. Then $a = 1_u *_{n-2} a'$ is an n-arrow of G from u to v.

$3 \Rightarrow 1)$ Obvious.

$4 \Rightarrow 3)$ Let $n \geq 1$, let u, v be two parallel $(n-1)$-arrows of G and let a, b be two n-arrows from u to v. Suppose we have $f(a) = f(b)$ in $\pi_n(H, f(u), f(v))$. Then there exists an $(n+1)$-arrow of H from $f(a)$ to $f(b)$. By surjectivity of the map

$$\pi_{n+1}(G,a,b) \to \pi_{n+1}(H,f(a),f(b)),$$

there exists an $(n+1)$-arrow in G from a to b. Thus $a = b$ in $\pi_n(G,u,v)$.

$3 \Rightarrow 4)$ Obvious.

□

2 The Brown-Golasiński model structure

In [4], Brown and Golasiński introduce a model category structure on the category of crossed complexes. By the equivalence of categories between crossed complexes and strict ∞-groupoids constructed in [5], this model structure induces a model structure on strict ∞-groupoids. The purpose of this section is to describe this model structure.

2.1 Crossed complexes

Let us denote by **Grp** the category of groups and by $\mathbf{CGrp}_{\geq 2}$ the category of homological complexes of (not necessarily commutative) groups in dimension greater or equal to 2, that is of sequences of morphisms of groups

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_3 \xrightarrow{d_3} C_2$$

such that for every $n \geq 4$, we have $d_{n-1}d_n = 1$, where 1 denotes the unit element of C_{n-2}. We have an inclusion functor $i_2 \colon \mathbf{Grp} \to \mathbf{CGrp}_{\geq 2}$ which sends a group G to the complex concentrated in degree 2 on G.

Let $C_{\leq 1}$ be a groupoid. We will denote by C_0 its set of objects and by $C_1(x, y)$ the set of morphisms from an object x to an object y in $C_{\leq 1}$. Let $C_1 \colon C_{\leq 1} \to \mathbf{Grp}$ be the functor defined in the following way: an object x of $C_{\leq 1}$ is sent to the group $C_1(x) = C_1(x, x)$; a morphism $u \colon x \to y$ of $C_{\leq 1}$ is sent to the morphism of conjugation by u, i.e.,

$$\begin{aligned} C_1(x) &\to C_1(y) \\ v &\mapsto uvu^{-1}. \end{aligned}$$

Definition 2.1. *A* precrossed complex *consists of*

- *a groupoid $C_{\leq 1}$;*
- *a functor $C_{\geq 2} \colon C_{\leq 1} \to \mathbf{CGrp}_{\geq 2}$;*
- *an augmentation of $C_{\geq 2}$ over C_1, that is a natural transformation $d_2 \colon C_{\geq 2} \to i_2 C_1$.*

Explicitly, a precrossed complex is given by

- for every x in C_0, a complex

$$C_{\geq 2}(x) = \cdots \to C_n(x) \xrightarrow{d_n} C_{n-1}(x) \to \cdots \to C_3(x) \xrightarrow{d_3} C_2(x);$$

- for every x in C_0, a morphism $d_2 \colon C_2(x) \to C_1(x)$;
- for every $n \geq 2$ and every morphism $u \colon x \to y$ of $C_{\leq 1}$, a morphism $C_n(x) \to C_n(y)$ of groups functorial in u,

such that for every morphism $u \colon x \to y$ of $C_{\leq 1}$, the diagram

$$\begin{tikzcd}
\arrow[r] & C_n(x) \arrow[r, "d_n"] \arrow[d] & C_{n-1}(x) \arrow[r] \arrow[d] & \cdots \arrow[r] & C_2(x) \arrow[r, "d_2"] \arrow[d] & C_1(x) \arrow[d] \\
\arrow[r] & C_n(y) \arrow[r, "d_n"] & C_{n-1}(y) \arrow[r] & \cdots \arrow[r] & C_2(y) \arrow[r, "d_2"] & C_1(y),
\end{tikzcd}$$

where $C_1(x) \to C_1(y)$ is the conjugation by u, is commutative.

If C is a precrossed complex we will denote by $C(x)$ the augmented complex

$$\cdots \to C_n(x) \xrightarrow{d_n} C_{n-1}(x) \to \cdots \to C_3(x) \xrightarrow{d_3} C_2(x) \xrightarrow{d_2} C_1(x).$$

If $u\colon x \to y$ is a morphism of $C_{\leq 1}$ and $n \geq 1$, we will call the map $C_n(x) \to C_n(y)$ the *action* of u on $C_n(x)$. In particular, an element of $C_1(x)$ *acts* on $C_n(x)$ for all $n \geq 1$.

Definition 2.2. *A precrossed complex C is a* crossed complex *if for all x in C_0 the following conditions are satisfied:*

- *for every $n \geq 3$, the group $C_n(x)$ is abelian;*
- *for every u in $C_2(x)$, the element $d_2(u)$ of $C_1(x)$ acts*
  - *by conjugation by u on $C_2(x)$;*
  - *trivially on $C_n(x)$ for $n \geq 3$.*

Definition 2.3. *Let C and D be two crossed complexes. A* morphism of crossed complexes *$f\colon C \to D$ consists of*

- *a functor $f_{\leq 1}\colon C_{\leq 1} \to D_{\leq 1}$;*
- *a natural transformation $f_{\geq 2}\colon C_{\geq 2} \to D_{\geq 2} f_{\leq 1}$,*

such that $f_{\geq 2}$ is compatible with the augmentation, i.e., the square

$$\begin{tikzcd} C_{\geq 2} \arrow[r] \arrow[d, "d_2"'] & D_{\geq 2} f_{\leq 1} \arrow[d, "d_2 f_{\leq 1}"] \\ i_2 C_1 \arrow[r] & i_2 D_1 f_{\leq 1} \end{tikzcd}$$

is commutative.

