



**HAL**  
open science

## Derived categories and rationality of conic bundles.

Marcello Bernardara, Michele Bolognesi

► **To cite this version:**

Marcello Bernardara, Michele Bolognesi. Derived categories and rationality of conic bundles.. 2010.  
hal-00525711v1

**HAL Id: hal-00525711**

**<https://hal.science/hal-00525711v1>**

Preprint submitted on 12 Oct 2010 (v1), last revised 10 Dec 2012 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# DERIVED CATEGORIES AND RATIONALITY OF CONIC BUNDLES

MARCELLO BERNARDARA AND MICHELE BOLOGNESI

ABSTRACT. We show that a standard conic bundle over a minimal rational surface is rational and its Jacobian splits as the direct sum of Jacobians of curves if and only if its derived category admits a semiorthogonal decomposition by exceptional objects and the derived categories of those curves. Moreover, such a decomposition gives the splitting of the intermediate Jacobian also when the surface is not minimal.

## 1. INTRODUCTION

One of the main fields of research in the theory of derived category is understanding how the geometry of a smooth projective variety  $X$  is encoded in the bounded derived category  $D^b(X)$  of coherent sheaves on it. One of the main ideas, first developed by Bondal and Orlov, is to understand to which extent this category contains interesting information about birational geometry.

The biggest problem is to understand how this information can be traced out. The most promising and, so far, prolific approach is studying semiorthogonal decompositions

$$D^b(X) = \langle \mathbf{A}_1, \dots, \mathbf{A}_k \rangle.$$

In many interesting situations, one has such a decomposition with all or almost all of the  $\mathbf{A}_i$  equivalent to the derived category of a point. If  $X$  is a projective space or a smooth quadric, all of the  $\mathbf{A}_i$  are like this. It is expected that if a non-trivial subcategory appears in such decomposition, then it has to carry informations about the birational geometry of  $X$ . For example, if  $X$  is a  $V_{14}$  Fano threefold, then  $D^b(X)$  admits a semiorthogonal decomposition with only one non-trivial component, say  $\mathbf{A}_X$ . A similar decomposition holds for any smooth cubic threefold. Kuznetsov showed that if  $Y$  is the unique cubic threefold birational to  $X$  (see [22]),  $\mathbf{A}_X$  is equivalent to the non-trivial component  $\mathbf{A}_Y$  of  $D^b(Y)$ , and then it is a birational invariant for  $X$  [28]. Moreover it has been shown in [10], by reconstructing the Fano variety of lines on  $Y$  from  $\mathbf{A}_Y$ , that  $\mathbf{A}_Y$  determines the isomorphism class of  $Y$ . Similar correspondences between the non-trivial components of semiorthogonal decompositions of pairs of Fano threefolds are described in [31]. The derived category of a smooth cubic fourfold also admits such a decomposition, and it is conjectured that the non-trivial component determines its rationality [30].

It is a classical and still open problem in complex algebraic geometry to study the rationality of a standard conic bundle  $\pi : X \rightarrow S$  over a smooth projective surface. A necessary condition for rationality is that the intermediate Jacobian  $J(X)$  is isomorphic, as principally polarized abelian variety, to the direct sum of Jacobians of smooth projective curves. This allowed to prove the non rationality of smooth cubic threefolds [17]. The discriminant locus of the conic bundle is a curve  $C \subset S$ , with at most double points. The smooth points of  $C$  correspond to two intersecting lines, and the nodes to double lines. There is then a natural étale double cover (an admissible cover if  $C$  is singular [5])  $\tilde{C} \rightarrow C$  of the curve  $C$  associated to  $X$ . The intermediate Jacobian  $J(X)$  is then isomorphic to the Prym variety  $P(\tilde{C}/C)$  as principally polarized abelian variety [5]. This allows to show the non-rationality of conic bundles over  $\mathbb{P}^2$  with discriminant curve of degree  $\geq 6$  [5]. Remark that if  $S$  is not rational or  $C$  disconnected, then  $X$  cannot be rational. We will then not consider these cases.

If  $S$  is a minimal rational surface, then Shokurov [45] has shown that  $X$  is rational if and only if  $J(X)$  splits as the direct sum of Jacobians of smooth projective curves and that this happens

only in 4 cases: if  $S = \mathbb{P}^2$ , either  $C$  is a quartic, or  $C$  is a quintic and  $\tilde{C} \rightarrow C$  is given by an even theta-characteristic; if  $S = \mathbb{F}_n$ , either  $C$  is hyperelliptic or  $C$  is trigonal and in both cases the map to  $\mathbb{P}^1$  is induced by the ruling of  $S$ . If  $S$  is not minimal, it is conjectured that there are essentially no more cases [24].

Our aim is to give a categorical approach to this problem, using semiorthogonal decompositions. Indeed, in [29] Kuznetsov considers the sheaf  $\mathcal{B}_0$  of even parts of Clifford algebras associated to the quadratic form defining the conic fibration, and  $D^b(S, \mathcal{B}_0)$  the bounded derived category of coherent  $\mathcal{B}_0$ -algebras over  $S$ . He describes a fully faithful functor  $\Phi : D^b(S, \mathcal{B}_0) \rightarrow D^b(X)$  and gives a semiorthogonal decomposition for the derived category of  $X$  as follows:

$$D^b(X) = \langle \Phi D^b(S, \mathcal{B}_0), \pi^* D^b(S) \rangle.$$

If  $S$  is a rational surface, its derived category admits a full exceptional sequence, which leads to the following semiorthogonal decomposition

$$(1.1) \quad D^b(X) = \langle \Phi D^b(S, \mathcal{B}_0), E_1, \dots, E_s \rangle,$$

where  $\{E_i\}_{i=1}^s$  are exceptional objects. The non-trivial information about the geometry of the conic bundle is contained in the category  $D^b(S, \mathcal{B}_0)$ . Note that in the case where  $X$  is the blow-up of a smooth cubic threefold  $Y$  along a line,  $D^b(S, \mathcal{B}_0)$  contains  $\mathbf{A}_Y$ , which identifies the isomorphism class of  $Y$  [10]. Remark that a different approach to the same problem, via generalized homological mirror symmetry, leads to the conjectures stated in [26, 27]. Anyway we do not establish any link with the results described here.

**Theorem 1.1.** *Let  $\pi : X \rightarrow S$  be a standard conic bundle over a rational surface. Suppose that  $\{\Gamma_i\}_{i=1}^k$  are smooth projective curves and  $k \geq 0$ , with fully faithful functors  $\Psi_i : D^b(\Gamma_i) \rightarrow D^b(S, \mathcal{B}_0)$  for  $i = 1, \dots, k$ , such that  $D^b(S, \mathcal{B}_0)$  admits a semiorthogonal decomposition:*

$$(1.2) \quad D^b(S, \mathcal{B}_0) = \langle \Psi_1 D^b(\Gamma_1), \dots, \Psi_k D^b(\Gamma_k), E_1, \dots, E_l \rangle,$$

where  $E_i$  are exceptional objects and  $l \geq 0$ . Then  $J(X) = \bigoplus_{i=1}^k J(\Gamma_i)$  as principally polarized abelian variety.

If  $S$  is non-rational, and then so is  $X$ , Theorem 1.1 fails; its proof relies indeed strictly on the rationality of  $S$ . In 6.3 we provide an example of a standard conic bundle over a non-rational surface with  $D^b(S, \mathcal{B}_0)$  decomposing in derived categories of smooth projective curves.

The interest of Theorem 1.1 is twofold: first it is the first non-trivial example where informations on the birational properties and on algebraically trivial cycles are obtained directly from a semiorthogonal decomposition. Secondly it gives a categorical criterion of rationality for conic bundles over minimal surfaces, thanks to Shokurov result [45]. We can also prove the other implication by a case by case analysis.

**Theorem 1.2.** *If  $S$  is minimal, then  $X$  is rational and  $J(X) = \bigoplus_{i=1}^k J(\Gamma_i)$  if and only if there are fully faithful functors  $\Psi_i : D^b(\Gamma_i) \rightarrow D^b(S, \mathcal{B}_0)$  and a semiorthogonal decomposition*

$$D^b(S, \mathcal{B}_0) = \langle \Psi_1 D^b(\Gamma_1), \dots, \Psi_k D^b(\Gamma_k), E_1, \dots, E_l \rangle,$$

where  $E_i$  are exceptional objects and  $l \geq 0$ .

The key of the proof of Theorem 1.1 is the study of the maps induced by a fully faithful functor  $\Psi : D^b(\Gamma) \rightarrow D^b(X)$  on the rational Chow motives, as explained in [39], where  $\Gamma$  is a smooth projective curve of positive genus. In particular, the biggest step consists in proving that such a functor induces an injective morphism  $\psi : J(\Gamma) \rightarrow J(X)$  preserving the principal polarization. The existence of the required semiorthogonal decomposition implies then the bijectivity of the sum of the  $\psi_i$ 's.

The paper is organized as follows: in Sections 2 and 3 we recall respectively basic facts about motives and derived categories and the construction from [39], and the description of motive,

derived category and intermediate Jacobian of a conic bundle. In Section 4 we prove Theorem 1.1, and in Sections 5 and 6 we finish the proof of Theorem 1.2, analyzing respectively the case  $S = \mathbb{P}^2$  and  $S = \mathbb{F}_n$ .

**Notations.** Except for Section 2, we work over the complex field  $\mathbb{C}$ . Any triangulated category is assumed to be essentially small. Given a smooth projective variety  $X$ , we denote  $D^b(X)$  the bounded derived category of coherent sheaves on it,  $K_0(X)$  its Grothendieck group,  $CH^d(X)$  the Chow group of codimension  $d$  cycles and  $A^d(X)$  the subgroup of algebraically trivial cycles in  $CH^d(X)$ . The subscript  $\mathbb{Q}$  is used there whenever we consider  $\mathbb{Q}$ -coefficients, while  $h(X)$  already denotes the rational Chow motive. We will denote  $\text{Prym}(\tilde{C}/C)$  the Prym motive and  $P(\tilde{C}/C)$  the Prym variety for an admissible double cover  $\tilde{C} \rightarrow C$ . Whenever a functor between derived categories is given, it will be denoted as underived, for example for  $f : X \rightarrow Y$ ,  $f^*$  and  $f_*$  denote respectively the derived pull-back and push-forward.

**Acknowledgements.** Part of this work has been developed during short visits of the authors at the Humboldt University of Berlin, the University of Duisburg–Essen, and the Roma III University, that are warmly acknowledged. M.Be. is grateful to H.Esnault for pointing him out [39], and to her and A.Chatzistamatiou for useful discussions. Moreover, it is a pleasure to thank the people that shared their views with us and encouraged us during the writing of this paper. In alphabetical order: A.Beauville, G.Casnati, I.Dolgachev, V.Kanev, L.Katzarkov, A.Kuznetsov, M.Mella, A.Verra. M.Be. was supported by the SFB/TR 45 ‘Periods, moduli spaces, and arithmetic of algebraic varieties’.

## 2. PRELIMINARIES

In this Section, we recall some basic facts about motives, derived categories, semiorthogonal decompositions and Fourier–Mukai functors. The experienced reader can easily skip subsections 2.1 and 2.2. In 2.3, we explain how a Fourier–Mukai functor induces a motivic map, following [39], and we retrace the results from [9] under this point of view to give a baby example clarifying some of the arguments we will use later.

**2.1. Motives.** We give a brief introduction to rational Chow motives, following [44]. The most important results we will need are the correspondence between the submotive  $h^1(C) \subset h(C)$  of a smooth projective curve and its Jacobian, and the Chow–Künneth decomposition of the motive of a smooth surface.

Let  $X$  be a smooth projective scheme over a field  $\kappa$ . For any integer  $d$ , let  $\mathcal{Z}^d(X)$  be the free abelian group generated by irreducible subvarieties of  $X$  of codimension  $d$ . We denote by  $CH^d(X) = \mathcal{Z}^d(X)/\sim_{rat}$  the codimension  $d$  Chow group and by  $CH_{\mathbb{Q}}^d(X) := CH^d(X) \otimes \mathbb{Q}$ . In this section, we are only concerned with rational coefficients. Let  $Y$  be a smooth projective scheme. If  $X$  is purely  $d$ -dimensional, we put, for any integer  $r$ ,

$$\text{Corr}^r(X, Y) := CH_{\mathbb{Q}}^{d+r}(X \times Y).$$

If  $X = \coprod X_i$ , where  $X_i$  is connected, we put

$$\text{Corr}^r(X, Y) := \bigoplus \text{Corr}^r(X_i, Y) \subset CH_{\mathbb{Q}}^*(X \times Y).$$

If  $Z$  is a smooth projective scheme, the composition of correspondences is defined by a map

$$(2.1) \quad \begin{aligned} \text{Corr}^r(X, Y) \otimes \text{Corr}^s(Y, Z) &\longrightarrow \text{Corr}^{r+s}(X, Z) \\ f \otimes g &\longmapsto p_{13*}(p_{12}^* f \cdot p_{23}^* g), \end{aligned}$$

where  $p_{ij}$  are the projections from  $X \times Y \times Z$  onto products of two factors.

The category  $\mathcal{M}_{\kappa}$  of Chow motives over  $\kappa$  with rational coefficients is defined as follows: an object of  $\mathcal{M}_{\kappa}$  is a triple  $(X, p, m)$ , where  $X$  is a variety,  $m$  an integer and  $p \in \text{Corr}^0(X, X)$  an

idempotent, called a *projector*. Morphisms from  $(X, p, m)$  to  $(Y, q, n)$  are given by elements of  $\text{Corr}^{n-m}(X, Y)$  precomposed with  $p$  and composed with  $q$ .

There is a natural functor  $h$  from the category of smooth projective schemes to the category of motives, defined by  $h(X) = (X, \text{Id}, 0)$ , and, for any morphism  $\phi : X \rightarrow Y$ ,  $h(\phi)$  being the correspondence given by the graph of  $\phi$ . We write  $\mathbb{Q} := (\text{Spec} \kappa, \text{Id}, 0)$  for the unit motive and  $\mathbb{Q}(-1) := (\text{Spec} \kappa, \text{Id}, -1)$  for the Tate (or Lefschetz) motive, and  $\mathbb{Q}(-i) := \mathbb{Q}(-1)^{\otimes i}$  for  $i > 0$ . We denote  $h(X)(-i) := h(X) \otimes \mathbb{Q}(-i)$ . Finally, we have  $\text{Hom}(\mathbb{Q}(-d), h(X)) = CH_{\mathbb{Q}}^d(X)$  for all smooth projective schemes  $X$  and all integers  $d$ .

