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# A BIFURCATION FOR A GENERALIZED BURGER'S EQUATION IN DIMENSION ONE

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ABSTRACT. We consider the generalized Burger's equation

$$\begin{cases} \partial_t u = \partial_x^2 u - u \partial_x u + u^p - \lambda u & \text{in } \overline{\Omega} \times (0, T), \\ \mathcal{B}(u) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = \varphi \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

with  $p > 1$ ,  $\lambda \in \mathbb{R}$ ,  $T \in (0, \infty]$ ,  $\Omega$  a subdomain of  $\mathbb{R}$ , and where  $\mathcal{B}(u) = 0$  designates some boundary conditions. First, using some phase plane arguments, we study the existence of stationary solutions under Dirichlet or Neumann boundary conditions and prove a bifurcation depending on the parameter  $\lambda$ . Then, we compare positive solutions of the parabolic equation with appropriate stationary solutions to prove that global existence can occur when  $\mathcal{B}(u) = 0$  stands for the Dirichlet, the Neumann or the dissipative dynamical boundary conditions  $\sigma \partial_t u + \partial_\nu u = 0$ . Finally, for many boundary conditions, global existence and blow up phenomena for solutions of the nonlinear parabolic problem in an unbounded domain  $\Omega$  are investigated by using some standard super-solutions and some weighted  $L^1$ -norms.

**1. Introduction.** Let  $\Omega$  be a domain of the real line  $\mathbb{R}$ , not necessarily bounded. Let  $p$  be a real number with  $p > 1$ ,  $\lambda \in \mathbb{R}$  and  $\varphi$  a non-negative continuous function in  $\overline{\Omega}$ . Consider the following nonlinear parabolic problem

$$\begin{cases} \partial_t u = \partial_x^2 u - u \partial_x u + u^p - \lambda u & \text{in } \overline{\Omega} \times (0, \infty), \\ \mathcal{B}(u) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}, \end{cases} \quad (1)$$

where  $\mathcal{B}(u) = 0$  stands for the Dirichlet boundary conditions ( $u = 0$ ), the Neumann boundary conditions ( $\partial_\nu u = 0$ ) or the dynamical boundary conditions ( $\sigma \partial_t u + \partial_\nu u = 0$  with  $\sigma$  a non-negative smooth function). In the first section, we study the stationary equation

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \quad (2)$$

stemming from Problem 1. We aim to prove the existence of positive and sign-changing solutions using phase plane arguments and dealing with the first order system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ uv - u|u|^{p-1} + \lambda u \end{pmatrix}. \quad (3)$$

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We prove a bifurcation in the phase plane of this system, depending on the parameters  $\lambda$  and  $p$ , which influences the resolution of Equation 2 under the Dirichlet, Neumann and mixed boundary conditions. Then in a second section, using the comparison method from [2], we deduce from the solutions of the stationary Equation 2 some regular super-solutions for the Problem 1. Dealing with these super-solutions and with the blow-up results from [4], we investigate global existence and blow-up phenomena for the Problem 1 for different values of  $\lambda$  and  $p$ , and for the Dirichlet, the Neumann and the dynamical boundary conditions. We also examine both phenomena in unbounded domains: we obtain global existence results with the comparison method and using some well-known super-solutions (we mean explicit functions) for the Dirichlet, the Neumann and the dynamical boundary conditions. The blowing-up concerns the regular solutions of Problem 1 satisfying some growth order at infinity and some boundary conditions such that

- $\partial_\nu u = 0$  (Neumann b.c.),
- $\partial_\nu u = g(u)$  with  $g$  a polynomial of degree 2 (nonlinear b.c.).

We use some weighted  $L^1$ -norms: our technique is to prove the blowing-up of the solution by proving the blowing-up of appropriate  $L^1$ -norms.

Before starting, let us define the kind of solution we look for:

**Definition 1.1.** A function  $u$  is called a solution (or regular solution) of Equation 2 in  $\Omega$  if  $u$  is of class  $\mathcal{C}^2(\Omega)$  and satisfies the equation in the classical sense.

A function  $u$  is called a solution (or regular solution) of Problem 1 in  $\Omega$  if  $u$  is of class  $\mathcal{C}(\bar{\Omega} \times [0, T]) \cap \mathcal{C}^{2,1}(\bar{\Omega} \times (0, T))$ , where denotes  $T$  its maximal existence time, and satisfies the equations in the classical sense.

**2. Stationary equation.** In this section, we study the existence of positive and sign-changing solutions of Equation 2 using a phase plane method. For the theory of phase planes (nature of equilibrium, regularity, behaviour and uniqueness of trajectory), we refer to H.Amann's book [1]. Unless otherwise stated, we suppose  $p \in (1, \infty)$ . First, we can note that System 3 has three equilibrium points if  $\lambda > 0$ :  $(0, 0)$ ,  $(\lambda^{\frac{1}{p-1}}, 0)$  and  $(-\lambda^{\frac{1}{p-1}}, 0)$ . Using Hartman-Grobman's linearization theorem (see Reference [1]), we can state that  $(0, 0)$  is a saddle point,  $(\lambda^{\frac{1}{p-1}}, 0)$  is an unstable and repulsive vortex (if  $1 - 4(p-1)\lambda^{\frac{p-3}{p-1}} < 0$ ), a node (if  $1 - 4(p-1)\lambda^{\frac{p-3}{p-1}} \geq 0$ , which degenerates when  $1 - 4(p-1)\lambda^{\frac{p-3}{p-1}} = 0$ ). And  $(-\lambda^{\frac{1}{p-1}}, 0)$  is a stable and attractive vortex (for  $1 - 4(p-1)\lambda^{\frac{p-3}{p-1}} < 0$ ), a node (for  $1 - 4(p-1)\lambda^{\frac{p-3}{p-1}} \geq 0$  with degeneracy when equality occurs). If  $\lambda \leq 0$ , then  $(0, 0)$  is the only equilibrium point of System 3. We will prove later that  $(0, 0)$  is a center.