Explicitly, a morphism $f\colon C \to D$ is given by

- a functor $f_{\leq 1}\colon C_{\leq 1} \to D_{\leq 1}$;
- for every $n \geq 2$ and every x in C_0, a group morphism $f_n\colon C_n(x) \to D_n(f_0(x))$, where f_0 denotes the restriction of $f_{\leq 1}$ to objects,

such that for every x in C_0, the diagram

$$\begin{tikzcd} \arrow[r] & C_n(x) \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1}(x) \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & C_1(x) \arrow[d, "f_1"'] \\ \arrow[r] & D_n(f_0(x)) \arrow[r, "d_n"] & D_{n-1}(f_0(x)) \arrow[r] & \cdots \arrow[r] & D_1(f_0(x)), \end{tikzcd}$$

where f_1 denotes the restriction of $f_{\leq 1}$ to $C_1(x)$, is commutative. We will often simply write f for f_n.

We will denote the category of crossed complexes by **CrC** .

2.2 The equivalence with strict ∞-groupoids

Let G be a strict ∞-groupoid. One can associate to G a precrossed complex C in the following way:

- the groupoid $C_{\leq 1}$ is the 1-truncation of G obtained from G by throwing out the n-arrows for $n \geq 2$;
- $C_n(x)$ is the set of n-arrows u of G whose source is an iterated unit of an object, that is such that there exists an object x of G such that $s_{n-1}(u) = 1_x$;
- the group law on $C_n(x)$ is induced by the composition $*_0^n \colon G_n \times_{G_0} G_n \to G_n$;
- $d_n \colon C_n(x) \to C_{n-1}(x)$ is induced by the target map $t_{n-1} \colon G_n \to G_{n-1}$;
- if $u \colon x \to y$ is a morphism in $C_{\leq 1}$ and v a morphism in $C_n(x)$ for $n \geq 2$, the action of u on v is
$$u *_0 v *_0 w_0(u).$$

This precrossed complex is a crossed complex (see paragraph 3 of [5]). Moreover this construction defines a functor $\mathrm{A} \colon \infty\text{-}\mathbf{Grp} \to \mathbf{CrC}$.

Theorem 2.4 (Brown-Higgins)**.** *The functor* $\mathrm{A} \colon \infty\text{-}\mathbf{Grp} \to \mathbf{CrC}$ *is an equivalence of categories.*

Proof. See Theorem 4.1 of [5]. □

2.3 The model structure on crossed complexes

Definition 2.5. *Let C be a crossed complex. The set of* connected components *of C is*

$$\pi_0(C) = \pi_0(C_{\leq 1}).$$

For x in C_0, the fundamental group of C at x *is*

$$\pi_1(C, x) = \mathrm{Coker}(d_2 \colon C_2(x) \to C_1(x)),$$

and for $n \geq 2$, the n-th homotopy group of C at x *is*

$$\pi_n(C, x) = H_n(C(x)).$$

It is clear that π_0 defines a functor from the category of crossed complexes to the category of sets and that for $n \geq 1$, π_n defines a functor from the category of pointed crossed complexes to the category of groups.

Definition 2.6. *A morphism $f \colon C \to D$ of crossed complexes is a* weak equivalence of crossed complexes *(see [4]) if the map $\pi_0(f) \colon \pi_0(C) \to \pi_0(D)$ is a bijection and if for every x in C_0 and every $n \geq 1$, the morphism $\pi_n(f, x) \colon \pi_n(C, x) \to \pi_n(D, f(x))$ is an isomorphism.*

Definition 2.7. *Let $f\colon C \to D$ be a morphism of crossed complexes. The morphism f is a* trivial fibration of crossed complexes *(see Proposition 2.2 (ii) of [4]) if the following conditions are satisfied:*

- *for every y in D_0, there exists x in C_0 such that $f(x) = y$;*
- *for every x, y in C_0 and every morphism $v\colon f(x) \to f(y)$ in $D_{\leq 1}$, there exists a morphism $u\colon x \to y$ in $C_{\leq 1}$ such that $f(u) = v$;*
- *for every $n \geq 2$, x in C_0, t in $C_{n-1}(x)$ and every v in $D_n(f(x))$ such that $d_n(v) = f(t)$, there exists u in $C_n(x)$ such that $d_n(u) = t$ and $f(u) = v$.*

Theorem 2.8 (Brown-Golasiński)**.** *The weak equivalences and trivial fibrations of crossed complexes define a model category structure on the category of crossed complexes.*

Proof. See Theorem 2.12 of [4]. □

2.4 The model structure on strict ∞-groupoids

One obtains a model category structure on strict ∞-groupoids by transferring the model structure on crossed complexes defined in the previous section via the equivalence of categories $\mathrm{A}\colon \infty\text{-}\mathbf{Grp} \to \mathbf{CrC}$. We will call this model structure the Brown-Golasiński model structure. A morphism f of strict ∞-groupoids is a weak equivalence (respectively a trivial fibration) for the Brown-Golasiński model structure if and only if $\mathrm{A}(f)$ is a weak equivalence (respectively a trivial fibration) of crossed complexes. We will denote these classes by $\mathcal{W}_{\mathrm{cc}}$ and $\mathcal{TF}_{\mathrm{cc}}$ respectively.

We now describe these two classes more explicitly.

Proposition 2.9. *We have $\mathcal{W}_{\mathrm{cc}} = \mathcal{W}_{\mathrm{gr}}$. In other words, a morphism $f\colon G \to H$ of strict ∞-groupoids is a weak equivalence of strict ∞-groupoids if and only if the morphism $\mathrm{A}(f)\colon \mathrm{A}(G) \to \mathrm{A}(H)$ is a weak equivalence of crossed complexes.*

Proof. Since the two notions of weak equivalences are defined in terms of homotopy groups, it suffices to show that the two notions of homotopy groups coincide.

Let G be a groupoid. By definition, $\pi_0(G) = \pi_0(\mathrm{A}(G))$.

Let x be an object of G. By definition, $\pi_1(\mathrm{A}(G), x)$ is the cokernel of the morphism $\mathrm{A}(G)_2(x) \to \mathrm{A}(G)_1(x)$. The set $\mathrm{A}(G)_1(x)$ is the set of 1-arrows $u\colon x \to x$ in G and two such arrows u, u' are identified in the cokernel if and only if there exists a 2-arrow from 1_x to $u *_0 u'^{-1}$ in G. This condition is equivalent to the existence of a 2-arrow from u to u'. Hence $\pi_1(G, x) = \pi_1(\mathrm{A}(G), x)$.

