# The Brown-Golasinski model structure on $\infty$-groupoids revisited

Dimitri Ara, François Métayer

# The Brown-Golasiński model structure on $\infty$-groupoids revisited

Dimitri Ara* & François Métayer†

October 12, 2010

**Abstract**

We prove that the folk model structure on strict $\infty$-categories transfers to $\infty$-groupoids, and that the resulting model structure coincides with the one defined by Brown and Golasiński via crossed complexes.

## 1 Introduction

In [4], Brown and Golasiński build a model structure on the category of crossed complexes. In an earlier work [5], Brown and Higgins established an equivalence of categories between crossed complexes and strict $\infty$-groupoids, whence a model structure on the latter category.

On the other hand, there is a "folk" model structure on strict $\infty$-categories recently discovered by Lafont, Worytkiewicz and the second author [11], which extends previously known model structures on categories [9] and 2-categories [10].

This immediately raised the questions whether the folk model structure on $\infty$-categories may be transferred to $\infty$-groupoids by inclusion, and in that case whether the Brown-Golasiński model structure may be recovered this way.

The purpose of the present work is to show that both questions have affirmative answers.

We first recall the basic definitions of strict $\infty$-categories and $\infty$-groupoids (Section 2). Then, we describe the Brown-Golasiński model structure on crossed complexes and $\infty$-groupoids (Section 3). Section 4 proves the transfer theorem leading to the definition of the folk model structure on strict $\infty$-groupoids. Finally, Section 5 shows that the two model structures previously defined on $\infty$-groupoids are in fact the same.

## 2 Strict $\infty$-groupoids

The purpose of this section is to introduce the definitions and notations about strict $\infty$-groupoids and their weak equivalences that we will use in the sequel of this paper. Our presentation is essentially the same as the one given in [2].

### 2.1 Globular sets

Let us denote by $\mathbf{O}$ the *globular category*, that is the category generated by the graph

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} \cdots \underset{\tau_{i-1}}{\overset{\sigma_{i-1}}{\rightrightarrows}} i \underset{\tau_i}{\overset{\sigma_i}{\rightrightarrows}} i+1 \underset{\tau_{i+1}}{\overset{\sigma_{i+1}}{\rightrightarrows}} \cdots$$

and the coglobular relations

$$\sigma_{i+1}\sigma_i = \tau_{i+1}\sigma_i \quad \text{and} \quad \sigma_{i+1}\tau_i = \tau_{i+1}\tau_i, \qquad i \geq 0.$$

*Institut Mathématique de Jussieu Université Paris 7
†Laboratoire PPS Université Paris 7 - CNRS

A *globular set* or $\infty$*-graph* is a presheaf on $\mathbf{O}$. A globular set $X$ amounts to a diagram of sets

$$\cdots \underset{t_{n+1}}{\overset{s_{n+1}}{\rightrightarrows}} X_{n+1} \underset{t_n}{\overset{s_n}{\rightrightarrows}} X_n \underset{t_{n-1}}{\overset{s_{n-1}}{\rightrightarrows}} \cdots \underset{t_1}{\overset{s_1}{\rightrightarrows}} X_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} X_0$$

satisfying the globular relations

$$s_i s_{i+1} = s_i t_{i+1} \quad \text{and} \quad t_i s_{i+1} = t_i t_{i+1}, \qquad i \geq 0.$$

For $i \geq j \geq 0$, we will denote by $s^i_j$ and $t^i_j$ the maps from $X_i$ to $X_j$ defined by

$$s^i_j = s_j \cdots s_{i-2} s_{i-1} \quad \text{and} \quad t^i_j = t_j \cdots t_{i-2} t_{i-1}.$$

If $X$ is a globular set, we will call $X_0$ the set of *objects* of $X$ and $X_n$ for $n \geq 0$ the set of $n$*-arrows* or $n$*-cells*. The notation $u : x \to y$ will mean that $u$ is a $n$-arrow for $n \geq 1$ whose source is an $(n-1)$-arrow $x$ (that is $s_n(u) = x$) and whose target is an $(n-1)$-arrow $y$ (that is $t_n(u) = y$). We will say that two $n$-arrows $u$ and $v$ are *parallel* if either $n = 0$, or $n \geq 1$ and $u, v$ have same source and same target. For $i \geq j \geq 0$, if $u$ is an $i$-arrow, we will often denote by $s_j(u)$ (respectively $t_j(u)$) the $j$-arrow $s^i_j(u)$ (respectively $t^i_j(u)$).

If $u$ and $v$ are $n$-arrows, $X(u, v)$ will denote the globular set whose $k$-arrows are the $(n + k + 1)$-arrows $a$ of $G$ such that $s_n(a) = u$ and $t_n(a) = v$. In particular, $X(u, v)_0$ is the set of $(n + 1)$-arrows $a : u \to v$ in $X$.

## 2.2 Strict $\infty$-categories

An $\infty$*-precategory* is a globular set $C$ endowed with maps

$$*^i_j : (X_i, s^i_j) \times_{X_j} (t^i_j, X_i) \to X_i, \quad i > j \geq 0,$$
$$k_i : X_{i-1} \to X_i, \quad i \geq 1,$$

such that

1. for every couple $(u, v)$ in $(X_i, s^i_j) \times_{X_j} (t^i_j, X_i)$ with $i > j \geq 0$, we have

$$s_{i-1}(u *^i_j v) = \begin{cases} s_{i-1}(v), & j = i-1 \\ s_{i-1}(u) *^{i-1}_j s_{i-1}(v), & j < i-1 \end{cases};$$

2. for every couple $(u, v)$ in $(X_i, s^i_j) \times_{X_j} (t^i_j, X_i)$ with $i > j \geq 0$, we have

$$t_{i-1}(u *^i_j v) = \begin{cases} t_{i-1}(u), & j = i-1 \\ t_{i-1}(u) *^{i-1}_j t_{i-1}(v), & j < i-1 \end{cases};$$

3. for every $u$ in $X_i$ with $i \geq 0$, we have

$$s_i k_{i+1}(u) = u = t_i k_{i+1}(u).$$

For $i \geq j \geq 0$, we will denote by $k^j_i$ the map from $X_j \to X_i$ defined by

$$k^j_i = k_i \cdots k_{j+2} k_{j+1}.$$

If $u$ and $v$ are $n$-arrows for $n \geq 1$ of an $\infty$-precategory, we will often denote the composition $u *^n_k v$ by $u *_k v$. If $u$ is an $n$-arrow, $1_u$ will denote the iterated identity $k^n_m(u)$ of $u$ in a dimension $m \geq n$ clear by the context.

**Definition 2.1.** *A* strict $\infty$-category *is an $\infty$-precategory $X$ such that the following axioms are satisfied:*

1. *Associativity*
   *for every triple $(u, v, w)$ in $(X_i, s^i_j) \times_{X_j} (t^i_j, X_i, s^i_j) \times_{X_j} (t^i_j, X_i)$ with $i > j \geq 0$, we have*

$$(u *_j v) *_j w = u *_j (v *_j w);$$

2. *Exchange law*
*for every 4-uple* $(u, u', v, v')$ *in*

$$(X_i, s^i_j) \times_{X_j} (t^i_j, X_i, s^i_k) \times_{X_k} (t^i_k, X_i, s^i_j) \times_{X_j} (t^i_j, X_i),$$

*with* $i > j > k \geq 0$*, we have*

$$(u *_j u') *_k (v *_j v') = (u *_k v) *_j (u' *_k v');$$

3. *Units*
*for every* $u$ *in* $X_i$ *with* $i \geq 1$ *and every* $j$ *such that* $i > j \geq 0$*, we have*

$$u *_j 1_{s_j(u)} = u = 1_{t_j(u)} *_j u;$$

4. *Functoriality of units*
*for every couple* $(u, v)$ *in* $(X_i, s^i_j) \times_{X_j} (t^i_j, X_i)$ *with* $i > j \geq 0$*, we have the following equality between* $(i+1)$*-arrows:*

$$1_{u *_j v} = 1_u *_j 1_v.$$

*A* morphism of strict $\infty$-categories *or* $\infty$-functor *is a morphism of globular sets compatible with the maps* $*^i_j$ *and* $k_i$*.*

*We will denote by* $\infty$-**Cat** *the category of* $\infty$*-categories.*

This category is a full reflexive subcategory of the presheaf category of globular sets. Moreover, it is stable under filtered colimits. Hence, by Theorem 1.46 of [1], $\infty$-**Cat** is locally presentable.

Note that if $u$ and $v$ are two $n$-arrows of a strict $\infty$-category $C$, the globular set $C(u, v)$ inherits a structure of strict $\infty$-category.