If  $X$  is irreducible of dimension  $d$  and has a rational point, the embedding  $\alpha : pt \rightarrow X$  of the point defines a motivic map  $\mathbb{Q} \rightarrow h(X)$ . We denote by  $h^0(X)$  its image and by  $h^{\geq 1}(X)$  the quotient of  $h(X)$  via  $h^0(X)$ . Similarly, we have that  $\mathbb{Q}(-d)$  is a quotient of  $h(X)$ , and we denote it by  $h^{2d}(X)$ . For example, if  $X = \mathbb{P}^1$ , we have that  $h^{\geq 1}(\mathbb{P}^1) = h^2(\mathbb{P}^1)$  and then  $h(\mathbb{P}^1) \simeq \mathbb{Q} \oplus \mathbb{Q}(-1)$ . In the case of smooth projective curves of positive genus another factor which corresponds to the Jacobian variety of the curve is appearing.

Let  $C$  be a smooth projective connected curve with a rational point. Then one can define a motive  $h^1(C)$  such that we have a direct sum:

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C).$$

The main fact is that the theory of the motives  $h^1(C)$  corresponds to that of Jacobian varieties (up to isogeny). Indeed we have

$$\text{Hom}(h^1(C), h^1(C')) = \text{Hom}(J(C), J(C')) \otimes \mathbb{Q}.$$

In particular, the full subcategory of  $\mathcal{M}_{\kappa}$  whose objects are direct summands of the motive  $h^1(C)$  is equivalent to the category of abelian subvarieties of  $J(C)$  up to isogeny. Finally, for all  $d$  there is no non-trivial map  $h^1(C) \rightarrow h^1(C)$  factoring through  $\mathbb{Q}(-d)$ . Indeed, we have  $\text{Hom}(h^1(C), \mathbb{Q}(-d)) = CH_{\mathbb{Q}}^{1-d}(C)_{\text{num}=0}$ , which is zero unless  $d = 0$ , while  $\text{Hom}(\mathbb{Q}(-d), h^1(C)) = CH_{\mathbb{Q}}^d(C)_{\text{num}=0}$ , which is zero unless  $d = 1$ .

Let  $S$  be a surface. Murre constructed [35] the motives  $h^i(S)$ , defined by projectors  $p_i$  in  $CH_{\mathbb{Q}}^i(S \times S)$  for  $i = 1, 2, 3$ , and described a decomposition

$$h(S) = h^0(S) \oplus h^1(S) \oplus h^2(S) \oplus h^3(S) \oplus h^4(S).$$

We already remarked that  $h^0(S) = \mathbb{Q}$  and  $h^4(S) = \mathbb{Q}(-2)$ . Roughly speaking, the submotive  $h^1(S)$  carries the Picard variety, the submotive  $h^3(S)$  the Albanese variety and the submotive  $h^2(S)$  carries the Néron–Severi group, the Albanese kernel and the transcendental cycles. If  $S$  is a smooth rational surface, then  $h^1(S)$  and  $h^3(S)$  are trivial, while  $h^2(S) \simeq \mathbb{Q}(-1)^{\rho}$ , where  $\rho$  is the rank of the Néron–Severi group. In particular, the motive of  $S$  splits in a finite direct sum of (differently twisted) Tate motives.

In general, it is expected that if  $X$  is a smooth projective  $d$ -dimensional variety, there exist projectors  $p_i$  in  $CH_{\mathbb{Q}}^i(X \times X)$  defining motives  $h^i(X)$  such that  $h(X) = \bigoplus_{i=0}^{2d} h^i(X)$ . Such a decomposition is called a *Chow–Künneth decomposition*. We have seen that the motive of any smooth projective curve or surface admits a Chow–Künneth decomposition. This is true also for the motive of a smooth uniruled complex threefold [2].

**2.2. Semiorthogonal decomposition, exceptional objects and mutations.** We introduce here semiorthogonal decompositions, exceptional objects and mutations in a  $\kappa$ -linear triangulated category  $\mathbf{T}$ , following [12, 13, 14], and give some examples which will be useful later on. Our only applications will be given in the case where  $\mathbf{T}$  is the bounded derived category of a smooth projective variety, but we stick to the more general context. A full triangulated category  $\mathbf{A}$  of  $\mathbf{T}$  is called *admissible* if the embedding functor admits a left and a right adjoint.

**Definition 2.1** ([13, 14]). A *semiorthogonal decomposition* of  $\mathbf{T}$  is a sequence of full admissible triangulated subcategories  $\mathbf{A}_1, \dots, \mathbf{A}_n$  of  $\mathbf{T}$  such that  $\text{Hom}_{\mathbf{T}}(A_i, A_j) = 0$  for all  $i > j$  and for

all objects  $A_i$  in  $\mathbf{A}_i$  and  $A_j$  in  $\mathbf{A}_j$ , and for every object  $T$  of  $\mathbf{T}$ , there is a chain of morphisms  $0 = T_n \rightarrow T_{n-1} \rightarrow \dots \rightarrow T_1 \rightarrow T_0 = T$  such that the cone of  $T_k \rightarrow T_{k-1}$  is an object of  $\mathbf{A}_k$  for all  $k = 1, \dots, n$ . Such a decomposition will be written

$$\mathbf{T} = \langle \mathbf{A}_1, \dots, \mathbf{A}_n \rangle.$$

**Definition 2.2** ([12]). An object  $E$  of  $\mathbf{T}$  is called *exceptional* if  $\mathrm{Hom}_{\mathbf{T}}(E, E) = \kappa$ , and  $\mathrm{Hom}_{\mathbf{T}}(E, E[i]) = 0$  for all  $i \neq 0$ . A collection  $(E_1, \dots, E_l)$  of exceptional objects is called *exceptional* if  $\mathrm{Hom}_{\mathbf{T}}(E_j, E_k[i]) = 0$  for all  $j > k$  and for all integer  $i$ .

If  $E$  in  $\mathbf{T}$  is an exceptional object, the triangulated category generated by  $E$  (that is, the smallest full triangulated subcategory of  $\mathbf{T}$  containing  $E$ ) is equivalent to the derived category of a point, seen as a smooth projective variety. The equivalence  $D^b(pt) \rightarrow \langle E \rangle \subset \mathbf{T}$  is indeed given by sending  $\mathcal{O}_{pt}$  to  $E$ . Given an exceptional collection  $(E_1, \dots, E_l)$  in the derived category  $D^b(X)$  of a smooth projective variety, there is a semiorthogonal decomposition [14]

$$D^b(X) = \langle \mathbf{A}, E_1, \dots, E_l \rangle,$$

where  $\mathbf{A}$  is the full triangulated subcategory whose objects are all the  $A$  satisfying  $\mathrm{Hom}(E_i, A) = 0$  for all  $i = 1, \dots, l$ , and we denote by  $E_i$  the category generated by  $E_i$ . We say that the exceptional sequence is *full* if the category  $\mathbf{A}$  is trivial.

There are many examples of smooth projective varieties admitting a full exceptional sequence. For example the sequence  $(\mathcal{O}(i), \dots, \mathcal{O}(i+n))$  is full exceptional in  $D^b(\mathbb{P}^n)$  for all  $i$  integer [6]. If  $X$  is an even-dimensional smooth quadric hypersurface in  $\mathbb{P}^n$  and  $\Sigma$  the spinor bundle, the sequence  $(\Sigma(i), \mathcal{O}(i+1), \dots, \mathcal{O}(i+n))$  is full exceptional in  $D^b(X)$  and a similar sequence (with two spinor bundles) exists for odd-dimensional smooth quadric hypersurfaces [25].

**Proposition 2.3** ([37]). *Let  $X$  be a smooth projective variety and  $F$  a locally free sheaf of rank  $r+1$  over it. Let  $p : \mathbb{P}(F) \rightarrow X$  be the associated projective bundle. The functor  $p^* : D^b(X) \rightarrow D^b(\mathbb{P}(F))$  is fully faithful and for all integer  $i$  we have the semiorthogonal decomposition:*

$$D^b(\mathbb{P}(F)) = \langle p^*D^b(X) \otimes \mathcal{O}_{\mathbb{P}/X}(i), \dots, p^*D^b(X) \otimes \mathcal{O}_{\mathbb{P}/X}(i+r) \rangle,$$

where  $\mathcal{O}_{\mathbb{P}/X}(1)$  is the relative ample line bundle.

**Proposition 2.4** ([37]). *Let  $X$  be a smooth projective variety,  $Y \hookrightarrow X$  a smooth projective subvariety of codimension  $d > 1$  and  $\varepsilon : \tilde{X} \rightarrow X$  the blow-up of  $X$  along  $Y$ . Let  $D \xrightarrow{\iota} \tilde{X}$  be the exceptional divisor and  $p : D \rightarrow Y$  the restriction of  $\varepsilon$ . Then the functors  $\varepsilon^* : D^b(X) \rightarrow D^b(\tilde{X})$  and  $\Psi^j := \iota_* \circ \otimes \mathcal{O}_{D/Y}(j) \circ p^* : D^b(Y) \rightarrow D^b(\tilde{X})$  are fully faithful for all  $j$  and we have the following semiorthogonal decomposition:*

$$D^b(\tilde{X}) = \langle \Psi^0 D^b(Y), \dots, \Psi^{d-1} D^b(Y), \varepsilon^* D^b(X) \rangle.$$

We will refer to Proposition 2.4 as the *Orlov formula* for blow ups. Notice that both Proposition 2.3 and 2.4 have motivic counterparts [32].

We finally remark that if  $X$  has dimension at most 2 and is rational, the derived category  $D^b(X)$  admits a full exceptional sequence. We have already seen this for  $\mathbb{P}^1$  and  $\mathbb{P}^2$ . If  $X$  is a Hirzebruch surface, then it has a 4-objects full exceptional sequence by Prop. 2.3 and the decomposition of  $\mathbb{P}^1$ . We conclude by the birational classification of smooth projective surfaces and the Orlov formula for blow-ups. In particular a rational surface with Picard number  $\rho$  has a full exceptional sequence of  $\rho + 2$  objects.

Given a semiorthogonal decomposition  $\langle \mathbf{A}_1, \dots, \mathbf{A}_n \rangle$  of  $\mathbf{T}$ , we can define an operation called *mutation* (called originally, in Russian, *perestroika*) which allows to give new semiorthogonal decompositions with equivalent components. What we need here is the following fact, gathering different results from [12].

**Proposition 2.5.** *Suppose that  $\mathbf{T}$  admits a semiorthogonal decomposition  $\langle \mathbf{A}_1, \dots, \mathbf{A}_n \rangle$ . Then for each  $1 \leq k \leq n-1$ , there is a semiorthogonal decomposition*

$$\mathbf{T} = \langle \mathbf{A}_1, \dots, \mathbf{A}_{k-1}, L_{\mathbf{A}_k}(\mathbf{A}_{k+1}), \mathbf{A}_k, \mathbf{A}_{k+2}, \dots, \mathbf{A}_n \rangle,$$

where  $L_{\mathbf{A}_k} : \mathbf{A}_{k+1} \rightarrow L_{\mathbf{A}_k}(\mathbf{A}_{k+1})$  is an equivalence, called the left mutation through  $\mathbf{A}_k$ . Similarly, for each  $2 \leq k \leq n$ , there is a semiorthogonal decomposition

$$\mathbf{T} = \langle \mathbf{A}_1, \dots, \mathbf{A}_{k-2}, \mathbf{A}_k, R_{\mathbf{A}_k}(\mathbf{A}_{k-1}), \mathbf{A}_{k+1}, \dots, \mathbf{A}_n \rangle,$$

where  $R_{\mathbf{A}_k} : \mathbf{A}_{k-1} \rightarrow R_{\mathbf{A}_k}(\mathbf{A}_{k-1})$  is an equivalence, called the right mutation through  $\mathbf{A}_k$ .

Remark in particular that the mutation of an exceptional object is an exceptional object. If  $\mathbf{T}$  is the bounded derived category of a smooth projective variety and  $n = 2$ , there is a very useful explicit formula for left and right mutations.

**Lemma 2.6** ([13]). *Let  $X$  be a smooth projective variety and  $D^b(X) = \langle \mathbf{A}, \mathbf{B} \rangle$  a semiorthogonal decomposition. Then  $L_{\mathbf{A}}(\mathbf{B}) = \mathbf{B} \otimes \omega_X$  and  $R_{\mathbf{B}}(\mathbf{A}) = \mathbf{A} \otimes \omega_X^{-1}$ .*

**2.3. Fourier–Mukai functors, motives and Chow groups.** Fourier–Mukai functors are the main tool in studying derived categories of coherent sheaves. We recall here the main properties of a Fourier–Mukai functor and how it interacts with other theories, such as the Grothendieck group, Chow rings and motives. A more detailed treatment (except for motives, see [39]) can be found in [21, Chap. 5].

Let  $X$  and  $Y$  be smooth projective varieties of dimension  $n$  and  $m$  respectively and  $\mathcal{E}$  an object of  $D^b(X \times Y)$ . The *Fourier–Mukai functor*  $\Phi_{\mathcal{E}} : D^b(Y) \rightarrow D^b(X)$  with kernel  $\mathcal{E}$  is given by  $\Phi_{\mathcal{E}}(A) = p_*(q^*A \otimes \mathcal{E})$ , where  $p$  and  $q$  denote the projections from  $X \times Y$  onto  $X$  and  $Y$  respectively. We will sometimes drop the subscript  $\mathcal{E}$ . If  $Z$  is a smooth projective variety,  $\Phi_{\mathcal{E}} : D^b(Y) \rightarrow D^b(X)$  and  $\Phi_{\mathcal{F}} : D^b(X) \rightarrow D^b(Z)$ , then the composition  $\Phi_{\mathcal{F}} \circ \Phi_{\mathcal{E}}$  is the Fourier–Mukai functor with kernel

$$(2.2) \quad \mathcal{G} := p_{13*}(p_{12}^*\mathcal{E} \otimes p_{23}^*\mathcal{F}),$$

where  $p_{ij}$  are the projections from  $Y \times X \times Z$  onto products of two factors. It is worth noting the similarity between (2.2) and the composition of correspondences (2.1).

A Fourier–Mukai functor  $\Phi_{\mathcal{E}}$  always admits a left and right adjoint which are the Fourier–Mukai functors with kernel  $\mathcal{E}_L$  and  $\mathcal{E}_R$  resp., defined by

$$\mathcal{E}_L := \mathcal{E}^{\vee} \otimes p^*\omega_X[n] \quad \text{and} \quad \mathcal{E}_R := \mathcal{E}^{\vee} \otimes q^*\omega_Y[m].$$

A celebrated result from Orlov [38] shows that any fully faithful exact functor  $F : D^b(Y) \rightarrow D^b(X)$  with right and left adjoint is a Fourier–Mukai functor whose kernel is uniquely determined up to isomorphism.