Let $n \geq 2$. The kernel of the map $d_n\colon \mathrm{A}(G)_n(x) \to \mathrm{A}(G)_{n-1}(x)$ is the set $G_n(x, x)$. Thus the same argument as in dimension 1 shows that $\pi_n(G, x) = \pi_n(\mathrm{A}(G), x)$. □

Proposition 2.10. *A morphism $f\colon G \to H$ of strict ∞-groupoids is in $\mathcal{TF}_{\mathrm{cc}}$ if and only if it satisfies the following conditions:*

- *for every object y of H, there exists an object x of G such that $f(x) = y$;*

- *for every pair (x, y) of objects of G, the map*

$$G(x, y)_0 \to G(f(x), f(y))_0$$

induced by f is a surjection;

- *for all $n \geq 2$, every object x of G and every $(n-1)$-arrow $u\colon 1_x \to 1_x$, the map*

$$G(1_x, u)_0 \to H(1_{f(x)}, f(u))_0$$

is surjective.

Proof. By definition, f is in $\mathcal{TF}_{\mathrm{cc}}$ if and only if $\mathrm{A}(f)$ is a trivial fibration of crossed complexes. This proposition is then just a matter of translation using the definition of the functor A. □

3 The folk model structure on ∞-Grp

This section shows that the folk model structure on ∞-**Cat** defined in [10] transfers to ∞-**Grp** via the inclusion functor

$$U\colon \infty\text{-}\mathbf{Grp} \to \infty\text{-}\mathbf{Cat}.$$

We first give a brief review of the main results of [10], and introduce the material needed to prove the transfer theorem.

3.1 The folk model structure on ∞-Cat

Given an ∞-category C, we define *reversible* cells in C and the relation of *ω-equivalence* between cells of C by mutual coinduction on $n \geq 0$.

Definition 3.1. *Let $n \in \mathbb{N}$:*

- *an n-cell x is* ω-equivalent *to an n-cell y if there is a reversible $(n+1)$-cell $u\colon x \to y$;*
- *an $(n+1)$-cell $u\colon x \to y$ is* reversible *if there is an $(n+1)$-cell $\bar{u}\colon y \to x$ such that $\bar{u} *_n u$ is ω-equivalent to 1_x and $u *_n \bar{u}$ is ω-equivalent to 1_y.*

Note that, for each $r \geq 0$, if two cells are r-equivalent in the sense of [12], then they are ω-equivalent, with the converse being false. We also refer to [7] for a gentle introduction to coinductive methods. Remark also that if G is an ∞-groupoid, any $(n+1)$-cell u of G is reversible and the cell $\bar{u}$ whose existence is stated in Definition 3.1 is of course just u^{-1}.

Let $\mathcal{W}$ denote the class of those morphisms $f\colon C \to D$ satisfying the following two conditions:

1. for each 0-cell y in D, there is a 0-cell x in C such that fx is ω-equivalent to y;

2. for each pair (x, x') of parallel n-cells in C, where $n \geq 0$, and each $(n+1)$-cell $v\colon fx \to fx'$, there is an $(n+1)$-cell $u\colon x \to x'$ such that fu is ω-equivalent to v.

Now for each $n \geq 0$, we define the *n-globe* $\mathbf{O}^n$ as the free ∞-category generated by the representable globular set $Y(n) = \mathbf{O}(n, -)$. Thus $Y(n)$ has exactly one n-cell, two k-cells for each $k < n$ and no k-cell for $k > n$. Let also $\partial Y(n)$ be the globular set having the same cells as $Y(n)$ except in dimension n where $(\partial Y(n))_n = \emptyset$. We denote by $\partial \mathbf{O}^n$ the free ∞-category on $\partial Y(n)$. We finally have, for each $n \geq 0$, an inclusion morphism

$$\mathbf{i}_n\colon \partial \mathbf{O}^n \to \mathbf{O}^n.$$

The set $\{\mathbf{i}_n \mid n \in \mathbb{N}\}$ is denoted by I.

A map is a *trivial fibration* if it has the right-lifting property with respect to I and a *cofibration* if it has the left-lifting property with respect to all trivial fibrations.

Proposition 3.2. *Any ∞-functor f factors as $f = p \circ k$ where p is a trivial fibration and k is a cofibration.*

Proof. By a standard application of the small object argument, using the fact that ∞-**Cat** is locally presentable. □

On the other hand the maps $\sigma_n, \tau_n\colon n \to n+1$ in the globular category $\mathbf{O}$ (see Section 1) induce corresponding maps from $\mathbf{O}^n$ to $\mathbf{O}^{n+1}$, of the form $\mathbf{i}_{n+1} \circ \hat{\sigma}_n$ and $\mathbf{i}_{n+1} \circ \hat{\tau}_n$ respectively, where $\hat{\sigma}_n, \hat{\tau}_n\colon \mathbf{O}^n \to \partial \mathbf{O}^{n+1}$. Moreover, we get a pushout diagram

$$\begin{tikzcd} \partial \mathbf{O}^n \arrow[r, "\mathbf{i}_n"] \arrow[d, "\mathbf{i}_n"'] & \mathbf{O}^n \arrow[d, "\hat{\sigma}_n"] \\ \mathbf{O}^n \arrow[r, "\hat{\tau}_n"'] & \partial \mathbf{O}^{n+1}. \end{tikzcd}$$

