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Large solutions of elliptic systems of second order and applications to the biharmonic equation*

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Abstract

In this work we study the nonnegative solutions of the elliptic system

$$\Delta u = |x|^a v^\delta, \qquad \Delta v = |x|^b u^\mu$$

in the superlinear case $\mu\delta > 1$, which blow up near the boundary of a domain of $\mathbb{R}^N$, or at one isolated point. In the radial case we give the precise behavior of the large solutions near the boundary in any dimension N. We also show the existence of infinitely many solutions blowing up at 0. Furthermore, we show that there exists a global positive solution in $\mathbb{R}^N \setminus \{0\}$, large at 0, and we describe its behavior. We apply the results to the sign changing solutions of the biharmonic equation

$$\Delta^2 u = |x|^b |u|^\mu .$$

Our results are based on a new dynamical approach of the radial system by means of a quadratic system of order 4, introduced in [4], combined with the nonradial upper estimates of [5].

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1 Introduction

This article is concerned with the nonnegative large solutions of the elliptic system

$$\begin{cases} \Delta u = |x|^a v^\delta \\ \Delta v = |x|^b u^\mu, \end{cases} \tag{1.1}$$

in two cases: solutions in a bounded domain Ω in $\mathbb{R}^N$, which blow up at the boundary, that is

$$\lim_{d(x,\partial\Omega)\to 0} u(x) = \lim_{d(x,\partial\Omega)\to 0} v(x) = \infty, \tag{1.2}$$

where $d(x,\partial\Omega)$ is the distance from x to $\partial\Omega$; or solutions in $\Omega\setminus\{0\}$ which blow up at 0 :

$$\lim_{x\to 0} u(x) = \infty \quad \text{or} \quad \lim_{x\to 0} v(x) = \infty. \tag{1.3}$$

We study the superlinear case, where $\mu, \delta > 0$, and

$$D = \mu\delta - 1 > 0, \tag{1.4}$$

and a, b are real numbers such that

$$a, b > \max\{-2, -N\}. \tag{1.5}$$

First we recall some well-known results in the scalar case of the Emden-Fowler equation

$$\Delta U = U^Q \tag{1.6}$$

with $Q > 1$. Concerning the boundary blow-up problem, there exists a unique solution U in Ω such that $\lim_{d(x,\partial\Omega)\to 0} U(x) = \infty$, and near $\partial\Omega$

$$U(x) = Cd(x,\partial\Omega)^{-2/(Q-1)}(1 + o(1)),$$

where $C = C(Q)$. Several researchs on the more general equation

$$\Delta U = p(x)f(U)$$

have been done with different assumptions on f and on the weight p, with asymptotic expansions near $\partial\Omega$, see for instance [2], [3], [7], [9], [16], [17], [19], [20], [22]; see also [1], [10] for quasilinear equations. These results rely essentially on the *comparison principle* valid for this equation, and the construction of supersolutions and subsolutions.

The existence and the behavior of solutions of (1.6) in $\Omega\setminus\{0\}$ which blow up at 0:

$$\lim_{x\to 0} U(x) = \infty,$$

called large (or singular) at 0, have also been widely investigated during the last decades, see for example [23], and the references therein. There exists a particular solution in $\mathbb{R}^N\setminus\{0\}$ whenever $Q < N/(N-2)$ or $N = 1, 2$, given by $U^*(x) = C^* |x|^{-2/(Q-1)}$, with $C^* = C^*(Q, N)$.

If $Q \geq N/(N-2)$, there is no large solution at 0, and the singularity is removable. If $Q < N/(N-2)$ or $N = 2$, any large solution satisfies $\lim_{|x|\to 0} |x|^{2/(Q-1)} U = C^*$, or

$$\lim_{|x|\to 0} |x|^{N-2} U = \alpha > 0 \quad \text{if } N > 2, \qquad \lim_{|x|\to 0} |\ln |x|| \, U = \alpha > 0, \quad \text{if } N = 2. \tag{1.7}$$

There exist solutions of each type, distinct from U^*. Moreover, up to a scaling, there exists a unique positive radial solution in $\mathbb{R}^N \setminus \{0\}$, such that (1.7) holds and $\lim_{|x|\to\infty} |x|^{2/(Q-1)} U = C^*$, see [23] and also [4].

In **Section 2** we consider the blow up problem of system (1.1) at the boundary.

Up to our knowledge all the known results for systems are related with systems for which some comparison properties hold, for example

$$\begin{cases} \Delta u = u^s v^\delta, \\ \Delta v = u^\mu v^m, \end{cases}$$

where $s, m > 1$, $\delta, \mu > 0$, and $\delta\mu \leq (s-1)(m-1)$, of competitive type, see [13], or $\delta, \mu < 0$, of cooperative type, see [8]; see also some extensions to problems with weights in [21], or with quasilinear operators in [14], [24], [25], and cooperative systems of Lotka-Volterra in [12].

On the contrary the problem (1.1)-(1.2) has been the object of very few works, because it brings many difficulties. The main one is the *lack of a comparison principle* for the system. As a consequence all the methods of supersolutions, subsolutions and comparison, valid for the case of a single equation fail.

Until now the existence of large solutions is an open question in the nonradial case. In the radial case the problem was studied in [15], without weights: $a = b = 0$. It was shown that there are infinitely many nonnegative radial solutions to (1.1) which blow up at the boundary of a ball provided that (1.4) holds, and no blow up occurs otherwise. In particular, there exist solutions even in the case where either u or v vanishes at 0. This shows *the lack of a Harnack inequality,* even in the radial case. The precise behavior of the solutions was obtained in [15] for $N = 1$, $a = b = 0$, where system (1.1) is autonomous, with an elaborate proof wich could not be extended to higher dimension.

Our first main result *solves this question in any dimension*, with possible weights, and moreover we give an expansion of order 1 of the solutions:

Theorem 1.1 *Let (u, v) be any radial nonnegative solution of (1.1) defined for $r \in (r_0, R)$, $r_0 \geq 0$, unbounded at $r = R$. Then $\lim_{r\to R} u(r) = \lim_{r\to R} v(r) = \infty$, and u, v admit the following expansions near R :*

$$u(r) = A_1 d(r)^{-\gamma}(1 + O(d(r))), \qquad v(r) = B_1 d(r)^{-\xi}(1 + O(d(r))), \tag{1.8}$$

where $d(r) = R - r$ is the distance to the boundary, and

$$\gamma = \frac{2(1+\delta)}{D}, \qquad \xi = \frac{2(1+\mu)}{D}, \tag{1.9}$$

$$A_1 = (\gamma(\gamma+1)(\xi(\xi+1))^\delta)^{1/D}, \qquad B_1 = (\xi(\xi+1)(\gamma(\gamma+1))^\mu)^{1/D}. \tag{1.10}$$

Our proof is essentially based on a new dynamical approach of system (1.1), initiated in [4]: we reduce the problem to a quadratic, in general nonautonomous, system of order 4, which, under the assumptions of Theorem 1.1, can be reduced to a nonautonomous perturbation of a quadratic system of order 2. We then show the convergence of the solution of the original system to a suitable fixed point by using the perturbation arguments of [18].

Theorem 1.1 can be applied to *sign changing* solutions of some elliptic systems, in particular to the biharmonic equation, where $\delta = 1$:

Corollary 1.2 *Let $\mu > 1$, $b \in \mathbb{R}$. Then any radial solution u of the problem*

$$\Delta^2 u = |x|^b |u|^\mu \quad in \quad (r_0, R), \qquad u(R) = \infty, \tag{1.11}$$

satisfies

$$u(r) = Ad(r)^{-4/(\mu-1)}(1 + O(d(r))), \tag{1.12}$$

with $A^{\mu-1} = 8(\mu+3)(\mu+1)(3\mu-1)(\mu-1)^{-4}$.

We notice here a case where we find an *explicit* solution: for $N > 4$ and $\mu = \frac{N+4}{N-4}$, equation $\Delta^2 u = u^\mu$ admits the solution in the ball $B(0,1)$,

$$u(r) = C(1-r^2)^{(4-N)/2}, \qquad C^{8/(N-4)} = N(N-4)(N^2-4),$$

and $v = \Delta u = C(N-4)(1-r^2)^{-N/2}(N-2r^2) \geq 0$, and (1.8) and (1.12) hold with $\gamma = \frac{N-4}{2}, \xi = \frac{N}{2}$.

In **Section 3** we consider the problem of large solutions at the origin, that is (1.1)-(1.3).

System (1.1) admits a particular radial positive solution (u^*, v^*), given by

$$u^*(r) = A_N r^{-\gamma_{a,b}}, \qquad v^*(r) = B_N r^{-\xi_{a,b}}, \quad r = |x|, \tag{1.13}$$

where

$$\gamma_{a,b} = \frac{(2+a)+(2+b)\delta}{D} > 0, \qquad \xi_{a,b} = \frac{(2+b)+(2+a)\mu}{D} > 0, \tag{1.14}$$

$$A_N^D = \gamma_{a,b}(\gamma_{a,b} - N + 2)\left(\xi_{a,b}(\xi_{a,b} - N + 2)\right)^\delta, \quad B_N^D = \xi_{a,b}(\xi_{a,b} - N + 2)\left(\gamma_{a,b}(\gamma_{a,b} - N + 2)\right)^\mu,$$

whenever

$$\min\{\gamma_{a,b}, \xi_{a,b}\} > N - 2, \quad \text{or } N = 1, 2. \tag{1.15}$$

Note that in particular $\gamma_{0,0} = \gamma, \xi_{0,0} = \xi$.