**2.1. Case  $\lambda > 0$ .** Let  $\lambda$  be a positive real number and  $p > 1$ . We want to study the phase plane of the System 3. First we prove a lemma on the symmetry of the trajectories:

**Lemma 2.1.** *The support of the trajectories of the System 3 are symmetric with respect to the ordinates axis.*

*Proof.* Let  $(u, v)$  denote a solution of the System 3 in  $(-a, a)$  for some  $a \in (0, \infty]$ , and define

$$\begin{cases} w(x) &= -u(-x) \\ z(x) &= v(-x) \end{cases} \text{ for all } x \in (-a, a).$$

A simple calculus of the derivatives implies

$$w'(x) = u'(-x) = v(-x) = z(x),$$

and

$$\begin{aligned} z'(x) &= -u(-x) \\ &= -v'(-x) \\ &= -\left[u(-x)v(-x) - u(-x)|u(-x)|^{p-1} + \lambda u(-x)\right] \\ &= w(x)z(x) - w(x)|w(x)|^{p-1} + \lambda w(x). \end{aligned}$$

Then  $(w, z)$  is also a trajectory of the System 3, and it is symmetric to  $(u, v)$  with respect to the ordinates axis.  $\square$

Thus, we can reduce our phase plane analysis to the half plane  $\mathbb{R}^+ \times \mathbb{R}$ . In order to draw the phase plane of the System 3, we write the ordinate  $v$  as a function depending on the abscissa  $u$ :  $v = f(u)$ . We do not know the function  $f$ , but we can deduce its variations and convexity using the equations 3. For the variations, we have

$$\frac{dv}{du} = \frac{uv - u|u|^{p-1} + \lambda u}{v} = \frac{u}{v} \left( v - |u|^{p-1} + \lambda \right), \quad (4)$$

in particular, it vanishes along the axis  $\{u = 0\}$  and along the curve  $\{v = |u|^{p-1} - \lambda\}$ . For  $u < \lambda^{\frac{1}{p-1}}$ , we have

$$\left. \frac{dv}{du} \right|_{v=0} = \infty$$

whereas for  $u > \lambda^{\frac{1}{p-1}}$

$$\left. \frac{dv}{du} \right|_{v=0} = -\infty.$$

Then we have  $\frac{dv}{du} > 0$  in the sets  $\{u > 0, v > 0, v > |u|^{p-1} - \lambda\}$  and  $\{u > 0, v < 0, v < |u|^{p-1} - \lambda\}$ . On the other hand,  $\frac{dv}{du} < 0$  in the sets  $\{u > 0, v < 0, v > |u|^{p-1} - \lambda\}$  and  $\{u > 0, v > 0, v < |u|^{p-1} - \lambda\}$ . Next, we compute the convexity of the function  $f$  and we obtain

$$\frac{d^2v}{du^2} = 1 + \frac{1}{v^2} \left[ (\lambda - pu^{p-1})v - u(\lambda - u^{p-1}) \frac{dv}{du} \right]. \quad (5)$$

We have  $\frac{d^2v}{du^2} < 0$  in  $\{u > 0, v > 0, v < |u|^{p-1} - \lambda\}$  and  $\frac{d^2v}{du^2} > 0$  in  $\{u > 0, v < 0, v < |u|^{p-1} - \lambda\}$ . Since

$$\left. \frac{d^2v}{du^2} \right|_{u=0} = 1 + \frac{\lambda}{v} \quad \text{and} \quad \left. \frac{d^2v}{du^2} \right|_{v=|u|^{p-1}-\lambda} = (1-p) \frac{|u|^{p-1}}{v},$$

the convexity is sign-changing in  $\{u > 0, v > |u|^{p-1} - \lambda\}$ . These arguments are sufficient to know the profile of the trajectories in the half plane  $\{v < |u|^{p-1} - \lambda\}$ . We do not need to know how the trajectories behave in  $\{u > 0, v < 0, v > |u|^{p-1} - \lambda\}$  to solve Equation 2. In  $\{u > 0, v > 0, v > |u|^{p-1} - \lambda\}$ , things are different: unbounded trajectories can appear (see §2.3). To ensure the occurrence of bounded trajectories, we need an additional hypothesis:

$$p \geq 3. \quad (6)$$

**Lemma 2.2.** *Under hypothesis 6, all the trajectories of the System 3 are bounded in  $A = \{u > 0, v > 0, v > |u|^{p-1} - \lambda\}$ .*

*Proof.* Let  $v_0 > 0$  and consider  $(u, v)$  the solution of the System 3 with initial data  $(u(0), v(0)) = (0, v_0)$ . The calculus of the variations (see Equation 4) ensures that  $(u(t), v(t)) \in A$  for small  $t > 0$ . We prove that there exist  $0 < \tau < \infty$  such that  $v(\tau) = |u(\tau)|^{p-1} - \lambda$ . It means that  $(u, v)$  is bounded in  $A$ . Since  $(u, v)$  belongs to  $A$ , we have

$$\frac{dv}{du} = u + \frac{\lambda u}{v} - \frac{u|u|^{p-1}}{v} \leq u + \frac{\lambda u}{v}.$$

Then  $\frac{dv}{du} \geq 0$  in  $A$  implies  $v > v_0$  as long as  $(u, v) \in A$ , and we obtain

$$\frac{dv}{du} \leq u \left(1 + \frac{\lambda}{v_0}\right).$$

Integration gives

$$v \leq \frac{1}{2} \left(1 + \frac{\lambda}{v_0}\right) u^2 + v_0.$$

If  $p > 3$ , the intersection  $\{v = |u|^{p-1} - \lambda\} \cap \{v = \frac{1}{2} \left(1 + \frac{\lambda}{v_0}\right) u^2 + v_0\}$  is non-empty for all  $v_0 > 0$ . If  $p = 3$ , we need to choose  $v_0$  sufficiently big such that

$$\frac{1}{2} \left(1 + \frac{\lambda}{v_0}\right) < 1.$$

Then, the trajectory  $(u, v)$  belongs to the compact

$$\{u \geq 0, v \geq |u|^{p-1} - \lambda, v \leq \frac{1}{2} \left(1 + \frac{\lambda}{v_0}\right) u^2 + v_0\},$$

and, using  $\frac{dv}{du} \geq 0$ , we know that there exist  $0 < \tau < \infty$  such that  $v(\tau) = |u(\tau)|^{p-1} - \lambda$ . This argument proves that each solution of the System 3 with initial data  $(u(0), v(0)) = (0, v_0)$  is bounded in  $A$  if  $v_0$  is big enough. Thanks to uniqueness of solution, it also proves the result for all the solutions initiated in  $A$ .  $\square$

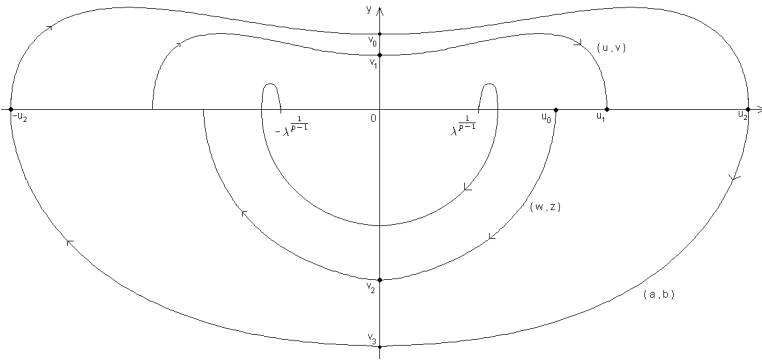


FIGURE 1. Phase plane for  $p \geq 3$  and  $\lambda > 0$ .