Now the above pushout determines a canonical map

$$\mathbf{o}_n\colon \partial \mathbf{O}^{n+1} \to \mathbf{O}^n$$

such that $\mathbf{o}_n \circ \hat{\sigma}_n = \mathbf{o}_n \circ \hat{\tau}_n = \mathrm{id}_{\mathbf{O}^n}$. Proposition 3.2 applies to $\mathbf{o}_n$, yielding an object $\mathbf{P}^n$ together with a trivial fibration $\mathbf{p}_n\colon \mathbf{P}^n \to \mathbf{O}^n$ and a cofibration $\mathbf{k}_n\colon \partial \mathbf{O}^{n+1} \to \mathbf{P}^n$ satisfying $\mathbf{o}_n = \mathbf{p}_n \circ \mathbf{k}_n$. We finally define $\mathbf{j}_n\colon \mathbf{O}^n \to \mathbf{P}^n$ as $\mathbf{k}_n \circ \hat{\sigma}_n$ and

$$J = \{\mathbf{j}_n \mid n \in \mathbb{N}\}.$$

Theorem 3.3. *There is a cofibrantly generated model structure on ∞-**Cat** where $\mathcal{W}$ is the class of weak equivalences, I a set of generating cofibrations and J a set of generating trivial cofibrations.*

This statement is in fact [10, Theorem 4.39] and the main result of that article.

$$\begin{tikzcd} x \arrow[d, "U^\natural"] \\ y \end{tikzcd} \qquad \begin{tikzcd} x^\sharp \arrow[d, "U^\sharp"'] \arrow[dr, Rightarrow, "U^\natural"] & x^\flat \arrow[l, "x"'] \arrow[d, "U^\flat"] \\ y^\sharp & y^\flat \arrow[l, "y"] \end{tikzcd}$$

Figure 1: n-cylinders for $n = 0, 1$

3.2 Path object

Let C be an object in a model category and $\Delta_C \colon C \to C \times C$ be the diagonal map. A *path object* for C consists in an object P_C together with a factorization of Δ_C of the form

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[rr, bend right, "\Delta_C"'] & P_C \arrow[r, "p"] & C \times C \end{tikzcd}$$

where p is a fibration and j is a weak equivalence. Such a P_C is not unique: in the case of ∞-**Cat**, one particular choice is given by the functor Γ we now describe. We first define, by induction on n, the notion of *n-cylinder between n-cells x, y of an ∞-category C*. A few notations will be useful: for each n-cell x we denote by $x^\flat$, respectively $x^\sharp$ its 0-source $s_0 x$, respectively 0-target $t_0 x$. Now let C be an ∞-category and x, y two 0-cells in it. There is an ∞-category $C(x, y)$ whose n-cells are the $(n+1)$-cells u of C such that $u^\flat = x$ and $u^\sharp = y$. Whenever u is such an $(n+1)$-cell of C, we denote by $[u]$ the corresponding n-cell of $C(x, y)$. Finally, let x, y, z be 0-cells of C. Each 1-cell $u \colon x \to y$ determines an ∞-functor $- \cdot u \colon C(y, z) \to C(x, z)$ given by $[v] \cdot u = [v *_0 u]$. Likewise $u \colon y \to z$ determines an ∞-functor $u \cdot - \colon C(x, y) \to C(x, z)$ by $u \cdot [v] = [u *_0 v]$.

Definition 3.4. *1. A* 0-cylinder *$U \colon x \curvearrowright y$ in C is given by a reversible* 1-*cell $U^\natural \colon x \to y$;*

2. If $n > 0$, an n-cylinder *$U \colon x \curvearrowright y$ in C is given by two reversible 1-cells $U^\flat \colon x^\flat \to y^\flat$ and $U^\sharp \colon x^\sharp \to y^\sharp$, together with some $(n-1)$-cylinder $[U] \colon U^\sharp \cdot [x] \curvearrowright [y] \cdot U^\flat$ in the ∞-category $[x^\flat, y^\sharp] = C(x^\flat, y^\sharp)$.*

If $U \colon x \curvearrowright y$ is an n-cylinder in C, we write $\pi^1_C U$ and $\pi^2_C U$ for the n-cells x and y, or simply $\pi^1 U$ and $\pi^2 U$. Figure 1 represents n-cylinders for $n = 0$ and $n = 1$. For each $n \in \mathbb{N}$, any $(n+1)$-cylinder $W \colon z \curvearrowright z'$ in an ∞-category C determines a pair of n-cylinders in C:

Definition 3.5. *The* source n-cylinder *$U \colon x \curvearrowright x'$ and the* target n-cylinder *$V \colon y \curvearrowright y'$ of the $(n+1)$-cylinder $W \colon z \curvearrowright z'$ between $(n+1)$-cells $z \colon x \to y$ and $z' \colon x' \to y'$ are defined inductively by:*

- *if $n = 0$, then $U^\natural = W^\flat$ and $V^\natural = W^\sharp$;*

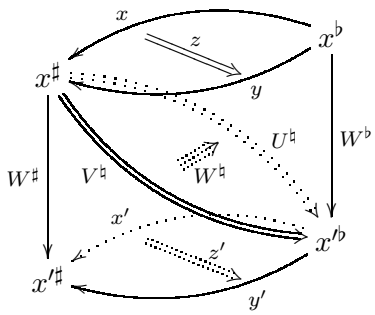

Figure 2: source and target of a 2-cylinder

- *if $n > 0$, then $U^\flat = V^\flat = W^\flat$ and $U^\sharp = V^\sharp = W^\sharp$, whereas the two $(n-1)$-cylinders $[U]$ and $[V]$ are respectively defined as the source and the target of the n-cylinder $[W]$ in the ∞-category $[z^\flat, z'^\sharp]$.*

If W has source U and target V we write $W\colon U \to V$ or $W\colon U \to V \mid z \curvearrowright z'$ (see Figure 2). It turns out that the source and target maps so defined satisfy the globular relations, so that the correspondence

$$n \mapsto \{U \mid U \text{ is an } n\text{-cylinder in } C\}$$

determines a globular set $\Gamma(C)$. We now turn to *trivial n-cylinders*:

Definition 3.6. *The* trivial n-cylinder $\tau\, x\colon x \curvearrowright x$ *on the n-cell x is defined inductively by:*

- *if $n = 0$, then $(\tau\, x)^\natural = 1_x$;*
- *if $n > 0$, then $(\tau\, x)^\flat = 1_{x^\flat}$ and $(\tau\, x)^\sharp = 1_{x^\sharp}$, whereas $[\tau\, x]$ is the trivial cylinder $\tau[x]$ in $[x^\flat, x^\sharp]$.*

We write $\tau_C\, x$ in case we need to mention the ambient ∞-category C.