The problem has been initiated in [26] and [5], see also [27]. Let us recall an important result of [5] giving upper estimates for system (1.1) in the *nonradial* case, stated for $N \geq 3$, but its proof is valid for any $N \geq 1$. It is not based on supersolutions, but on estimates of the mean value of u, v on spheres:

Keller-Osserman type estimates [5]. *Let Ω be a domain of $\mathbb{R}^N (N \geq 1)$, containing 0, and $u, v \in C^2(\Omega \setminus \{0\})$ be any nonnegative subsolutions of (1.1), that is,*

$$\begin{cases} -\Delta u + |x|^a v^\delta \leq 0, \\ -\Delta v + |x|^b u^\mu \leq 0, \end{cases}$$

with μ, δ satisfying (1.4). Then there exists $C = C(a, b, \delta, \mu, N)$ such that near $x = 0$,

$$u(x) \leq C\,|x|^{-\gamma_{a,b}}\,, \qquad v(x) \leq C\,|x|^{-\xi_{a,b}}\,. \tag{1.16}$$

Moreover, one finds in [5] a quite exhaustive study about all the *possible* behaviors of the solutions (radial or not) in $\Omega\setminus\{0\}$.

Here we complete those results by proving the existence of local radial solutions large at 0 of each of the types described in [5], see Propositions 3.2, 3.4 in Section 3. By using these results, we obtain our second main result in this work, which is the following global existence theorem:

Theorem 1.3 *Assume that $N \geq 2$ and that (1.15) holds. Then there exists a radial positive global solution of system (1.1) in $\mathbb{R}^N\setminus\{0\}$, large near 0, unique up to a scaling, such that*

$$\lim_{r\to\infty} r^{\gamma_{a,b}} u = A_N, \qquad \lim_{r\to\infty} r^{\xi_{a,b}} v = B_N; \tag{1.17}$$

and, for $N > 2$, and up to a change of u, μ, a, into v, δ, b, when $\delta < \frac{N+a}{N-2}$, it satisfies

$$\lim_{r\to 0} r^{N-2} u = \alpha > 0, \qquad \begin{cases} \lim\limits_{r\to 0} r^{N-2} v &= \beta > 0, \qquad \text{if } \mu < \frac{N+b}{N-2}, \\ \lim\limits_{r\to 0} r^{(N-2)\mu-(2+b)} v &= \beta > 0, \qquad \text{if } \mu > \frac{N+b}{N-2}, \\ \lim\limits_{r\to 0} r^{N-2}\,|\ln r|^{-1}\, v &= \beta > 0, \qquad \text{if } \mu = \frac{N+b}{N-2}, \end{cases}$$

and for $N = 2$,

$$\lim_{r\to 0} |\ln r|^{-1}\, u = \alpha > 0, \qquad \lim_{r\to 0} |\ln r|^{-1}\, v = \beta > 0.$$

Our proof also relies on the dynamical approach of system (1.1) in dimension N by a quadratic autonomous system of order 4, given in [4]. Finally we give an application to the biharmonic equation:

Corollary 1.4 *Let $N > 2$. Assume that $1 < \mu < \frac{N+2+b}{N-2}$. There exists a positive global solution, unique up to a scaling, of equation*

$$\Delta^2 u = |x|^b u^\mu$$

in $\mathbb{R}^N\setminus\{0\}$, such that

$$\lim_{r\to 0} r^{N-2} u = \alpha > 0, \qquad \lim_{r\to\infty} r^{(4+b)/(\mu-1)} u = C,$$

where $C^{\mu-1} = (4+b)(N+2+b-(N-2)\mu)\,(2\mu+2+b)(N+b-(N-4)\mu)\,(\mu-1)^{-4}$.

2 Large solutions at the boundary

This section is devoted to the study of the boundary blow up problem for nonnegative radial solutions of (1.1). We begin by observing that system (1.1) admits a scaling invariance: if (u, v) is a solution, then for any $\theta > 0$,

$$r \mapsto (\theta^{\gamma_{a,b}} u(\theta r), \theta^{\xi_{a,b}} v(\theta r)), \tag{2.1}$$

where $\gamma_{a,b}, \xi_{a,b}$ are defined in (1.14), is also a solution.

2.1 Existence and estimates of large solutions

We say that a nonnegative solution (u,v) of (1.1) defined in $(0,R)$ is *regular* at 0 if $u,v \in C^2(0,R) \cap C([0,R))$. Then $u,v \in C^1([0,R))$ when $a,b \geq -1$, and moreover $u'(0) = v'(0) = 0$ when $a,b > -1$, and $u,v \in C^2([0,R))$ when $a,b \geq 0$.

We first give an existence and uniqueness result for regular solutions:

Proposition 2.1 *Assume (1.5) and only that $D = \delta\mu - 1 \neq 0$. Then for any $u_0,\ v_0 \geq 0$, there exists a unique local regular solution (u,v) with initial data (u_0,v_0).*

The result follows from classical fixed point theorem when $u_0, v_0 > 0$, by writing the problem in an integral form:

$$u(r) = u_0 + \int_0^r \tau^{1-N} \int_0^\tau \theta^{N-1+a} v^\delta(\theta) d\theta, \qquad v(r) = v_0 + \int_0^r \tau^{1-N} \int_0^\tau \theta^{N-1+b} u^\mu(\theta) d\theta.$$

In the case $u_0 > 0 = v_0$, the existence can be obtained from the Schauder fixed point theorem, and the uniqueness by using monotonicity arguments as in [15]. We give an alternative proof in Section 3, using the dynamical system approach introduced in [4], which can be extended to more general operators.

Next we show that all the nontrivial regular solutions blow up at some finite $R > 0$, and give the first upper estimates for any large solution. Our proofs are a direct consequence of estimates (1.16).

Proposition 2.2 *(i) Assume (1.4) and (1.5). For any regular nonnegative solution $(u,v) \not\equiv (0,0)$, there exists R such that u and v are unbounded near R.*

(ii) Any solution (u,v) which is nonnegative in an interval (r_0,R) and unbounded at R, satisfies

$$\lim_{r\to R} u = \lim_{r\to R} v = \lim_{r\to R} u' = \lim_{r\to R} v' = \infty. \tag{2.2}$$

and there exists $C = C(N,\delta,\mu) > 0$ such that near $r = R$,

$$u(r) \leq C(R-r)^{-\gamma}, \qquad v(r) \leq C(R-r)^{-\xi}. \tag{2.3}$$

Proof. (i) Let (u,v) be any nontrivial regular solution. Suppose first that $v_0 > 0$. Then from (1.1), $r^{N-1}u'$ is positive for small r, and nondecreasing, hence u is increasing. If the solution is entire, then it satisfies (1.16) near ∞: indeed by the Kelvin transform, the functions

$$\overline{u}(x) = |x|^{2-N}\, u(x/|x|^2), \qquad \overline{v}(x) = |x|^{2-N}\, v(x/|x|^2),$$

satisfy in $B(0,1)\setminus\{0\}$ the system

$$\begin{cases} -\Delta\overline{u} + |x|^{\overline{a}}\,\overline{v}^\delta = 0, \\ -\Delta\overline{v} + |x|^{\overline{b}}\,\overline{u}^\mu = 0, \end{cases}$$

where $\overline{a} = (N-2)\delta - (N+2+a), \overline{b} = (N-2)\mu - (N+2+b)$, and $\gamma_{a,b}, \xi_{a,b}$ are replaced by $N-2-\gamma_{a,b}, N-2-\xi_{a,b}$. Then the estimate (1.16) for $(\overline{u}, \overline{v})$ implies the one for (u, v) and thus u tends to 0 at ∞, which is contradictory. Furthermore, from

$$u \leq u_0 + \frac{r^{2+a}}{(2+a)(N+a)} v^{\delta}, \quad v \leq v_0 + \frac{r^{2+b}}{(2+b)(N+b)} u^{\mu},$$

u and v blow up at the same point $R > 0$.

(ii) Since $r^{N-1}u'$ is increasing, it has a limit as $r \to R$. If this limit is finite, then u' is bounded, implying that u has a finite limit; this contradicts our assumption. Thus (2.2) holds. By (2.1) we can assume $R = 1$ and make the transformation

$$r = \Psi(s) = \begin{cases} (1+(N-2)s)^{-1/(N-2)}, & \text{if } N \neq 2, \\ e^{-s}, & \text{if } N = 2, \end{cases} \tag{2.4}$$

(in particular $r = 1 - s$ if $N = 1$), so that s describes an interval $(0, s_0]$, $s_0 > 0$, and we get the system

$$\begin{cases} u_{ss} = F(s) v^{\delta} \\ v_{ss} = G(s) u^{\mu} \end{cases} \tag{2.5}$$

with

$$F(s) = r^{2N-2+a}, \qquad G(s) = r^{2N-2+b}; \tag{2.6}$$

hence $\lim_{s\to 0} F = \lim_{s \to 0} G = 1$. Then

$$\begin{cases} -u_{ss} + \frac{1}{2} v^{\delta} \leq 0 \\ -v_{ss} + \frac{1}{2} u^{\mu} \leq 0 \end{cases}$$

in some interval $(0, s_1]$, thus from the Keller-Osserman estimates (1.16), there exists $C = C(N, \delta, \mu) > 0$ such that $u(s) \leq Cs^{-\gamma}, v(s) \leq Cs^{-\xi}$, near $s = 0$ and (2.3) follows. ■

2.2 The precise behavior near the boundary

In this section we prove Theorem 1.1.