Then, we complete this phase plane analysis by proving the existence of periodic trajectories.

**Lemma 2.3.** *Assume that hypothesis 6 is fulfilled. Then, there exists periodic trajectories of the System 3.*

*Proof.* Thanks to the symmetry (see Lemma 2.1), we just need to prove that for some initial data belonging to  $\{0\} \times (0, \infty)$ , there exists a trajectory which attains a point belonging to  $\{0\} \times (-\infty, 0)$ . First, consider a trajectory  $(u, v)$  initiated at  $(0, v_1)$  with  $v_1 > 0$ . According to hypothesis 6, we know that  $(u, v)$  is bounded, and using its variations and its convexity (Equations 4 and 5), we can deduce that  $(u, v)$  attains the  $x$ -axis at a point  $(u_1, 0)$  with  $u_1 > \lambda^{\frac{1}{p-1}}$  (see Figure 1). Then, using the reverse system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -v \\ -uv + u|u|^{p-1} - \lambda u \end{pmatrix},$$

and one of its trajectories initiated at  $(0, v_2)$  with  $v_2 < -\lambda$  (trajectories of reverse system and of System 3 have same support), one can note that for  $u_0 > \lambda^{\frac{1}{p-1}}$ , there exist a trajectory  $(w, z)$  with  $w(0) = u_0$  and  $z(0) = 0$  (see Figure 1). Finally, let us consider the trajectory  $(a, b)$  of System 3 containing the point  $(u_2, 0)$ . Thanks to the uniqueness of the solutions, and using the information on the variations and the convexity, we deduce that there exist two real numbers  $s < t$  such that  $a(s) = a(t) = 0$ ,  $b(s) = v_0$  and  $b(t) = v_3$  (see Figure 1). Thus, the trajectory  $(a, b)$  is the periodic trajectory we look for.  $\square$

Now, analysing the phase plane of the System 3, we deduce the following results concerning the equation 1.

**Theorem 2.4.** *Assume hypothesis 6 and  $\lambda > 0$ . For some  $\alpha > 0$  and for each boundary conditions*

- $u(-\alpha) = u(\alpha) = 0$  (Dirichlet b.c.),
- $u'(-\alpha) = u'(\alpha) = 0$  (Neumann b.c.),
- $u(-\alpha) = u'(\alpha) = 0$  (mixed-1 b.c.),
- $u'(-\alpha) = u(\alpha) = 0$  (mixed-2 b.c.),

*there exists a unique positive solution of the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha).$$

*Proof.* We use the phase plane of System 3, see Figure 1. Consider the trajectory  $(a, b)$  between the points

- $(0, v_0)$  and  $(0, v_3)$ : we obtain the Dirichlet solution,
- $(0, v_0)$  and  $(u_2, 0)$ : we obtain the mixed-1 solution,
- $(u_2, 0)$  and  $(0, v_3)$ : we obtain the mixed-2 solution.

For the Neumann solution, consider  $0 < \mu_0 < \lambda^{\frac{1}{p-1}}$  and the trajectory  $(\mu, \nu)$  of System 3 initiated at  $(\mu_0, 0)$ . Since  $(\mu, \nu)$  can not cross the trajectory  $(u, v)$  (see Figure 1), it must cross the  $x$ -axis at  $(\mu_1, 0)$  with  $\lambda^{\frac{1}{p-1}} < \mu_1 < u_1$ . Thus, the abscissa of this trajectory is the Neumann solution we look for. Uniqueness of solution comes from standard ODE's theorems applied to the System 3. Finally, the length  $(2\alpha)$  of the existence interval is governed by the time needed by the trajectory to go from its initial data to its "final data".  $\square$

**Theorem 2.5.** *Assume hypothesis 6 and  $\lambda > 0$ . For some  $\alpha > 0$ , there exists a periodic sign-changing solution of the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } \mathbb{R}.$$

*Proof.* We just need to choose one the periodic trajectories of the System 3 built in Lemma 2.3.  $\square$

**Remark 1.** Using the periodic solutions in the previous theorem, we can build four sign-changing solutions satisfying the four boundary conditions: Dirichlet, Neumann, mixed-1 and mixed-2 (see Theorem 2.4).

Now, suppose that hypothesis 6 is not achieved. Then, we do not know if the solutions are bounded in  $\{v > |u|^{p-1} - \lambda\}$ : we will see in §2.3 that unbounded solutions appear. But in  $\{v < |u|^{p-1} - \lambda\}$ , the behaviour of the trajectories do not change.

**Theorem 2.6.** *Let  $\lambda > 0$ . For some  $\alpha > 0$ , there exists a unique positive solution of the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha)$$

*with the mixed boundary conditions  $u'(-\alpha) = u(\alpha) = 0$ . In addition, if*

$$1 - 4(p-1)\lambda^{\frac{p-3}{p-1}} < 0, \quad (7)$$

*then there exists a unique positive solution of the equation 1 under the Neumann boundary conditions.*

*Proof.* The first part of the statement comes from Theorem 2.4, the solution with mixed-2 boundary conditions is located in  $\{v < |u|^{p-1} - \lambda\}$ . The other part stems from Equation 7: in this case, the equilibrium  $(\lambda^{\frac{1}{p-1}}, 0)$  is an unstable vortex. If we consider  $u_0 > 0$  such that  $|\lambda^{\frac{1}{p-1}} - u_0|$  is sufficiently small, the trajectory  $(u, v)$  of the System 3, with  $u(0) = u_0$  and  $v(0) = 0$ , whirls around  $(\lambda^{\frac{1}{p-1}}, 0)$ . Thus, there exists  $\tau > 0$  such that  $v(\tau) = 0$  and  $u(t) > 0$  for all  $t \in [0, \tau]$ .  $\square$

Without hypothesis 6, we can not construct positive solutions anymore for the Dirichlet, Neumann or mixed-1 boundary conditions. If we do not impose the positivity, we obtain this result:

**Theorem 2.7.** *Let  $\lambda > 0$ . For some  $\alpha > 0$  and for each boundary conditions*

- $u'(-\alpha) = u'(\alpha) = 0$  (Neumann b.c.) ,
- $u(-\alpha) = u(\alpha) = 0$  (mixed-1 b.c.),

*there exists a solution of the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha).$$

*Proof.* As we mentioned before, we consider the part  $\{v < |u|^{p-1} - \lambda\}$  of the phase plane of the System 3 (see Figure 1). For the Neumann solution, we consider the trajectory  $(a, b)$  between  $(u_2, 0)$  and  $(-u_2, 0)$ . For the mixed-1 solution, we can also consider the trajectory  $(a, b)$ , but only between  $(0, v_3)$  and  $(-u_2, 0)$ .  $\square$

**Remark 2.** The Neumann solution built above is sign changing, whereas the mixed-1 solution is negative.

**Remark 3.** In the general case, we can not build any solution with the Dirichlet boundary conditions using our phase plane method. Indeed, we will give a criterion in Theorem 2.18 concerning nonexistence of the Dirichlet solution.