Let us finally recall from [10, Appendix A] that $\Gamma(C)$ becomes a strict ∞-category when defining units and compositions as follows:

Definition 3.7. *Let $U\colon x \curvearrowright y$ be an n-cylinder. We define the $(n+1)$-cylinder $1_U\colon U \to U \mid 1_x \curvearrowright 1_y$ by induction on n:*

- *if $n = 0$, then $(1_U)^\flat = (1_U)^\sharp = U^\natural$, whereas $[1_U] = \tau[1_{U^\natural}]$;*
- *if $n > 0$, then $(1_U)^\flat = U^\flat$ and $(1_U)^\sharp = U^\sharp$, whereas $[1_U] = 1_{[U]}$.*

In order to define composition, we first introduce the operation of *concatenation*:

Definition 3.8. *Let $U \mid x \curvearrowright y$ and $V \mid y \curvearrowright z$ be two n-cylinders. The* concatenation *$V * U \mid x \curvearrowright z$ of U and V is defined by induction on n:*

- *if $n = 0$, then $(V * U)^\natural = V^\natural *_0 U^\natural$;*

- *if* $n > 0$*, then* $(V*U)^\flat = V^\flat *_0 U^\flat$*,* $(V*U)^\sharp = V^\sharp *_0 U^\sharp$ *and* $[V*U] = [V]\cdot U^\flat * V^\sharp \cdot [U]$*.*

Definition 3.9. *Let* $m \geq 1$*,* $0 \leq n < m$ *and* $U \mid x \curvearrowright x'$*,* $V \mid y \curvearrowright y'$ *two* m*-cylinders such that* $t_n(U) = s_n(V)$*. The* composition $V *_n U \mid y *_n x \curvearrowright y' *_n x'$ *is defined by induction on* n *as follows:*

- $(V *_0 U)^\flat = U^\flat$*,* $(V *_0 U)^\sharp = V^\sharp$ *and* $[V *_0 U] = y' \cdot [U] * [V] \cdot x$*;*
- *if* $n > 0$*, then* $(V *_n U)^\flat = U^\flat = V^\flat$*,* $(U *_n V)^\sharp = U^\sharp = V^\sharp$ *and* $[V *_n U] = [V] *_{n-1} [U]$*.*

Note that explicit formulas may be found in [11].

For example, Figure 3 shows the composition $V *_0 U$ of two 1-cylinders $U\colon x \curvearrowright x'$ and $V\colon y \curvearrowright y'$ such that $U^\sharp = V^\flat$. Precisely, the composite $V *_0 U$ is the 1-cylinder $W\colon z \curvearrowright z'$ where $z = y *_0 x$, $z' = y' *_0 x'$, $W^\flat = U^\flat$, $W^\sharp = V^\sharp$, and the 0-cylinder $[W]$ of $C(z^\flat, z'^\sharp)$ is the reversible 1-cell of $C(z^\flat, z'^\sharp)$ given by the following corresponding reversible 2-cell of C:

$$W^\natural = (y' *_0 U^\natural) *_1 (V^\natural *_0 x).$$

$$\begin{tikzcd}
y^\sharp \arrow[d, "V^\sharp"'] \arrow[dr, Rightarrow, "V^\natural"] & y^\flat \arrow[l, "y"'] \arrow[r, equal] \arrow[d, "V^\flat = U^\sharp"] & x^\sharp \arrow[d] \arrow[dr, Rightarrow, "U^\natural"] & x^\flat \arrow[l, "x"'] \arrow[d, "U^\flat"] \\
y'^\sharp & y'^\flat \arrow[l, "y'"] \arrow[r, equal] & x'^\sharp & x'^\flat \arrow[l, "x'"]
\end{tikzcd}$$

Figure 3: composition of 1-cylinders

The following result summarizes the main properties of Γ:

Theorem 3.10. *The correspondence* $C \mapsto \Gamma(C)$ *induces an endofunctor on* ∞-**Cat***, and* $\pi^1, \pi^2\colon \Gamma \to \mathrm{id}$*,* $\tau\colon \mathrm{id} \to \Gamma$ *are natural transformations.*

An additional property, of particular importance here, is that the functor Γ preserves ∞-groupoids:

Lemma 3.11. *If* G *is an* ∞*-groupoid, so is* $\Gamma(G)$*.*

Proof. We show, by induction on $n \geq 1$, that if G is an ∞-groupoid and $W\colon U \to V \mid x \curvearrowright y$ is an n-cylinder of G, there is an n-cylinder $W'\colon V \to U \mid x^{-1} \curvearrowright y^{-1}$ such that $W' *_{n-1} W = 1_U$ and $W *_{n-1} W' = 1_V$.

- Let G be an ∞-groupoid and $W\colon U \to V \mid x \curvearrowright y$ a 1-cylinder of G. By definition, we get two 1-cells $U^\natural\colon x^\flat \to y^\flat$, $V^\natural\colon x^\sharp \to y^\sharp$ and a 2-cell $W^\natural\colon V^\natural *_0 x \to y *_0 U^\natural$ in G. Consider $W^{\natural -1}\colon y *_0 U^\natural \to V^\natural *_0 x$ the $*_1$-inverse of $W^\natural$ and build

$$y^{-1} *_0 W^{\natural -1} *_0 x^{-1}\colon U^\natural *_0 x^{-1} \to y^{-1} *_0 V^\natural.$$

If $W' \colon V \to U \mid x^{-1} \curvearrowright y^{-1}$ is the 1-cylinder of G defined by

$$W'^{\natural} = y^{-1} *_0 W^{\natural^{-1}} *_0 x^{-1},$$

we get

$$W' *_0 W = 1_U \quad \text{and} \quad W *_0 W' = 1_V,$$

which proves the case $n = 1$.