2.2.1 Scheme of the proof

Consider a solution blowing up at $R = 1$. In the case of dimension $N = 1$, and $a = b = 0$, we have that $F \equiv G \equiv 1$ in (2.6), and we are concerned with the system

$$\begin{cases} u_{ss} = v^{\delta} \\ v_{ss} = u^{\mu}. \end{cases} \tag{2.7}$$

Following the ideas of [4], we are led to make the substitution

$$X(t) = -\frac{s u_s}{u}, \qquad Y(t) = -\frac{s v_s}{v}, \qquad Z(t) = \frac{s v^{\delta}}{u_s}, \qquad W(t) = \frac{s u^{\mu}}{v_s},$$

where $t = \ln s$, t describes $(-\infty, t_0]$, and we obtain the autonomous system

$$\left\{ \begin{array}{rcl} X_t & = & X\left[X+1+Z\right], \\ Y_t & = & Y\left[Y+1+W\right], \\ Z_t & = & Z\left[1-\delta Y-Z\right], \\ W_t & = & W\left[1-\mu X-W\right]. \end{array} \right. \tag{2.8}$$

We study the solutions in the region where $X, Y \geq 0$ and $Z, W \leq 0$. In this region system (2.8) admits two fixed points

$$O = (0,0,0,0), \quad M_{0,1} = (\gamma, \xi, -1-\gamma, -1-\xi) \tag{2.9}$$

where γ and ξ are defined in (1.9). We intend to show that trajectories associated to the large solutions converge to $M_{0,1}$. Observe that system (2.7) has a first integral, which is a *crucial point* in what follows:

$$u_s v_s - \frac{u^{\mu+1}}{\mu+1} - \frac{v^{\delta+1}}{\delta+1} = C,$$

equivalently

$$e^{-2t} uv(XY + \frac{XZ}{\delta+1} + \frac{YW}{\mu+1}) = C.$$

Since any large solution at $r = 1$ satisfies $\lim_{r\to 1} u = \lim_{r\to 1} v = \infty$, we obtain

$$XY + \frac{XZ}{\delta+1} + \frac{YW}{\mu+1} = o(e^{2t})$$

as $t \to -\infty$. Thus, eliminating W, we get the nonautonomous system of order 3

$$\left\{ \begin{array}{rcl} X_t & = & X\left[X+1+Z\right], \\ Y_t & = & Y\left[Y+1\right] - (\mu+1)X(Y + \frac{Z}{\delta+1}) + o(e^{2t}), \\ Z_t & = & Z\left[1-\delta Y-Z\right]. \end{array} \right. \tag{2.10}$$

which appears as a perturbation of system

$$\left\{ \begin{array}{rcl} X_t & = & X\left[X+1+Z\right], \\ Y_t & = & Y\left[Y+1\right] - (\mu+1)X(Y + \frac{Z}{\delta+1}), \\ Z_t & = & Z\left[1-\delta Y-Z\right]. \end{array} \right. \tag{2.11}$$

Moreover, by using a suitable change of variables, system (2.10) *reduces to a nonautonomous system of order 2*, and we can show that the last system behaves like an autonomous one. Then we come back to the initial system and deduce the convergence.

In the case $N \geq 1$ or $a,\ b$ not necessarily equal to 0, we first reduce the problem to a system similar to (2.8), but nonautonomous, and we prove that it is a perturbation of (2.8). Moreover we produce an identity that plays the role of a first integral, allowing us to reduce to a double perturbation of (2.11). We manage with the two perturbations in order to conclude.

2.2.2 Steps of the proof

Our proof relies strongly in a result due to Logemann and Ryan, see [18]. We state it below for the convenience of the reader.

Theorem 2.3 *[18, Corollary 4.1] Let $h : \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^M$ be of Carathéodory class. Assume that there exists a locally Lipschitz continuous function $h^* : \mathbb{R}^M \to \mathbb{R}^M$ such that for all compact $C \subset \mathbb{R}^M$ and all $\varepsilon > 0$, there exists $T \geq 0$ such that*

$$\sup_{c\in C} ess \sup_{\tau\geq T} ||h(\tau, x) - h^*(x)|| < \varepsilon$$

Assume that x is a bounded solution of equation $x_\tau = h(\tau, x)$ on $\mathbb{R}_+$ such that $x(0) = x_0$. Then the ω-limit set of x is non empty, compact and connected, and invariant under the flow generated by h^*.

The proof of Theorem 1.1 requires some important lemmas. By scaling we still assume that $R = 1$.

Lemma 2.4 *Let (u, v) be any fixed solution of system (1.1) in $[r_0, 1)$, unbounded at 1. Let us set $t = \log s$, where $s = \Psi^{-1}(r)$ is defined in (2.4). Let F, G be defined by (2.6). Then the functions*

$$X(t) = -\frac{su_s}{u} > 0,\ Y(t) = -\frac{sv_s}{v} > 0,\ Z(t) = \frac{sF(s)v^\delta}{u_s} < 0,\ W(t) = \frac{sG(s)u^\mu}{v_s} < 0, \tag{2.12}$$

satisfy the (in general nonautonomous) system

$$\begin{cases} X_t &= X\left[X + 1 + Z\right], \\ Y_t &= Y\left[Y + 1 + W\right], \\ Z_t &= Z\left[1 - \delta Y - Z - \alpha(t)\right], \\ W_t &= W\left[1 - \mu X - W - \beta(t)\right], \end{cases} \tag{2.13}$$

where

$$\alpha(t) = \frac{2N - 2 + a}{1 + (N - 2)e^t}e^t, \qquad \beta(t) = \frac{2N - 2 + b}{1 + (N - 2)e^t}e^t. \tag{2.14}$$

Moreover we recover u, v by the relations

$$u = s^{-\gamma}F^{-\frac{1}{D}}G^{-\frac{\delta}{D}}(|Z|X)^{\frac{1}{D}}(|W|Y)^{\frac{\delta}{D}}, \qquad v = s^{-\xi}F^{-\frac{\mu}{D}}G^{-\frac{1}{D}}(|W|Y)^{\frac{1}{D}}(|Z|X)^{\frac{\mu}{D}}. \tag{2.15}$$

Proof. Since (u, v) is unbounded, (2.2) holds. We make the substitution (2.4), which leads to system (2.5), with F, G given by (2.6). Clearly we can assume that $u_s < 0$ and $v_s < 0$ on $(0, s_0]$, $\lim_{s\to 0} |u_s| = \lim_{s\to 0} |v_s| = \lim_{s\to 0} u = \lim_{s\to 0} v = \infty$. Then we can define X, Y, Z, W by (2.12) and we obtain system (2.13) with

$$\alpha(t) = -s\frac{F'(s)}{F(s)}, \qquad \beta(t) = -s\frac{G'(s)}{G(s)};$$

then (2.14) follows, and we deduce (2.15) by straight computation. ■

Next we prove that system (2.13) is a perturbation of the corresponding autonomous system (2.8):

Lemma 2.5 *Let $N \geq 1$. Under the assumptions of Theorem 1.1, there exist $k > 0$ and $\bar{t} < t_0$ such that*

$$1/k \leq X, Y, |Z|, |W| \leq k \qquad \text{for} \quad t \leq \bar{t}. \tag{2.16}$$

Moreover, setting

$$XY + \frac{XZ}{\delta+1} + \frac{YW}{\mu+1} = \frac{\varpi(t)}{\mu+1}, \tag{2.17}$$

we have $\varpi(t) = O(e^t)$ as $t \to -\infty$.

Proof. We establish some integral inequalities, playing the role of a first integral, then we use them to prove (2.16), and finally we deduce the behavior of ϖ.

(i) *Integral inequalities.* Let σ, $\theta \in \mathbb{R}$ and set

$$\begin{aligned} H_{\sigma,\theta}(s) &= r^{2-N}\left(u_s v_s - F(s)\frac{v^{\delta+1}}{\delta+1} - G(s)\frac{u^{\mu+1}}{\mu+1} - \frac{\sigma v u_s + \theta u v_s}{1+(N-2)s}\right) \\ &= r^{2-N} u v e^{-2t}\left(XY + \frac{X(Z+\bar{\alpha}(t))}{\delta+1} + \frac{Y(W+\bar{\beta}(t))}{\mu+1}\right), \end{aligned}$$

where

$$\bar{\alpha}(t) = \frac{\sigma(\delta+1)s}{1+(N-2)s} \quad \text{and} \quad \bar{\beta}(t) = \frac{\theta(\mu+1)s}{1+(N-2)s}.$$

It can be easily verified that

$$H'_{\sigma,\theta}(s) = (N-2-\sigma-\theta)u_s v_s + F(s)\frac{v^{\delta+1}}{\delta+1}(N+a-\sigma(\delta+1)) + G(s)\frac{u^{\mu+1}}{\mu+1}(N+b-\theta(\mu+1)). \tag{2.18}$$

By choosing first the constants $\sigma = \sigma_1 > 0$ and $\theta = \theta_1 > 0$ large enough, we obtain that $H'_{\sigma_1,\theta_1}(s) < 0$ and thus $H_{\sigma_1,\theta_1}(s) \geq -C_1$ for some t $C_1 > 0$; next choosing $\sigma = \sigma_2 < 0$ and $\theta = \theta_2 < 0$ and large enough in absolute value, we obtain that $H'_{\sigma_2,\theta_2}(s) > 0$ and thus $H_{\sigma_2,\theta_2}(s) \leq C_2$ for some $C_2 > 0$. Hence, there exists functions $\bar{\alpha}_i(t), \bar{\beta}_i(t), i = 1, 2$, which are $O(e^t)$ as $t \to -\infty$, and such that

$$XY + \frac{X(Z+\bar{\alpha}_1(t))}{\delta+1} + \frac{Y(W+\bar{\beta}_1(t))}{\mu+1} \geq -C_1 r^{N-2}\frac{e^{2t}}{uv} \tag{2.19}$$

$$XY + \frac{X(Z+\bar{\alpha}_2(t))}{\delta+1} + \frac{Y(W+\bar{\beta}_2(t))}{\mu+1} \leq C_2 r^{N-2}\frac{e^{2t}}{uv}. \tag{2.20}$$