## 2.3 Strict $\infty$-groupoids

Let $C$ be a strict $\infty$-category and $u$ an $i$-arrow for $i \geq 1$. For $j$ such that $0 \leq j < i$, a $*^i_j$*-inverse* $v$ of $u$ is an $i$-arrow such that $s_j(v) = t_j(u)$ and $t_j(v) = s_j(u)$ satisfying

$$u *_j v = 1_{t_j(u)} \quad \text{and} \quad v *_j u = 1_{s_j(u)}.$$

It is easy to see that if it exists, such an inverse is unique. For $i > j \geq 0$, we will say that $C$ *admits* $*^i_j$*-inverses* if every $i$-arrow of $C$ admits a $*^i_j$-inverse.

**Definition 2.2.** *A* strict $\infty$-groupoid *is a strict* $\infty$*-category which admits* $*^i_j$*-inverses for every* $i > j \geq 0$*. We will denote by* $\infty$-**Grp** *the full subcategory of* $\infty$-**Cat** *whose objects are strict* $\infty$*-groupoids.*

The same argument as for $\infty$-**Cat** shows that $\infty$-**Grp** is a locally presentable category.

If $G$ is a strict $\infty$-groupoid and $u$ is an $i$-arrow of $G$ for $i \geq 1$, we will denote by $w^i_j(u)$ or simply by $w_j(u)$ the $*^i_j$-inverse of $u$ and by $u^{-1}$ the $*^i_{i-1}$-inverse. Note that if $u$ and $v$ are two $n$-arrows of a strict $\infty$-groupoid $G$, the strict $\infty$-category $G(u, v)$ is a strict $\infty$-groupoid.

**Proposition 2.3.** *Let* $C$ *be a strict* $\infty$*-category. Then the following assertions are equivalent:*

1. $C$ *is a strict* $\infty$*-groupoid;*
2. $C$ *admits* $*^i_{i-1}$*-inverses for every* $i \geq 1$*;*
3. $C$ *admits* $*^i_0$*-inverses for every* $i \geq 1$*;*
4. *for all* $i \geq 1$*, there exists* $j$ *satisfying* $0 \leq j < i$ *such that* $C$ *admits* $*^i_j$*-inverses.*

*Proof.* By induction, it is enough to show that if $i > j > k \geq 0$ and $C$ admits $*^j_k$-inverses, then $C$ admits $*^i_k$-inverses if and only if it admits $*^i_j$-inverses. By using the fact that 2-graph

$$C_i \underset{t^i_j}{\overset{s^i_j}{\rightrightarrows}} C_j \underset{t^j_k}{\overset{s^j_k}{\rightrightarrows}} C_k$$

has a natural structure of 2-category, one can assume that $k = 0$, $j = 1$ and $i = 2$. The result is thus a consequence of the following lemma. □

**Lemma 2.4.** *Let $C$ be a 2-category whose 1-arrows are invertible. Then a 2-arrow is invertible for horizontal composition (i.e. $*_0^2$) if and only if it is invertible for vertical composition (i.e. $*_1^2$).*

*Proof.* Let $a : u \to v$ be a 2-arrow. Suppose $a$ admits a horizontal inverse $a^*$. Then $v *_0 a^* *_0 u$ is a vertical inverse. Conversely, suppose $a$ admits a vertical inverse $a^{-1}$. Then $v^{-1} *_0 a^{-1} *_0 u^{-1}$ is a horizontal inverse. □

## 2.4 Weak equivalences of strict $\infty$-groupoids

Let $G$ be a strict $\infty$-groupoid. An $n$-arrow $u$ of $G$ is *homotopic* to another $n$-arrow $v$ if there exists a $(n+1)$-arrow from $u$ to $v$. This implies that the arrows $u$ and $v$ are parallel. If $u$ is homotopic to $v$, we will write $u \sim v$. The relation $\sim$ is an equivalence relation on $G_n$: the properties with respect to source and target of the maps $k_{n+1}$, $w_{n-1}^n$ and $*_{n-1}^n$ imply respectively that $\sim$ is reflexive, symmetric and transitive.

Let us denote by $\overline{G_n}$ the quotient of $G_n$ by $\sim$. The composition

$$*_{n-1}^n : G_n \times_{G_{n-1}} G_n \to G_n$$

induces a map

$$*_{n-1}^n : \overline{G_n} \times_{G_{n-1}} \overline{G_n} \to \overline{G_n},$$

thanks to the properties with respect to source and target of the composition $*_{n-1}^{n+1}$. For $n \geq 1$, we can thus define a groupoid $\varpi_n(G)$ whose objects are $(n-1)$-arrows of $G$ and whose morphisms are elements of $\overline{G_n}$. It is clear that $\varpi_n$ defines a functor from the category of strict $\infty$-groupoids to the category of groupoids.

**Definition 2.5.** *The set of* connected components *of $G$ is*

$$\pi_0(G) = \pi_0(\varpi_1(G)) = \overline{G_0}.$$

*For $n \geq 1$ and $x$ an object of $G$, the* $n$-th homotopy group *of $G$ at $x$ is*

$$\pi_n(G, x) = \pi_1(\varpi_n(G), 1_x) = \mathsf{Aut}_{\varpi_n(G)}(1_x).$$

By functoriality of the $\varpi_n$'s, $\pi_0$ induces a functor from the category of strict $\infty$-groupoids to the category of sets and $\pi_n$, for $n \geq 1$, induces a functor from the category of pointed strict $\infty$-groupoids to the category of groups. By the Eckmann-Hilton argument, the groups $\pi_n(G, x)$ are abelian for $n \geq 2$. More generally, if $u, v$ are two $(n-1)$-arrows for $n \geq 1$ we set

$$\pi_n(G, u, v) = \mathsf{Hom}_{\varpi_n(G)}(u, v) \quad \text{and} \quad \pi_n(G, u) = \pi_n(G, u, u).$$

**Definition 2.6.** *A morphism $f : G \to H$ of strict $\infty$-groupoids is a* weak equivalence of strict $\infty$-groupoids *if the map $\pi_0(f) : \pi_0(G) \to \pi_0(H)$ is a bijection and if for all $n \geq 1$ and all object $x$ of $G$, the morphism $\pi_n(f, x) : \pi_n(G, x) \to \pi_n(H, f(x))$ is a group isomorphism. We will denote by $\mathcal{W}_{\mathrm{gr}}$ the class of such weak equivalences.*

**Proposition 2.7.** *Let $f : G \to H$ a morphism of strict $\infty$-groupoids. The following conditions are equivalent:*

1. *$f$ is a weak equivalence of strict $\infty$-groupoids;*
2. *$\pi_0(f) : \pi_0(G) \to \pi_0(H)$ is a bijection and for all $n \geq 1$ and every $(n-1)$-arrow $u$ of $G$, $f$ induces a bijection*
$$\pi_n(G, u) \to \pi_n(H, f(u));$$
3. *$\varpi_1(f) : \varpi_1(G) \to \varpi_1(H)$ is an equivalence of categories and for all $n \geq 2$ and every couple $(u, v)$ of parallel $(n-1)$-arrows of $G$, $f$ induces a bijection*
$$\pi_n(G, u, v) \to \pi_n(H, f(u), f(v));$$

4. *$\varpi_1(f) : \varpi_1(G) \to \varpi_1(H)$ is full and essentially surjective, and for all $n \geq 2$ and every couple $(u, v)$ of parallel $(n-1)$-arrows of $G$, $f$ induces a surjection*

$$\pi_n(G, u, v) \to \pi_n(H, f(u), f(v)).$$

*Proof.* $1 \Rightarrow 2$) The case $n = 1$ is obvious. Let $n \geq 2$ and $u$ be an $(n-1)$-arrow of $G$. Set $x = s_0(u)$. The map

$$\pi_n(G, x) \to \pi_n(G, u)$$

which sends an $n$-arrow $a : 1_x \to 1_x$ on the $n$-arrow $1_u *_0 a : u \to u$, is an isomorphism. Moreover $f$ commutes to this isomorphism, that is the square

$$\begin{array}{ccc} \pi_n(G, x) & \longrightarrow & \pi_n(G, u) \\ \downarrow & & \downarrow \\ \pi_n(H, f(x)) & \longrightarrow & \pi_n(H, f(u)) \end{array}$$

is commutative. The map $\pi_n(G, u) \to \pi_n(H, f(u))$ is thus a bijection for $n \geq 2$.