Concerning the solutions in infinite interval, we can state:

**Theorem 2.8.** *Let  $\lambda > 0$ . Then the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0$$

*admits*

- a positive solution  $u$  in  $(-\infty, 0]$  satisfying  $u'(-\infty) = u'(0) = 0$  (Neumann).
- a positive solution  $v$  in  $(-\infty, 0]$  satisfying  $v'(-\infty) = v(0) = 0$  (mixed-2).
- a sign-changing solution  $w$  in  $\mathbb{R}$  satisfying  $w'(-\infty) = w'(\infty) = 0$  (Neumann).
- a negative solution  $u$  in  $[0, \infty)$  satisfying  $z(0) = z'(\infty) = 0$  (mixed-1).

*Proof.* Consider  $\mu_0 > 0$  with  $\mu_0 > \lambda^{\frac{1}{p-1}}$  and with  $|\lambda^{\frac{1}{p-1}} - u_0|$  small enough such that there exists a trajectory  $(\mu, \nu)$  of the System 3 satisfying

$$\mu(-\infty) = \lambda^{\frac{1}{p-1}}, \quad \nu(-\infty) = 0 \quad \text{and} \quad \mu(0) = \mu_0, \quad \nu(0) = 0$$

Hence,  $u = \mu$  in  $(-\infty, 0]$  is suitable for the first statement. Then, the trajectory  $(\mu, \nu)$  can be continued in the part  $\{u > 0, v < 0\}$  using the information on its behaviour (see Equations 4 and 5) until  $(\mu, \nu)$  attains the ordinate axis. Denote  $t_1 > 0$  the time such that  $\mu(t_1) = 0$  and  $\nu(t_1) < 0$ . We obtain the second statement setting  $v(t) = \mu(t + t_1)$  for all  $t \in (-\infty, 0]$ . Finally, these results and the symmetry of the trajectories (see Lemma 2.1) imply the third and the fourth statements with the following definitions:

$$w(t) = \begin{cases} v(t) & \forall t \leq 0 \\ -v(-t) & \forall t > 0 \end{cases} \quad \text{and} \quad z(t) = -v(-t) \quad \text{for all } t \geq 0.$$

□

**2.2. Case  $\lambda \leq 0$ .** First note that the System 3 has only one equilibrium point  $(0, 0)$ . As in the previous case, we can reduce our phase plane analysis to the half-plane  $\mathbb{R}^+ \times \mathbb{R}$  since of Lemma 2.1. Again, we obtain some information on the variations of the trajectories of the System 3 using Equation 4. We have  $\frac{dv}{du} = 0$  along the curves  $\{u = 0\}$  and  $\{v = |u|^{p-1} - \lambda\}$ . For  $u > 0$

$$\left. \frac{dv}{du} \right|_{v=0} = -\infty$$

whereas for  $u < 0$

$$\left. \frac{dv}{du} \right|_{v=0} = +\infty.$$

Then, we have  $\frac{dv}{du} \geq 0$  in  $\{u > 0, v < 0\} \cup \{v \geq |u|^{p-1} - \lambda\}$  and  $\frac{dv}{du} \leq 0$  in  $\{u > 0, v > 0, v \leq |u|^{p-1} - \lambda\}$ . In addition, thanks to Equation 5, we know that  $\frac{d^2v}{du^2} \leq 0$  in  $\{u > 0, v > 0, v \leq |u|^{p-1} - \lambda\}$ ,  $\frac{dv}{du} \geq 0$  in  $\{u > 0, v < 0\}$  while it is sign-changing in  $\{u > 0, v \geq |u|^{p-1} - \lambda\}$ . In this last part of the plane, we use the following lemma, similar to Lemma 2.2:

**Lemma 2.9.** *Let  $\lambda \leq 0$  and  $(u, v)$  be a trajectory of the System 3 with initial data  $(0, v_0)$ . If  $v_0 > -\lambda$  satisfies*

$$\begin{cases} v_0 > -\lambda & \text{if } p \geq 3, \\ v_0 \leq -\lambda + (p-1)^{\frac{p-1}{3-p}} - \frac{1}{2}(p-1)^{\frac{2}{3-p}} & \text{if } p < 3, \end{cases} \quad (8)$$

*then the trajectory  $(u, v)$  is bounded in  $A = \{u > 0, v \geq |u|^{p-1} - \lambda\}$ .*

*Proof.* The calculus of the variations (see Equation 4) ensures that  $(u(t), v(t)) \in A$  for small  $t > 0$ . We prove that there exist  $0 < \tau < \infty$  such that  $v(\tau) = |u(\tau)|^{p-1} - \lambda$ . It means that  $(u, v)$  is bounded in  $A$ . Since  $(u, v)$  belongs to  $A$  and thanks to  $\lambda \leq 0$ , we have

$$0 \leq \frac{dv}{du} = u + \frac{\lambda u}{v} - \frac{u|u|^{p-1}}{v} \leq u.$$



Then, integration between 0 and  $u$  gives

$$v \leq \frac{1}{2}u^2 + v_0.$$

Hypothesis 8 implies that  $\{u > 0, v = |u|^{p-1} - \lambda\} \cap \{u > 0, v = \frac{1}{2}u^2 + v_0\}$  is not empty. Thus, the trajectory  $(u, v)$  belongs to the compact

$$\{u \geq 0, v \geq |u|^{p-1} - \lambda, v \leq \frac{1}{2}u^2 + v_0\}.$$

Using  $\frac{dv}{du} \geq 0$ , we know that there exist  $\tau > 0$  such that  $v(\tau) = |u(\tau)|^{p-1} - \lambda$ .  $\square$

Now, the phase plane of the System 3 can be drawn, see Figure 2.

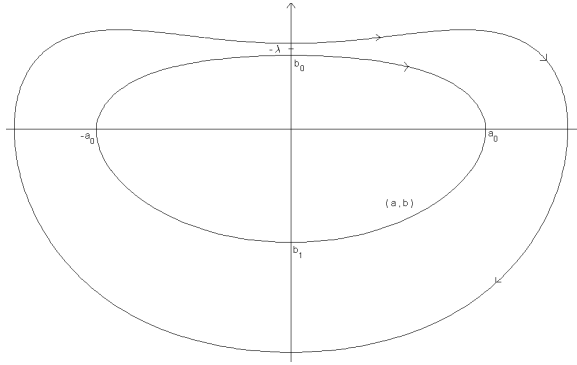


FIGURE 2. Phase plane for  $\lambda \leq 0$ .