Given the Fourier–Mukai functor  $\Phi_{\mathcal{E}} : D^b(Y) \rightarrow D^b(X)$ , consider the element  $[\mathcal{E}]$  in  $K_0(X \times Y)$ , given by the alternate sum of the cohomologies of  $\mathcal{E}$ . Then we have a commutative diagram

$$(2.3) \quad \begin{array}{ccc} D^b(Y) & \xrightarrow{\Phi_{\mathcal{E}}} & D^b(X) \\ \downarrow [\ ] & & \downarrow [\ ] \\ K_0(Y) & \xrightarrow{\Phi_{\mathcal{E}}^K} & K_0(X), \end{array}$$

where  $\Phi_{\mathcal{E}}^K$  is the  $K$ -theoretical Fourier–Mukai transform defined by  $\Phi_{\mathcal{E}}^K(A) = p_!(q^*A \otimes [\mathcal{E}])$ . If  $\Phi_{\mathcal{E}}$  is fully faithful, we have  $\Phi_{\mathcal{E}} \circ \Phi_{\mathcal{E}_R} = \text{Id}_{D^b(Y)}$ . This implies  $\Phi_{\mathcal{E}}^K \circ \Phi_{\mathcal{E}_R}^K = \text{Id}_{K_0(Y)}$  and then  $K_0(Y)$  is a direct summand of  $K_0(X)$ .

**Lemma 2.7.** *Let  $X$ ,  $\{Y_i\}_{i=1, \dots, k}$  be smooth projective varieties,  $\Phi_i : D^b(Y_i) \rightarrow D^b(X)$  fully faithful functors and  $D^b(X) = \langle \Phi_1 D^b(Y_1), \dots, \Phi_k D^b(Y_k) \rangle$  a semiorthogonal decomposition. Then  $K_0(X) = \bigoplus_{i=1}^k K_0(Y_i)$ , and  $CH_{\mathbb{Q}}^*(X) = \bigoplus_{i=1}^k CH_{\mathbb{Q}}^*(Y_i)$ .*

*Proof.* The full and faithful functors  $\Phi_i : D^b(Y_i) \rightarrow D^b(X)$  have to be of Fourier–Mukai type and then  $K_0(Y_i)$  are direct summands of  $K_0(X)$ . The generation follows from the definition of a semiorthogonal decomposition. The decomposition of the rational Chow ring is a straightforward consequence of Grothendieck–Riemann–Roch Theorem.  $\square$

Consider the element  $e := \text{ch}([\mathcal{E}]).\text{Td}(Y)$  in  $CH_{\mathbb{Q}}^*(X \times Y)$ . This gives a correspondence  $e : CH_{\mathbb{Q}}^*(Y) \rightarrow CH_{\mathbb{Q}}^*(X)$  and we have a commutative diagram

$$(2.4) \quad \begin{array}{ccc} D^b(Y) & \xrightarrow{\Phi_{\mathcal{E}}} & D^b(X) \\ \downarrow & & \downarrow \\ CH_{\mathbb{Q}}^*(Y) & \xrightarrow{e} & CH_{\mathbb{Q}}^*(X), \end{array}$$

where the vertical arrows are obtained by taking the Chern character and multiplying with the Todd class. The commutativity of the diagram follows from the Grothendieck–Riemann–Roch formula. Remark that here we used that the relative Todd class of the projection  $X \times Y \rightarrow X$  is  $\text{Td}(Y)$ .

As for the Grothendieck groups, the Chow ring and the rational cohomology (see [21, Chapt. 5]), one can find a functorial correspondence between derived Fourier–Mukai functors and motivic maps. This was first carried out by Orlov [39]. Indeed, the cycle  $e$  is of mixed type in  $CH_{\mathbb{Q}}^*(X \times Y)$ . Its components  $e_i$  in  $CH_{\mathbb{Q}}^i(X \times Y)$  give motivic maps  $e_i : h(Y) \rightarrow h(X)(i - n)$ . Denote by  $\mathcal{F} := \mathcal{E}_R$  the kernel of the right adjoint of  $\Phi_{\mathcal{E}}$ , and  $f = \text{ch}([\mathcal{F}]).\text{Td}(X)$  the associated cycle in  $CH_{\mathbb{Q}}^*(X \times Y)$ . Then we get motivic maps  $f_i : h(X)(i - n) \rightarrow h(Y)$ . If we consider the cycles  $e$  and  $f$ , the Grothendieck–Riemann–Roch formula implies that  $f.e$  induces the identity  $\text{Id} : h(Y) \rightarrow h(Y)$ .

**Example 2.8.** As an example, we describe the result in [9] from the motivic point of view. This turns out to be very useful in understanding the relationship between the derived category, the motive and the Jacobian of a smooth projective curve, and contains some ideas that we will use in the proof of Theorem 1.1

Let  $C_1$  and  $C_2$  be smooth projective curves and  $\Phi_{\mathcal{E}} : D^b(C_1) \rightarrow D^b(C_2)$  a Fourier–Mukai functor. In [9] it is shown that the map  $\phi : J(C_1) \rightarrow J(C_2)$  induced by  $\Phi_{\mathcal{E}}$  preserves the principal polarization if and only if  $\Phi_{\mathcal{E}}$  is an equivalence.

We could describe such result in the following way: consider the motivic maps  $e_i : h(C_1) \rightarrow h(C_2)(i - 1)$  where  $e$  is the cycle associated to  $\mathcal{E}$ . We define  $f$  as before via the right adjoint. If  $\Phi_{\mathcal{E}}$  is fully faithful, then we have  $f.e = \bigoplus_{i=0}^2 f_i.e_{2-i} = \text{Id}$ . Since  $h^0(C_j) = \mathbb{Q}$ , and  $h^2(C_j) = \mathbb{Q}(-1)$  for  $j = 1, 2$ , if we restrict to  $h^1(C_1)$ , we get that  $(f_i.e_{2-i})|_{h^1(C_1)} = 0$  unless  $i = 1$ . In particular we obtain that  $(e_1.f_1)|_{h^1(C_1)} = \text{Id}_{h^1(C_1)}$  and then  $h^1(C_1)$  is a direct summand of  $h^1(C_2)$ . Every fully faithful functor between the derived categories of smooth projective curves is an equivalence, and we can apply the same argument to the adjoint of  $\Phi_{\mathcal{E}}$ , obtaining an isomorphism  $h^1(C_1) \simeq h^1(C_2)$ . This gives an isogeny  $J_{\mathbb{Q}}(C_1) \simeq J_{\mathbb{Q}}(C_2)$ .

Moreover, the maps  $e_1$  and  $f_1$  are given both by  $c_1([\mathcal{E}])$ , and they define a morphism  $\phi : J(C_1) \rightarrow J(C_2)$  of abelian varieties, with finite kernel. The key point to prove the preservation of the principal polarization is the fact that that the dual map  $\hat{\phi}$  of  $\phi$  is induced by the adjoint of  $\Phi_{\mathcal{E}}$ . Being  $\Phi_{\mathcal{E}}$  a Fourier–Mukai functor carries indeed a deep amount of geometrical information.

### 3. DERIVED CATEGORIES, MOTIVES AND CHOW GROUPS OF CONIC BUNDLES

From now on, we only consider varieties defined over  $\mathbb{C}$ . Let  $S$  be a smooth projective surface, and  $\pi : X \rightarrow S$  a smooth standard conic bundle. By this, we mean a surjective morphism whose scheme theoretic fibers are isomorphic to plane conics, such that for any curve  $D \subset S$  the surface  $\pi^{-1}(D)$  is irreducible (this second condition is also called relative minimality). The discriminant



locus of the conic bundle is a curve  $C \subset S$ , which can be possibly empty, with at most double points. The fiber of  $\pi$  over a smooth point of  $C$  is the union of two lines intersecting in a single point, while the fiber over a node is a double line. Recall that any conic bundle is birationally equivalent to a standard one via elementary transformations [42].

In this section, we recall known results about the geometry of  $\pi : X \rightarrow S$ . In section 3.1 we deal with the decomposition of  $h(X)$  described by Nagel and Saito [36] and with the semiorthogonal decomposition of  $D^b(X)$  described by Kuznetsov [29]. In section 3.2 we recall the description of the intermediate Jacobian and the algebraically trivial part  $A^2(X) := CH^2(X)_{alg=0}$  of the Chow group given by [5, 8]. The order of the two sections reverses history, but the decompositions of  $h(X)$  and  $D^b(X)$  hold in a more general frame.

Before that, recall that to any standard conic bundle, we can associate an admissible double covering  $\tilde{C} \rightarrow C$  of the curve  $C$ , ramified along the singular points of  $C$ . The set of vertical lines of  $X$  (that is, the ones contained in a fiber) is then a  $\mathbb{P}^1$ -bundle over  $\tilde{C}$  [5]. In the results recalled here, if  $C$  is not smooth, then it has to be replaced by its normalization and the corresponding double covering. Anyway, with no risk of misunderstanding, we will tacitly assume this replacement when needed, and keep the notation  $\tilde{C} \rightarrow C$ .

**3.1. The decompositions of  $h(X)$  and  $D^b(X)$ .** Consider the rational Chow motive  $h(X)$ . Nagel and Saito [36] provide a relative Chow–Künneth decomposition for  $h(X)$ . First of all, for a given double covering  $\tilde{C} \rightarrow C$  of an irreducible curve with at most double points, they define the *Prym motive*  $\text{Prym}(\tilde{C}/C)$  as a submotive of  $h(\tilde{C})$  via the involution associated to the covering. In particular  $\text{Prym}^1(\tilde{C}/C)$  is a submotive of  $h^1(\tilde{C})$ ,  $\text{Prym}(\tilde{C}/C) = \text{Prym}^1(\tilde{C}/C)$  if the double covering is not trivial and  $\text{Prym}(\tilde{C}/C) = h(\tilde{C})$  otherwise. We refrain here to give the details of the construction, for which the reader can consult [36]. Moreover they show how  $h(S)$  and  $h(S)(-1)$  are direct summands of  $h(X)$ . Any conic bundle (non necessarily standard) is uniruled and  $h(X) = \bigoplus_{i=0}^6 h^i(X)$  is the Chow–Künneth decomposition [2]. We have the following description:

$$h^i(X) = h^i(S) \oplus h^{i-2}(S)(-1) \oplus \bigoplus_{j=1}^r \text{Prym}^{i-2}(\tilde{C}_j/C_j)(-1),$$

where  $C_j$ , for  $j = 1, \dots, r$ , are the irreducible components of the discriminant curve  $C$ .

If  $\pi : X \rightarrow S$  is standard, then there is no component of  $C$  over which the double cover is trivial. It follows that  $h^i(X) = h^i(S) \oplus h^{i-2}(S)(-1)$  for  $i \neq 3$  and

$$h^3(X) = h^3(S) \oplus h^1(S)(-1) \oplus \bigoplus_{j=1}^r \text{Prym}^1(\tilde{C}_j/C_j)(-1).$$

We will focus on the case where  $S$  is a rational surface and  $C$  is connected (in any other case, the conic bundle is not rational). We finally end up, recalling section 2.1, with:

$$(3.1) \quad \begin{aligned} h^i(X) &= h^i(S) \oplus h^{i-2}(S)(-1) & \text{if } i \neq 3, \\ h^3(X) &= \text{Prym}^1(\tilde{C}/C)(-1), \end{aligned}$$

and in particular, for  $i \neq 3$ ,  $h^i(X)$  is either trivial or a finite sum of Tate motives (with different twists).

Consider the derived category  $D^b(X)$ . It is well-known (see [5, 43]) that the fibers of  $\pi$  are plane conics and that there is a locally free rank 3 vector bundle  $\mathcal{E}$  on  $S$ , such that  $X \subset \mathbb{P}(\mathcal{E})$  is the zero locus of a section  $s : \mathcal{O}_S(-1) \rightarrow \text{Sym}^2(\mathcal{E})$  and the map  $\pi$  is the restriction of the fibration  $\mathbb{P}(\mathcal{E}) \rightarrow S$ . Kuznetsov defines, in the more general frame of any quadric fibration over any smooth projective manifold, the sheaf of even parts  $\mathcal{B}_0$  of the Clifford algebra associated to the section  $s$ . One can consider the abelian category  $\mathbf{Coh}(S, \mathcal{B}_0)$  of coherent sheaves with a structure of  $\mathcal{B}_0$ -algebra and its bounded derived category  $D^b(S, \mathcal{B}_0)$ . In the case of a standard conic bundle,  $\mathcal{B}_0 = \mathcal{O}_S \oplus (\Lambda^2(\mathcal{E}) \otimes \mathcal{O}_S(-1))$  is a locally free sheaf of rank 4.

**Proposition 3.1** ([29]). *Let  $\pi : X \rightarrow S$  be a conic bundle and  $\mathcal{B}_0$  the sheaf of even parts of the Clifford algebra associated to it. Then  $\pi^* : D^b(S) \rightarrow D^b(X)$  is fully faithful and there is a fully faithful functor  $\Phi : D^b(S, \mathcal{B}_0) \rightarrow D^b(X)$  such that*

$$D^b(X) = \langle \Phi D^b(S, \mathcal{B}_0), \pi^* D^b(S) \rangle.$$

We will refer to Proposition 3.1 as the *Kuznetsov formula* for conic bundles. Remark that Kuznetsov actually gives a similar semiorthogonal decomposition for any quadric fibration over any smooth projective manifold. If in particular  $S$  is a smooth rational surface with Picard number  $\rho$ , its derived category admits a full exceptional sequence. It follows that

$$D^b(X) = \langle \Phi D^b(S, \mathcal{B}_0), E_1, \dots, E_{\rho+2} \rangle,$$

where  $(E_1, \dots, E_{\rho+2})$  is the the pull back of the full exceptional sequence of  $D^b(S)$ .

**Remark 3.2.** Let  $S$  be a smooth projective surface and  $K(S)$  its residue field, and  $\text{Br}(-)$  denote the Brauer group. A quaternion algebra  $\mathcal{A}_\eta$  is an element of order 2 of  $\text{Br}(K(S))$ . There is an exact sequence [3]:

$$0 \longrightarrow \text{Br}(S) \longrightarrow \text{Br}(K(S)) \xrightarrow{\alpha} \bigoplus_{D \subset S} H_{\text{et}}^1(D, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} \bigoplus_{x \in S} \mu^{-1},$$

where in the third (resp. fourth) term the sum runs over curves in (resp. points of)  $S$ . Recall that all elements of order two in  $\text{Br}(K(S))$  are quaternion algebras [33]. The exact sequence sets up a 1-1 correspondence between maximal orders  $\mathcal{A}$  in  $\mathcal{A}_\eta$  and standard conic bundles with associated double covering  $\tilde{C} \rightarrow C$  [3, 43].