$2 \Rightarrow 3$) Let $n \geq 1$ and $u, v$ two parallel $(n-1)$-arrows of $G$. Suppose there exists an $n$-arrow $a : u \to v$ in $G$. The map

$$\pi_n(G, u) \to \pi_n(G, u, v)$$

which sends an $n$-arrow $b : u \to u$ to the $n$-arrow $a *_{n-1} b : u \to v$, is a bijection. Moreover $f$ commutes to this bijection, that is the square

$$\begin{array}{ccc} \pi_n(G, u) & \longrightarrow & \pi_n(G, u, v) \\ \downarrow & & \downarrow \\ \pi_n(H, f(u))) & \longrightarrow & \pi_n(H, f(u), f(v)) \end{array}$$

is commutative.

Thus to conclude it is enough to show that if there exists an $n$-arrow $b : f(u) \to f(v)$ in $H$, then there exists an $n$-arrow $a : u \to v$ in $G$. It is clear when $n = 1$ by injectivity of $\pi_0(F)$. Let $n \geq 2$ and $b : f(u) \to f(v)$ an $n$-arrow of $H$. Set $x = s_{n-2}(u)$. The arrow $1_{f(u)^{-1}} *_{n-2} b$ is an $n$-arrow of $H$ from $1_{f(x)} : f(x) \to f(x)$ to $f(u)^{-1} *_{n-2} f(v) : f(x) \to f(x)$. Since the map

$$\pi_{n-1}(G, x) \to \pi_{n-1}(H, f(x))$$

is injective, there exists an $n$-arrow $a'$ of $G$ from $1_x$ to $u^{-1} *_{n-2} v$. Then $a = 1_u *_{n-2} a'$ is an $n$-arrow of $G$ from $u$ to $v$.

$3 \Rightarrow 1$) Obvious.

$4 \Rightarrow 3$) Let $n \geq 1$, let $u, v$ be two parallel $(n-1)$-arrows of $G$ and $a, b$ be two $n$-arrows from $u$ to $v$. Suppose we have $f(a) = f(b)$ in $\pi_n(H, f(u), f(v))$. Then there exists a $(n+1)$-arrow in $H$ from $f(a)$ to $f(b)$. By surjectivity of the map

$$\pi_{n+1}(G, a, b) \to \pi_{n+1}(H, f(a), f(b)),$$

there exists an $(n+1)$-arrow in $G$ from $a$ to $b$. Thus $a = b$ in $\pi_n(G, u, v)$.

$3 \Rightarrow 4$) Obvious.

□

# 3 The Brown-Golasiński model structure

In [4], Brown and Golasiński introduce a model category structure on the category of crossed complexes. By the equivalence of categories between crossed complexes and strict $\infty$-groupoids constructed in [5], this model structure induces a model structure on strict $\infty$-groupoids. The purpose of this section is to describe this model structure.

## 3.1 Crossed complexes

Let us denote by **Grp** the category of groups and by $\mathbf{CGrp}_{\geq 2}$ the category of homological complexes of (not necessarily commutative) groups in dimension greater or equal to 2, that is of sequences of maps of groups

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_3 \xrightarrow{d_3} C_2$$

such that for every $n \geq 4$, we have $d_n d_{n-1} = 1$, where 1 denotes the unit element of $C_{n-2}$. We have an inclusion functor $i_2 : \mathbf{Grp} \to \mathbf{CGrp}_{\geq 2}$ which sends a group $G$ to the complex concentrated in degree 2 of value $G$.

Let $C_{\leq 1}$ be a groupoid. We will denote by $C_0$ its set of objects and by $C_1(x, y)$ the set of morphisms from an object $x$ to an object $y$ in $C_{\leq 1}$. Let $C_1 : C_{\leq 1} \to \mathbf{Grp}$ be the functor defined in the following way: an object $x$ of $C_{\leq 1}$ is sent to the group $C_1(x) = C_1(x, x)$; a morphism $u : x \to y$ of $C_{\leq 1}$ is sent to the morphism of conjugation by $u$, i.e.

$$\begin{aligned} C_1(x) &\to C_1(y) \\ v &\mapsto uvu^{-1}. \end{aligned}$$

**Definition 3.1.** *A* precrossed complex *is the data of*

- *a groupoid $C_{\leq 1}$;*
- *a functor $C_{\geq 2} : C_{\leq 1} \to \mathbf{CGrp}_{\geq 2}$;*
- *an augmentation of $C_{\geq 2}$ over $C_1$, that is a natural transformation $d_2 : C_{\geq 2} \to i_2 C_1$.*

Explicitly, a precrossed complex is given by

- for every $x$ in $C_0$, a complex

$$C_{\geq 2}(x) = \cdots \to C_n(x) \xrightarrow{d_n} C_{n-1}(x) \to \cdots \to C_3(x) \xrightarrow{d_3} C_2(x);$$

- for every $x$ in $C_0$, a morphism $d_2 : C_2(x) \to C_1(x)$;
- for every $n \geq 2$ and every morphism $u : x \to y$ of $C_{\leq 1}$, a morphism $C_n(x) \to C_n(y)$ of groups functorial in $u$,

such that for every morphism $u : x \to y$ of $C_{\leq 1}$, the diagram

$$\begin{array}{ccccccccc} \longrightarrow & C_n(x) & \xrightarrow{d_n} & C_{n-1}(x) & \longrightarrow & \cdots & \longrightarrow & C_2(x) & \xrightarrow{d_2} & C_1(x) \\ & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \longrightarrow & C_n(y) & \xrightarrow{d_n} & C_{n-1}(y) & \longrightarrow & \cdots & \longrightarrow & C_2(y) & \xrightarrow{d_2} & C_1(y), \end{array}$$

where $C_1(x) \to C_1(y)$ is the conjugation by $u$, is commutative.

If $C$ is a precrossed complex we will denote by $C(x)$ the augmented complex

$$\cdots \to C_n(x) \xrightarrow{d_n} C_{n-1}(x) \to \cdots \to C_3(x) \xrightarrow{d_3} C_2(x) \xrightarrow{d_2} C_1(x).$$

If $u : x \to y$ is a morphism of $C_{\leq 1}$ and $n \geq 1$, we will call the map $C_n(x) \to C_n(y)$ the *action* of $u$ on $C_n(x)$. In particular, an element of $C_1(x)$ *acts* on $C_n(x)$ for all $n \geq 1$.

**Definition 3.2.** *A precrossed complex $C$ is a* crossed complex *if for all $x$ in $C_0$ the following conditions are satisfied:*

- *for every $n \geq 3$, the group $C_n(x)$ is abelian;*
- *for every $u$ in $C_2(x)$, the element $d_2(u)$ of $C_1(x)$ acts*
  - *by conjugation by $u$ on $C_2(x)$;*
  - *trivially on $C_n(x)$ for $n \geq 3$.*

**Definition 3.3.** *Let $C$ and $D$ be two crossed complexes. A morphism $f : C \to D$ is the data of*

- *a functor* $f_{\leq 1} : C_{\leq 1} \to D_{\leq 1}$*;*
- *a natural transformation* $f_{\geq 2} : C_{\geq 2} \to D_{\geq 2} f_{\leq 1}$*,*

*such that* $f_{\geq 2}$ *is compatible with the augmentation, i.e. the square*

$$\begin{array}{ccc} C_{\geq 2} & \longrightarrow & D_{\geq 2} f_{\leq 1} \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_2 f_{\leq 1}} \\ i_2 C_1 & \longrightarrow & i_2 D_1 f_{\leq 1} \end{array}$$

*is commutative.*

Explicitly, a morphism $f : C \to D$ is given by

- a functor $f_{\leq 1} : C_{\leq 1} \to D_{\leq 1}$;
- for every $n \geq 2$ and every $x$ in $C_0$, a group morphism $f_n : C_n(x) \to D_n(f_0(x))$, where $f_0$ denotes the restriction of $f_{\leq 1}$ to objects,

such that for every $x$ in $C_0$, the diagram

$$\begin{array}{ccccccccc} \longrightarrow & C_n(x) & \xrightarrow{d_n} & C_{n-1}(x) & \longrightarrow & \cdots & \longrightarrow & C_1(x) \\ & {\scriptstyle f_n}\downarrow & & {\scriptstyle f_{n-1}}\downarrow & & & & {\scriptstyle f_1}\downarrow \\ \longrightarrow & D_n(f_0(x)) & \xrightarrow{d_n} & D_{n-1}(f_0(x)) & \longrightarrow & \cdots & \longrightarrow & D_1(f_0(x)), \end{array}$$

where $f_1$ denotes the restriction of $f_{\leq 1}$ to $C_1(x)$, is commutative. We will often simply denote $f_n$ by $f$.

We will denote the category of crossed complexes by **CrC** .