**Corollary 1.** *The equilibrium point  $(0, 0)$  is a center for the System 3.*

Now, we use these information on the trajectories of the System 3 to obtain some results concerning the solutions of Equation 1.

**Theorem 2.10.** *Let  $\lambda \leq 0$ . For some  $\alpha > 0$  and for each boundary conditions*

- $u(-\alpha) = u(\alpha) = 0$  (*Dirichlet b.c.*),
- $u(-\alpha) = u'(\alpha) = 0$  (*mixed-1 b.c.*),
- $u'(-\alpha) = u(\alpha) = 0$  (*mixed-2 b.c.*),

*there exists a unique positive solution of the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha).$$

*Proof.* We use the phase plane of System 3, see Figure 2. Consider the trajectory  $(a, b)$  between the points

- $(0, b_0)$  and  $(0, b_1)$ : we obtain the Dirichlet solution,
- $(0, b_0)$  and  $(a_0, 0)$ : we obtain the mixed-1 solution,
- $(a_0, 0)$  and  $(0, b_1)$ : we obtain the mixed-2 solution.

$\square$

**Theorem 2.11.** *Let  $\lambda \leq 0$ . For all  $\alpha > 0$ , the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha)$$

*admits no positive solution under the Neumann boundary conditions.*

*Proof.* Ab absurdo, suppose that there exists  $u$  a positive solution of 1 under the Neumann boundary conditions, and denote  $v = u'$ . Then the curve  $(u, v)$  is a trajectory of the System 3 located in  $\mathbb{R}^+ \times \mathbb{R}$  with initial data on the axis  $\{v = 0\}$ . Then Equations 4 and 5 prove that  $(u, v)$  can not cross the axis  $\{v = 0\}$  once again without going into  $\mathbb{R}^- \times \mathbb{R}$ . A contradiction with the positivity of  $u$ .  $\square$

**Theorem 2.12.** *Let  $\lambda \leq 0$ . For some  $\alpha > 0$ , the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha)$$

*admits a sign changing solution under the Neumann boundary conditions.*

*Proof.* Using the phase plane of System 3 (see Figure 2), consider the trajectory  $(a, b)$  between the points  $(a_0, 0)$  and  $(-a_0, 0)$ .  $\square$

To conclude this section, let us give this result concerning the periodic solutions:

**Theorem 2.13.** *Let  $\lambda \leq 0$ . For some  $\alpha > 0$ , there exists a sign-changing periodic solution of the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } \mathbb{R}.$$

*Proof.* As in Lemma 2.3, we can build periodic trajectories of 3 using the symmetry (Lemma 2.1).  $\square$

**2.3. Unbounded solutions.** In the above paragraphs, we proved that all the trajectories of the System 3 are bounded for  $p \geq 3$ , but if  $1 < p < 3$  we do not have a general answer: for example, we obtain some bounded trajectories when  $\lambda \leq 0$  (see Lemma 2.9), but with our method, we do not have (yet) any result when  $\lambda > 0$ . In this paragraph, we show that there exists unbounded trajectories for every  $\lambda \in \mathbb{R}$  and for all  $p \in (1, 3)$ . We start with a trajectory  $(u, v)$  with an initial data  $(0, v_0)$ .

**Lemma 2.14.** *Let  $p \in (1, 3)$  and  $\lambda \in \mathbb{R}$ . Suppose that*

$$v_0 > 2 \max\{-\lambda, 0\} + 2 \cdot (8)^{\frac{p-1}{3-p}}. \tag{9}$$

*Then the trajectory  $(u, v)$  is not bounded.*

*Proof.* We will show that under hypothesis 9, the trajectory  $(u, v)$  always lies above the curve  $\left\{v = 2u^{p-1} + 2 \max\{-\lambda, 0\}\right\}$ . Thus, using  $\frac{dv}{du} \geq 0$  (Equation 4), we obtain that  $(u, v)$  is not bounded. Ab absurdo, suppose that there exists  $x_* > 0$  such that  $u(x_*) = u_1 > 0$  and  $v(x_*) = v_1 > 0$  satisfy

$$v_1 = 2u_1^{p-1} + 2 \max\{-\lambda, 0\}, \tag{10}$$

and

$$v(x) > 2u(x)^{p-1} + 2 \max\{-\lambda, 0\} \forall x \in [0, x_*].$$

Thus in  $[0, x_*)$ , we have

$$\frac{\lambda - u^{p-1}}{v} > -\frac{1}{2}. \tag{11}$$

On the other hand, Equation 4 gives

$$\frac{dv}{du} = u + u \frac{\lambda - u^{p-1}}{v},$$

and thanks to condition 11, we obtain

$$\frac{dv}{du} \geq \frac{1}{2}u \geq 0. \tag{12}$$

Then  $v(u) \geq \frac{u^2}{4} + v_0$ . Hence, for  $u = u_1$ , we have:

$$v_1 = v(u_1) \geq \frac{u_1^2}{4} + v_0,$$

and by definition 10 of  $u_1$ , we have

$$2u_1^{p-1} + 2 \max\{-\lambda, 0\} \geq \frac{u_1^2}{4} + v_0.$$

Hypothesis 9 implies

$$-2 \cdot (8)^{\frac{p-1}{3-p}} > \frac{u_1^2}{4} - 2u_1^{p-1}. \quad (13)$$

Meanwhile, if we study both cases  $u_1 < 8^{\frac{1}{3-p}}$  and  $u_1 > 8^{\frac{1}{3-p}}$ , we remark that

$$\frac{u_1^2}{4} - 2u_1^{p-1} = \frac{u_1^{p-1}}{4} (u_1^{3-p} - 8) \geq -2 \cdot (8)^{\frac{p-1}{3-p}}. \quad (14)$$

Equations 13 and 14 are not compatible. Thus, the trajectory  $(u, v)$  can not attain the curve  $\left\{v = 2u^{p-1} + 2 \max\{-\lambda, 0\}\right\}$ .  $\square$

Concerning Equation 1, we obtain the following results:

**Theorem 2.15.** *Let  $p \in (2, 3)$  and  $\lambda \in \mathbb{R}$ . For some  $\alpha > 0$ , there exists a positive and unbounded solution of the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (-\alpha, \alpha).$$

satisfying

$$u(-\alpha) = 0 \text{ and } \lim_{x \rightarrow \alpha} u(x) = \infty.$$

*Proof.* The existence comes from the previous lemma. We just need to prove that the length of the existence interval is finite. Ab absurdo, suppose that there exists a positive and unbounded solution  $u$  of the Equation 1 in  $[0, \infty)$ . Let  $b > 0$  such that  $u > 2|\lambda|^{\frac{1}{p-1}}$  in  $[b, \infty)$ , and define  $w(x, t) = u(x+t)$  for all  $x \in [b, b+1]$  and for all  $t \in [0, \infty)$ . Thanks to the choice of  $b$ , we have

$$\partial_x^2 u - u \partial_x u + u^p - \lambda u \geq \partial_x^2 u - u \partial_x u + \frac{u^p}{2}$$

in  $[b, b+1] \times [0, \infty)$ . Because the solution  $u$  corresponds to a trajectory of the System 3 located in  $\mathbb{R} \times \mathbb{R}^+$ , we have  $\partial_t w > 0$ . Thus,  $w$  is super-solution of the following problem

$$\begin{cases} \partial_t v = \partial_x^2 u - u \partial_x u + \frac{1}{2} u^p - \lambda u & \text{in } [b, b+1] \times (0, \infty), \\ \partial_t v + \partial_v v = 0 & \text{on } \{\pm b\} \times (0, \infty), \\ v(\cdot, 0) = 2|\lambda|^{\frac{1}{p-1}} & \text{in } [b, b+1]. \end{cases}$$

By the comparison principle from [2],  $w \geq v$  where  $v$  is the solution of the previous problem. But, according to [4], the solution  $v$  blows up in finite time. A contradiction between  $w \geq v$  and the global existence of  $w$ . Thus,  $w$  can not exist on  $[b, b+1] \times (0, \infty)$ , and the solution  $u$  exists only in a finite interval.  $\square$

For  $1 < p \leq 2$ , we do not have the blowing-up argument and we are not sure that the existence interval of the solution is finite.

**Theorem 2.16.** *Let  $p \in (1, 2]$  and  $\lambda \in \mathbb{R}$ . For some  $\alpha \in (0, \infty]$ , there exists a positive and unbounded solution of the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (0, \alpha).$$

satisfying

$$u(-a) = 0 \text{ and } \lim_{x \rightarrow a} u(x) = \infty.$$

With some assumption on the parameter  $\lambda$ , we can also build a trajectory  $(u, v)$  with an initial data  $(u_0, 0)$  belonging to the abscissa axis.

**Lemma 2.17.** *Let  $p \in (1, 3)$  and  $\lambda \in \mathbb{R}$ . Suppose that there exists  $\beta > 1$  such that*

$$\lambda > \max \left\{ \frac{\beta - 1}{2\beta} \left( \frac{2\beta^2}{\beta - 1} \right)^{\frac{1}{3-p}}, \beta \left( \frac{2\beta^2}{\beta - 1} \right)^{\frac{p-1}{3-p}} \right\} \quad (15)$$

If

$$u_0 = \left( \frac{2\beta^2}{\beta - 1} \right)^{\frac{1}{3-p}}, \quad (16)$$

then the trajectory  $(u, v)$  is not bounded.

*Proof.* We use the same method as in Lemma 2.14: we prove that, under hypotheses 15 and 16, the trajectory  $(u, v)$  always lies above the curve  $\{v = \beta u^{p-1} - \lambda\}$ . Ab absurdo, suppose that there exist  $x_* > 0$  such that  $u(x_*) = u_1$  and  $v_1 = v(x_*)$  verify

$$v_1 = \beta u_1^{p-1} - \lambda, \quad (17)$$

and

$$v(x_*) > \beta u(x_*)^{p-1} - \lambda \quad \forall 0 < x < x_*.$$

Thus, in  $[0, x_*)$ , we have

$$\frac{\lambda - u^{p-1}}{v} \geq \frac{-1}{\beta}. \quad (18)$$

Equation 4 gives

$$\frac{dv}{du} = u + u \frac{\lambda - u^{p-1}}{v},$$

and condition 18 implies

$$\frac{dv}{du} \geq \frac{\beta - 1}{\beta} u \geq 0.$$

Integration between  $u_0$  and  $u_1$  leads to

$$v(u_1) \geq \frac{\beta - 1}{2\beta} (u_1 - u_0),$$

definition 17 gives

$$\beta u_1^{p-1} - \lambda \geq \frac{\beta - 1}{2\beta} (u_1 - u_0),$$

and we obtain

$$u_1^{p-1} \left( 1 - \frac{\beta - 1}{2\beta^2} u_1^{3-p} \right) \geq \frac{1}{\beta} \left( \lambda - u_0 \frac{\beta - 1}{2\beta} \right). \quad (19)$$

Since of  $u_0 < u_1$ , Equations 15 and 16 imply

$$\lambda - u_0 \frac{\beta - 1}{2\beta} > 0 \text{ and } 1 - \frac{\beta - 1}{2\beta^2} u_1^{3-p} < 0.$$

Hence, Equation 19 is a contradiction.  $\square$

Concerning Equation 1, and reasoning as in Theorem 2.15, we obtain the following result.

**Theorem 2.18.** *Let  $p \in (1, 3)$  and  $\lambda \in \mathbb{R}$  verifying Equation 15. For some  $\alpha \in (0, \infty]$ , there exists a positive and unbounded solution of the Equation 1*

$$u'' - uu' + u|u|^{p-1} - \lambda u = 0 \text{ in } (0, \alpha).$$

satisfying

$$u'(0) = 0 \text{ and } \lim_{x \rightarrow \alpha} u(x) = \infty.$$

In addition, if  $p \in (2, 3)$ , then  $\alpha$  is finite.

**2.4. Limiting case  $p = 1$ .** In this paragraph, we study the case where the exponent  $p$  attains the limit 1. Then, Equation 1 becomes

$$u'' - uu' + (1 - \lambda)u = 0 \text{ in } \mathbb{R},$$

and the System 3 is written

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ u(v + \lambda - 1) \end{pmatrix}. \quad (20)$$

For  $\lambda \neq 1$ ,  $(0, 0)$  is the only equilibrium point, while for  $\lambda = 1$  the axis  $\{v = 0\}$  is a continuum of equilibria. We begin with the case  $\lambda = 1$ . Here, we have  $\frac{dv}{du} = u$ , then

$$v(u) = \frac{1}{2}u^2 + c,$$

where  $c$  depends on the initial data. Thus, the phase plane is easily drawn, see Figure 3. Now, suppose  $\lambda \neq 1$ . One can compute the explicit trajectory

$$\begin{cases} u_e(x) = (1 - \lambda)x \\ v_e(x) = (1 - \lambda) \end{cases} \quad \forall x \in \mathbb{R}$$

Then, using the following equations

$$\frac{dv}{du} = u + \frac{u}{v}(\lambda - 1) \quad \text{and} \quad \frac{d^2v}{du^2} = 1 + \frac{\lambda - 1}{v^2} \left( v - u \frac{dv}{du} \right)$$

we can draw the phase plane of the System 20, see Figure 3.