- Let $n > 1$ and suppose that the property holds for $n-1$. Let G be an ∞-groupoid and $W \colon U \to V \mid x \curvearrowright y$ an n-cylinder of G. We get 1-cells $W^{\flat} \colon x^{\flat} \to y^{\flat}$, $W^{\sharp} \colon x^{\sharp} \to y^{\sharp}$ and an $(n-1)$-cylinder $[W] \colon [U] \to [V] \mid W^{\sharp} \cdot [x] \curvearrowright [y] \cdot W^{\flat}$ in $H = [x^{\flat}, y^{\sharp}]$. Now H is an ∞-groupoid, so that the induction hypothesis applies and there is an $(n-1)$-cylinder in H

$$[W]' \colon [V] \to [U] \mid W^{\sharp} \cdot \left[x^{-1}\right] \curvearrowright \left[y^{-1}\right] \cdot W^{\flat}$$

such that $[W]' *_{n-2} [W] = 1_{[U]}$ and $[W] *_{n-2} [W]' = 1_{[V]}$. Hence we may define an n-cylinder W' of G by $W'^{\flat} = W^{\flat}$, $W'^{\sharp} = W^{\sharp}$ and $[W'] = [W]'$. By construction

$$W' *_{n-1} W = 1_U \quad \text{and} \quad W *_{n-1} W' = 1_V.$$

□

Remark 3.12. *Let $n \geq 0$. The proof of the previous lemma actually shows that if G is a strict (∞, n)-category, then so is $\Gamma(G)$.*

3.3 Immersions

We now introduce a class of morphisms which plays an important part in the proof of the transfer theorem.

Definition 3.13. *An ∞-functor $f \colon C \to D$ belongs to the class $\mathcal{Z}$ of* immersions *if and only if there exist ∞-functors $g \colon D \to C$ and $h \colon D \to \Gamma(D)$ satisfying the following properties:*

1. *g is a retraction of f, that is $g \circ f = \mathrm{id}_C$;*
2. *$\pi^1_D \circ h = f \circ g$ and $\pi^2_D \circ h = \mathrm{id}_D$;*
3. *$h \circ f = \tau_D \circ f$. In other words, h is trivial on $f(C)$.*

The following proposition summarizes the properties of immersions we need here.

Proposition 3.14. *The class $\mathcal{Z}$ of immersions satisfies the following properties:*

(i) $\mathcal{Z}$ is closed by pushout;

(ii) all trivial cofibrations are immersions;

(iii) all immersions are weak equivalences.

We refer to [10, Section 4.6] for the proofs of these statements.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}_C"'] & D \arrow[r, "g"] & C \end{tikzcd} \qquad \begin{tikzcd} & D \arrow[dl, bend right, "f\circ g"'] \arrow[d, "h"] \arrow[dr, bend left, "\mathrm{id}_D"] & \\ D & \Gamma(D) \arrow[l, "\pi^1_D"] \arrow[r, "\pi^2_D"'] & D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "\tau_D\circ f"'] & D \arrow[d, "h"] \\ & \Gamma(D) \end{tikzcd}$$

Figure 4: Immersions

3.4 Transfer

Let $\mathbf{C}$, $\mathbf{D}$ be two categories and $L\colon \mathbf{C} \to \mathbf{D}$, $R\colon \mathbf{D} \to \mathbf{C}$ be a pair of functors with L left adjoint to R, and suppose that $\mathbf{C}$ is equipped with a model structure. We may define three classes of maps of $\mathbf{D}$ as follows:

- for each morphism f of $\mathbf{D}$, $f \in \mathcal{W}_{\mathbf{D}}$ if and only if $R(f)$ is a weak equivalence in $\mathbf{C}$;
- for each morphism f of $\mathbf{D}$, $f \in \mathcal{F}_{\mathbf{D}}$ if and only if $R(f)$ is a weak equivalence in $\mathbf{C}$;
- $\mathcal{C}_{\mathbf{D}}$ is the class of maps having the left-lifting property with respect to $\mathcal{W}_{\mathbf{D}} \cap \mathcal{F}_{\mathbf{D}}$.

We say that R *creates* a model structure on $\mathbf{D}$ if $\mathcal{W}_{\mathbf{D}}$, $\mathcal{F}_{\mathbf{D}}$ and $\mathcal{C}_{\mathbf{D}}$ are respectively the weak equivalences, fibrations and cofibrations of a model structure on $\mathbf{D}$. Sufficient conditions for this transfer to hold are given by [6, Theorem 3.3] or [3, Proposition 2.3]. The latter result immediately specializes to the following statement:

Proposition 3.15. *Let $\mathbf{C}$ a cofibrantly generated model structure, with I a set of generating cofibrations and J a set of generating trivial cofibrations. If $\mathbf{D}$ is locally presentable, then the following conditions are sufficient for R to create a model structure on $\mathbf{D}$:*

(C1) the weak equivalences of $\mathbf{C}$ are closed under filtered colimits;

(C2) R preserves filtered colimits;

(C3) for each generating trivial cofibration j of $\mathbf{C}$, if g is a pushout of $L(j)$ in $\mathbf{D}$, then $R(g)$ is a weak equivalence of $\mathbf{C}$.

Moreover, if these conditions hold, the model structure so defined is cofibrantly generated and has $L(I)$ as a set of generating cofibrations and $L(J)$ as a set of generating trivial cofibrations.

We now turn to the particular case where $\mathbf{C} = \infty\text{-}\mathbf{Cat}$, $\mathbf{D} = \infty\text{-}\mathbf{Grp}$ and R is the inclusion functor $U\colon \infty\text{-}\mathbf{Grp} \to \infty\text{-}\mathbf{Cat}$. Note that U has a left-adjoint $F\colon \infty\text{-}\mathbf{Cat} \to \infty\text{-}\mathbf{Grp}$ building the free ∞-groupoid on an ∞-category, as well as a right-adjoint M, building to the maximal ∞-groupoid in an ∞-category. Let us first establish a few properties about the adjunction $F \dashv U$.