If  $S$  is rational, then  $\text{Br}(S) = 0$ , the map  $\alpha$  is injective and there is unique maximal order  $\mathcal{A}$  for a given quaternion algebra  $\mathcal{A}_\eta$ . In this case, we have a 1-1 correspondence between quaternion algebras  $\mathcal{A}_\eta$  and standard conic bundles, as explained in [43, 24]. Consider the quadratic form defining the conic bundle over the generic point of  $S$ . Then the even part of its Clifford algebra is isomorphic to a quaternion algebra. With this in mind, we obtain that the algebra  $\mathcal{B}_0$  and the derived category  $D^b(S, \mathcal{B}_0)$  are fixed once fixed the admissible double cover  $\tilde{C} \rightarrow C$ . A similar argument was first developed by Panin ([40] page 450-51) in the case of conic bundles on  $\mathbb{P}^2$  with a quintic discriminant curve.

**3.2. Algebraically trivial cycles on  $X$  and Prym varieties.** Given a curve  $C$  with at most double points and an admissible double covering  $\tilde{C} \rightarrow C$  one can define the *Prym variety*  $P(\tilde{C}/C)$  as the connected component containing 0 of the kernel of  $Nm : J(\tilde{C}) \rightarrow J(C)$ . Remark that if  $C$  is singular, one has to go through the normalizations of  $C$  and  $\tilde{C}$ . The Prym variety is a principally polarized abelian subvariety of  $J(\tilde{C})$  of index 2 ([34, 4]).

Let  $\pi : X \rightarrow S$  be a standard conic bundle with associated double covering  $\tilde{C} \rightarrow C$ . If  $S = \mathbb{P}^2$ , Beauville showed that the intermediate Jacobian  $J(X)$  is isomorphic as a principally polarized abelian variety to  $P(\tilde{C}/C)$  [5]. Moreover, he shows that the algebraically trivial part  $A^2(X)$  of  $CH^2(X)$  is isomorphic to the Prym variety  $P(\tilde{C}/C)$ . The key geometric point is that the family of vertical lines (that is, lines contained in a fiber of  $\pi$ ) in  $X$  is a  $\mathbb{P}^1$ -bundle over the curve  $\tilde{C}$ . There is then a surjective morphism  $g : J(\tilde{C}) \rightarrow A^2(X)$  extending the map associating to a point  $c$  of  $\tilde{C}$  the line  $l_c$  over it. The isomorphism  $\xi : P(\tilde{C}/C) \rightarrow A^2(X)$  is obtained by taking the quotient via  $\ker(g)$ . The inverse isomorphism  $G = \xi^{-1}$  is a regular map making of  $P(\tilde{C}/C)$  the algebraic representative of  $A^2(X)$  (for more details, see [5, Ch. III]). Similar techniques prove the same results for any  $S$  rational [7, 8].

**Definition 3.3** ([5], Déf 3.4.2). Let  $Y$  be a smooth projective variety of odd dimension  $2n + 1$  and  $A$  (an abelian variety) the algebraic representative of  $A^{n+1}(Y)$  via the canonical map  $G : A^{n+1}(Y) \rightarrow A$ . A polarization of  $A$  with class  $\theta_A$  in  $\text{Corr}(A)$ , is the *incidence polarization* with

respect to  $Y$  if for all algebraic maps  $f : T \rightarrow A^{n+1}(Y)$  defined by a cycle  $z$  in  $CH^{n+1}(Y \times T)$ , we have

$$(G \circ f)^* \theta_A = (-1)^{n+1} I(z),$$

where  $I(z)$  in  $\text{Corr}(T)$  is the composition of the correspondences  $z \in \text{Corr}(T, X)$  and  $z \in \text{Corr}(X, T)$ .

**Proposition 3.4.** *Let  $\pi : X \rightarrow S$  be a standard conic bundle over a smooth rational surface. The principal polarization  $\Theta_P$  of  $P(\tilde{C}/C)$  is the incidence polarization with respect to  $X$ .*

*Proof.* We prove the statement in the case where  $C$  is smooth. In the case of nodal curves, one has to go through the normalization, and this is just rewriting the proof of [5, Thm. 3.6, (iii)].

If  $S = \mathbb{P}^2$ , this is [5, Prop. 3.5]. If  $S$  is not  $\mathbb{P}^2$ , consider the isomorphism  $\xi$ . The proof of [5, Prop. 3.3] can be rephrased in this setting, in particular, recalling the diagram in [7, Pag. 83], one can check that the map  $2\xi$  is described by a cycle  $y$  in  $CH^2(X \times P(\tilde{C}/C))$ . Let  $f : T \rightarrow A^2(X)$  be an algebraic map defined by a cycle  $z$  in  $CH^2(X \times T)$ . Denoting by  $u := G \circ f$  and  $u' := (\text{Id}_X, u)$ , the map  $2f$  is defined by the cycle  $(u')^* y$ . The proof is now the same as the one of [5, Prop. 3.5].  $\square$

#### 4. RECONSTRUCTING THE INTERMEDIATE JACOBIAN

The first main result of this paper is the reconstruction of  $J(X)$  as the direct sum of Jacobians of smooth projective curves, starting from a semiorthogonal decomposition of  $D^b(S, \mathcal{B}_0)$ . This Section is entirely dedicated to the proof of Theorem 1.1.

**Theorem 1.1.** *Let  $\pi : X \rightarrow S$  be a standard conic bundle over a rational surface. Suppose that  $\{\Gamma_i\}_{i=1}^k$  are smooth projective curves and  $k \geq 0$ , with fully faithful functors  $\Psi_i : D^b(\Gamma_i) \rightarrow D^b(S, \mathcal{B}_0)$  for  $i = 1, \dots, k$ , such that  $D^b(S, \mathcal{B}_0)$  admits a semiorthogonal decomposition:*

$$(4.1) \quad D^b(S, \mathcal{B}_0) = \langle \Psi_1 D^b(\Gamma_1), \dots, \Psi_k D^b(\Gamma_k), E_1, \dots, E_l \rangle,$$

where  $E_i$  are exceptional objects and  $l \geq 0$ . Then  $J(X) = \bigoplus_{i=1}^k J(\Gamma_i)$  as principally polarized abelian variety.

If  $S$  is minimal, we obtain the “if” part of Theorem 1.2 from [45, Thm. 10.1].

**Corollary 4.1.** *If  $\pi : X \rightarrow S$  is a standard conic bundle over a minimal rational surface and*

$$D^b(S, \mathcal{B}_0) = \langle \Psi_1 D^b(\Gamma_1), \dots, \Psi_k D^b(\Gamma_k), E_1, \dots, E_l \rangle,$$

where  $\Gamma_i$  are smooth projective curves,  $\Psi_i : D^b(\Gamma_i) \rightarrow D^b(S, \mathcal{B}_0)$  are full and faithful functors,  $E_i$  are exceptional objects and  $l, k \geq 0$ , then  $X$  is rational and  $J(X) = \bigoplus_{i=1}^k J(\Gamma_i)$ .

If we have the decomposition (4.1), using Prop. 3.1 and that  $S$  is rational, we get

$$(4.2) \quad D^b(X) = \langle \Psi_1 D^b(\Gamma_1), \dots, \Psi_k D^b(\Gamma_k), E_1, \dots, E_r \rangle,$$

where  $E_i$  are exceptional objects,  $r = l + \rho + 2 > 0$ , and we denote by  $\Psi_i$ , by abuse of notation, the composition of the full and faithful functor  $\Psi_i$  with the full and faithful functor  $D^b(S, \mathcal{B}_0) \rightarrow D^b(X)$ . Remark that we can suppose that  $\Gamma_i$  has positive genus for all  $i = 1, \dots, k$ . Indeed, the derived category of the projective line admits a semiorthogonal decomposition by two exceptional objects. Then if there exists an  $i$  such that  $\Gamma_i \simeq \mathbb{P}^1$ , it is enough to perform some mutation to get a semiorthogonal decomposition like (4.2) with  $g(\Gamma_i) > 0$  for all  $i$  (recall we do not exclude the case  $k = 0$ ). By Lemma 2.7 we have:

$$(4.3) \quad CH_{\mathbb{Q}}^*(X) = \bigoplus_{i=1}^k CH_{\mathbb{Q}}^*(\Gamma_i) \oplus \mathbb{Q}^r,$$

where we used the fact that the category generated by a single exceptional object is equivalent to the derived category of a point, and  $CH_{\mathbb{Q}}^*(pt) = \mathbb{Q}$ . We are interested in understanding how the

decomposition (4.3) projects onto the codimension 2 cycle group  $CH_{\mathbb{Q}}^2(X)$  and in particular onto the algebraically trivial part.

The proof is in two parts: first if  $\Psi : D^b(\Gamma) \rightarrow D^b(X)$  is fully faithful and  $\Gamma$  has positive genus, we get that  $J(\Gamma)$  is isomorphic to a principally polarized abelian subvariety of  $J(X)$  (Prop. 4.4). This is essentially based on constructions from [39] and results from [5]. In the second part, starting from the semiorthogonal decomposition we deduce the required isomorphism.

**Lemma 4.2.** *Let  $\Gamma$  be a smooth projective curve of positive genus. Suppose there is a fully faithful functor  $\Psi : D^b(\Gamma) \rightarrow D^b(X)$ . Then  $J(\Gamma)$  is isogenous to an abelian subvariety of  $J(X) \simeq P(\tilde{C}/C)$ .*

*Proof.* Let  $\mathcal{E}$  be the kernel of the fully faithful functor  $\Psi : D^b(\Gamma) \rightarrow D^b(X)$ , and  $\mathcal{F}$  the kernel of its right adjoint. If we consider the cycles  $e$  and  $f$  described in Section 2.3, the Grothendieck–Riemann–Roch formula implies that  $f.e$  induces the identity  $\text{Id} : h(\Gamma) \rightarrow h(\Gamma)$ . If  $e_i$  and  $f_i$  are the  $i$ -th codimensional components of  $e$  resp. of  $f$  in  $CH_{\mathbb{Q}}^*(X \times \Gamma)$ , then  $f.e = \oplus f_i.e_{4-i}$ . Remark that  $e_i$  gives a map  $h(\Gamma) \rightarrow h(X)(i-3)$ . If we restrict to  $h^1(\Gamma)$ , then the motivic decomposition 3.1, together with the fact that  $S$  is rational, gives us  $(f_i.e_{4-i})|_{h^1(\Gamma)} = 0$  for all  $i \neq 2$ . This implies that  $\text{Id}_{h^1(\Gamma)} = (f_2.e_2)|_{h^1(\Gamma)}$ , and then that  $h^1(\Gamma)$  is a direct summand of  $h(X)(-1)$  and in particular it is a direct summand of  $\text{Prym}^1(\tilde{C}/C)(-1)$ , which proves the claim.  $\square$

Remark that we can describe explicitly the map  $\psi_{\mathbb{Q}} : J_{\mathbb{Q}}(\Gamma) \rightarrow J_{\mathbb{Q}}(X)$  induced by  $\Psi$ , following the ideas in [9]. Indeed the map  $\psi_{\mathbb{Q}}$  is given by  $e_2$ , the codimension 2 component of the cycle associated to the kernel  $\mathcal{E}$ . Then  $\psi_{\mathbb{Q}}$  can be calculated just applying the Grothendieck–Riemann–Roch Theorem.

Let  $p : \Gamma \times X \rightarrow X$  and  $q : \Gamma \times X \rightarrow \Gamma$  be the two projections. For  $M$  in  $J(\Gamma)$  we calculate the second Chern character  $(\text{ch}(\Psi(M)))_2$ , since we know that the image of  $M$  lies in  $J(X)$ , that is in codimension 2. Applying Grothendieck–Riemann–Roch and using multiplicativity of Chern characters, we have the following:

$$(\text{ch}(p_*(q^*M \otimes \mathcal{E})))_2 = p_*(\text{ch}(q^*M).\text{ch}(e).(1 - (1/2)q^*K_{\Gamma}))_3,$$

since the relative dimension of  $p$  is 1 and the relative Todd class is  $1 - (1/2)q^*K_{\Gamma}$ . Recalling that  $\text{ch}(q^*M) = 1 + q^*M$  and  $q^*M.q^*K_{\Gamma} = 0$ , we get

$$(\text{ch}(p_*(q^*M \otimes \mathcal{E})))_2 = p_*(q^*M.\text{ch}_2(\mathcal{E}) - (1/2)q^*K_{\Gamma}.\text{ch}_2(\mathcal{E}) + \text{ch}_3(\mathcal{E})).$$

It is clear that this formula just defines an affine map  $\Psi^{CH} : CH_{\mathbb{Q}}^1(\Gamma) \rightarrow CH_{\mathbb{Q}}^2(X)$  of  $\mathbb{Q}$ -vector spaces. In order to get the isogeny  $\psi_{\mathbb{Q}}$ , we have to linearize and restrict to  $J_{\mathbb{Q}}(\Gamma)$ , to get finally:

$$\begin{aligned} \psi_{\mathbb{Q}} : J_{\mathbb{Q}}(\Gamma) &\longrightarrow J_{\mathbb{Q}}(X) \\ M &\mapsto p_*(q^*M.\text{ch}_2(\mathcal{E})) \end{aligned}$$

Now that we have the cycle describing the map  $\psi_{\mathbb{Q}}$ , we obtain a unique morphism  $\psi : J(\Gamma) \rightarrow J(X)$ , whose kernel can only be torsion. That is, we have an isogeny  $\psi$  between  $J(\Gamma)$  and an abelian subvariety of  $J(X)$ .

**Remark 4.3.** Arguing as in [9, Sect. 2.3], we can show that the correspondence between  $\Psi$  and  $\psi$  is functorial. Moreover, the functor with kernel  $\mathcal{E}[n]$  induces the map  $(-1)^n\psi$ . The functor with kernel  $\mathcal{E}^{\vee}$  induces the map  $\psi$ . Given line bundles  $L$  and  $L'$  on  $\Gamma$  and  $X$  respectively, the functor with kernel  $\mathcal{E} \otimes p^*L \otimes q^*L'$  induces the map  $\psi$ . The adjoint functor of  $\Psi$  is a Fourier–Mukai functor whose kernel is  $\mathcal{E}^{\vee} \otimes q^*\omega_X[3]$ . Its composition with  $\Psi$  gives the identity of  $D^b(\Gamma)$ . The motivic map  $f_2 : \text{Prym}^1(\tilde{C}/C)(-1) \rightarrow h^1(\Gamma)$  is then given by the cycle  $-\text{ch}_2(e)$ . Then, by functoriality and (2.2), the cycle  $I(\text{ch}_2(e))$ , as defined in Def. 3.3, is  $-\text{Id}$  in  $\text{Corr}(J(\Gamma))$ .