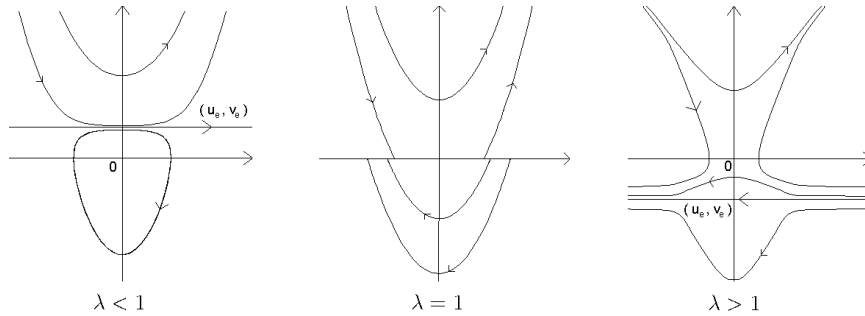


FIGURE 3. Phase planes for  $p = 1$ .

**2.5. Bifurcation.** According to the previous paragraphs, we can state that there exists a bifurcation of the phase plane of the System 3. First, we note that, for a fixed exponent  $p$ , the value of  $\lambda$  influences the phase plane of the System 3: for  $\lambda > 0$ , the System 3 admits three equilibrium points (a saddle point, an attractive equilibrium and a repulsive equilibrium). The distance between these equilibria goes to 0 when  $\lambda \rightarrow 0$ , and for  $\lambda = 0$ , they collapse and generate a unique center, which persists for all negative  $\lambda$  (see Figure 4).

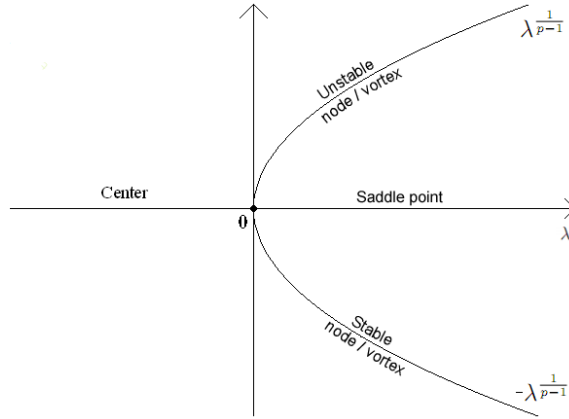


FIGURE 4. Bifurcation of the phase plane of the System 3 with different parameters.

Now, for a fixed  $\lambda$ , the value of the exponent  $p$  has an important role. With  $\lambda$ , the value of  $p$  governs the type of the equilibrium points (node, improper node, vortex). The exponent  $p$  also establishes if all the trajectories of the System 3 are bounded ( $p \geq 3$ ) or if there exists unbounded trajectories ( $1 \leq p < 3$ ). Moreover, when  $p$  attains the limit 1, the critical value of  $\lambda$  changes from 0 (if  $p > 1$ ) to 1 (for  $p = 1$ ). The case  $\lambda = 1$  is special because when  $p \rightarrow 1$ , the three equilibria of the System 3 (a saddle point, an attractive vortex and a repulsive vortex) generate a continuum of equilibria when  $p$  attains the limit 1 (see Figure 5).

**3. Parabolic problem.** In this section, we study the parabolic Problem 1 for many boundary conditions. First, we use the results concerning the stationary Equation 1 when the domain  $\Omega$  is bounded. Then, we consider the case of unbounded domains: we investigate global existence using the comparison method, and blow-up phenomenon thanks to a  $L^1$ -norm technique.

**3.1. Comparison.** We begin with the Dirichlet problem

$$\begin{cases} \partial_t u = \partial_x^2 u - u \partial_x u + u^p - \lambda u & \text{in } [-\alpha, \alpha] \times (0, \infty), \\ u = 0 & \text{on } \{\pm\alpha\} \times (0, \infty), \\ u(\cdot, 0) = \varphi & \text{in } [-\alpha, \alpha], \end{cases} \quad (21)$$

where  $\alpha > 0$ ,  $p > 1$ ,  $\lambda \in \mathbb{R}$  and  $\varphi \in \mathcal{C}_0([-\alpha, \alpha])$  is non-negative. Thanks to the comparison principle [2] and with the results of the previous sections, we have:

**Theorem 3.1.** *Let  $p > 1$  and  $\lambda \in \mathbb{R}$ . For some  $\alpha > 0$ , there exists a global positive solution*

$$u \in \mathcal{C}([-\alpha, \alpha] \times [0, \infty)) \cap \mathcal{C}^{2,1}([-\alpha, \alpha] \times (0, \infty))$$