(ii) *Estimates from below in (2.16).* Using that $u_{ss} \leq v^\delta$ and multiplying by $2u_s < 0$ we obtain

$$(u_s^2)_s \geq 2v^\delta u_s = (2v^\delta u)_s - 2\delta v^{\delta-1} u v_s > (2v^\delta u)_s$$

since $v_s < 0$ in $(0, s_0]$, hence $u_s^2 - 2v^\delta u \leq C = (u_s^2 - 2v^\delta u)(s_0)$; since $\lim_{s\to 0} v^\delta u = \infty$, it follows that $u_s^2 \leq (5/2)v^\delta u$ on $(0, s_1]$, for sufficiently small s_1. Using the same method for the second equation, we obtain from (2.12) that

$$X(t) \leq 3\,|Z(t)|\,, \qquad Y(t) \leq 3\,|W(t)|\,, \qquad \text{on } (-\infty, t_1]\,. \tag{2.21}$$

Also, from the generalized L'Hôpital's rule,

$$\overline{\lim_{s\to 0}}\frac{|Z|}{F}=\overline{\lim_{s\to 0}}\frac{sv^\delta}{-u_s}\le\overline{\lim_{s\to 0}}\frac{\delta sv^{\delta-1}v_s+v^\delta}{-u_{ss}}=\overline{\lim_{s\to 0}}\left(\frac{\delta Y-1}{F}\right),$$

and by symmetry

$$\delta\overline{\lim_{s\to 0}}Y\ge 1+\overline{\lim_{s\to 0}}|Z|,\qquad \mu\overline{\lim_{s\to 0}}X\ge 1+\overline{\lim_{s\to 0}}|W|. \tag{2.22}$$

Suppose now that $\underline{\lim}_{t\to-\infty}X=0$. From (2.22), $\overline{\lim}_{t\to-\infty}X\ge 1/\mu$, hence there is a sequence $\{t_n\}\to-\infty$ of local minima of X such that $\lim_{n\to\infty}X(t_n)=0$, and from the definition of X in (2.12), $X(t_n)>0$ for all n sufficiently large. At each t_n we have that $X_t(t_n)=0$ and $X_{tt}(t_n)\ge 0$. From (2.13), using that $X(t_n)\ne 0$, we have that $X(t_n)+1=|Z(t_n)|$ and hence $|Z(t_n)|>1$. Since $X_{tt}(t_n)=X(t_n)Z_t(t_n)$, it follows that $Z_t(t_n)\ge 0$, and thus, from the third equation in (2.13), $1-\delta Y(t_n)+|Z(t_n)|\le 0$, implying

$$1\le 1+|Z(t_n)|\le\delta Y(t_n)+\alpha(t_n). \tag{2.23}$$

From (2.19) and (2.21), we deduce

$$Y^2\le 3Y(\bar\beta_1(t)+(\mu+1)X)+3(\mu+1)\frac{X\bar\alpha_1(t)}{\delta+1}+O(e^{2t}),$$

hence $\lim_{n\to\infty}Y(t_n)=0$, which contradicts (2.23). We conclude that $\underline{\lim}_{t\to-\infty}X>0$, and similarly for Y, thus $X,Y,|Z|,|W|$ are bounded from below.

(iii) *Estimates from above.* From (2.3), $s^\gamma u$ and $s^\xi v$ are bounded as $s\to 0$, thus from (2.15) and (2.21), $X^2Y^{2\delta}$ is bounded as $t\to\infty$. Since X,Y are bounded from below, they are bounded from above, and then also $|Z|$ and $|W|$, from (2.22), hence (2.16) holds.

(iv) *Conclusion.* From (2.19), (2.20), since X,Y are bounded and $|\bar\alpha_i(t)|,\ |\bar\beta_i(t)|\le Ce^t$,

$$XY\ge\frac{X|Z|}{\delta+1}+\frac{Y|W|}{\mu+1}-C_3e^t\quad\text{and}\quad XY\le\frac{X|Z|}{\delta+1}+\frac{Y|W|}{\mu+1}+C_4e^t, \tag{2.24}$$

for some $C_3,C_4>0$. Then we deduce (2.17). ■

Next we show that a convenient combination of our solution (X,Y,Z,W) satisfies a system of order 2. We have

Lemma 2.6 *Under the assumptions of Theorem 1.1, and with the above notations, let*

$$x(\tau)=-\frac{X(t)}{Z(t)},\quad y=-\frac{Y(t)}{Z(t)},\quad \tau=-\int_t^{\bar t}Z(\sigma)d\sigma. \tag{2.25}$$

Then (x,y) lies in the region

$$\mathcal{R}_0:=\{(x,y)\mid 1/k^2\le x\le k^2,\quad \frac{1}{\delta+1}+\frac{1}{2(\mu+1)k^4}\le y\le k^2\}$$

for $\tau\ge\tilde\tau>0$, and satisfies

$$\left\{\begin{array}{lcl} x_\tau &=& x(-x-\delta y+2)+\varpi_1(\tau)\\ y_\tau &=& (\frac{1}{\delta+1}-y)((\delta+1)y-(\mu+1)x)+\varpi_2(\tau),\end{array}\right. \tag{2.26}$$

where $\varpi_1(\tau)=O(e^{-K\tau})$ and $\varpi_2(\tau)=O(e^{-K\tau})$ for some $K>0$, as $\tau\to\infty$.

Proof. We first reduce system (2.13) to a system of order 3: from relation (2.17) we eliminate W in the system (2.13) and obtain

$$\left\{\begin{array}{rcl} X_t &=& X\left[X+1+Z\right], \\ Y_t &=& Y\left[Y+1\right]-(\mu+1)X(Y+\frac{Z}{\delta+1})+\varpi(t), \\ Z_t &=& Z\left[1-\delta Y-Z-\alpha(t)\right], \end{array}\right.$$

which is a perturbation of system (2.11). Next, defining $x=-\frac{X}{Z}, \quad y=-\frac{Y}{Z}$, we get the system

$$\left\{\begin{array}{rcl} x_t &=& Z\left[x(2-x-\delta y)+\varpi_1\right] \\ y_t &=& Z\left[(\frac{1}{\delta+1}-y)((\delta+1)y-(\mu+1)x)+\varpi_2\right] \end{array}\right.$$

with

$$\varpi_1=-\frac{\alpha(t)X}{Z^2}=O(e^t), \qquad \varpi_2=\frac{\varpi(t)-\alpha(t)Y}{Z^2}=O(e^t), \tag{2.27}$$

from Lemma 2.5, and then

$$Z_t=Z(1+Z(\delta y-1)+\alpha(t)). \tag{2.28}$$

and $\tau(t)$ defined by (2.25) for $t\leq\bar{t}$ describes $[0,\infty)$ as t describes $(-\infty,\bar{t}]$, and $\tau/2k\leq|t|\leq 2k\tau$ for $t\leq\bar{t}$. Hence we deduce (2.26), and the estimates of ϖ_1,ϖ_2. Notice that $1/k^2\leq x,y\leq k^2$ for any $\tau\geq 0$ from (2.16), and from (2.17), for $\tau\geq\tilde{\tau}>0$,

$$y-\frac{1}{\delta+1}=\frac{1}{XZ}(\frac{YW}{\mu+1}+o(1))\geq\frac{1}{2(\mu+1)k^4},$$

ending the proof. ■

Hence system (2.26) appears as an exponential perturbation of an autonomous system that we study now:

Lemma 2.7 *Consider the system*

$$\left\{\begin{array}{rcl} x_\tau &=& x(2-x-\delta y) \\ y_\tau &=& (y-\frac{1}{\delta+1})((\mu+1)x-(\delta+1)y). \end{array}\right. \tag{2.29}$$

The fixed points of system (2.29) are $O=(0,0)$, and

$$j_0=\left(0,\frac{1}{\delta+1}\right), \quad \ell_0=\left(\frac{\delta+2}{\delta+1},\frac{1}{\delta+1}\right), \quad m_0=(x_0,y_0)=\left(\frac{2(\delta+1)}{\mu\delta+2\delta+1},\frac{2(\mu+1)}{\mu\delta+2\delta+1}\right),$$

and m_0 is a sink. Any solution of the system (2.29) which stays in the region $\mathcal{R}_0$ converges to the fixed point m_0 as $\tau\to\infty$.

Proof. The point m_0 is a sink: the eigenvalues of the linearized system of (2.29) at m_0 are the roots ℓ_1,ℓ_2 of equation

$$\ell^2+\frac{\delta\mu+3+2\mu+2\delta}{\mu\delta+2\delta+1}\ell+2\frac{\mu\delta+2\mu+1}{\mu\delta+2\delta+1}=0,$$

equivalently

$$(\gamma+1)\ell^2+(\gamma+\xi+1)\ell+2(\xi+1)=0, \tag{2.30}$$

and they have negative real part. Next we show that (2.29) has no limit cycle in $(0,\infty)\times(1/(\delta+1),\infty)$. Let $\mathcal{B}=x^p(y-\frac{1}{\delta+1})^{-q}$, where p,q are parameters. Writing (2.29) under the form $x_\tau=\mathcal{F}(x,y)$, $y_t=\mathcal{G}(x,y)$, we obtain

$$\nabla\cdot(\mathcal{B}(\mathcal{F},\mathcal{G}))=\mathcal{B}_x x_\tau+\mathcal{B}_\eta y_\tau+\mathcal{B}(\mathcal{F}_x+\mathcal{G}_y):=M\mathcal{B},\qquad \text{where}$$

$$M=(\mu-1-p-q(\mu+1))x-(p\delta-q(\delta+1)+3\delta+2)\,(y-\frac{1}{\delta+1})+\frac{p(\delta+2)+q(\delta+1)+1}{\delta+1}.$$

Choosing $q=\frac{\mu\delta+2\delta+2}{\mu\delta+2\delta+1}$ and $p=\mu-1-q(\mu+1)$, we find that

$$(\delta+1)M=-(\frac{\mu\delta+2\mu+1}{\mu\delta+2\delta+1}+\delta+2)<0.$$

Hence, by the Bendixson-Dulac Theorem, system (2.29) has no limit cycle. From the Poincaré-Bendixon Theorem, the ω-limit set Γ of any solution of (2.29) lying in $\mathcal{R}_0$ is fixed point, of a union of fixed points and connecting orbits. But m_0 is the unique fixed point in $\mathcal{R}_0$. Then any solution in $\mathcal{R}_0$ converges to m_0 as $\tau\to\infty$. ■

Remark 2.8 *It is easy to prove that there exists a connecting orbit joining the two points ℓ_0 and m_0, but it is not located in $\mathcal{R}_0$.*

We can now conclude.