Recall that, by [5, Sec. 3], [7, 8] and Proposition 3.4,  $P(\tilde{C}/C)$  is the algebraic representative of  $A^2(X)$  and the principal polarization  $\Theta_P$  of  $P(\tilde{C}/C)$  is the incidence polarization with respect to  $X$ . In particular, we have an isomorphism  $\xi : P(\tilde{C}/C) \rightarrow A^2(X)$  whose inverse  $G$  makes the Prym

variety the algebraic representative of  $A^2(X)$ . Moreover, if  $f : T \rightarrow A^2(X)$  is an algebraic map defined by a cycle  $z$  in  $CH_{\mathbb{Q}}^2(X \times T)$ , then, according to Definition 3.3, we have

$$(G \circ f)^* \theta_P = I(z).$$

The map  $\psi$  is defined by the cycle  $\text{ch}_2(e)$  in  $CH_{\mathbb{Q}}^2(X \times \Gamma)$ . Following Remark 4.3, the cycle  $I(\text{ch}_2(e))$  in  $CH_{\mathbb{Q}}^1(\Gamma \times \Gamma)$  gives the correspondence  $-\text{Id}$ , that is

$$(G \circ \psi)^* \theta_P = -\text{Id}.$$

Now going through the proof of [5, Prop. 3.3], it is clear that

$$\psi^* \theta_{J(X)} = \text{Id},$$

where  $\theta_{J(X)}$  is the class of principal polarization of  $J(X)$ . Hence we get an injective morphism  $\psi : J(\Gamma) \rightarrow J(X)$  preserving the principal polarization. We can state the following result.

**Proposition 4.4.** *Let  $\pi : X \rightarrow S$  a standard conic bundle over a rational surface. Suppose that there is a smooth projective curve  $\Gamma$  of positive genus and a fully faithful functor  $\Psi : D^b(\Gamma) \rightarrow D^b(S, \mathcal{B}_0)$ . Then there is an injective morphism  $\psi : J(\Gamma) \rightarrow J(X)$  of abelian varieties, preserving the principal polarization.*

Consider the projection  $pr : CH_{\mathbb{Q}}^*(X) \rightarrow CH_{\mathbb{Q}}^2(X)$ . The decomposition (4.3) is rewritten as:

$$CH_{\mathbb{Q}}^*(X) = \bigoplus_{i=0}^k \text{Pic}_{\mathbb{Q}}(\Gamma_i) \oplus \mathbb{Q}^{r+k},$$

where we used that  $CH_{\mathbb{Q}}^*(\Gamma_i) = \text{Pic}_{\mathbb{Q}}(\Gamma_i) \oplus \mathbb{Q}$ . The previous arguments show that  $pr$  restricted to  $\bigoplus_{i=1}^k \text{Pic}_{\mathbb{Q}}^0(\Gamma_i)$  is injective and has image in  $A_{\mathbb{Q}}^2(X)$ . This map corresponds on each direct summand to the injective map  $\psi_{i,\mathbb{Q}}$  obtained as in Lemma 4.2. Then the restriction of  $pr$  to  $\bigoplus_{i=1}^k \text{Pic}_{\mathbb{Q}}^0(\Gamma_i)$  corresponds to the sum of all those maps, and we denote it by  $\psi_{\mathbb{Q}}$ . Consider now the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{i=0}^k \text{Pic}_{\mathbb{Q}}^0(\Gamma_i) & \longrightarrow & \bigoplus_{i=0}^k \text{Pic}_{\mathbb{Q}}(\Gamma) \oplus \mathbb{Q}^{k+r} & & \\ & & \downarrow \scriptstyle pr = \psi_{\mathbb{Q}} & & \downarrow \scriptstyle pr & \searrow \scriptstyle \bar{pr} & \\ 0 & \longrightarrow & A_{\mathbb{Q}}^2(X) & \longrightarrow & CH_{\mathbb{Q}}^2(X) & \longrightarrow & CH_{\mathbb{Q}}^2(X)/A_{\mathbb{Q}}^2(X) \longrightarrow 0, \end{array}$$

where  $\bar{pr}$  denotes the composition of  $pr$  with the projection onto the the quotient.

Denote by  $J := \psi(\bigoplus \text{Pic}^0(\Gamma_i))$  the image of  $\Psi$  and  $J_{\mathbb{Q}} := J \otimes \mathbb{Q}$ . We have that the cokernel  $A_{\mathbb{Q}}^2(X)/J_{\mathbb{Q}}$  is a finite dimensional  $\mathbb{Q}$ -vector space. Since  $\psi$  is a morphism of abelian varieties, its cokernel is also an abelian variety, and then it has to be trivial. This gives the surjectivity of  $\psi$  and proves Theorem 1.1.

**Remark 4.5.** Let  $\rho$  be the rank of the Picard group of  $S$ . The numbers  $l$  and  $k$  satisfy a linear equation: using the decomposition (4.3), we obtain  $l = 2 + \rho - 2k$ .

## 5. RATIONAL CONIC BUNDLES OVER THE PLANE

Let  $\pi : X \rightarrow \mathbb{P}^2$  be a rational standard conic bundle. There are only two non-trivial possibilities for the discriminant curve ([5, 45, 23]). In fact,  $X$  is rational if and only if  $C + \omega_{\mathbb{P}^2}$  is non-effective, thus either  $C$  is a quintic and the double covering  $\tilde{C} \rightarrow C$  is given by an even theta characteristic or  $C$  is a quartic curve. As we have seen in Remark 3.2, once we fix the discriminant curve and the associated double cover, we fix the Clifford algebra  $\mathcal{B}_0$ . We then construct for any such plane curve and associated double cover a model of rational standard conic bundle  $X$  for which we provide the required semiorthogonal decomposition. We analyze the two cases separately.

**5.1. Degree five degeneration.** Suppose  $C$  is a degree 5 curve and  $\tilde{C} \rightarrow C$  is given by an even theta-characteristic. Recall the description of a birational map  $\chi : X \rightarrow \mathbb{P}^3$  from [40] (see also [24]). There is a smooth curve  $\Gamma$  of genus 5 and degree 7 in  $\mathbb{P}^3$  such that  $\chi : X \rightarrow \mathbb{P}^3$  is the blow-up of  $\mathbb{P}^3$  along  $\Gamma$ . In fact the conic bundle  $X \rightarrow \mathbb{P}^2$  is obtained [24] by resolving the linear system of cubics in  $\mathbb{P}^3$  vanishing on  $\Gamma$ . Let us denote by  $H$  the pull-back of  $\mathcal{O}_{\mathbb{P}^3}(1)$  via  $\chi$ , and by  $D$  the exceptional divisor. Remark that  $J(X)$  is isomorphic to  $J(\Gamma)$  as a principally polarized abelian variety.

Let  $\pi : X \rightarrow \mathbb{P}^2$  be the conic bundle structure. We denote by  $h$  the pull back of  $\mathcal{O}_{\mathbb{P}^2}(1)$  via  $\pi$ . The construction of the map  $\pi$  gives  $h = 3H - D$ , then we have  $D = 3H - h$ . The canonical bundle  $\omega_X$  is given by  $-4H + D = -H - h$ .

**Proposition 5.1.** *Let  $\pi : X \rightarrow \mathbb{P}^2$  be a standard conic bundle whose discriminant curve  $C$  is a degree 5 curve and  $\tilde{C} \rightarrow C$  is given by an even theta-characteristic. Then there exists an exceptional object  $E$  in  $\mathbb{D}^b(\mathbb{P}^2, \mathcal{B}_0)$  such that (up to equivalences):*

$$\mathbb{D}^b(\mathbb{P}^2, \mathcal{B}_0) = \langle \mathbb{D}^b(\Gamma), E \rangle,$$

where  $\Gamma$  is a smooth projective curve such that  $J(X) \simeq J(\Gamma)$  as a principally polarized abelian variety.

*Proof.* Consider the blow-up  $\chi : X \rightarrow \mathbb{P}^3$ . Orlov formula (see Prop. 2.4) provides a fully faithful functor  $\Psi : \mathbb{D}^b(\Gamma) \rightarrow \mathbb{D}^b(X)$  and a semiorthogonal decomposition:

$$\mathbb{D}^b(X) = \langle \Psi \mathbb{D}^b(\Gamma), \chi^* \mathbb{D}^b(\mathbb{P}^3) \rangle.$$

The derived category  $\mathbb{D}^b(\mathbb{P}^3)$  has a full exceptional sequence  $\langle \mathcal{O}_{\mathbb{P}^3}(-2), \dots, \mathcal{O}_{\mathbb{P}^3}(1) \rangle$ . We get then the semiorthogonal decomposition:

$$(5.1) \quad \mathbb{D}^b(X) = \langle \Psi \mathbb{D}^b(\Gamma), -2H, -H, \mathcal{O}, H \rangle.$$

Kuznetsov formula (see Prop. 3.1) provides the decomposition:

$$\mathbb{D}^b(X) = \langle \Phi \mathbb{D}^b(\mathbb{P}^2, \mathcal{B}_0), \pi^* \mathbb{D}^b(\mathbb{P}^2) \rangle.$$

The derived category  $\mathbb{D}^b(\mathbb{P}^2)$  has a full exceptional sequence  $\langle \mathcal{O}_{\mathbb{P}^2}(-1), \dots, \mathcal{O}_{\mathbb{P}^2}(1) \rangle$ . We get then the semiorthogonal decomposition

$$(5.2) \quad \mathbb{D}^b(X) = \langle \Phi \mathbb{D}^b(\mathbb{P}^2, \mathcal{B}_0), -h, \mathcal{O}, h \rangle.$$

We perform now some mutation to compare the decompositions 5.1 and 5.2.

Consider the decomposition 5.2 and mutate  $\Phi \mathbb{D}^b(\mathbb{P}^2, \mathcal{B}_0)$  to the right through  $-h$ . The functor  $\Phi' = \Phi \circ R_{-h}$  is full and faithful and we have the semiorthogonal decomposition:

$$\mathbb{D}^b(X) = \langle -h, \Phi' \mathbb{D}^b(\mathbb{P}^2, \mathcal{B}_0), \mathcal{O}, h \rangle.$$

Perform the left mutation of  $h$  through its left orthogonal, which gives

$$\mathbb{D}^b(X) = \langle -H, -h, \Phi' \mathbb{D}^b(\mathbb{P}^2, \mathcal{B}_0), \mathcal{O} \rangle,$$

using Lemma 2.6 and  $\omega_X = -H - h$ .

**Lemma 5.2.** *The pair  $\langle -H, -h \rangle$  is completely orthogonal.*

*Proof.* Consider the semiorthogonal decomposition 5.1 and perform the left mutation of  $H$  through its left orthogonal. By Lemma 2.6 we get

$$\mathbb{D}^b(X) = \langle -h, \Psi \mathbb{D}^b(\Gamma), -2H, -H, \mathcal{O} \rangle,$$

which gives us  $\text{Hom}^\bullet(-H, -h) = 0$  by semiorthogonality.  $\square$

We can now exchange  $-H$  and  $-h$ , obtaining a semiorthogonal decomposition

$$D^b(X) = \langle -h, -H, \Phi'D^b(\mathbb{P}^2, \mathcal{B}_0), \mathcal{O} \rangle.$$

The right mutation of  $-h$  through its right orthogonal gives (with Lemma 2.6) the semiorthogonal decomposition:

$$D^b(X) = \langle -H, \Phi'D^b(\mathbb{P}^2, \mathcal{B}_0), \mathcal{O}, H \rangle.$$

Perform the left mutation of  $\Phi'D^b(\mathbb{P}^2, \mathcal{B}_0)$  through  $-H$ . The functor  $\Phi'' = \Phi' \circ L_{-H}$  is full and faithful and we have the semiorthogonal decomposition:

$$D^b(X) = \langle \Phi''D^b(\mathbb{P}^2, \mathcal{B}_0), -H, \mathcal{O}, H \rangle.$$

This shows, by comparison with (5.1), that  $\Phi''D^b(\mathbb{P}^2, \mathcal{B}_0) = \langle \Psi D^b(\Gamma), -2H \rangle$ .  $\square$

**5.2. Degree four degeneration.** Suppose  $C \subset \mathbb{P}^2$  is a degree four curve with at most double points. We are going to describe  $X$  as a hyperplane section of a conic bundle over (a blow-up of)  $\mathbb{P}^3$ , basing upon a construction from [11]. Let  $\Gamma$  be a smooth genus 2 curve, and  $\text{Pic}^n(\Gamma)$  the Picard variety of  $\Gamma$  that parametrizes degree  $n$  line bundles, up to linear equivalence. Since  $g(\Gamma) = 2$ ,  $\text{Pic}^1(\Gamma)$  contains the canonical Riemann theta divisor  $\Theta := \{L \in \text{Pic}^1(\Gamma) \mid h^0(\Gamma, L) \neq 0\}$ . It is well known that the Kummer surface  $Kum(\Gamma) := \text{Pic}^0(\Gamma)/\pm \text{Id}$  is naturally embedded in the linear system  $|2\Theta| = \mathbb{P}^3$ . The surface  $Kum(\Gamma)$  sits in  $\mathbb{P}^3$  as a quartic surface with 16 double points. Note that the point corresponding to the line bundle  $\mathcal{O}_\Gamma$  is a node, and we will call it *the origin* or simply  $\mathcal{O}_\Gamma$ .

Now we remark that  $\Gamma$  is tri-canonically embedded in  $\mathbb{P}^4 = |\omega_\Gamma^3|^*$ , moreover we have a rational map

$$\varphi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3 := |\mathcal{I}_\Gamma(2)|^*$$

given by quadrics in the ideal of  $\Gamma$ . In [11] it is shown that there exists an isomorphism  $|\mathcal{I}_\Gamma(2)|^* \cong |2\Theta|$ . Let now  $\widetilde{Kum}(\Gamma)$  be the blow-up of  $Kum(\Gamma)$  in the origin  $\mathcal{O}_\Gamma$  and  $\mathbb{P}_\mathcal{O}^3$  the corresponding blow up of  $\mathbb{P}^3$ , so that we have  $\widetilde{Kum}(\Gamma) \subset \mathbb{P}_\mathcal{O}^3$ . Consider now the curve  $\Gamma$  in  $\mathbb{P}^4$  and any point  $p \in \Gamma$ . We denote by  $q_p$  the only effective divisor in the linear system  $|\omega_\Gamma(-p)|$ . The ruled surface

$$S := \{x \in \mathbb{P}^4 \mid x \in \overline{pq_p}, \forall p \in \Gamma\}$$

is a cone over a twisted cubic  $Y$  in  $\mathbb{P}^3$ . Let  $Bl_S \mathbb{P}^4$  the blow-up of  $\mathbb{P}^4$  along the cubic cone, then the main result of [11] can be phrased as follows.