## 3.2 The equivalence with strict $\infty$-groupoids

Let $G$ be an $\infty$-groupoid. One can associate to $G$ a precrossed complex $C$ in the following way:

- the groupoid $C_{\leq 1}$ is the 1-truncation of $G$ obtained from $G$ by throwing out the $n$-arrows for $n \geq 2$;
- $C_n(x)$ is the set of $n$-arrows $u$ of $G$ such that their source is an iterated unit of an object, that is such that there exists an object $x$ of $G$ such that $s_{n-1}(u) = 1_x$;
- the group law on $C_n(x)$ is induced by the composition $*_0^n : G_n \times_{G_0} G_n \to G_n$;
- $d_n : C_n(x) \to C_{n-1}(x)$ is induced by the target map $t_{n-1} : G_n \to G_{n-1}$;
- if $u : x \to y$ is a morphism in $C_{\leq 1}$ and $v$ a morphism in $C_n(x)$ for $n \geq 2$, the action of $u$ on $v$ is
$$u *_0 v *_0 w_0(u).$$

This precrossed complex is a crossed complex (see Paragraph 3 of [5]). Moreover this construction defines a functor $\mathrm{A} : \infty\text{-}\mathbf{Grp} \to \mathbf{CrC}$.

**Theorem 3.4** (Brown-Higgins)**.** *The functor* $\mathrm{A} : \infty\text{-}\mathbf{Grp} \to \mathbf{CrC}$ *is an equivalence of categories.*

*Proof.* See Theorem 4.1 of [5]. □

## 3.3 The model structure on crossed complexes

**Definition 3.5.** *Let $C$ be a crossed complex. The set of* connected components *of $C$ is*

$$\pi_0(C) = \pi_0(C_{\leq 1}).$$

*For $x$ in $C_0$, the* fundamental group of $C$ at $x$ *is*

$$\pi_1(C, x) = \mathrm{Coker}(d_2 : C_2(x) \to C_1(x)),$$

*and for $n \geq 2$, the* $n$-th homotopy group of $C$ at $x$ *is*

$$\pi_n(C, x) = H_n(C(x)).$$

It is clear that $\pi_0$ defines a functor from the category of crossed complexes to the category of sets and that for $n \geq 1$, $\pi_n$ defines a functor from the category of pointed crossed complexes to the category of groups.

**Definition 3.6.** *A morphism $f : C \to D$ of crossed complexes is a* weak equivalence of crossed complexes *(see [4]) if the map $\pi_0(f) : \pi_0(C) \to \pi_0(D)$ is bijection and if for every $x$ in $C_0$ and every $n \geq 1$, the morphism $\pi_n(f, x) : \pi_n(C, x) \to \pi_n(D, f(x)))$ is an isomorphism.*

**Definition 3.7.** *Let $f : C \to D$ be a morphism of crossed complexes. The morphism $f$ is a* trivial fibration of crossed complexes *(see Proposition 2.2 (ii) of [4]) if the following condition are satisfied:*

- *for every $y$ in $D_0$, there exists $x$ in $C_0$ such that $f(x) = y$;*
- *for every $x$, $y$ in $C_0$ and every morphism $v : f(x) \to f(y)$ in $D_{\leq 1}$, there exists a morphism $u : x \to y$ in $C_{\leq 1}$ such that $f(u) = v$;*
- *for every $n \geq 2$, $x$ in $C_0$, $t$ in $C_{n-1}(x)$ and every $v$ in $D_n(f(x))$ such that $d_n(v) = f(t)$, there exists $u$ in $C_n(x)$ such that $d_n(u) = t$ and $f(u) = v$.*

**Theorem 3.8** (Brown-Golasiński)**.** *The weak equivalences and trivial fibrations of crossed complexes define a model category structure on the category of crossed complexes.*

*Proof.* See Theorem 2.12 of [4]. □

## 3.4 The model structure on strict $\infty$-groupoids

One obtains a model category structure on strict $\infty$-groupoids by transferring the model structure on crossed complexes defined in the previous section via the equivalence of categories $\mathrm{A} : \infty\textbf{-Grp} \to \textbf{CrC}$. We will call this model structure the Brown-Golasiński model structure. A morphism $f$ of strict $\infty$-groupoids is a weak equivalence (respectively a trivial fibration) for the Brown-Golasiński model structure if and only if $\mathrm{A}(f)$ is weak equivalence (respectively a trivial fibration) of crossed complexes. We will denote this classes by $\mathcal{W}_{\mathrm{cc}}$ and $\mathcal{TF}_{\mathrm{cc}}$ respectively.

We now describe these two classes more explicitly.

**Proposition 3.9.** *We have $\mathcal{W}_{\mathrm{cc}} = \mathcal{W}_{\mathrm{gr}}$. In a other words, a morphism of strict $\infty$-groupoids $f : G \to H$ is a weak equivalence of strict $\infty$-groupoids if and only if $\mathrm{A}(f) : \mathrm{A}(G) \to \mathrm{A}(H)$ is a weak equivalence of crossed complexes.*

*Proof.* Since the two notions of weak equivalences are defined in terms of homotopy groups, it is enough to show that the two notions of homotopy groups coincide.

Let $G$ be a groupoid. By definition, $\pi_0(G) = \pi_0(\mathrm{A}(G))$.

Let $x$ be an object of $G$. By definition, $\pi_1(\mathrm{A}(G), x) = \mathrm{Coker}(\mathrm{A}(G)_2(x) \to \mathrm{A}(G)_1(x))$. The set $\mathrm{A}(G)_1(x)$ is the set of 1-arrow $u : x \to x$ in $G$ and two such arrows $u, u'$ are identified in the cokernel if and only if there exists a 2-arrow from $1_x$ to $u *_0 u'^{-1}$ in $G$. This condition is equivalent to the existence of a 2-arrow from $u$ to $u'$. Hence $\pi_1(G, x) = \pi_1(\mathrm{A}(G), x)$.

Let $n \geq 2$. The kernel of the map $d_n : \mathrm{A}(G)_n(x) \to \mathrm{A}(G)_{n-1}(x)$ is the set $G_n(x, x)$. Thus the same argument as in dimension 1 shows that $\pi_n(G, x) = \pi_n(\mathrm{A}(G), x)$. □

**Proposition 3.10.** *A morphism $f : G \to H$ of strict $\infty$-groupoids is in $\mathcal{TF}_{\mathrm{cc}}$ if and only if it satisfies the following conditions:*

- *for every object $y$ of $H$, there exists an object $x$ of $G$ such that $f(x) = y$;*
- *for every objects $x, y$ of $G$, the map*

$$G(x, y)_0 \to G(f(x), f(y))_0$$

*induced by $f$ is a surjection;*

- *for all $n \geq 2$, every object $x$ of $G$ and every $(n-1)$-arrow $u : 1_x \to 1_x$, the map*

$$G(1_x, u)_0 \to H(1_{f(x)}, f(u))_0$$

*is surjective.*

*Proof.* By definition, $f$ is in $\mathcal{TF}_{\mathrm{cc}}$ if and only if $\mathrm{A}(f)$ is a trivial fibration of crossed complexes. This proposition is then just a matter of translation using the definition of the functor A. □

# 4 The folk model structure on $\infty$-Grp

This section shows that the folk model structure on $\infty$-**Cat** defined in [11] transfers to $\infty$-**Grp** via the inclusion functor

$$U : \infty\text{-}\mathbf{Grp} \to \infty\text{-}\mathbf{Cat}.$$

We first give a brief review of the main results of [11], and introduce the material needed to prove the transfer theorem.

## 4.1 The folk model structure on $\infty$-Cat

Given an $\infty$-category $C$, we define *reversible* cells in $C$ and the relation of *$\omega$-equivalence* between cells of $C$ by mutual coinduction on $n \geq 0$:

**Definition 4.1.** *Let $n \in \mathbb{N}$:*

- *an $n$-cell $x$ is* $\omega$-equivalent *to an $n$-cell $y$ if there is a reversible $(n+1)$-cell $u : x \to y$;*
- *an $(n+1)$-cell $u : x \to y$ is* reversible *if there is an $(n+1)$-cell $\bar{u} : y \to x$ such that $\bar{u} *_n u$ is $\omega$-equivalent to $1_x$ and $u *_n \bar{u}$ is $\omega$-equivalent to $1_y$.*

Note that, for each $r \geq 0$, if two cells are $r$-equivalent in the sense of [13], then they are $\omega$-equivalent, the converse being false. We also refer to [8] for a gentle introduction to coinductive methods. Remark also that if $G$ is an $\infty$-groupoid, any $(n+1)$-cell $u$ of $G$ is reversible and the cell $\bar{u}$ whose existence is stated in Definition 4.1 is of course just $u^{-1}$.