Let T be the monad UF on ∞-**Cat**. Remark that, for any ∞-groupoid G, the free ∞-groupoid on the underlying ∞-category $U(G)$ is naturally isomorphic to G itself. In other words, the counit $\varepsilon\colon FU \to 1$ is a natural isomorphism. It follows that, for any ∞-groupoid G, we get an isomorphism

$$\eta_{U(G)}\colon U(G) \to UFU(G), \tag{1}$$

where η denotes the unit of the adjunction. Now, for each ∞-category C, $T(C)$ is of the form $U(G)$ where G is an ∞-groupoid, and so is $\Gamma T(C)$ by Lemma 3.11, so that

$$\eta_{\Gamma T(C)}\colon \Gamma T(C) \to T\Gamma T(C) \tag{2}$$

is an isomorphism, as a special case of (1). Thus, we may define a natural transformation

$$\lambda\colon T\Gamma \to \Gamma T$$

by

$$\lambda_C = \eta^{-1}_{\Gamma T(C)} \circ T\Gamma(\eta_C). \tag{3}$$

Note also that the monad multiplication $\mu\colon T^2 \to T$ is also a natural isomorphism, and we get

$$T(\eta_C) = \eta_{T(C)} = \mu_C^{-1}. \tag{4}$$

We may now state the following result:

Lemma 3.16. *The monad T preserves immersions.*

Proof. Let $f\colon C \to D$ be an immersion, and $f' = T(f)$. By Definition 3.13, there are $g\colon D \to C$ and $h\colon D \to \Gamma(D)$ such that

$$\begin{aligned} g \circ f &= \mathrm{id}_C; && (5)\\ \pi^1_D \circ h &= f \circ g; && (6)\\ \pi^2_D \circ h &= \mathrm{id}_D; && (7)\\ h \circ f &= \tau_D \circ f. && (8) \end{aligned}$$

Let $g' = T(g)$ and $h' = \lambda_D \circ T(h)$. We need to establish the following equations:

$$\begin{aligned} g' \circ f' &= \mathrm{id}_{T(C)}; && (9)\\ \pi^1_{T(D)} \circ h' &= f' \circ g'; && (10)\\ \pi^2_{T(D)} \circ h' &= \mathrm{id}_{T(D)}; && (11)\\ h' \circ f' &= \tau_{T(D)} \circ f'. && (12) \end{aligned}$$

Equation (9) is just functoriality. Let us prove (10). First remark that π^1 is a natural transformation, so that the following diagram commutes:

$$\begin{tikzcd} \Gamma(D) \arrow[r, "\pi^1_D"] \arrow[d, "\Gamma(\eta_D)"'] & D \arrow[d, "\eta_D"] \\ \Gamma T(D) \arrow[r, "\pi^1_{T(D)}"'] & T(D). \end{tikzcd} \tag{13}$$

We may now build the following commutative diagram:

$$\begin{tikzcd}
T\Gamma(D) \arrow[r, "T(\pi^1_D)"] \arrow[d, "T\Gamma(\eta_D)"'] \arrow[dd, bend right=60, "\lambda_D"'] & T(D) \arrow[d, "T(\eta_D)"] \arrow[dd, bend left=60, "\mathrm{id}_{T(D)}"] \\
T\Gamma T(D) \arrow[r, "T(\pi^1_{T(D)})"] \arrow[d, "\eta^{-1}_{\Gamma T(D)}"'] & T^2(D) \arrow[d, "\eta^{-1}_{T(D)}"] \\
\Gamma T(D) \arrow[r, "\pi^1_{T(D)}"'] & T(D).
\end{tikzcd} \tag{14}$$

In fact the upper square is the image of (13) by T and the lower square commutes by naturality of η. Hence

$$\begin{aligned}
\pi^1_{T(D)} \circ h' &= \pi^1_{T(D)} \circ \lambda_D \circ T(h) \\
&= T(\pi^1_D) \circ T(h) \\
&= T(\pi^1_D \circ h) \\
&= T(f \circ g) \\
&= f' \circ g'
\end{aligned}$$

which gives (10). Likewise, we get the following commutative diagram:

$$\begin{tikzcd}
T(D) \arrow[r, "T(h)"] \arrow[rr, bend left, "h'"] \arrow[dr, "\mathrm{id}_{T(D)}"'] & T\Gamma(D) \arrow[r, "\lambda_D"] \arrow[d, "T(\pi^2_D)"] & \Gamma T(D), \arrow[dl, "\pi^2_{T(D)}"] \\
& T(D) &
\end{tikzcd} \tag{15}$$

where the left hand triangle commutes by applying T to (7), and the right hand triangle commutes by replacing π^1 with π^2 in (14). Hence $\pi^2_{T(D)} \circ h' = \mathrm{id}_{T(D)}$ and (11) is proved. Finally, by using the naturality of τ instead of π^1, we get a commutative diagram analogue to (14):

$$\begin{tikzcd}
T(D) \arrow[r, "T(\tau_D)"] \arrow[d, "T(\eta_D)"'] \arrow[dd, bend right=60, "\mathrm{id}_{T(D)}"'] & T\Gamma(D) \arrow[d, "T\Gamma(\eta_D)"] \arrow[dd, bend left=60, "\lambda_D"] \\
T^2(D) \arrow[r, "T(\tau_{T(D)})"] \arrow[d, "\eta^{-1}_{T(D)}"'] & T\Gamma T(D) \arrow[d, "\eta^{-1}_{\Gamma T(D)}"] \\
T(D) \arrow[r, "\tau_{T(D)}"'] & \Gamma T(D).
\end{tikzcd} \tag{16}$$

Hence

$$\begin{aligned} h' \circ f' &= \lambda_D \circ T(h) \circ T(f) \\ &= \lambda_D \circ T(h \circ f) \\ &= \lambda_D \circ T(\tau_D \circ f) \\ &= \lambda_D \circ T(\tau_D) \circ f' \\ &= \tau_{T(D)} \circ f' \end{aligned}$$

which gives (12) and ends the proof. □

Lemma 3.17. *Let $f\colon C \to D$ be an immersion, and suppose that the following square is a pushout in ∞-**Grp**:*

$$\begin{tikzcd} FC \arrow[r, "u"] \arrow[d, "F(f)"'] & G \arrow[d, "g"] \\ FD \arrow[r, "v"'] & H. \end{tikzcd}$$

Then $U(g)$ is an immersion.