Proof of Theorem 1.1. (i) *Convergence for system (2.26).* From Proposition 2.3, the ω-limit set Σ of our solution (x,y) of (2.26) is nonempty, compact, connected and contained in $\mathcal{R}_0$, and $\Sigma=\bigcup_{\ell\in\Sigma,\tau\geq\tilde\tau}\varphi(\tau,\ell)$, where $\varphi(\tau,\ell)$ denotes the trajectory of (2.29) such that $\varphi(\tilde\tau,\ell)=\ell$. Since $\lim_{\tau\to\infty}\varphi(\tau,\ell)=m_0$, there holds $m_0\in\Sigma$. Since m_0 is a sink of (2.29), then from the standard stability theory, see for example [6, Theorem 3.1, page 327], (x,y) converges to m_0.

(ii) *Convergence for system (2.13).* By setting $g(t)=1/Z$, we find from (2.28) that $g'+(1-\alpha)g=1-\delta y$, hence by L'Hôpital's rule,

$$\lim_{t\to-\infty}Z=\lim_{t\to-\infty}\frac{(e^{\int_{\tilde t}^t(1-\alpha)})'}{(ge^{\int_{\tilde t}^t(1-\alpha)})'}=\lim_{t\to-\infty}\frac{1-\alpha}{1-\delta y}=-\frac{\mu\delta+2\delta+1}{\mu\delta-1}=-(1+\gamma)=Z_0.$$

Hence

$$\lim_{t\to-\infty}X=-\lim_{t\to-\infty}xZ=\frac{2(\delta+1)}{\mu\delta-1}=\gamma=X_0,\qquad \lim_{t\to-\infty}Y=\lim_{t\to-\infty}yZ=\frac{2(\mu+1)}{\mu\delta-1}=\xi=Y_0.$$

Finally, from (2.17), we obtain $\lim_{t\to-\infty}W=-(1+\xi)=W_0$. That means (X,Y,Z,W) converges to $M_{0,1}$ defined at (2.9). Then from (2.15) we deduce the estimates

$$u(r)=A_1d^{-\gamma}(1+o(1)),\qquad v(r)=B_1d^{-\xi}(1+o(1))$$

where A_1,B_1 are given by and (1.10).

(iii) *Expansion of u and v.* We first consider system (2.26). Setting $x=x_0+\tilde x, y_0+\tilde y$, we find a system of the form

$$(\tilde x_\tau,\tilde y_\tau)=\mathcal{A}(\tilde x,\tilde y)+\mathcal{Q}(\tilde x,\tilde y)+(\varpi_1,\varpi_2)$$

where $(\tilde{x},\tilde{y}) \to (0,0)$, the eigenvalues ℓ_1, ℓ_2 of $\mathcal{A}$ satisfy $\max(\mathrm{Re}(\ell_1,\ell_2)) = -m < -1/(\gamma+1)$, and $\mathcal{Q}$ is quadratic and $\varpi_1(\tau), \varpi_2(\tau) = O(e^{-K\tau})$. There exists an euclidian structure with a scalar product where $\langle \mathcal{A}(\tilde{x},\tilde{y}),(\tilde{x},\tilde{y})\rangle \leq -m \left\|(\tilde{x},\tilde{y})\right\|^2$. Then the function $\tau \mapsto \eta(\tau) = \left\|(\tilde{x},\tilde{y})\right\|(\tau)$ satisfies an inequality of the type $\eta_\tau \leq -(m-\varepsilon)\eta + Ce^{-K\tau}$ for any $\varepsilon > 0$ and τ large enough. Then

$$\eta(\tau) = O(e^{-K\tau}) + O(e^{-(m-\varepsilon)\tau}). \tag{2.31}$$

Then the convergence of (x,y) to (x_0,y_0) is exponential. From (2.28), the convergence of Z to Z_0 is exponential. Writing τ under the form

$$\tau = \overline{c} + Z_0 t + \int_t^\infty (Z_0 - Z),$$

we deduce that $\tau = \overline{c} + Z_0 t + O(e^{kt})$ for some $k > 0$. From (2.27) we obtain that $\varpi_1, \varpi_2 = O(e^{-K_0\tau})$ with $K_0 = 1/\left|Z_0\right|$; taking $K = K_0 = 1/(\gamma+1)$ in (2.31), we find that $\eta(\tau) = O(e^{-K_0\tau}) = O(e^t)$, because $m > K_0$. Then from (2.28) we deduce that $|Z - Z_0| = O(e^t)$, and then from (2.25), $|X - X_0| + |Y - Y_0| = O(e^t)$, and in turn $|W - W_0| = O(e^t)$ from (2.17). Finally we come back to u and v by means of (2.15): recalling that $s = e^t$ and $r = 1 + O(s)$ as $s \to 0$, we deduce that

$$u(r) = A_1 s^{-\gamma}(1 + O(s)), \qquad v(r) = B_1 s^{-\xi}(1 + O(s))$$

and the expansion (1.8) follows from (2.4). ■

Proof of Corollary 1.2. Let u be a radial solution of (1.11). Then u and $v = \Delta u$ satisfy

$$\begin{cases} \Delta u = v \\ \Delta v = |x|^b \, |u|^\mu \end{cases}$$

and then $u(r) > 0$ in (r_0, R) and $u(R) = \infty$. Integrating twice the second equation in this system, we have that $\lim\limits_{r\to R} v(r) = \infty$ and Theorem 1.1 applies. ■

2.3 The set of initial data for blow up

Here we suppose $a = b = 0$. By scaling, for any $\rho > 0$ there exists solutions which blow up at ρ. Let us call $\rho(u_0, v_0)$ the blow-up radius of a regular solution with initial data (u_0, v_0). From (2.1), we find

$$\rho(\lambda^\gamma u_0, \lambda^\xi v_0) = \lambda^{-1}\rho(u_0, v_0).$$

Then for any $(u_0, v_0) \in S^1$ there is a unique λ such that $\rho(\lambda^\gamma u_0, \lambda^\xi v_0) = 1$. Thus there exist infinitely many solutions blowing up at $R = 1$, including in particular two unique solutions with respective initial data $(\bar{u}_0, 0)$ and $(0, \bar{v}_0)$. Using monotonicity properties, it was shown in [15] that the set

$$\mathcal{S} = \left\{(u_0, v_0) \in [0,\infty) \times [0,\infty) : \ \lim_{r\to 1} u = \lim_{r\to 1} v = \infty\right\}$$

is contained in $[0, \bar{u}_0] \times [0, \bar{v}_0]$. Next we give some properties of $\mathcal{S}$ extending some results of [15] to higher dimensions.

Proposition 2.9 *Let $N \geq 1$. If $\min\{\delta, \mu\} \geq 1$, then $\mathcal{S}$ is a simple curve joining the two points $(\bar{u}_0, 0)$ and $(0, \bar{v}_0)$.*

Proof. We claim that the mapping $(u_0, v_0) \in [0,\infty)\times[0,\infty)\setminus\{0,0\} \longmapsto \rho(u_0,v_0)$ is continuous. As in [11] this will follow from our global estimates.

(i) *The function ρ is lower semi-continuous.* Indeed the local existence is obtained by the fixed point theorem of a strict contraction, since $\min\{\delta,\mu\}\geq 1$, then we have local continuous dependence of the initial conditions, even if $u_0 = 0$ or $v_0 = 0$, and the result follows classically.

(ii) *The function ρ is upper semi-continuous.* We can start from a point $r_0 > 0$ instead of 0. We prove that for any positive $(\tilde{u}_0, \tilde{v}_0)$, considering any solution $(\tilde{u}, \tilde{v})$ equal to $(\tilde{u}_0, \tilde{v}_0)$ at r_0, with blow-up point $\tilde{\rho}$, for any $\tilde{r} > \tilde{\rho}$, any solution (u, v) starting from r_0 with data sufficiently close to $(\tilde{u}_0, \tilde{v}_0)$, blows up before $\tilde{r}$: suppose that it is false, then there exists a sequence of positive solutions (u_n, v_n), with data $(\tilde{u}_n, \tilde{v}_n)$ at r_0, tending to $(\tilde{u}_0, \tilde{v}_0)$, increasing, and blowing up at $\rho_n \geq \tilde{r}$. We can assume $\tilde{r} = 1$. Making the change of variables (2.4) we get solutions of system (2.5) in $(0, s_0]$, satisfying $C_0 = C_0(r_0, N, a, b)$

$$\begin{cases} -u_{ss} + C_0 v^\delta \leq 0 \\ -v_{ss} + C_0 u^\mu \leq 0 \end{cases}$$

with u and v decreasing. In fact estimates (1.16) hold with a universal constant, in any $B(0,k)\setminus\{0\} \subset \Omega$ such that the mean values of u and v on $\partial B(0,r)$ are strictly monotone. Then there exists a constant $C = C(C_0, N, \delta, \mu)$ such that

$$u_n(s) \leq Cs^{-\gamma}, \quad v_n(s) \leq Cs^{-\xi} \quad \text{for } s \leq s_0,$$

that means

$$u_n(r) \leq C(r^{2-N} - 1)^{-\gamma}, \quad v_n(r) \leq C(r^{2-N} - 1)^{-\xi} \quad \text{for } r \in [r_0, 1)\,.$$