Let $\mathcal{W}$ denote the class of those morphisms $f : C \to D$ satisfying the following two conditions:

1. for each 0-cell $y$ in $D$, there is a 0-cell $x$ in $C$ such that $fx$ is $\omega$-equivalent to $y$;
2. for each pair $\langle x, x' \rangle$ of parallel $n$-cells in $C$, where $n \geq 0$, and each $(n+1)$-cell $v : fx \to fx'$, there is an $(n+1)$-cell $u : x \to x'$ such that $fu$ is $\omega$-equivalent to $v$.

Now for each $n \geq 0$, we define the *$n$-globe* $\mathbf{O}^n$ as the free $\infty$-category generated by the representable globular set $Y(n) = \mathbf{O}(n, -)$. Thus $Y(n)$ has exactly one $n$-cell, two $k$-cells for each $k < n$ and no $k$-cell for $k > n$. Let also $\partial Y(n)$ be the globular set having the same cells as $Y(n)$ except in dimension $n$ where $(\partial Y(n))_n = \emptyset$. We denote by $\partial \mathbf{O}^n$ the free $\infty$-category on $\partial Y(n)$. We finally have, for each $n \geq 0$, an inclusion morphism

$$\mathbf{i}_n : \partial \mathbf{O}^n \to \mathbf{O}^n.$$

The set $\{\mathbf{i}_n \mid n \in \mathbb{N}\}$ is denoted by $I$.

A map is a *trivial fibration* if it has the right-lifting property with respect to $I$ and a *cofibration* if it has the left-lifting property with respect to all trivial fibrations.

**Proposition 4.2.** *Any $\infty$-functor $f$ factors as $f = p \circ k$ where $p$ is a trivial fibration and $k$ is a cofibration.*

*Proof.* By a standard application of the small object argument, using the fact that $\infty$-**Cat** is locally presentable (see [7]). □

On the other hand the maps $\sigma_n, \tau_n : n \to n+1$ in the globular category $\mathbf{O}$ (see Section 2) induce corresponding maps from $\mathbf{O}^n$ to $\mathbf{O}^{n+1}$, of the form $\mathbf{i}_{n+1} \circ \hat{\sigma}_n$ and $\mathbf{i}_{n+1} \circ \hat{\tau}_n$ respectively, where $\hat{\sigma}_n, \hat{\tau}_n : \mathbf{O}^n \to \partial\mathbf{O}^{n+1}$. Moreover, we get a pushout diagram

$$\begin{array}{ccc} \partial\mathbf{O}^n & \xrightarrow{\mathbf{i}_n} & \mathbf{O}^n \\ {\scriptstyle \mathbf{i}_n}\downarrow & & \downarrow{\scriptstyle \hat{\sigma}_n} \\ \mathbf{O}^n & \xrightarrow[\hat{\tau}_n]{} & \partial\mathbf{O}^{n+1}. \end{array}$$

Now the above pushout determines a canonical map

$$\mathbf{o}_n : \partial\mathbf{O}^{n+1} \to \mathbf{O}^n$$

such that $\mathbf{o}_n \circ \hat{\sigma}_n = \mathbf{o}_n \circ \hat{\tau}_n = \mathrm{id}_{\mathbf{O}^n}$. Proposition 4.2 applies to $\mathbf{o}_n$, yielding an object $\mathbf{P}^n$ together with a trivial fibration $\mathbf{p}_n : \mathbf{P}^n \to \mathbf{O}^n$ and a cofibration $\mathbf{k}_n : \partial\mathbf{O}^{n+1} \to \mathbf{P}^n$ satisfying $\mathbf{o}_n = \mathbf{p}_n \circ \mathbf{k}_n$. We finally define $\mathbf{j}_n : \mathbf{O}^n \to \mathbf{P}^n$ as $\mathbf{k}_n \circ \hat{\sigma}_n$ and

$$J = \{\mathbf{j}_n \mid n \in \mathbb{N}\}.$$

**Theorem 4.3.** *There is a cofibrantly generated model structure on $\infty$-**Cat** where $\mathcal{W}$ is the class of weak equivalences, $I$ a set of generating cofibrations and $J$ a set of generating trivial cofibrations.*

This statement is in fact [11, Theorem 4.39] and the main result of that article.

## 4.2 Path object

Let $C$ be an object in a model category and $\Delta_C : C \to C \times C$ be the diagonal map. A *path object* for $C$ consists in an object $P_C$ together with a factorization of $\Delta_C$ of the form

$$C \xrightarrow{j} P_C \xrightarrow{p} C \times C, \qquad \Delta_C : C \to C \times C$$

where $p$ is a fibration and $j$ is a weak equivalence. Such a $P_C$ is not unique: in the case of $\infty$-**Cat**, one particular choice is given by the functor $\Gamma$ we now describe. We first define, by induction on $n$, the notion of *$n$-cylinder between $n$-cells $x$, $y$ of an $\infty$-category $C$*. A few notations will be useful: for each $n$-cell $x$ we denote by $x^\flat$, respectively $x^\sharp$ its 0-source $s_0 x$, respectively 0-target $t_0 x$. Now let $C$ be an $\infty$-category and $x$, $y$ two 0-cells in it, there is an $\infty$-category $C(x, y)$ whose $n$-cells are the $(n+1)$-cells $u$ of $C$ such that $u^\flat = x$ and $u^\sharp = y$. Whenever $u$ is such an $(n+1)$-cell of $C$, we denote by $[u]$ the corresponding $n$-cell of $C(x, y)$. Finally, let $x$, $y$, $z$ be 0-cells of $C$, each 1-cell $u : x \to y$ determines an $\infty$-functor $- \cdot u : C(y, z) \to C(x, z)$ given by $[v] \cdot u = [v *_0 u]$. Likewise $u : y \to z$ determines an $\infty$-functor $u \cdot - : C(x, y) \to C(x, z)$ by $u \cdot [v] = [u *_0 v]$.

**Definition 4.4.** *1. a* 0-cylinder *$U : x \curvearrowright y$ in $C$ is given by a reversible* 1-*cell $U^\natural : x \to y$;*

*2. if $n > 0$, an* $n$-cylinder *$U : x \curvearrowright y$ in $C$ is given by two reversible 1-cells $U^\flat : x^\flat \to y^\flat$ and $U^\sharp : x^\sharp \to y^\sharp$, together with some $n-1$-cylinder $[U] : U^\sharp \cdot [x] \curvearrowright [y] \cdot U^\flat$ in the $\infty$-category $[x^\flat, y^\sharp] = C(x^\flat, y^\sharp)$.*

If $U : x \curvearrowright y$ is an $n$-cylinder in $C$, we write $\pi^1_C U$ and $\pi^2_C U$ for the $n$-cells $x$ and $y$, or simply $\pi^1 U$ and $\pi^2 U$. Figure 1 represents $n$-cylinders for $n = 0$ and $n = 1$. For each $n \in \mathbb{N}$, any $n+1$-cylinder $W : z \curvearrowright z'$ in an $\infty$-category $C$ determines a pair of $n$-cylinders in $C$:

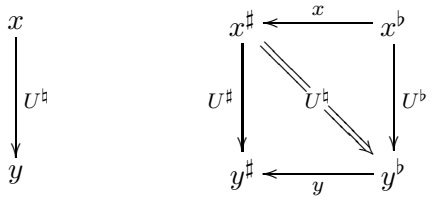



Figure 1: $n$-cylinders for $n = 0, 1$

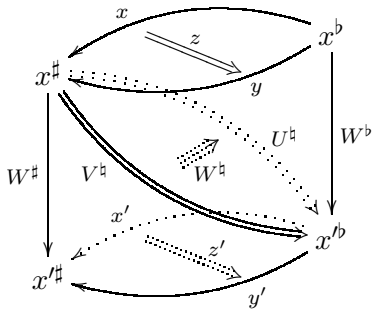



Figure 2: source and target of a 2-cylinder

**Definition 4.5.** *The* source $n$-cylinder $U : x \curvearrowright x'$ *and the* target $n$-cylinder $V : y \curvearrowright y'$ *of the* $n{+}1$*-cylinder* $W : z \curvearrowright z'$ *between* $n{+}1$*-cells* $z : x \to y$ *and* $z' : x' \to y'$ *are defined inductively by:*

- *if* $n = 0$*, then* $U^\natural = W^\flat$ *and* $V^\natural = W^\sharp$*;*
- *if* $n > 0$*, then* $U^\flat = V^\flat = W^\flat$ *and* $U^\sharp = V^\sharp = W^\sharp$*, whereas the two* $n{-}1$*-cylinders* $[U]$ *and* $[V]$ *are respectively defined as the source and the target of the* $n$*-cylinder* $[W]$ *in the* $\infty$*-category* $[z^\flat, z'^\sharp]$*.*

If $W$ has source $U$ and target $V$ we write $W : U \to V$ or $W : U \to V \mid z \curvearrowright z'$ (see Figure 2). It turns out that the source and target maps so defined satisfy the globular relations, so that the correspondence

$$n \mapsto \{U \mid U \text{ is an } n{-}\text{cylinder in } C\}$$

determines a globular set $\Gamma(C)$. We now turn to *trivial* $n$*-cylinders*:

**Definition 4.6.** *The* trivial $n$-cylinder $\tau\, x : x \curvearrowright x$ *on the* $n$*-cell* $x$ *is defined inductively by:*

- *if* $n = 0$*, then* $(\tau\, x)^\natural = 1_x$*;*
- *if* $n > 0$*, then* $(\tau\, x)^\flat = 1_{x^\flat}$ *and* $(\tau\, x)^\sharp = 1_{x^\sharp}$*, whereas* $[\tau\, x]$ *is the trivial cylinder* $\tau[x]$ *in* $[x^\flat, x^\sharp]$*.*

We write $\tau_C\, x$ in case we need to mention the ambient $\infty$-category $C$.