**Theorem 5.3.** *The rational map  $\varphi$  resolves to a morphism  $\tilde{\varphi} : Bl_S \mathbb{P}^4 \rightarrow \mathbb{P}_\mathcal{O}^3$  that is a conic bundle degenerating on  $\widetilde{Kum}(\Gamma)$ . Hence we have the following commutative diagram.*

$$\begin{array}{ccc} Bl_S \mathbb{P}^4 & \xrightarrow{\tilde{\varphi}} & \mathbb{P}_\mathcal{O}^3 \\ \downarrow & & \downarrow \\ \mathbb{P}^4 & \dashrightarrow_{\varphi} & \mathbb{P}^3 \end{array}$$

**Remark 5.4.** The conic bundle described in Thm. 5.3 is standard. This is straightforward from the description in [11].

For any plane quartic curve  $C$  with at most double points, we are going to obtain a structure of a standard conic bundle on  $\mathbb{P}^2$  degenerating on  $C$  by taking the restriction of  $\tilde{\varphi}$  to suitable hyperplanes of  $\mathbb{P}^3$  for suitable choices of the genus two curve  $\Gamma$ . In fact every such quartic curve can be obtained via hyperplane intersection with an appropriate Jacobian Kummer surface, see [1, Rem. 2.2] and [46].

More precisely consider the composition  $\phi : Bl_S \mathbb{P}^4 \rightarrow \mathbb{P}_\mathcal{O}^3 \rightarrow \mathbb{P}^3$  of the conic bundle of Thm. 5.3 with the blow-down map. Consider a hyperplane  $N \subset \mathbb{P}^3$  not containing the origin  $\mathcal{O}_\Gamma$  and denote

by  $X := \phi^{-1}(N)$  and by  $\pi$  the restriction of  $\phi$  to  $X$ . Then the induced map  $\pi : X \rightarrow N \simeq \mathbb{P}^2$  defines a standard conic bundle that degenerates on the intersection  $N \cap Kum(\Gamma)$ . Then it is easy to see that  $X$  is isomorphic to the blow-up along  $\Gamma$  of a smooth (since  $N$  does not contain  $\mathcal{O}_\Gamma$ ) quadric hypersurface  $Z \subset \mathbb{P}^4$  in the ideal of  $\Gamma \subset \mathbb{P}^4$ .

It is also known ([1], [46]) that the admissible double cover of  $N \cap Kum(\Gamma)$  induced by the degree 2 cover  $J(\Gamma)/\pm Id$  has Prym variety isomorphic to  $J(\Gamma)$ . Remark that this is indeed the double cover of the plane quartic induced by the corresponding conic bundle. This means that the intermediate Jacobian  $J(X)$  is isomorphic to  $J(\Gamma)$  as a principally polarized abelian variety.

Finally, remark that we can always assume that the quadric hypersurface  $Z$  we are considering is smooth. The locus of singular quadric hypersurfaces corresponds via  $\phi$  to hyperplanes in  $\mathbb{P}^3$  passing through the origin  $\mathcal{O}_\Gamma$  of the Kummer surface. Notably these correspond to the quadric cones over the quadrics in  $\mathbb{P}^3$  vanishing on the twisted cubic  $Y$ . It is easy to see, using the invariance of  $Kum(\Gamma)$  under the action of  $(\mathbb{Z}/2\mathbb{Z})^4$ , that one can get any plane quartic with at most double points to us by considering hyperplanes in  $\mathbb{P}^3$  that do not contain the origin.

Resuming, let  $\chi : X \rightarrow Z$  be the blow-up of  $Z$  along  $\Gamma$ . Let us denote by  $H$  both the restriction of  $\mathcal{O}_{\mathbb{P}^4}(1)$  to  $Z$  and its pull-back to  $X$  via  $\chi$ , and by  $D$  the exceptional divisor. Remark that  $\omega_Z = -3H$ . Let  $\Sigma$  be the spinor bundle on the quadric  $Z$ . Let  $\pi : X \rightarrow \mathbb{P}^2$  be the conic bundle structure. We denote by  $h$  the pull back of  $\mathcal{O}_{\mathbb{P}^2}(1)$  via  $\pi$ . The construction of the map  $\pi$  gives  $h = 2H - D$ , then we have  $D = 2H - h$ . The canonical bundle  $\omega_X$  is given by  $-3H + D = -H - h$ .

**Proposition 5.5.** *Let  $\pi : X \rightarrow \mathbb{P}^2$  be a standard conic bundle whose discriminant locus  $C$  is a degree 4 curve. Then there exists an exceptional object  $E$  in  $D^b(\mathbb{P}^2, \mathcal{B}_0)$  such that (up to equivalences):*

$$D^b(\mathbb{P}^2, \mathcal{B}_0) = \langle D^b(\Gamma), E \rangle,$$

where  $\Gamma$  is a smooth projective curve such that  $J(X) \simeq J(\Gamma)$  as a principally polarized abelian variety.

*Proof.* Consider the blow-up  $\chi : X \rightarrow Z$ . Orlov formula (see Prop. 2.4) provides a fully faithful functor  $\Psi : D^b(\Gamma) \rightarrow D^b(X)$  and a semiorthogonal decomposition:

$$D^b(X) = \langle \Psi D^b(\Gamma), \chi^* D^b(Z) \rangle.$$

By [25], the derived category  $D^b(Z)$  has a full exceptional sequence  $\langle \Sigma - 2H, -H, \mathcal{O}, H \rangle$ . We get then the semiorthogonal decomposition:

$$(5.3) \quad D^b(X) = \langle \Psi D^b(\Gamma), \Sigma - 2H, -H, \mathcal{O}, H \rangle.$$

Kuznetsov formula (see Prop. 3.1) provides the decomposition:

$$D^b(X) = \langle \Phi D^b(\mathbb{P}^2, \mathcal{B}_0), \pi^* D^b(\mathbb{P}^2) \rangle.$$

The derived category  $D^b(\mathbb{P}^2)$  has a full exceptional sequence  $\langle \mathcal{O}_{\mathbb{P}^2}(-1), \dots, \mathcal{O}_{\mathbb{P}^2}(1) \rangle$ . We get then the semiorthogonal decomposition:

$$(5.4) \quad D^b(X) = \langle \Phi D^b(\mathbb{P}^2, \mathcal{B}_0), -h, \mathcal{O}, h \rangle.$$

We perform now some mutation to compare the decompositions (5.3) and (5.4). Surprisingly to us, we will follow the same path as in the proof of Proposition 5.1.

Consider the decomposition (5.4) and mutate  $\Phi D^b(\mathbb{P}^2, \mathcal{B}_0)$  to the right through  $-h$ . The functor  $\Phi' = \Phi \circ R_{-h}$  is full and faithful and we have the semiorthogonal decomposition:

$$D^b(X) = \langle -h, \Phi' D^b(\mathbb{P}^2, \mathcal{B}_0), \mathcal{O}, h \rangle.$$

Perform the left mutation of  $h$  through its left orthogonal, which gives

$$D^b(X) = \langle -H, -h, \Phi' D^b(\mathbb{P}^2, \mathcal{B}_0), \mathcal{O} \rangle,$$

using Lemma 2.6 and  $\omega_X = -H - h$ . We can prove the following Lemma in the same way we proved Lemma 5.2.



**Lemma 5.6.** *The pair  $\langle -H, -h \rangle$  is completely orthogonal.*

We can now exchange  $-H$  and  $-h$ , obtaining a semiorthogonal decomposition

$$D^b(X) = \langle -h, -H, \Phi'D^b(\mathbb{P}^2, \mathcal{B}_0), \mathcal{O} \rangle.$$

The right mutation of  $-h$  through its right orthogonal gives (with Lemma 2.6) the semiorthogonal decomposition:

$$D^b(X) = \langle -H, \Phi'D^b(\mathbb{P}^2, \mathcal{B}_0), \mathcal{O}, H \rangle.$$

Perform the left mutation of  $\Phi'D^b(\mathbb{P}^2, \mathcal{B}_0)$  through  $-H$ . The functor  $\Phi'' = \Phi' \circ L_{-H}$  is full and faithful and we have the semiorthogonal decomposition:

$$D^b(X) = \langle \Phi''D^b(\mathbb{P}^2, \mathcal{B}_0), -H, \mathcal{O}, H \rangle.$$

This shows, by comparison with (5.3) that  $\Phi''D^b(\mathbb{P}^2, \mathcal{B}_0) = \langle \Psi D^b(\Gamma), \Sigma - 2H \rangle$ .  $\square$

## 6. RATIONAL CONIC BUNDLES OVER HIRZEBRUCH SURFACES

Let us consider now the case  $S = \mathbb{F}_n$  for  $n \neq 1$ . In this case, following ([24, 45]), we have only two non-trivial possibilities for a standard conic bundle  $\pi : X \rightarrow S$  to be rational: there must exist a base point free pencil  $L_0$  of rational curves such that either  $L_0 \cdot C = 3$  or  $L_0 \cdot C = 2$ . In the first case  $C$  is trigonal, and in the second one  $C$  is hyperelliptic. In both instances, the only such pencil is the natural ruling of  $S$ . Hence, if we let  $q : S \rightarrow \mathbb{P}^1$  be the ruling map, the trigonal or hyperelliptic structure is induced by the fibers of  $q$ . As we have seen in Remark 3.2, once we fix the discriminant curve and the associated double cover, we fix the Clifford algebra  $\mathcal{B}_0$ . We then construct for any such curve and associated double cover a model of rational standard conic bundle  $X$  for which we provide the required semiorthogonal decomposition.

We will proceed as follows: fixed the discriminant curve and the double cover  $\tilde{C} \rightarrow C$ , we describe a structure of conic bundle  $\pi : X \rightarrow S$  following Casnati [16] as the blow-up of a  $\mathbb{P}^2$ -bundle over  $\mathbb{P}^1$  along a certain tetragonal curve (in the case of hyperelliptic degeneration this requires a little more work and the tetragonal curve splits into two hyperelliptic curves) given by Recillas' construction ([41] for the trigonal case) and one of its degenerations (for the hyperelliptic case). These constructions can be performed for all the trigonal or hyperelliptic discriminant curves with at most nodes as singularities. We describe the case of trigonal and hyperelliptic degeneration separately, following anyway the same path.

The trigonal construction had already been used in the framework of conic bundles, in a slightly different context, in [19].

**6.1. Trigonal degeneration.** In the case where  $C$  is a trigonal curve on  $S$ , we can give an explicit description of the conic bundle  $\pi : X \rightarrow S$  degenerating along  $C$ , exploiting Recillas' trigonal construction [41]. We will develop the trigonal construction in the more general framework presented by Casnati in [16], that emphasizes the conic bundle structure. For a detailed account in the curve case, with emphasis on the beautiful consequences on the structure of the Prym map, see also [18].

Before going through details let us recall from [15] that any Gorenstein degree 3 cover  $t' : C \rightarrow \mathbb{P}^1$  can be obtained inside a suitable  $\mathbb{P}^1$ -bundle  $S := \mathbb{P}(\mathcal{F})$  over  $\mathbb{P}^1$  as the zeros of a relative cubic form in two variables. On the other hand each Gorenstein degree 4 cover  $t : \Gamma \rightarrow \mathbb{P}^1$  is obtained [15] as the base locus of a relative pencil of conics over  $\mathbb{P}^1$  contained in a  $\mathbb{P}^2$ -bundle  $Z := \mathbb{P}(\mathcal{E})$  over  $\mathbb{P}^1$ . Moreover, the restriction, both to  $C$  and  $\Gamma$ , of the natural projection of each projective bundle give the respective finite cover map to  $\mathbb{P}^1$ . For instance, the  $\mathbb{P}^2$ -fiber  $Z_x$  contains the four points of  $\Gamma$  over  $x \in \mathbb{P}^1$ .

In particular, fix a point  $x$  of  $\mathbb{P}^1$  and the corresponding plane  $Z_x$ , that is the fiber of the natural projection  $p : Z \rightarrow \mathbb{P}^1$  over  $x$ . Then consider the pencil of conics through the 4 points given by

intersecting  $Z_x$  with  $\Gamma$ . What we got is then a pencil of conics with three degenerate conics for each point of  $\mathbb{P}^1$ . We then have a pencil of such conic pencils (parameterized by the ruled surface  $S$ ), which can be described as the 2-dimensional family of vertical conics in  $Z$  intersecting  $\Gamma$  in all the four points of  $\Gamma \cap Z_x$ . The standard conic bundle over  $S$  is then given by resolving the linear system  $|\mathcal{O}_{Z/\mathbb{P}^1}(2) - \Gamma|$ .

This natural constructions for degree 3 and 4 covers naturally lead us to the result that matters the most to us, that is Thm. 6.5 of [16] (see also Thm 2.9 of [18]). This theorem basically says that to any trigonal Gorenstein curve  $C$  we can associate a smooth tetragonal curve  $\Gamma$  such that  $C$  is the discriminant locus of the conic bundle that defines  $\Gamma$ . That is: we consider the relative pencil of conics in the projective bundle  $Z \rightarrow \mathbb{P}^1$  that define  $\Gamma$ , this gives a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^1$  such that the locus of degenerate conics is exactly the curve  $C$  in its natural embedding as a relative cubic form. This Theorem ensures that all trigonal curves with at most double points that sit in some ruled surface  $S$  are discriminant divisors of a conic bundle (for details see [16], Sect. 5 and 6). The reader can easily see that the conic bundle  $X$  degenerating on  $C$  is isomorphic to the blow-up of  $Z$  along  $\Gamma$ . This tight connection between trigonal and tetragonal curves is reflected also when considering the corresponding Prym and Jacobian varieties. The Prym variety of the admissible cover of  $C$  induced by the conic bundle is in fact isomorphic to the Jacobian of  $\Gamma$  [41] and to the intermediate Jacobian of the conic bundle  $X$ .

In the following we will stick to the notation we used here above:  $C$  will indicate any trigonal curve, and  $\Gamma$  the tetragonal curve corresponding to  $C$  via the Casnati-Recillas construction. Both curves will be considered in their natural projective bundle embeddings.

Summarizing, we end up with the following commutative diagram:

$$\begin{array}{ccccc}
 & & X & \xleftrightarrow{\quad} & D \\
 & \swarrow \pi & \searrow \chi & & \searrow \chi \\
 S & & & & Z \\
 & \searrow q & & & \swarrow p \\
 & & & & \mathbb{P}^1,
 \end{array}$$

where  $p : Z \rightarrow \mathbb{P}^1$  is a  $\mathbb{P}^2$ -bundle,  $\Gamma \subset Z$  the tetragonal curve,  $\chi : X \rightarrow Z$  the blow-up of  $\Gamma$  with exceptional divisor  $D$ . The surface  $q : S \rightarrow \mathbb{P}^1$  is ruled and  $\pi : X \rightarrow S$  is the conic bundle structure degenerating along the trigonal curve  $C$ .