Passing to the limit we find that u, v are bounded at the point $\tilde{\rho} < 1$, which is contradictory. Then the claim is proved. Thus $\mathcal{S}$ is a curve with

$$(u_0, v_0) = \left[\rho^\gamma(\cos\theta, \sin\theta)\cos\theta, \rho^\xi(\cos\theta, \sin\theta)\sin\theta\right], \qquad \theta \in [0, \pi/2]\,,$$

as a parametric representation. ■

3 Behavior of system (1.1) near the origin

3.1 Formulation as a dynamical system

In [4] the authors study general quasilinear elliptic systems, and in particular the system

$$\begin{cases} -\Delta u = -(u_{rr} + \frac{N-1}{r}u_r) = \varepsilon_1 r^a v^\delta, \\ -\Delta v = -(v_{rr} + \frac{N-1}{r}v_r) = \varepsilon_2 r^b u^\mu, \end{cases} \tag{3.1}$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. Near any point r where $u(r) \neq 0, u'(r) \neq 0$ and $v(r) \neq 0$, $v'(r) \neq 0$, they define

$$X(t) = -\frac{ru_r}{u}, \qquad Y(t) = -\frac{rv_r}{v}, \qquad Z(t) = -\varepsilon_1\frac{r^{1+a}v^\delta}{u_r}, \qquad W(t) = -\varepsilon_2\frac{r^{1+b}u^\mu}{v_r}, \tag{3.2}$$

with $t = \ln r$, so system (3.1) becomes

$$\left\{\begin{array}{rcl} X_t &=& X\left[X-(N-2)+Z\right], \\ Y_t &=& Y\left[Y-(N-2)+W\right], \\ Z_t &=& Z\left[N+a-\delta Y-Z\right], \\ W_t &=& W\left[N+b-\mu X-W\right]. \end{array}\right. \tag{3.3}$$

One recovers u and v by the formulas

$$u{=}r^{-\gamma_{a,b}}(\,|ZX|)^{1/D}(\,|WY|)^{\delta/D}, \qquad v{=}r^{-\xi_{a,b}}\,|WY|)^{1/D}\,|ZX|^{\mu/D}\,, \tag{3.4}$$

and we notice the relations $\gamma_{a,b}+2+a=\delta\xi_{a,b}$ and $\xi_{a,b}+2+b=\mu\gamma_{a,b}$.

As mentioned in [4], system (3.3) is independent of ε_i, $i=1,2$, and thus it allows to study system (3.1) in a unified way. In our case $\varepsilon_1=\varepsilon_2=-1$, then $XZ=-r^{a+2}v^{\delta}/u$ and $YW=r^{b+2}u^{\mu}/v$, thus we are led to study (3.3) in the region

$$\mathcal{R}=\{(X,Y,Z,W)\mid XZ\le 0,\ YW\le 0\}.$$

This system is *quadratic*, and it admits four invariant hyperplanes: $X=0, Y=0, Z=0, W=0$. The trajectories located on these hyperplanes do not correspond to a solution of system (3.1), and they are called *nonadmissible*. System (3.3) has sixteen fixed points, including $O=(0,0,0,0)$. The main one is

$$M_0=(X_0,Y_0,Z_0,W_0)=(\gamma_{a,b},\xi_{a,b},N-2-\gamma_{a,b},N-2-\xi_{a,b})\,,$$

which is interior to $\mathcal{R}$ whenever (1.15) holds; it corresponds to the particular solution (u^*,v^*) given in (1.13). Among the other fixed points, as we see below,

$$\begin{array}{l} N_0=(0,0,N+a,N+b), \\ R_0=(0,-(2+b),N+a+(2+b)\delta,N+b)\,, \quad S_0=(-(2+a),0,N+a,N+b+(2+a)\mu)\,, \end{array}$$

are linked to the regular solutions, and

$$A_0=(N{-}2,N{-}2,0,0),\ G_0=(N{-}2,0,0,N{+}b{-}(N{-}2)\mu),\ H_0=(0,N{-}2,N{+}a{-}(N{-}2)\delta,0),$$

$$\begin{array}{c} P_0=(N-2,(N-2)\mu-2-b,0,(N+b-(N-2)\mu)), \\ Q_0=((N-2)\delta-2-a,N-2,N+a-(N-2)\delta,0), \end{array}$$

and $M_0$are linked to the large solutions near 0. Notice that $P_0\notin\mathcal{R}$ for $\frac{2+b}{N-2}<\mu<\frac{N+b}{N-2}$ and $Q_0\notin\mathcal{R}$ for $\frac{2+a}{N-2}<\delta<\frac{N+a}{N-2}$. We are not concerned by the other fixed points

$$I_0=(N-2,0,0,0),J_0=(0,N-2,0,0),K_0=(0,0,N+a,0),L_0=(0,0,0,N+b),$$

which correspond to non admissible solutions, from [4], and

$$C_0=(0,-(2+b),0,N+b)\,, \qquad D_0=(-(2+a),0,N+a,0)\,,$$

which can be shown as non admissible as $t\to-\infty$.

3.2 Regular solutions

First we give an alternative proof of Proposition 2.1.

Proposition 3.1 *Assume (1.5) and $D \neq 0$. Then a solution (u, v) is regular with initial data $(u_0, v_0), u_0, v_0 > 0$ (resp. $(u_0, 0)$,$u_0 > 0$, resp. $(0, v_0), v_0 > 0$), if and only the corresponding solution (X, Y, Z, W) converges to N_0 (resp. R_0, resp. S_0) as $t \to -\infty$. For any $u_0, v_0 \geq 0$, not both 0, there exists a unique local regular solution (u, v) with initial data (u_0, v_0).*

Proof. The proof in the case $u_0, v_0 > 0$ is done in [4, Proposition 4.4]. Suppose $u_0 > 0 = v_0$, and consider any regular solution (u, v) with initial data $(u_0, 0)$. We find

$$v' = \frac{u_0^\mu}{N+b} r^{1+b}(1+o(1)), \qquad v = \frac{u_0^\mu}{(N+b)(2+b)} r^{2+b}(1+o(1)),$$

$$(r^{N-1}u')' = \frac{u_0^{\mu^\delta} r^{N-1+a+(2+b)\delta}}{((N+b)(2+b))^\delta}(1+o(1)), \; u' = \frac{u_0^{\delta\mu} r^{1+a+(2+b)\delta}}{((N+b)(2+b))^\delta (N+a+(2+b)\delta)}(1+o(1));$$

then from (3.2) the corresponding trajectory (X, Y, Z, W) converges to R_0 as $t \to -\infty$. Next we show that there exists a unique trajectory converging to R_0. We write

$$R_0 = \left(0, \bar{Y}, \bar{Z}, \bar{W}\right) = (0, -(2+b), N+a+(2+b)\delta, N+b)\,.$$

Under our assumptions it lies in $\mathcal{R}$. Setting $Y = \bar{Y} + \tilde{Y}, Z = \bar{Z} + \tilde{Z}, W = \bar{W} + \tilde{W}$, the linearization at R_0 gives

$$X_t = \lambda_1 X, \quad \tilde{Y}_t = \bar{Y}\left[\tilde{Y} + \tilde{W}\right], \quad Z_t = \bar{Z}\left[-\delta\tilde{Y} - \tilde{Z}\right], \quad W_t = \bar{W}\left[-\mu X - \tilde{W}\right];$$

the eigenvalues are

$$\lambda_1 = 2 + a + \delta(2+b) > 0, \quad \lambda_2 = -(2+b) < 0, \quad \lambda_3 = -\bar{Z} < 0, \quad \lambda_4 = -(N+b) < 0.$$

The unstable manifold $\mathcal{V}_u$ has dimension 1 and $\mathcal{V}_u \cap \{X = 0\} = \emptyset$, hence there exist precisely one admissible trajectory such that $X < 0$ and $Z > 0$. Moreover it satisfies

$$\lim_{t\to-\infty} e^{-\lambda_1 t} X = C_1 > 0, \lim_{t\to-\infty} Y = \bar{Y}, \lim_{t\to-\infty} Z = \bar{Z}, \lim_{t\to-\infty} W = \bar{W}.$$

Then from (3.4) u has a positive limit $\overline{u}_0$, and v=$O(e^{2t})$, thus v tends to 0; then (u, v) is regular with initial data $(\overline{u}_0, 0)$. By (2.1) we obtain existence for any $(u_0, 0)$ and the uniqueness still holds. Similarly the solutions with initial data $(0, v_0)$ correspond to S_0. ■

3.3 Local existence of large solutions near 0

Next we prove the existence of different types of local solutions large at 0, by linearization around the fixed points A_0, G_0, H_0, P_0, Q_0. For simplicity we do not consider the limit cases, where one of the eigenvalues of the linearization is 0, corresponding to behaviors of u, v of logarithmic type. All the following results extend by symmetry, after exchanging $u, \delta, a, \gamma_{a,b}$ and $v, \mu, b, \xi_{a,b}$.

Proposition 3.2 *Assume $N > 2$.*

(i) If $\delta < \frac{N+a}{N-2}$ and $\mu < \frac{N+b}{N-2}$, then there exist solutions (u, v) to (1.1) such that

$$\lim_{r\to 0} r^{N-2}u = \alpha > 0, \qquad \lim_{r\to 0} r^{N-2}v = \beta > 0. \tag{3.5}$$

If $\delta > \frac{N+a}{N-2}$ or $\mu > \frac{N+b}{N-2}$, there exist no such solutions.

(ii) Let $\gamma_{a,b} > N-2$ and let $\mu < \frac{2+b}{N-2}$ or $\mu > \frac{N+b}{N-2}$. Then there exist solutions (u, v) of (1.1) such that

$$\lim_{r\to 0} r^{N-2}u = \alpha > 0, \qquad \lim_{r\to 0} r^{(N-2)\mu-(2+b)}v = \beta(\alpha) > 0, \tag{3.6}$$

with $\beta(\alpha) = \alpha^\mu/((N-2)\mu - N - b)((N-2)\mu - 2 - b)$. If $\gamma_{a,b} < N-2$, there exist no such solutions.