Let us finally recall from [11, Appendix A] that $\Gamma(C)$ becomes a strict $\infty$-category when defining units and compositions as follows :

**Definition 4.7.** *Let* $U : x \curvearrowright y$ *be an* $n$*-cylinder. We define the* $(n+1)$*-cylinder* $1_U : U \to U \mid 1_x \curvearrowright 1_y$ *by induction on* $n$*:*

- *if* $n = 0$*, then* $(1_U)^\flat = (1_U)^\sharp = U^\natural$*, whereas* $[1_U] = \tau[1_{U^\natural}]$*;*
- *if* $n > 0$*, then* $(1_U)^\flat = U^\flat$ *and* $(1_U)^\sharp = U^\sharp$*, whereas* $[1_U] = 1_{[U]}$*.*

In order to define composition, we first introduce the operation of *concatenation*:

**Definition 4.8.** *Let* $U \mid x \curvearrowright y$ *and* $V \mid y \curvearrowright z$ *be two* $n$*-cylinders. The* concatenation $V * U \mid x \curvearrowright z$ *of* $U$ *and* $V$ *is defined by induction on* $n$*:*

- *if* $n = 0$*, then* $(V * U)^\natural = V^\natural *_0 U^\natural$*;*

- *if $n > 0$, then $(V * U)^\flat = V^\flat *_0 U^\flat$, $(V * U)^\sharp = V^\sharp *_0 U^\sharp$ and $[V * U] = [V] \cdot U^\flat * V^\sharp \cdot [U]$.*

**Definition 4.9.** *Let $m \geq 1$, $0 \leq n < m$ and $U \mid x \curvearrowright x'$, $V \mid y \curvearrowright y'$ two $m$-cylinders such that $t_n(U) = s_n(V)$. The* composition *$V *_n U \mid y *_n x \curvearrowright y' *_n x'$ is defined by induction on $n$ as follows:*

- *$(V *_0 U)^\flat = U^\flat$ and $(V *_0 U)^\sharp = V^\sharp$, whereas $[V *_0 U] = y' \cdot [U] * [V] \cdot x$;*
- *if $n > 0$, then $(V *_n U)^\flat = U^\flat = V^\flat$ and $(U *_n V)^\sharp = U^\sharp = V^\sharp$, whereas $[V *_n U] = [V] *_{n-1} [U]$.*

Note that explicit formulas may be found in [12].

For example, Figure 3 shows the composition $V *_0 U$ of two 1-cylinders $U : x \curvearrowright x'$ and $V : y \curvearrowright y'$ such that $U^\sharp = V^\flat$. Precisely, the composite $V *_0 U$ is the 1-cylinder $W : z \curvearrowright z'$ where $z = y *_0 x$, $z' = y' *_0 x'$, $W^\flat = U^\flat$, $W^\sharp = V^\sharp$, and the 0-cylinder $[W]$ of $C(z^\flat, z'^\sharp)$ is the reversible 1-cell of $C(z^\flat, z'^\sharp)$ given by the following corresponding reversible 2-cell of $C$:

$$W^\natural = (y' *_0 U^\natural) *_1 (V^\natural *_0 x).$$

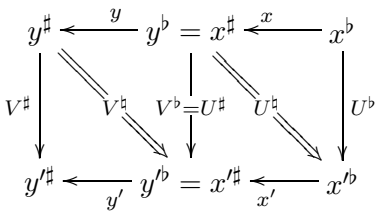



Figure 3: composition of 1-cylinders

The following result summarizes the main properties of $\Gamma$:

**Theorem 4.10.** *The correspondence $C \mapsto \Gamma(C)$ induces an endofunctor on $\infty$-**Cat**, and $\pi^1, \pi^2 : \Gamma \to \mathrm{id}$, $\tau : \mathrm{id} \to \Gamma$ are natural transformations.*

An additional property, of particular importance here, is that $\Gamma$ preserves $\infty$-groupoids:

**Lemma 4.11.** *If $G$ is an $\infty$-groupoid, so is $\Gamma(G)$.*

*Proof.* We show, by induction on $n \geq 1$, that if $G$ is an $\infty$-groupoid and $W : U \to V \mid x \curvearrowright y$ is an $n$-cylinder of $G$, there is an $n$-cylinder $W' : V \to U \mid x^{-1} \curvearrowright y^{-1}$ such that $W' *_{n-1} W = 1_U$ and $W *_{n-1} W' = 1_V$.

- Let $G$ be an $\infty$-groupoid and $W : U \to V \mid x \curvearrowright y$ a 1-cylinder of $G$. By definition, we get two 1-cells $U^\natural : x^\flat \to y^\flat$, $V^\natural : x^\sharp \to y^\sharp$ and a 2-cell $W^\natural : V^\natural *_0 x \to y *_0 U^\natural$ in $G$. Consider $W^{\natural^{-1}} : y *_0 U^\natural \to V^\natural *_0 x$ the $*_1$-inverse of $W^\natural$ and build

$$y^{-1} *_0 W^{\natural^{-1}} *_0 x^{-1} : U^\natural *_0 x^{-1} \to y^{-1} *_0 V^\natural.$$

  If $W' : V \to U \mid x^{-1} \curvearrowright y^{-1}$ is the 1-cylinder of $G$ defined by

$$W'^\natural = y^{-1} *_0 W^{\natural^{-1}} *_0 x^{-1},$$

  we get

$$W' *_0 W = 1_U \quad \text{and} \quad W *_0 W' = 1_V,$$

  which proves the case $n = 1$.

- Let $n > 1$ and suppose that the property holds for $n-1$. Let $G$ be an $\infty$-groupoid and $W : U \to V \mid x \curvearrowright y$ an $n$-cylinder of $G$. We get 1-cells $W^\flat : x^\flat \to y^\flat$, $W^\sharp : x^\sharp \to y^\sharp$ and an $n-1$-cylinder $[W] : [U] \to [V] \mid W^\sharp \cdot [x] \curvearrowright [y] \cdot W^\flat$ in $H = [x^\flat, y^\sharp]$. Now $H$ is an $\infty$-groupoid, so that the induction hypothesis applies and there is an $n-1$-cylinder in $H$

$$[W]' : [V] \to [U] \mid W^\sharp \cdot \left[x^{-1}\right] \curvearrowright \left[y^{-1}\right] \cdot W^\flat$$

such that $[W]' *_{n-2} [W] = 1_{[U]}$ and $[W] *_{n-2} [W]' = 1_{[V]}$. Hence we may define an $n$-cylinder $W'$ of $G$ by $W'^{\flat} = W^{\flat}$, $W'^{\sharp} = W^{\sharp}$ and $[W'] = [W]'$. By construction

$$W' *_{n-1} W = 1_U \quad \text{and} \quad W *_{n-1} W' = 1_V,$$

□

## 4.3 Immersions

We now introduce a class of morphisms which plays an important part in the proof of the transfer theorem.

**Definition 4.12.** *An $\infty$-functor $f : C \to D$ belongs to the class $\mathcal{Z}$ of* immersions *if and only if there exist $\infty$-functors $g : D \to C$ and $h : D \to \Gamma(D)$ satisfying the following properties:*

1. *$g$ is a retraction of $f$, that is $g \circ f = \mathrm{id}_C$;*
2. *$\pi^1_D \circ h = f \circ g$ and $\pi^2_D \circ h = \mathrm{id}_D$;*
3. *$h \circ f = \tau_D \circ f$. In other words, $h$ is trivial on $f(C)$.*

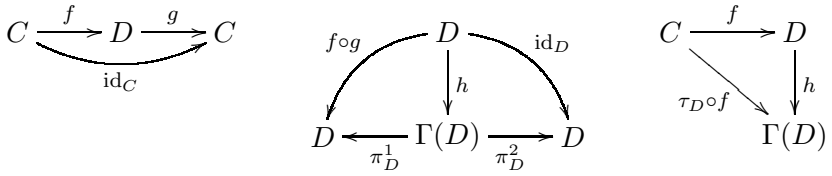



Figure 4: Immersions

The following proposition summarizes the properties of immersions we need here.