Proof. As U is left adjoint to M, it preserves pushouts, so that the following square is a pushout in ∞-**Cat**:

$$\begin{tikzcd} TC \arrow[r, "U(u)"] \arrow[d, "T(f)"'] & UG \arrow[d, "U(g)"] \\ TD \arrow[r, "U(v)"'] & UH. \end{tikzcd}$$

By Lemma 3.16, $T(f)$ is an immersion, and so is its pushout $U(g)$, by Proposition 3.14(i). □

Lemma 3.18. *If j is a generating trivial cofibration of ∞-**Cat**, and g is a pushout of $F(j)$ in ∞-**Grp**, then $U(g)$ is a weak equivalence of ∞-**Cat**.*

Proof. Let j be a generating trivial cofibration of ∞-**Cat**, and g be a pushout of $F(j)$ in ∞-**Grp**. By Proposition 3.14(ii), j is an immersion, and so is $U(g)$, by Lemma 3.17. By Proposition 3.14(iii), $U(g)$ is a weak equivalence. □

We may finally state the main result of this section:

Theorem 3.19. *The forgetful functor $U\colon \infty$-**Grp** $\to \infty$-**Cat** creates a model structure on ∞-**Grp** in which the weak equivalences are the morphisms f such that $U(f)$ belongs to $\mathcal{W}$. Moreover, the model structure so defined has $(F(\mathbf{i}_k))_{k\in\mathbb{N}}$ as a family of generating cofibrations, and $(F(\mathbf{j}_k))_{k\in\mathbb{N}}$ as a family of generating trivial cofibrations.*

Proof. As the model structure on ∞-**Cat** is cofibrantly generated and ∞-**Grp** is locally presentable, Proposition 3.15 applies, and it suffices to check conditions (C1), (C2) and (C3). Condition (C1) is proved in [10], and condition (C2) follows from the fact

that U has a right-adjoint M, hence preserves colimits, and in particular filtered ones. Condition (C3) is Lemma 3.18. The statement about generating families follows from Proposition 3.15. □

Remark 3.20. *Using Remark 3.12, one can easily adapt the proof of the previous theorem to show that a similar theorem holds for strict (∞, n)-categories. In particular, the inclusion functor (∞, n)-**Cat** $\to \infty$-**Cat** creates a model structure on (∞, n)-**Cat**.*

We call the model structure just defined the *folk model structure* on ∞-groupoids. We denote its weak equivalences by $\mathcal{W}_{\text{folk}}$ and its trivial fibrations by $\mathcal{TF}_{\text{folk}}$. Note that a morphism f is in $\mathcal{TF}_{\text{folk}}$ if and only if $U(f)$ is a trivial fibration of ∞-**Cat**.

Proposition 3.21. *A morphism $f\colon G \to H$ of ∞-groupoids belongs to $\mathcal{TF}_{\text{folk}}$ if and only if the following conditions are satisfied:*

1. *for every object y of H, there exists an object x of G such that $f(x) = y$;*
2. *for all $n \geq 1$ and every pair (u, v) of parallel $(n-1)$-arrows of G, the map*

$$G(u, v)_0 \to H(f(u), f(v))_0$$

is surjective.

Proof. By definition, f belongs to $\mathcal{TF}_{\text{folk}}$ if and only if $U(f)$ has the right lifting property with respect to I. This proposition is then just a matter of translation. □

4 Comparison

In this section, we show that the folk model structure on strict ∞-groupoids defined in the previous section coincides with the Brown-Golasiński model structure. To see this, it suffices to prove that they have the same weak equivalences and the same trivial fibrations.

Proposition 4.1. *We have $\mathcal{W}_{\text{gr}} = \mathcal{W}_{\text{folk}} = \mathcal{W}_{\text{cc}}$.*

Proof. We first show that $\mathcal{W}_{\text{folk}} = \mathcal{W}_{\text{gr}}$. In a strict ∞-groupoid, two n-arrows f and g are ω-equivalent if and only if there exists an $(n+1)$-arrow $a\colon f \to g$, that is if and only if f and g are homotopic. Therefore a morphism of strict ∞-groupoids is in $\mathcal{W}_{\text{folk}}$ if and only if it satisfies condition 4 of Proposition 1.7. The statement is thus exactly the equivalence between conditions 1 and 4 of this very proposition.

By Proposition 2.9, we have $\mathcal{W}_{\text{cc}} = \mathcal{W}_{\text{gr}}$, hence the result. □

Proposition 4.2. *We have $\mathcal{TF}_{\text{folk}} = \mathcal{TF}_{\text{cc}}$.*

Proof. To prove the equivalence between the two notions of trivial fibrations, we will use the descriptions of these notions provided by Propositions 2.10 and 3.21. The conditions for being in $\mathcal{TF}_{\text{folk}}$ are a priori stronger. Let $f\colon G \to H$ be a in $\mathcal{TF}_{\text{cc}}$. Let us prove it

is actually in $\mathcal{TF}_{\text{folk}}$. There is nothing to prove for the conditions in dimension 0 and 1. Let $n \geq 2$ and let u, v be two parallel $(n-1)$-arrows. We want to show that the map

$$G(u,v)_0 \to H(f(u), f(v))_0$$

is surjective. Let b be an n-arrow from $f(u)$ to $f(v)$ in H. Set $x = s_0(u)$. Then $b' = 1_{w_0(f(u))} *_0 b$ is an n-arrow of H from $1_{f(x)}$ to $w_0(f(u)) *_0 f(v)$. Since the map

$$G(1_x, w_0(u) *_0 v)_0 \to H(f(1_x), f(w_0(u) *_0 v))_0$$

is surjective, there exists an n-arrow a' of G from 1_x to $w_0(u) *_0 v$ such that $f(a') = b'$. Then, the n-arrow $a = 1_u *_0 a'$ is from u to v and we have

$$\begin{aligned} f(a) &= f(1_u *_0 a') \\ &= 1_{f(u)} *_0 b' \\ &= 1_{f(u)} *_0 1_{w_0(f(u))} *_0 b \\ &= b. \end{aligned}$$

□

Theorem 4.3. *The Brown-Golasiński model structure and the folk model structure on strict ∞-groupoids coincide.*

Proof. By the two previous propositions, these model structures have the same weak equivalences and the same trivial fibrations. □

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