(iii) If $\mu < \frac{2+b}{N-2}$ then there exist solutions (u, v) of (1.1) such that

$$\lim_{r\to 0} r^{N-2}u = \alpha > 0, \qquad \lim_{r\to 0} v = \beta > 0. \tag{3.7}$$

If $\mu > \frac{2+b}{N-2}$ there exist no such solutions.

Proof. (i) We study the behaviour of the solutions of (3.3) near A_0 as $t \to -\infty$. The linearization at A_0 gives, with $X = N-2+\tilde{X}, Y = N-2+\tilde{Y}$,

$$\tilde{X}_t = (N-2)\left[\tilde{X} + Z\right], \quad \tilde{Y}_t = (N-2)\left[\tilde{Y} + W\right], \quad Z_t = \lambda_3 Z, \quad W_t = \lambda_4 W,$$

with eigenvalues

$$\lambda_1 = \lambda_2 = (N-2) > 0,\ \lambda_3 = N + a - (N-2)\delta,\ \lambda_4 = N + b - (N-2)\mu.$$

If $\delta < \frac{N+a}{N-2}$ and $\mu < \frac{N+b}{N-2}$, then we have $\lambda_3, \lambda_4 > 0$; the unstable manifold $\mathcal{V}_u$ has dimension 4, then there exists an infinity of trajectories converging to A_0 as $t \to -\infty$, interior to $\mathcal{R}$, then admissible, with $Z, W < 0$. The solutions satisfy $\lim_{t\to-\infty} e^{-\lambda_3 t}Z = Z_0 < 0$ and $\lim_{t\to-\infty} e^{-\lambda_4 t}W = W_0 < 0$, with $\lim_{t\to-\infty} X = \lim_{t\to-\infty} Y = N-2$. Hence from (3.4), the corresponding solutions (u, v) of (1.1) satisfy (3.5). If $\delta > \frac{N+a}{N-2}$ or $\mu > \frac{N+b}{N-2}$, then $\lambda_3 < 0$ or $\lambda_4 < 0$, respectively, and $\mathcal{V}_u$ has at most dimension 3, and it satisfies $Z = 0$ or $W = 0$ respectively. Therefore there is no admissible trajectory converging at $-\infty$.

(ii) Here we study the behaviour near P_0. Setting $P_0 = (N-2, Y_*, 0, W_*)$, with

$$Y_* = (N-2)\mu - 2 - b, \qquad W_* = N + b - (N-2)\mu,$$

the linearization at P_0 gives, with $X = N-2+\tilde{X}, Y = Y_* + \tilde{Y}$, $W = W_* + \tilde{W}$,

$$\tilde{X}_t = (N-2)\left[\tilde{X} + Z\right], \quad \tilde{Y}_t = Y_*\left[\tilde{Y} + \tilde{W}\right], \quad Z_t = \lambda_3 Z, \quad \tilde{W}_t = W_*\left[-\mu\tilde{X} - \tilde{W}\right].$$

By direct computation we obtain that the eigenvalues are

$$\lambda_1 = N-2 > 0, \quad \lambda_2 = Y_*, \quad \lambda_3 = N + a - \delta Y_* = D(\gamma_{a,b} - (N-2)), \quad \lambda_4 = -W_*.$$

Assume first that $\gamma_{a,b} > N-2$. Then $\lambda_3 > 0$. If $\mu > \frac{N+b}{N-2}$, then also $\lambda_2, \lambda_4 > 0$ and thus $\mathcal{V}_u$ has dimension 4, then there exist an infinity of admissible trajectories, with $Z < 0$, converging as $t \to -\infty$. If $\mu < \frac{2+b}{N-2}$, then $\lambda_2, \lambda_4 < 0$, thus $\mathcal{V}_u$ has dimension 2, and $\mathcal{V}_u \cap \{Z = 0\}$ has dimension 1, thus there also exist an infinity of admissible trajectories with $Z < 0$ converging when $t \to -\infty$. Then $\lim_{t\to-\infty} e^{-\lambda_3 t} Z = C_3 < 0$, $\lim_{t\to-\infty} X = N-2$, $\lim_{t\to-\infty} Y = Y_*$ and $\lim_{t\to-\infty} W = W_*$, thus (3.4), (u, v) satisfy (3.6). If $\gamma_{a,b} < N-2$, then $\lambda_3 < 0$ and $\mathcal{V}_u = \mathcal{V}_u \cap \{Z = 0\}$ and there is no admissible trajectory converging when $t \to -\infty$.

(iii) We consider the behaviour near G_0. The linearization at G_0 gives, with $X = N-2+\tilde{X}, W = N+b-(N-2)\mu+\tilde{W}$,

$$\tilde{X}_t = (N-2)\left[\tilde{X} + Z\right], \quad Y_t = (2+b-(N-2)\mu)Y,$$
$$Z_t = (N+a)Z, \quad W_t = (N+b-(N-2)\mu)\left[-\mu\tilde{X} - \tilde{W}\right],$$

and the eigenvalues are

$$\lambda_1 = N-2 > 0,\ \lambda_2 = 2+b-(N-2)\mu,\ \lambda_3 = N+a > 0,\ \lambda_4 = (N-2)\mu - N - b.$$

If $\mu < \frac{2+b}{N-2}$, then $\lambda_2, \lambda_4 < 0$. Then $\mathcal{V}_u$ has dimension 3, and $\mathcal{V}_u \cap \{Y = 0\}$ and $\mathcal{V}_u \cap \{Z = 0\}$ have dimension 2. This implies that $\mathcal{V}_u$ must contain admissible trajectories such that $X > 0$ (because $N-2 > 0$), $Y < 0$, $Z < 0$ and $W > 0$ (because $N+b-(N-2)\mu > 0$). Clearly, $\lim_{t\to-\infty} X = N-2$ and $\lim_{t\to-\infty} W = N+b-(N-2)\mu > 0$. Moreover, $\lim_{t\to-\infty} e^{-\lambda_2 t} Y = C_2 < 0$ and $\lim_{t\to-\infty} e^{-\lambda_3 t} Z = C_3 < 0$, thus (3.7) follows from (3.4). Let now $\mu > \frac{2+b}{N-2}$, so that $\lambda_2 < 0$. If $\mu < \frac{N+b}{N-2}$, then $\lambda_4 < 0$, $\mathcal{V}_u$ has dimension 2, and also $\mathcal{V}_u \cap \{Y = 0\}$, hence $\mathcal{V}_u = \mathcal{V}_u \cap \{Y = 0\}$, and there exists no admissible trajectory. If $\mu > \frac{N+b}{N-2}$, then $\lambda_4 > 0$, $\mathcal{V}_u$ has dimension 3 and also $\mathcal{V}_u \cap \{Y = 0\}$, there is no admissible trajectory. ■

Remark 3.3 *If $\mu > \frac{N+b}{N-2}$, in (ii) the two functions u, v are large near 0. If $\mu < \frac{2+b}{N-2}$, then u is large near 0 and v tends to 0.*

Next we study the behavior near M_0, which is the most interesting one.

Proposition 3.4 *Assume $N \geq 1$ and (1.15). Then (up to a scaling) there exist infinitely many solutions defined near $r = 0$ such that*

$$\lim_{r\to 0} r^{\gamma_{a,b}} u = A_N, \lim_{r\to 0} r^{\xi_{a,b}} v = B_N.$$

Proof. Setting $X = X_0 + \tilde{X}, Y = Y_0 + \tilde{Y}, Z = Z_0 + \tilde{Z}, W = W_0 + \tilde{W}$, the linearized system is

$$\begin{cases} \tilde{X}_t &= X_0(\tilde{X} + \tilde{Z}), \\ \tilde{Y}_t &= Y_0(\tilde{Y} + \tilde{W}), \\ \tilde{Z}_t &= Z_0(-\delta\tilde{Y} - \tilde{Z}), \\ \tilde{W}_t &= W_0(-\mu\tilde{X} - \tilde{W}). \end{cases}$$

As described in [4], the eigenvalues are the roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, of the characteristic polynomial

$$\begin{aligned} f(\lambda) &= \det \begin{pmatrix} X_0 - \lambda & 0 & X_0 & 0 \\ 0 & Y_0 - \lambda & 0 & Y_0 \\ 0 & \delta\,|Z_0| & |Z_0| - \lambda & 0 \\ \mu\,|W_0| & 0 & 0 & |W_0| - \lambda \end{pmatrix} \\ &= (\lambda - X_0)(\lambda + Z_0)(\lambda - Y_0)(\lambda + W_0) - \delta\mu X_0 Y_0 Z_0 W_0, \end{aligned} \tag{3.8}$$

where we recall that $X_0, Y_0 > 0$ and $Z_0, W_0 < 0$. We write f in the form

$$f(\lambda) = \lambda^4 + E_0\lambda^3 + F_0\lambda^2 + G_0\lambda - H_0,$$

with

$$\begin{cases} E_0 &= Z_0 - X_0 + W_0 - Y_0, \\ F_0 &= (Z_0 - X_0)(W_0 - Y_0) - X_0 Z_0 - Y_0 W_0, \\ G_0 &= -Y_0 W_0 (Z_0 - X_0) - X_0 Z_0 (W_0 - Y_0), \\ H_0 &= D X_0 Y_0 Z_0 W_0. \end{cases}$$