**Proposition 4.13.** *The class of immersions satisfies the following properties:*

*(i) $\mathcal{Z}$ is closed by pushout;*

*(ii) all trivial cofibrations are immersions;*

*(iii) all immersions are weak equivalences;*

We refer to [11, Section 4.6] for the proofs of these statements.

## 4.4 Transfer

Let $\mathbf{C}$, $\mathbf{D}$ be two categories and $L : \mathbf{C} \to \mathbf{D}$, $R : \mathbf{D} \to \mathbf{C}$ be a pair of functors with $L$ left adjoint to $R$, and suppose that $\mathbf{C}$ is equipped with a model structure. We may define three classes of maps of $\mathbf{D}$ as follows:

- for each morphism $f$ of $\mathbf{D}$, $f \in \mathcal{W}_{\mathbf{D}}$ if and only if $R(f)$ is a weak equivalence in $\mathbf{C}$;
- for each morphism $f$ of $\mathbf{D}$, $f \in \mathcal{F}_{\mathbf{D}}$ if and only if $R(f)$ is a weak equivalence in $\mathbf{C}$;
- $\mathcal{C}_{\mathbf{D}}$ is the class of maps having the left-lifting property with respect to $\mathcal{W}_{\mathbf{D}} \cap \mathcal{F}_{\mathbf{D}}$.

We say that $R$ *creates* a model structure on $\mathbf{D}$ if $\mathcal{W}_{\mathbf{D}}$, $\mathcal{F}_{\mathbf{D}}$ and $\mathcal{C}_{\mathbf{D}}$ are respectively the weak equivalences, fibrations and cofibrations of a model structure on $\mathbf{D}$. Sufficient conditions for this transfer to hold are given by [6, Theorem 3.3] or [3, Proposition 2.3]. The latter result immediately specializes to the following statement:

**Proposition 4.14.** *Let $\mathbf{C}$ a cofibrantly generated model structure, with $I$ a set of generating cofibrations and $J$ a set of generating trivial cofibrations. If $\mathbf{D}$ is locally presentable, then the following conditions are sufficient for $R$ to create a model structure on $\mathbf{D}$:*

*(C1) the weak equivalences of $\mathbf{C}$ are closed under filtered colimits;*

*(C2) $R$ preserves filtered colimits;*

*(C3) for each generating trivial cofibration $j$ of $\mathbf{C}$, if $g$ is a pushout of $L(j)$ in $\mathbf{D}$, then $R(g)$ is a weak equivalence of $\mathbf{C}$.*

*Moreover, if these conditions hold, the model structure so defined is cofibrantly generated and has $L(I)$ as a set of generating cofibrations and $L(J)$ as a set of generating trivial cofibrations.*

We now turn to the particular case where $\mathbf{C} = \infty\text{-}\mathbf{Cat}$, $\mathbf{D} = \infty\text{-}\mathbf{Grp}$ and $R$ is the inclusion functor $U : \infty\text{-}\mathbf{Grp} \to \infty\text{-}\mathbf{Cat}$. Note that $U$ has a left-adjoint $F : \infty\text{-}\mathbf{Cat} \to \infty\text{-}\mathbf{Grp}$ building the free $\infty$-groupoid on an $\infty$-category, as well as a right-adjoint $M$, building to the maximal $\infty$-groupoid in an $\infty$-category. Let us first establish a few properties about the adjunction $F \dashv U$.

Let $T$ be the monad $UF$ on $\infty\text{-}\mathbf{Cat}$. Remark that, for any $\infty$-groupoid $G$, the free $\infty$-groupoid on the underlying $\infty$-category $U(G)$ is naturally isomorphic to $G$ itself. In other words, the counit $\epsilon : FU \to 1$ is a natural isomorphism. It follows that, for any $\infty$-groupoid $G$, we get an isomorphism

$$\eta_{U(G)} : U(G) \to UFU(G), \tag{1}$$

where $\eta$ denotes the unit of the adjunction. Now, for each $\infty$-category $C$, $T(C)$ is of the form $U(G)$ where $G$ is an $\infty$-groupoid, and so is $\Gamma T(C)$ by Lemma 4.11, so that

$$\eta_{\Gamma T(C)} : \Gamma T(C) \to T\Gamma T(C) \tag{2}$$

is an isomorphism, as a special case of (1). Thus, we may define a natural transformation

$$\lambda : T\Gamma \to \Gamma T$$

by

$$\lambda_C = \eta^{-1}_{\Gamma T(C)} \circ T\Gamma(\eta_C). \tag{3}$$

Note also that the monad multiplication $\mu : T^2 \to T$ is also a natural isomorphism, and we get

$$T(\eta_C) = \eta_{T(C)} = \mu_C^{-1}. \tag{4}$$

We may now state the following result:

**Lemma 4.15.** *The monad $T$ preserves immersions.*

*Proof.* Let $f : C \to D$ be an immersion, and $f' = T(f)$. By Definition 4.12, there are $g : D \to C$ and $h : D \to \Gamma(D)$ such that

$$\begin{aligned} g \circ f &= \mathrm{id}_C; && (5)\\ \pi^1_D \circ h &= f \circ g; && (6)\\ \pi^2_D \circ h &= \mathrm{id}_D; && (7)\\ h \circ f &= \tau_D \circ f. && (8) \end{aligned}$$

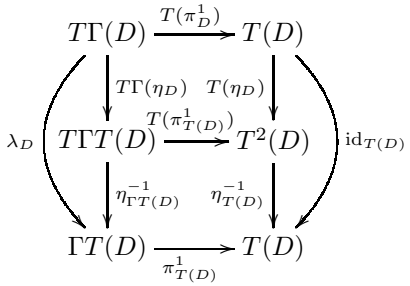
Let $g' = T(g)$ and $h' = \lambda_D \circ T(h)$. We need to establish the following equations:

$$\begin{aligned} g' \circ f' &= \mathrm{id}_{T(C)}; && (9)\\ \pi^1_{T(D)} \circ h' &= f' \circ g'; && (10)\\ \pi^2_{T(D)} \circ h' &= \mathrm{id}_{T(D)}; && (11)\\ h' \circ f' &= \tau_{T(D)} \circ f'. && (12) \end{aligned}$$

Equation (9) is just functoriality. Let us prove (10). First remark that $\pi^1$ is a natural transformation, so that the following diagram commutes:

$$\begin{array}{ccc} \Gamma(D) & \xrightarrow{\pi^1_D} & D \\ {\scriptstyle \Gamma(\eta_D)}\downarrow & & \downarrow{\scriptstyle \eta_D} \\ \Gamma T(D) & \xrightarrow[\pi^1_{T(D)}]{} & T(D) \end{array} . \tag{13}$$

We may now build the following commutative diagram:

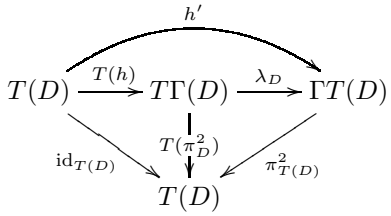



(14)

In fact the upper square is the image of (13) by $T$ and the lower square commutes by naturality of $\eta$. Hence

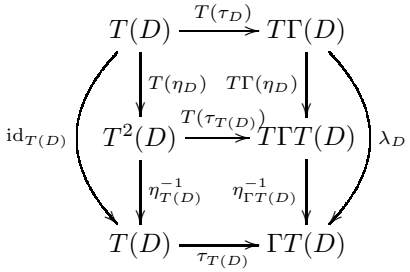
$$\begin{aligned}
\pi^1_{T(D)} \circ h' &= \pi^1_{T(D)} \circ \lambda_D \circ T(h) \\
&= T(\pi^1_D) \circ T(h) \\
&= T(\pi^1_D \circ h) \\
&= T(f \circ g) \\
&= f' \circ g'
\end{aligned}$$

which gives (10). Likewise, we get the following commutative diagram




(15)

where the left hand triangle commutes by applying $T$ to (7), and the right hand triangle commutes by replacing $\pi^1$ with $\pi^2$ in (14). Hence $\pi^2_{T(D)} \circ h' = \mathrm{id}_{T(D)}$ and (11) is proved. Finally, by using the naturality of $\tau$ instead of $\pi^1$, we get a commutative diagram analogue to (14):




(16)