We denote by  $H := \mathcal{O}_{Z/\mathbb{P}^1}(1)$  the relative ample line bundle on  $Z$  and by  $h := \mathcal{O}_{S/\mathbb{P}^1}(1)$  the relative ample line bundle on  $S$ . By abuse of notation, we still denote by  $H$  and  $h$  the pull-back of  $H$  and  $h$  via  $\chi$  and  $\pi$  respectively. The construction of the map  $\pi$  gives  $h = 2H - D$ , from which we deduce that  $D = 2H - h$ . The canonical bundle  $\omega_X$  is given by  $\omega_X = \chi^*\omega_Z + D$ . Since we have  $\omega_Z = \omega_{Z/\mathbb{P}^1} + p^*\omega_{\mathbb{P}^1} = -3H + p^*\omega_{\mathbb{P}^1}$ , we finally get  $\omega_X = -H - h + \chi^*p^*\omega_{\mathbb{P}^1}$ .

**Proposition 6.1.** *Let  $\pi : X \rightarrow S$  be a conic bundle whose discriminant locus  $C$  is a trigonal curve whose trigonal structure is given by the intersection of  $C$  with the ruling  $S \rightarrow \mathbb{P}^1$ . Then there exist two exceptional objects  $E_1, E_2$  in  $D^b(S, \mathcal{B}_0)$  such that (up to equivalences):*

$$D^b(S, \mathcal{B}_0) = \langle D^b(\Gamma), E_1, E_2 \rangle,$$

where  $\Gamma$  is a smooth projective curve such that  $J(X) \simeq J(\Gamma)$  as a principally polarized abelian variety.

*Proof.* Consider the blow-up  $\chi : X \rightarrow Z$ . Orlov formula (see Prop. 2.4) provides a fully faithful functor  $\Psi : D^b(\Gamma) \rightarrow D^b(X)$  and a semiorthogonal decomposition:

$$D^b(X) = \langle \Psi D^b(\Gamma), \chi^* D^b(Z) \rangle.$$

By Prop. 2.3 we can choose the semiorthogonal decomposition  $\langle p^*D^b(\mathbb{P}^1) - H, p^*D^b(\mathbb{P}^1), p^*D^b(\mathbb{P}^1) + H \rangle$  of  $D^b(Z)$ , where the notation  $p^*D^b(\mathbb{P}^1) + iH$  stands for  $p^*D^b(\mathbb{P}^1) \otimes \mathcal{O}_{Z/\mathbb{P}^1}(i)$ . We then get:

$$(6.1) \quad D^b(X) = \langle \Psi D^b(\Gamma), \chi^* p^* D^b(\mathbb{P}^1) - H, \chi^* p^* D^b(\mathbb{P}^1), \chi^* p^* D^b(\mathbb{P}^1) + H \rangle.$$

Kuznetsov formula (see Prop. 3.1) provides the decomposition:

$$D^b(X) = \langle \Phi D^b(S, \mathcal{B}_0), \pi^* D^b(S) \rangle.$$

By Prop. 2.3 we can choose the semiorthogonal decomposition  $\langle q^*D^b(\mathbb{P}^1) - h, q^*D^b(\mathbb{P}^1) \rangle$  of  $D^b(S)$ . We then get:

$$(6.2) \quad D^b(X) = \langle \Phi D^b(S, \mathcal{B}_0), \pi^* q^* D^b(\mathbb{P}^1) - h, \pi^* q^* D^b(\mathbb{P}^1) \rangle.$$

We perform now some mutation to compare the decompositions (6.1) and (6.2). First of all, since  $\pi^* q^* = \chi^* p^*$ , we have  $\pi^* q^* D^b(\mathbb{P}^1) = \chi^* p^* D^b(\mathbb{P}^1)$  and we will denote this category simply by  $D^b(\mathbb{P}^1)$ .

Consider the decomposition (6.2) and mutate  $\Phi D^b(S, \mathcal{B}_0)$  to the right with respect to  $D^b(\mathbb{P}^1) - h$ . The functor  $\Phi' := \Phi \circ R_{D^b(\mathbb{P}^1) - h}$  is full and faithful and we have the semiorthogonal decomposition

$$D^b(X) = \langle D^b(\mathbb{P}^1) - h, \Phi' D^b(S, \mathcal{B}_0), D^b(\mathbb{P}^1) \rangle.$$

Perform the right mutation of  $D^b(\mathbb{P}^1) - h$  through its right orthogonal. We have

$$R_{\langle D^b(\mathbb{P}^1) - h \rangle^\perp} (D^b(\mathbb{P}^1) - h) = D^b(\mathbb{P}^1) - h - \omega_X \cong D^b(\mathbb{P}^1) + H.$$

Indeed  $\omega_X = -H - h + \chi^* p^* \omega_{\mathbb{P}^1}$  and the tensorization with  $\chi^* p^* \omega_{\mathbb{P}^1}$  gives an autoequivalence of  $D^b(\mathbb{P}^1)$ . We then have the decomposition

$$D^b(X) = \langle \Phi' D^b(S, \mathcal{B}_0), D^b(\mathbb{P}^1), D^b(\mathbb{P}^1) + H \rangle.$$

Comparing this last decomposition with (6.1) we get

$$\Phi' D^b(S, \mathcal{B}_0) = \langle \Psi(\Gamma), D^b(\mathbb{P}^1) - H \rangle,$$

and the proof now follows recalling that  $D^b(\mathbb{P}^1)$  has a two-objects full exceptional sequence.  $\square$

**6.2. Hyperelliptic degeneration.** Also in the case where  $C$  is a hyperelliptic curve on  $S$ , we can give an explicit description of the conic bundle  $\pi : X \rightarrow S$  degenerating along  $C$ . The key remark here is that  $X$  can be obtained via a birational transformation starting from a degenerate case of the Casnati-Recillas construction.

Let us consider the disconnected trigonal curve  $C' = C \amalg L \rightarrow \mathbb{P}^1$ , where  $L$  is isomorphic to  $\mathbb{P}^1$  and the degree 3 cover is the obvious one. Donagi pointed out [18, Ex. 2.10] that the Casnati-Recillas construction gives in this case a tetragonal curve  $\Gamma = \Gamma_0 \amalg \Gamma_1$  that splits into the disjoint union of two hyperelliptic curves. Let  $R_i$  be the ramification locus of  $\Gamma_i$ , and  $R$  the one of  $C$ .

As a tetragonal curve,  $\Gamma$  is naturally embedded in a  $\mathbb{P}^2$ -bundle  $Z := \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1$ . On the other hand the double cover  $\tilde{C}'$  of  $C'$  splits as  $\tilde{C} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1$ , where  $\tilde{C}$  is a double cover of  $C$ . The  $\mathbb{P}^1 \amalg \mathbb{P}^1$  part is of course the trivial disconnected double cover of  $L$ . This implies that  $P(\tilde{C}'/C') \cong P(\tilde{C}/C)$  and that the conic bundle  $Y \rightarrow S$  obtained as the blow up of  $Z$  along  $\Gamma$  is not standard. Indeed, being the double cover of  $L$  trivial implies that the preimage  $G$  of  $L$  is a reducible rank 2 quadric surface.

In order to fix this, we perform some birational transformation to find the standard conic bundle  $\pi : X \rightarrow S$  degenerating along the hyperelliptic curve  $C$ . Such birational transformation is a slight generalization of the elementary transformation described in [42, Sect. 2.1] and, roughly, it consists in contracting one of the two rational components of  $G$ . After that,  $L$  is no longer contained in the discriminant locus, hence the discriminant locus is the hyperelliptic curve. This transformation corresponds to a birational transformation of the projective bundle  $Z$ .

Consider the curve  $\Gamma_0$  in  $Z$  and blow it up. Let  $\tilde{Z} \rightarrow Z$  be the blow-up and  $\tilde{D}$  the exceptional divisor. Let  $T \subset \tilde{Z}$  be the strict transform of the ruled surface obtained by taking the closure of

the locus of lines spanned by each couple of points of  $\Gamma_0$  associated by the hyperelliptic involution. Let us denote  $Q$  the 3-fold obtained from  $\tilde{Z}$  by blowing down  $T$  to a line along the ruling. Remark that, since  $\Gamma_1$  is disjoint from  $\Gamma_0$ , then  $\Gamma_1$  is embedded in  $Q$ .

**Lemma 6.2.** (i) *There is a quadric bundle structure  $\tau : Q \rightarrow \mathbb{P}^1$  of relative dimension 2, with simple degeneration along the ramification set  $R_0$  of  $\Gamma_0$ .*

(ii) *There exists a full and faithful functor  $\bar{\Psi}_0 : D^b(\Gamma_0) \rightarrow D^b(Q)$  and a semiorthogonal decomposition*

$$D^b(Q) = \langle \bar{\Psi}_0 D^b(\Gamma_0), \tau^* D^b(\mathbb{P}^1), \tau^* D^b(\mathbb{P}^1) \otimes \mathcal{O}_{Q/\mathbb{P}^1}(1) \rangle .$$

*Proof.* (i) Consider a point  $x$  in  $\mathbb{P}^1$  and the fiber  $Z_x$ , which is a projective plane. Let  $a_i, b_i$  be the points where  $\Gamma_i$  intersects  $Z_x$ . Then if we blow-up  $a_0$  and  $b_0$  and we contract the line through them, we get a birational map  $Z_x \dashrightarrow Q_x$ , where  $Q_x$  is a quadric surface, which is smooth if and only if  $a_0 \neq b_0$  [20, pag. 85] and has simple degeneration otherwise, in fact  $\mathbb{F}_1$  is isomorphic to the blow up of a quadric cone in its node.

(ii) By [29], if we denote by  $\mathcal{C}_0$  the sheaf of even parts of the Clifford algebra associated to  $\tau$ , there is a fully faithful functor  $\bar{\Psi}_0 : D^b(\mathbb{P}^1, \mathcal{C}_0) \rightarrow D^b(Q)$  and a semiorthogonal decomposition

$$D^b(Q) = \langle \bar{\Psi}_0 D^b(\mathbb{P}^1, \mathcal{C}_0), \tau^* D^b(\mathbb{P}^1), \tau^* D^b(\mathbb{P}^1) \otimes \mathcal{O}_{Q/\mathbb{P}^1}(1) \rangle .$$

Now apply [29, Cor. 3.14] to get the equivalence  $D^b(\mathbb{P}^1, \mathcal{C}_0) \cong D^b(\Gamma_0)$ .  $\square$

Now we complete the frame by describing the conic bundle structure  $\pi : X \rightarrow S$  degenerating along  $C$ , where  $X$  is the blow-up of  $Q$  along  $\Gamma_1$ . Through the birational transformation just described, the  $\mathbb{P}^2$ -bundle  $Z$  has been transformed into the quadric bundle  $Q$  and the pencil of conics in  $Z_x$  passing through  $a_0, b_0, a_1, b_1$  has been transformed into a pencil of hyperplane sections of  $Q_x$  passing through  $a_1$  and  $b_1$ . Hence each line of the ruling of  $S$  corresponds to a pencil of quadratic hyperplane sections. Moreover the conics over the rational curve  $L \subset S$  had simple degeneration in  $Y$  and are smooth in  $X$ . On the rest of the ruled surface  $S$  the degeneration type of the conics is preserved.

This implies that  $\pi : X \rightarrow S$  is a standard conic bundle degenerating along the hyperelliptic  $C$ . It is given by resolving the relative linear system  $|\mathcal{O}_{Q/\mathbb{P}^1}(1) - \Gamma_1|$ .

Summarizing, we end up with the following diagram:

$$\begin{array}{ccccc}
 & & X & \xleftarrow{\quad} & D \\
 & \swarrow \pi & & \searrow \chi & \searrow \chi \\
 & & S & & Q \\
 & \searrow q & & \swarrow \tau & \xleftarrow{\quad} \Gamma_1 \\
 & & \mathbb{P}^1 & & 
 \end{array}$$

Where  $\tau : Q \rightarrow \mathbb{P}^1$  is a quadric bundle degenerating exactly in the ramification locus of  $\Gamma_0 \rightarrow \mathbb{P}^1$  and contains the hyperelliptic curve  $\Gamma_1$ . The map  $\chi$  is the blow-up of  $Q$  along  $\Gamma_1$  with exceptional divisor  $D$ . The surface  $q : S \rightarrow \mathbb{P}^1$  is ruled and  $\pi : X \rightarrow S$  is the conic bundle structure degenerating along the hyperelliptic curve  $C$ . Remark that  $J(X)$  is isomorphic to  $J(\Gamma_0) \oplus J(\Gamma_1)$  as principally polarized abelian variety. Since  $J(X) \cong P(\tilde{C}/C)$ , if  $C$  is smooth it can be shown that  $R_1 \cup R_0 = R$  and the configuration of Pryms and Jacobians is the one described by Mumford in [34].

We denote by  $H := \mathcal{O}_{Q/\mathbb{P}^1}(1)$  the relative ample line bundle on  $Q$ . We have  $\omega_{Q/\mathbb{P}^1} = -2H$ . Denote by  $h := \mathcal{O}_{S/\mathbb{P}^1}(1)$  the relative ample line bundle on  $S$ . By abuse of notation, we still denote by  $H$  and  $h$  the pull-backs of  $H$  and  $h$  via  $\chi$  and  $\pi$  respectively. The construction of the map  $\pi$  gives  $h = H - D$ , from which we deduce that  $D = H - h$ .

The canonical bundle  $\omega_X$  is given by  $\omega_X = \chi^*\omega_Q + D$ . Since we have  $\omega_Q = \omega_{Q/\mathbb{P}^1} + \tau^*\omega_{\mathbb{P}^1}$ , we finally get  $\omega_X = -H - h + \chi^*\tau^*\omega_{\mathbb{P}^1}$ .

**Proposition 6.3.** *Let  $\pi : X \rightarrow S$  be a conic bundle whose discriminant locus  $C$  is a hyperelliptic curve whose hyperelliptic structure is given by the intersection of  $C$  with the ruling  $S \rightarrow \mathbb{P}^1$ . Then (up to equivalences):*

$$D^b(S, \mathcal{B}_0) = \langle D^b(\Gamma_1), D^b(\Gamma_0) \rangle,$$

where  $\Gamma_0$  and  $\Gamma_1$  are smooth projective curves such that  $J(X) \simeq J(\Gamma_0) \oplus J(\Gamma_1)$  as a principally polarized abelian variety.