We note that $E_0 < 0,\ F_0 > 0$ and $2G_0 = -E_0\,[Y_0 Z_0 + X_0 W_0] < 0$. From (1.4) we have $H_0 > 0$, hence $\lambda_1\lambda_2\lambda_3\lambda_4 < 0$. Hence there exist two real roots $\lambda_3 < 0 < \lambda_4$, with

$$\lambda_4 > \max(\{X_0, Y_0, |Z_0|, |W_0|\}$$

from (3.8), and two roots λ_1, λ_2, which may be real or complex. From the form of $f(\lambda)$ in (3.8), we also see easily that if the roots λ_1, λ_2 are real, they are positive. Next we claim that $\operatorname{Re}\lambda_1 > 0$. Suppose $\operatorname{Re}\lambda_1 = 0$. Then $f(i\operatorname{Im}\lambda_1) = 0$, then $G_0^2 = E_0 F_0 G_0 + E_0^2 H_0$, and thus, dividing by E_0,

$$0 = G_0^2 - E_0 F_0 G_0 + E_0^2 H_0 = \frac{E_0^2}{4}\left([Y_0 Z_0 + X_0 W_0]^2 + 2\,[Y_0 Z_0 + X_0 W_0]\,F_0 - 4H_0\right),$$

hence $[Y_0 Z_0 + X_0 W_0 + F_0]^2 = F_0^2 + 4H_0 > F_0^2$; but

$$Y_0 Z_0 + X_0 W_0 + F_0 = (X_0 - W_0)(Y_0 - Z_0) \in (0, F_0)$$

which is a contradiction. Since $\operatorname{Re}\lambda_1$ is a continuous function of (δ, μ), it is sufficient to find a value (μ, δ) satisfying (1.15) for which it is positive. Taking $\delta = \mu$, the equation in λ reduces to two equations of order 2:

$$\begin{aligned} f(\lambda) &= (\lambda - X_0)^2(\lambda - |Z_0|)^2 - \delta^2 X_0^2 Z_0^2 \\ &= \left[\lambda^2 - (X_0 + |Z_0|)\lambda - (\delta - 1)X_0\,|Z_0|\right]\left[\lambda^2 - (X_0 + |Z_0|)\lambda + (1 + \delta)X_0\,|Z_0|\right], \end{aligned}$$

and $X_0 + |Z_0| > 0$, thus the claim is proved. Then $\mathcal{V}_u$ has dimension 3 and $\mathcal{V}_s$ has dimension 1. Hence the result follows. ■

Remark 3.5 *In the case $N = 1$, two roots are explicit: $\lambda_3 = -1, \lambda_4 = 2 + \gamma + \xi$, and λ_1, λ_2 are the roots of equation*

$$\lambda^2 - (1 + \gamma + \xi)\lambda + 2(1 + \gamma)(1 + \xi) = 0. \tag{3.9}$$

The 4 roots are real if $(1+\gamma+\xi)^2 - 8(1+\gamma)(1+\xi) \geq 0$*, that means*

$$(\delta\mu + 3 + 2\mu + 2\delta)^2 - 8(\mu\delta + 2\delta + 1)(\mu\delta + 2\mu + 1) \geq 0,$$

which is not true for $\delta = \mu$*, but is true for example when* δ/μ *is large enough. The roots of equation (3.9) and the roots of equation (2.30) relative to the linearization of system (2.29) at* m_0 *are linked by the relations* $\ell_1 = \lambda_1/|Z_0|$, $\ell_2 = \lambda_2/|Z_0|$. *Indeed* $M_0 = M_{0,1}$ *defined at (2.9) satisfies relation (2.17) with* $\varpi = 0$*, thus* (X_0, Y_0, Z_0) *is a fixed point of system (2.11) and the linearization of (2.11) at this point gives the eigenvalues* $-1, \lambda_1, \lambda_2$*. The point* m_0 *is the image of* (X_0, Y_0, Z_0) *by the transformation (2.25), which divides the eigenvalues by* $|Z_0|$*, due to the change in time* $t \mapsto \tau$*.*

3.4 Global results

Here we prove our second main result.

Proof of Theorem 1.3. From the proof of Proposition 3.4, the linearization at M_0 admits a unique real eigenvalue $\lambda_3 < 0$. From (3.8) a generating eigenvector (u_1, u_2, u_3, u_4) satisfies $u_1u_3 < 0$ and $u_2u_4 < 0$, and hence it is of the form $\vec{u} = (-\alpha^2, -\beta^2, \sigma^2, \rho^2)$, or $-\vec{u}$. There exist precisely two trajectories $\mathcal{T}_{\vec{u}}$ and $\mathcal{T}_{-\vec{u}}$ converging to M_0 as $t \to \infty$ and the convergence of X, Y, Z, W is monotone near $t = \infty$; from (3.4), the corresponding solutions (u, v) of system (1.1) satisfy (1.17).

We consider the trajectory $\mathcal{T}_{\vec{u}}$ corresponding to $\vec{u}$. Let us show that the convergence is monotone in all $\mathbb{R}$. Notice that neither of the components can vanish, since system (1.1) is of Kolmogorov type. Near $t = \infty$, X and Y are increasing, and Z, W are decreasing. Suppose that there exists a greatest value t_1 such that X has a minimum local at t_1, hence

$$X_{tt}(t_1) = X(t_1)Z_t(t_1) \geq 0, \qquad Z(t_1) = N - 2 - X(t_1),$$

thus $Z_t(t_1) \geq 0$. Then there exists $t_2 \geq t_1$ such that $Z_t(t_2) = 0$, and

$$Z_{tt}(t_2) = -\delta Z(t_2)Y_t(t_2) \leq 0, \qquad Z(t_2) = N + a - \delta Y(t_2),$$

then $Y_t(t_2) \leq 0$. There exists $t_3 \geq t_2$ such that $Y_t(t_3) = 0$, and

$$Y_{tt}(t_3) = Y(t_3)W_t(t_3) \geq 0, \qquad Y(t_3) = N - 2 - W(t_3).$$

There exists $t_4 \geq t_3$ such that $W_t(t_4) = 0$ and $W_{tt}(t_4) = -W(t_4)X_t(t_4) \leq 0$. From the definition of t_1, this implies $t_4 = t_1$, and then all the conditions above imply that $(X, Y, Z, W)(t_1) = M_0$, which is impossible. Hence X stays strictly monotone, and similarly Y, Z, W also stay strictly monotone. Since $X, Y > 0$, and $Y, Z < 0$, then $\mathcal{T}_{\vec{u}}$ is bounded, hence defined on $\mathbb{R}$ and converges to some fixed point $L = (l_1, l_2, l_3, l_4)$ of the system as $t \to -\infty$ and necessarily $l_1 < X_0, l_2 < Y_0, l_3 > Z_0, l_4 > W_0$.

• Case $N > 2$. First we note that along $\mathcal{T}_{\vec{u}}$ we always have $X, Y > N - 2$. Indeed, if at some point t we have $X(t) = N - 2$, then $X_t(t) = (N - 2)Z(t) < 0$, which is contradictory. Hence the possible values for L are A_0, or P_0 when $\mu \geq \frac{N+b}{N-2}$, or Q_0 when $\delta \geq \frac{N+a}{N-2}$, since I_0 is nonadmissible. By hypothesis, $\gamma_{a,b} > N - 2$, then either $\mu < \frac{N+b}{N-2}$ or $\delta < \frac{N+a}{N-2}$. We can assume

that $\delta < \frac{N+a}{N-2}$. Then $Q_0 \notin \mathcal{R}$, then $L = A_0$ or P_0. When $\mu < \frac{N+b}{N-2}$, then $L = A_0$. When $\mu > \frac{N+b}{N-2}$, from Proposition 3.2(i), we have $L \neq A_0$, thus $L = P_0$. In the limit case $\mu = \frac{N+b}{N-2}$, we find $P_0 = A_0$. From the linearization at A_0 we have

$$\lambda_1 = \lambda_2 = N - 2 > 0,\ \lambda_3 = N + a - (N-2)\delta > 0,\ \lambda_4 = 0.$$

Coming back to the proof of Proposition 3.2(i), we find that the convergence of Z and $\tilde{X} = X - (N-2)$ to 0 are exponential. From the fourth equation in (3.3) we see that $W_t + W^2 > 0$, hence $-1/W \leq C|t|$ near $-\infty$. Then, there exists $m > 0$ such that

$$W_t = W^2(-1 - \mu W^{-1}\tilde{X}) = W^2(-1 + O(e^{mt});$$

integrating over (t, t_0), $t_0 < 0$, we obtain that $W(t) = t^{-1} + O(t^{-2})$. In turn we estimate Y; setting $\overline{Y} = \tilde{Y} + W$, then $\overline{Y}_t = (N-2)\overline{Y} + \overline{Y}(\overline{Y} - W) + W(-\mu\tilde{X} - W)$, and thus

$$\overline{Y}_t = ((N-2) + \varepsilon(t))\overline{Y} + O(t^{-2}),$$

implying $\overline{Y} = O(t^{-2})$ and thus $Y = N - 2 - t^{-1} + O(t^{-2})$. Next we find that $Z_t/Z = \lambda_3 + t^{-1} + O(t^{-2})$, which yields $\lim_{t\to-\infty} e^{-\lambda_3 t}|t|^{-\delta}|Z| = C > 0$. Finally, by replacing in (3.4), and deduce the behavior of u and v as claimed:

$$\lim_{r\to 0} r^{N-2}u = C_1 > 0 \quad \text{and} \quad \lim_{r\to 0} r^{N-2}|\log(r)|^{-1}v = C_2 > 0.$$

• Case $N = 2$. Then necessarily $L = O = (0,0,0,0)$. The eigenvalues of the linearized problem at this point are $0, 0, 2+a, 2+b$. Since $Z_t = Z(2 + a - \delta Y - Z)$ and Y and Z tend to 0 as t tends to $-\infty$, Z converges exponentially to 0, and similarly W. Since $X_t \leq X^2$, it follows that $X \geq C\,|t|^{-1}$ near $-\infty$. Then

$$X_t = X^2(1 + Z/X) = X^2(1 + O(e^{mt}))$$

for some $m > 0$, hence $X = -1/t + O(t^{-2})$, then the function $t \mapsto \varphi = u(t)/t$ satisfies $\varphi_t/\varphi = O(t^{-2})$, then φ has a finite limit, hence $u(r)/\ln r$ has a finite positive limit, and similarly for v. ■

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