Hence

$$\begin{aligned}
h' \circ f' &= \lambda_D \circ T(h) \circ T(f) \\
&= \lambda_D \circ T(h \circ f) \\
&= \lambda_D \circ T(\tau_D \circ f) \\
&= \lambda_D \circ T(\tau_D) \circ f' \\
&= \tau_{T(D)} \circ f'
\end{aligned}$$

which gives (12) and ends the proof. □

**Lemma 4.16.** *Let $f : C \to D$ be an immersion, and suppose that the following square is a pushout in $\infty$-**Grp**:*

$$\begin{array}{ccc} FC & \xrightarrow{u} & G \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle g} \\ FD & \xrightarrow[v]{} & H \end{array}$$

*Then $U(g)$ is an immersion.*

*Proof.* As $U$ is left adjoint to $M$, it preserves pushouts, so that the following square is a pushout in $\infty$-**Cat**:

$$\begin{array}{ccc} TC & \xrightarrow{U(u)} & UG \\ {\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle U(g)} \\ TD & \xrightarrow[U(v)]{} & UH \end{array}$$

By Lemma 4.15, $T(f)$ is an immersion, and so is its pushout $U(g)$, by Proposition 4.13(i). □

**Lemma 4.17.** *If $j$ be a generating trivial cofibration of $\infty$-**Cat**, and $g$ be a pushout of $F(j)$ in $\infty$-**Grp**, then $U(g)$ is a weak equivalence of $\infty$-**Cat**.*

*Proof.* Let $j$ be a generating trivial cofibration of $\infty$-**Cat**, and $g$ be a pushout of $F(j)$ in $\infty$-**Grp**. By Proposition 4.13(ii), $j$ is an immersion, and so is $U(g)$, by Lemma 4.16. By Proposition 4.13(iii), $U(g)$ is a weak equivalence. □

We may finally state the main result of this section:

**Theorem 4.18.** *The forgetful functor $U : \infty$-**Grp** $\to \infty$-**Cat** creates a model structure on $\infty$-**Grp** in which the weak equivalences are the morphisms $f$ such that $U(f) \in \mathcal{W}$. Moreover, the model structure so defined has $(F(\mathbf{i}_k))_{k\in\mathbb{N}}$ as a family of generating cofibrations, and $(F(\mathbf{j}_k))_{k\in\mathbb{N}}$ as a family of generating trivial cofibrations.*

*Proof.* As the model structure on $\infty$-**Cat** is cofibrantly generated and $\infty$-**Grp** is locally presentable, Proposition 4.14 applies, and it suffices to check conditions *(C1)*, *(C2)* and *(C3)*. Condition*(C1)* is proved in [11], and condition*(C2)* follows from the fact that $U$ has a right-adjoint $M$, hence preserves colimits, and in particular filtered ones. Condition*(C3)* is Lemma 4.17. The statement about generating families follows from Proposition 4.14. □

We call the model structure just defined the *folk model structure* on $\infty$-groupoids. We denote its weak equivalences by $\mathcal{W}_{\text{folk}}$ and its trivial fibrations by $\mathcal{TF}_{\text{folk}}$. Note that a morphism $f$ is in $\mathcal{TF}_{\text{folk}}$ if and only if $U(f)$ is a trivial fibration of $\infty$-**Cat**.

**Corollary 4.19.** *A morphism $f : G \to H$ of $\infty$-groupoids belongs to $\mathcal{TF}_{\text{folk}}$ if and only if*

1. *for each 0-cell $y$ in $H$, there is a 0-cell $x$ in $G$ such that $fx = y$;*
2. *for each pair $\langle x, x'\rangle$ of parallel $n$-cells in $G$, where $n \geq 0$, and each $(n+1)$-cell $v : fx \to fx'$, there is an $(n+1)$-cell $u : x \to x'$ such that $fu = v$.*

# 5 Comparison

In this section, we show that the folk model structure on strict $\infty$-groupoids defined in the previous section coincides with the Brown-Golasiński model structure. To see this, it is sufficient to prove that they have the same weak equivalences and the same trivial fibrations.

**Proposition 5.1.** *We have $\mathcal{W}_{\text{gr}} = \mathcal{W}_{\text{folk}} = \mathcal{W}_{\text{cc}}$.*

*Proof.* We first show that $\mathcal{W}_{\text{folk}} = \mathcal{W}_{\text{gr}}$. In a strict $\infty$-groupoid, two $n$-arrows $f$ and $g$ are $\omega$-equivalent if and only if there exists an $(n+1)$-arrows $a : f \to g$, that is if and only $f$ and $g$ are homotopic. Therefore a morphism of strict $\infty$-groupoids is in $\mathcal{W}_{\text{folk}}$ if and only if it satisfies condition 4 of Proposition 2.7. The statement is thus the equivalence between condition 1 and 4 of this very proposition.

By Proposition 3.9, we have $\mathcal{W}_{\text{cc}} = \mathcal{W}_{\text{gr}}$, hence the result. □

**Proposition 5.2.** *We have* $\mathcal{TF}_{\text{folk}} = \mathcal{TF}_{\text{cc}}$.

*Proof.* To prove the equivalence between the two notions of trivial fibrations, we will use the descriptions of these notions provided by Proposition 3.10 and Corollary 4.19 . The conditions for being in $\mathcal{TF}_{\text{folk}}$ are a priori stronger. Let $f : G \to H$ be a in $\mathcal{TF}_{\text{cc}}$. Let us prove it is actually in $\mathcal{TF}_{\text{folk}}$. There is nothing to prove for the conditions in dimension 0 and 1. Let $n \geq 2$ and $u, v$ be two $(n-1)$-arrows. We want to show that the map

$$G(u,v)_0 \to H(f(u), f(v))_0$$

is surjective. Let $b$ be a $(n+1)$-morphism $f(u) \to f(v)$ in $H$. Set $x = s_0(u)$. Then $b' = 1_{w_0(f(u))} *_0^n b$ is an $(n+1)$-arrow of $H$ from $1_{f(x)}$ to $w_0(f(u)) *_0 f(v)$. Since the map

$$G(1_x, w_0(u) *_0 v)_0 \to H(f(1_x), f(w_0(u) *_0 v))_0$$

is surjective, there exists an $n$-arrow $a'$ of $G$ from $1_x$ to $w_0(u) *_0 v$ such that $f(a') = b'$. Then, the $n$-arrow $a = 1_u *_0^n a'$ is from $u$ to $v$ and we have

$$\begin{aligned} f(a) &= f(1_u *_0 a') \\ &= 1_{f(u)} *_0 b' \\ &= 1_{f(u)} *_0 1_{w_0(f(u))} *_0 b \\ &= b. \end{aligned}$$

□

**Theorem 5.3.** *The Brown-Golasiński model structure and the folk model structure on strict $\infty$-groupoids coincide.*

*Proof.* By the two previous propositions, these model structures have the same weak equivalences and the same trivial fibrations. □

# References


[1] J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*. Cambridge University Press, 1994.

[2] D. Ara. *Sur les $\infty$-groupoïdes de Grothendieck et une variante $\infty$-catégorique*. PhD thesis, Université Paris 7, 2010.

[3] T. Beke. Sheafifiable homotopy model categories. II. *J.Pure Appl.Algebra*, 164(3):307–324, 2001.

[4] R. Brown and M. Golasiński. A model structure for the homotopy theory of crossed complexes. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 30(1):61–82, 1989.

[5] R. Brown and P. J. Higgins. The equivalence of $\infty$-groupoids and crossed complexes. *Cahiers Topologie Géom. Différentielle*, 22(4):371–386, 1981.

[6] S. Crans. Quillen closed model structures for sheaves. *Journal of Pure and Applied Algebra*, 101:35–57, 1995.

[7] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory*. Springer, 1967.

[8] B. Jacobs and J. Rutten. A tutorial on (co)algebras and (co)induction. *Bulletin of the EATCS*, 62:222–259, 1997.

[9] A. Joyal and M. Tierney. Strong stacks and classifying spaces. In *Category Theory, Proc. Int. Conf., Como/Italy 1990*, number 1488 in Lect. Notes Math., pages 213–236, 1991.

[10] S. Lack. A Quillen model structure for 2-categories. *K-Theory*, 26(2):171–205, 2002.

[11] Y. Lafont, F. Métayer, and K. Worytkiewicz. A folk model structure on omega-cat. *Advances in Mathematics*, 224(3):1183–1231, 2010.

[12] F. Métayer. Resolutions by polygraphs. *Theory and Applications of Categories*, 11(7):148–184, 2003. http://www.tac.mta.ca/tac/.

[13] R. Street. The algebra of oriented simplexes. *Journal of Pure and Applied Algebra*, 49:283–335, 1987.