*Proof.* Consider the blow-up  $\chi : X \rightarrow Q$ . Orlov formula (see Prop. 2.4) provides a fully faithful functor  $\Psi_1 : D^b(\Gamma_1) \rightarrow D^b(X)$  and a semiorthogonal decomposition:

$$D^b(X) = \langle \Psi_1 D^b(\Gamma_1), \chi^* D^b(Q) \rangle.$$

Lemma 6.2 gives us

$$(6.3) \quad D^b(X) = \langle \Psi_1 D^b(\Gamma_1), \Psi_0 D^b(\Gamma_0), \chi^* \tau^* D^b(\mathbb{P}^1), \chi^* \tau^* D^b(\mathbb{P}^1) + H \rangle,$$

where  $\Psi_0 = \bar{\Psi}_0 \circ \chi^*$  is fully faithful.

Kuznetsov formula (see Prop. 3.1) provides the decomposition:

$$D^b(X) = \langle \Phi D^b(S, \mathcal{B}_0), \pi^* D^b(S) \rangle.$$

By Prop. 2.3 we can choose the semiorthogonal decomposition  $\langle q^* D^b(\mathbb{P}^1), q^* D^b(\mathbb{P}^1) - h \rangle$  of  $D^b(S)$ . We then get

$$(6.4) \quad D^b(X) = \langle \Phi D^b(S, \mathcal{B}_0), \pi^* q^* D^b(\mathbb{P}^1) - h, \pi^* q^* D^b(\mathbb{P}^1) \rangle.$$

We perform now some mutation to compare the decompositions (6.3) and (6.4). First of all, since  $\pi^* q^* = \chi^* \tau^*$ , we have  $\pi^* q^* D^b(\mathbb{P}^1) = \chi^* \tau^* D^b(\mathbb{P}^1)$  and we will denote this category simply by  $D^b(\mathbb{P}^1)$ .

Consider the decomposition (6.4) and mutate  $\Phi D^b(S, \mathcal{B}_0)$  to the right through  $D^b(\mathbb{P}^1) - h$ . The functor  $\Phi' := \Phi \circ R_{D^b(\mathbb{P}^1) - h}$  is full and faithful and we have the semiorthogonal decomposition

$$D^b(X) = \langle D^b(\mathbb{P}^1) - h, \Phi' D^b(S, \mathcal{B}_0), D^b(\mathbb{P}^1) \rangle.$$

Perform the right mutation of  $D^b(\mathbb{P}^1) - h$  through its right orthogonal. We have

$$R_{\langle D^b(\mathbb{P}^1) - h \rangle^\perp} (D^b(\mathbb{P}^1) - h) = D^b(\mathbb{P}^1) - h - \omega_X \cong D^b(\mathbb{P}^1) + H.$$

Indeed  $\omega_X = -H - h + \chi^* \tau^* \omega_{\mathbb{P}^1}$  and the tensorization with  $\chi^* \tau^* \omega_{\mathbb{P}^1}$  gives an autoequivalence of  $D^b(\mathbb{P}^1)$ . We then have the decomposition

$$D^b(X) = \langle \Phi' D^b(S, \mathcal{B}_0), D^b(\mathbb{P}^1), D^b(\mathbb{P}^1) + H \rangle.$$

Comparing this last decomposition with (6.4) we get

$$\Phi' D^b(S, \mathcal{B}_0) = \langle \Psi_1 D^b(\Gamma_1), \Psi_0 D^b(\Gamma_0) \rangle.$$

□

**Remark 6.4.** Remark that the choice of blowing up first  $\Gamma_0$  and then  $\Gamma_1$  has no influence (up to equivalence) on the statement of Proposition 6.3.

**6.3. A non-rational example.** Theorem 1.1 states that if  $\pi : X \rightarrow S$  is standard and  $S$  rational, then a semiorthogonal decomposition of  $D^b(S, \mathcal{B}_0)$  (and then of  $D^b(X)$ ) via derived categories of curves and exceptional objects allows to reconstruct the intermediate Jacobian  $J(X)$  as the direct sum of the Jacobians of the curves. It is clear by the technique used, that  $S$  being rational is crucial. Using the construction by Casnati [16], we provide here examples of standard conic bundles  $\pi : X \rightarrow S$  over a non-rational surface such that both  $D^b(X)$  and  $D^b(S, \mathcal{B}_0)$  admit a decomposition via derived categories of smooth projective curves. In these cases,  $X$  is clearly non-rational, and  $J(X)$  is only isogenous to  $P(\tilde{C}/C) \oplus A^2(S) \oplus A^1(S)$  [8].

Let  $G$  be a smooth projective curve of positive genus. Remark that  $D^b(G)$  contains no exceptional object, because of Serre duality. Consider a smooth degree four cover  $\Gamma \rightarrow G$ , and its embedding in a  $\mathbb{P}^2$ -bundle  $Z \rightarrow G$ . By [16], there is a unique degree 3 cover  $C \rightarrow G$  embedded in a ruled surface  $S \rightarrow G$ , and we suppose that  $C$  has at most double points. As in 6.1, we end up with a commutative diagram:

$$\begin{array}{ccccc}
 & & X & \xleftarrow{D} & \\
 & \swarrow \pi & & \searrow \chi & \\
 S & & & & Z \\
 & \searrow q & & \swarrow p & \\
 & & G & & \\
 & & & & \Gamma \\
 & & & & \xleftarrow{\quad}
 \end{array}$$

where  $X$  is the blow-up of  $Z$  along  $\Gamma$ ,  $D$  the exceptional divisor and  $\pi : X \rightarrow S$  a standard conic bundle degenerating along  $C$ , induced by the relative linear system  $|\mathcal{O}_{Z/G}(2) - \Gamma|$ . Orlov formula for blow-ups (see Prop. 2.4) and Prop. 2.3 give a semiorthogonal decomposition

$$D^b(X) = \langle \Psi D^b(\Gamma), D^b(G) - H, D^b(G), D^b(G) + H \rangle,$$

where we keep the notation of 6.1 and we write  $D^b(G) := \chi^* p^* D^b(G) = \pi^* q^* D^b(G)$ . Then  $D^b(X)$  is decomposed by derived categories of smooth projective curves. Going through the proof of Proposition 6.1, it is clear that replacing  $\mathbb{P}^1$  with  $G$  does not affect any calculation, except the fact that  $D^b(G)$  contains no exceptional object. Keeping the same notation, we end up with the semiorthogonal decomposition

$$\Phi' D^b(S, \mathcal{B}_0) = \langle \Psi D^b(\Gamma), D^b(G) - H \rangle.$$

#### REFERENCES

- [1] J. Almeida, L. Gruson, and N. Perrin, *Courbes de genre 5 munies d'une involution sans point fixe*, J. London Math. Soc. **72** (2005), no. 3, 545–570.
- [2] P. L. del Angel, and S. Müller-Stach, *Motives of uniruled 3-folds*, Comp. Math. **112** (1998), 1–16.
- [3] M. Artin, and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London math. soc. (3) **25** (1972), 75–95.
- [4] A. Beauville, *Prym varieties and Schottky problem*, Inventiones Math. **41** (1977), 149–196.
- [5] A. Beauville, *Variétés de Prym et jacobiniennes intermédiaires*, Ann. scient. ENS **10** (1977), 309–391.
- [6] A. A. Beilinson, *The derived category of coherent sheaves on  $\mathbb{P}^n$* , Sel. Math. Sov. **34** (1984), no. 3, 233–237.
- [7] M. Beltrametti, and P. Francia, *Conic bundles on non-rational surfaces*, in *Algebraic Geometry - Open problems (proceedings, Ravello 1982)*, Lect. Notes in Math. **997**, Springer Verlag, 34–89.
- [8] M. Beltrametti, *On the Chow group and the intermediate Jacobian of a conic bundle*, Annali di Mat. pura e applicata **41** (1985), no. 4, 331–351.
- [9] M. Bernardara, *Fourier–Mukai transforms of curves and principal polarizations*, C. R. Acad. Sci. Paris, Ser. I, **345** (2007), 203–208.
- [10] M. Bernardara, E. Macrì, S. Mehrotra, and P. Stellari *A categorical invariant for cubic threefolds*, preprint arXiv:0903.4414.
- [11] M. Bolognesi, *A conic bundle degenerating on the Kummer surface*, Math. Z. **261** (2009), no. 1, 149–168.

- [12] A. Bondal, *Representations of associative algebras and coherent sheaves*, Math. USSR-Izv. **34** (1990), no. 1, 23–42.
- [13] A. Bondal, and M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Math. USSR-Izv. **35** (1990), no. 3, 519–541.
- [14] A. Bondal, and D. Orlov, *Semiorthogonal decomposition for algebraic varieties*, preprint math.AG/9506012.
- [15] G. Casnati, and T. Ekedahl, *Covers of algebraic varieties. I. A general structure theorem, covers of degree 3, 4 and Enriques surfaces*, J. Algebraic Geom. **5** (1996), no. 3, 439–460.
- [16] G. Casnati, *Covers of algebraic varieties III: The discriminant of a cover of degree 4 and the trigonal construction*, Trans. Am. Math. Soc. **350** (1998), no. 4, 1359–1378.
- [17] C.H. Clemens, and P.A. Griffiths, *The intermedite Jacobian of the cubic threefold* Ann. Math. (2) **95**, 281–356 (1972).
- [18] R. Donagi, *The fibers of the Prym map*, in *Curves, Jacobians, and Abelian varieties*, Proc. AMS-IMS-SIAM Jt. Summer Res. Conf. Schottky Probl., Contemp. Math. **136** (1992), 55–125.
- [19] B. van Geemen *Some remarks on Brauer groups of K3 surfaces.*, Adv. in Math. **197** (2005), 222–247.
- [20] J. Harris, *Algebraic geometry. A first course*, Graduate Texts in Math. **133**, Springer-Verlag (1992).
- [21] D. Huybrechts, *Fourier-Mukai transforms in Algebraic Geometry*, Oxford Math. Monographs (2006).
- [22] A. Iliev, and D. Markushevich, *The Abel-Jacobi map for cubic threefold and periods of Fano threefolds of degree 14*, Doc. Math., J. DMV **5**, 23–47 (2000).
- [23] V. A. Iskovskikh, *Congruences of conics in  $P^3$*  (Russian. English summary), Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1982, no. 6, 57–62, 121; English translation: Moscow Univ. Math. Bull. **37** (1982), no. 6, 67–73.
- [24] V.A. Iskovskikh, *On the rationality problem for conic bundles*, Duke Math. J. **54** (1987), 271–294
- [25] M.M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), 479–508.
- [26] L. Katzarkov, *Generalized homological mirror symmetry, superschemes and nonrationality*, in *Special metrics and supersymmetry. Lectures given in the workshop on geometry and physics: special metrics and supersymmetry, Bilbao, Spain, 29–31 May 2008*, AIP Conference Proceedings **1093**, 92–131 (2009).
- [27] L. Katzarkov, *Generalized homological mirror symmetry and rationality questions*, in *Cohomological and geometric approaches to rationality problems*, Progr. Math. **282**, Birkhäuser Boston, 163–208 (2010).
- [28] A. Kuznetsov, *Derived categories of cubic and  $V_{14}$  threefolds* Proc. Steklov Inst. Math. **246**, 171–194 (2004); translation from Tr. Mat. Inst. Steklova **246**, 183–207 (2004).
- [29] A. Kuznetsov, *Derived categories of quadric fibrations and intersections of quadrics*, Adv. Math. **218** (2008), no. 5, 1340–1369.
- [30] A. Kuznetsov, *Derived categories of cubic fourfolds*, preprint arXiv:0808.3351.
- [31] A. Kuznetsov, *Derived categories of Fano threefolds*, preprint arXiv:0809.0225.
- [32] Y.I. Manin, *Correspondences, motives and monoidal transformations*, Math USSR Sb. **6** (1968), 439–470.
- [33] A. S. Merkur’ev, *On the norm residue symbol of degree 2*, Dokl. Akad. Nauk SSSR **261**, No. 3 (1981), 542–547, english translation in Soviet Math. Doklady **24** (1981), 546–551.
- [34] D. Mumford, *Prym Varieties I*, in *Contribution to Analysis*, Academic Press, 1974.
- [35] J.P. Murre, *On the motive of an algebraic surface*, J. reine angew. Math. **409** (1990), 190–204.
- [36] J. Nagel, and M. Saito, *Relative Chow-Künneth decomposition for conic bundles and Prym varieties*, Int. Math. Res. Not. **2009** (2009), 2978–3001.
- [37] D.O. Orlov, *Projective bundles, monoidal transformations and derived categories of coherent sheaves*, Russian Math. Izv. **41** (1993), 133–141.
- [38] D.O. Orlov, *Derived categories of coherent sheaves and equivalences between them*, Russian Math. Surveys **58** (2003), 511–591.
- [39] D.O. Orlov, *Derived categories of coherent sheaves and motives*, Russian Math. Surveys **60** (2005), 1242–1244.
- [40] I.A. Panin, *Rationality of bundles of conics with degenerate curve of degree five and even theta-characteristic*, J. Sov. Math. **24** (1984), 449–452; Russian original in Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova **103** (1980), 100–106.
- [41] S. Recillas, *Jacobians of curves with  $g_4^1$ ’s are the Pryms of trigonal curves*, Bol. Soc. Mat. Mex., II. Ser. **19** (1974), 9–13.
- [42] V.G. Sarkisov, *Birational automorphisms of conic bundles*, Math. USSR, Izv. **17** (1981), no. 1, 177–202.
- [43] V.G. Sarkisov, *On conic bundle structures*, Math. USSR Izv. **20**, No. 2 (1982), 355–390.
- [44] A.J. Scholl, *Classical motives*, in *Motives. Proceedings of the summer research conference on motives, held at the University of Washington, Seattle, WA, USA, July 20-August 2, 1991*, AMS Proc. Symp. Pure Math. **55**, (1994) Pt. 1, 163–187 (1994).
- [45] V.V. Shokurov, *Prym varieties: theory and applications*, Math. USSR-Izv. **23** (1984), 83–147.
- [46] A. Verra, *The fibre of the Prym map in genus three*, Math. Ann. **276** (1987), 433–448.

M.BE.: UNIVERSITÄT DUISBURG–ESSEN, FAKULTÄT FÜR MATHEMATIK. UNIVERSITÄTSTR. 2, 45117 ESSEN  
(GERMANY)

*E-mail address:* `marcello.bernardara@uni-due.de`

M.BO.: IRMAR, UNIVERSITÉ DE RENNES 1. 263 Av. GÉNÉRAL LECLERC, 35042 RENNES CEDEX (FRANCE)

*E-mail address:* `michele.bolognesi@univ-rennes1.fr`