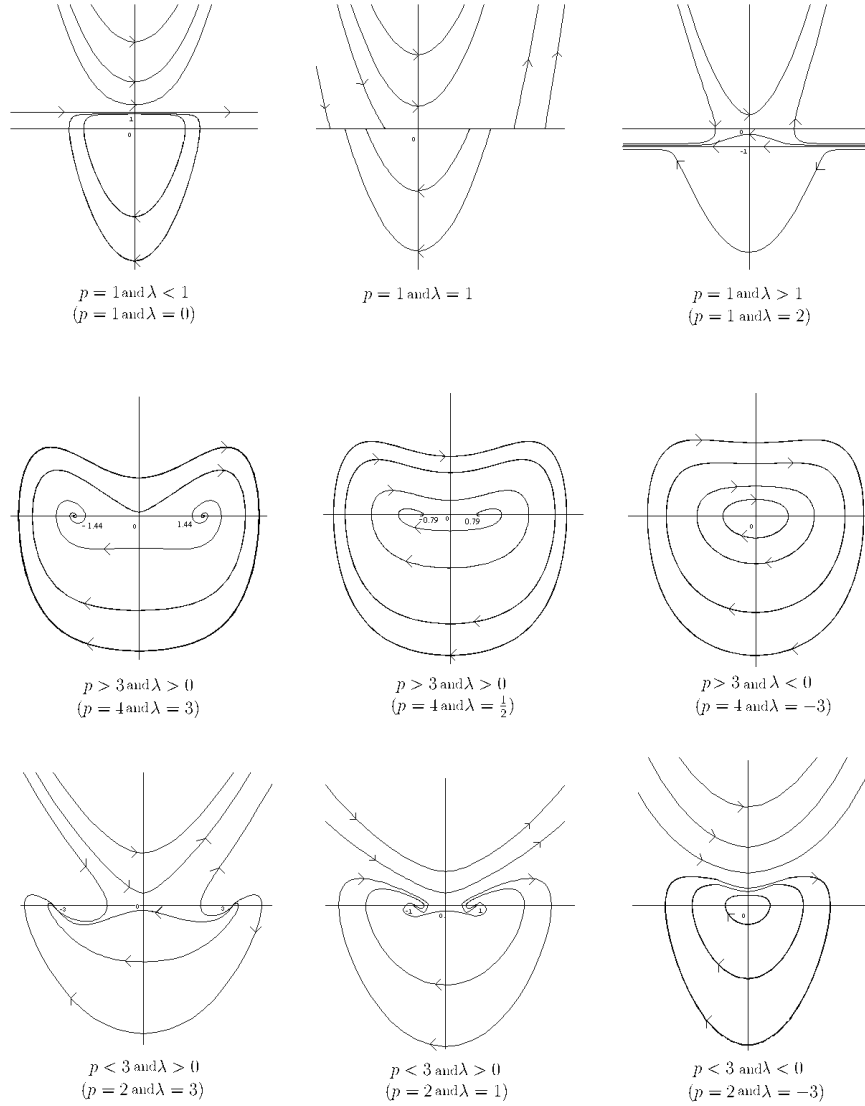


FIGURE 5. Phase planes of the System 3 with different parameters.

of Problem 21 if the initial data  $\varphi \in \mathcal{C}_0([-\alpha, \alpha])$  is sufficiently small.

*Proof.* If  $p \geq 3$  and  $\lambda > 0$ , consider  $\beta \in \mathcal{C}^2([-\alpha, \alpha])$  a solution of 1 with the Dirichlet boundary conditions (see Theorem 2.4). Suppose that  $\varphi$  is small enough:  $\varphi \leq \beta$  in  $[-\alpha, \alpha]$ . Then, we obtain

$$\begin{cases} \partial_t \beta = 0 = \partial_x^2 \beta - \beta \partial_x \beta + \beta^p - \lambda u & \text{in } [-\alpha, \alpha] \times (0, \infty), \\ \beta = 0 & \text{on } \{\pm \alpha\} \times (0, \infty), \\ \beta(\cdot, 0) \geq \varphi & \text{in } [-\alpha, \alpha]. \end{cases}$$

Using the comparison principle from [2], we prove that there exists a solution  $u$  of 21 satisfying  $0 \leq u \leq \beta$  for all  $(x, t) \in [-\alpha, \alpha] \times (0, \infty)$ . Thus,  $u$  is a global positive solution. If  $1 < p < 3$  and  $\lambda > 0$ , then we just need to choose a positive solution  $\beta$  given in Theorem 2.6 (even if  $\beta(\pm\alpha) > 0$ ). For  $\lambda \leq 0$ , we consider the Dirichlet solution given in Theorem 2.10.  $\square$

Now, we replace the Dirichlet boundary conditions by the dynamical boundary conditions. Consider the following problem

$$\begin{cases} \partial_t u = \partial_x^2 u - u \partial_x u + u^p - \lambda u & \text{in } [-\alpha, \alpha] \times (0, \infty), \\ \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \{\pm\alpha\} \times (0, \infty), \\ u(\cdot, 0) = \varphi & \text{in } [-\alpha, \alpha], \end{cases} \quad (22)$$

with  $\alpha > 0$ ,  $p > 1$ ,  $\lambda \in \mathbb{R}$  and where  $\varphi \in \mathcal{C}([-\alpha, \alpha])$  and  $\sigma \in \mathcal{C}^1(\{\pm\alpha\}[0, \infty))$  are non-negative. We obtain two results, depending on the sign of  $\lambda$ .

**Theorem 3.2.** *Let  $p > 1$  and  $\lambda > 0$ . There exists a global positive solution*

$$u \in \mathcal{C}([-\alpha, \alpha] \times [0, \infty)) \cap \mathcal{C}^{2,1}([-\alpha, \alpha] \times (0, \infty))$$

of Problem 22

- for all  $\alpha > 0$  if  $\varphi \leq \lambda^{\frac{1}{p-1}}$ .
- for some  $\alpha > 0$  if  $\varphi - \lambda^{\frac{1}{p-1}}$  is sign changing and  $\max\{\varphi - \lambda^{\frac{1}{p-1}}, 0\}$  is sufficiently close to 0.
- for no  $\alpha > 0$  if  $\varphi > \lambda^{\frac{1}{p-1}}$  and  $p > 2$ .

*Proof.* For the first statement, we just need to note that the constant function  $\lambda^{\frac{1}{p-1}}$  is a super-solution of 22 when  $0 \leq \varphi \leq \lambda^{\frac{1}{p-1}}$ . For the second statement, we consider two cases: when  $p \geq 3$ , we consider a positive solution  $w$  of 1 under the Neumann boundary conditions, see Theorem 2.4. Choosing  $0 \leq \varphi \leq w$ , we obtain a super-solution of 22. If  $1 < p < 3$ , we consider a trajectory  $(\mu, \gamma)$  of 3 with  $0 < \mu(0) < \lambda^{\frac{1}{p-1}}$  and  $\gamma(0) = 0$ . According to Equation 4, for a small  $x_* > 0$ , we have  $\gamma(-x) < 0$  and  $\gamma(x) > 0$  for all  $x \in (0, x_*)$ . Thus,  $\mu$  satisfies  $\partial_\nu \mu(-x_*) = -\gamma(-x_*) > 0$  and  $\partial_\nu \mu(x_*) = \gamma(x_*) > 0$ , and it is a super-solution of 22 when  $0 \leq \varphi \leq \mu$  in  $[-x_*, x_*]$ . Then, using these super-solutions and the comparison principle from [2], we prove first and second assertions. For the third statement, consider  $c > 0$  such that

$$\varphi > c > \lambda^{\frac{1}{p-1}}.$$

The comparison principle from [2] implies that  $u > c$ , where  $u$  denote the solution of 22 with the initial data  $\varphi$ . Hence, there exists  $d > 0$  such that

$$u^p - \lambda u \geq du^p \text{ for all } x \in [-\alpha, \alpha] \text{ and for all } t > 0.$$

Thus,  $u$  verifies

$$\begin{cases} \partial_t u \geq \partial_x^2 u - u \partial_x u + du^p & \text{in } [-\alpha, \alpha] \text{ for } t > 0, \\ \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \{-\alpha, \alpha\} \text{ for } t > 0, \\ u(\cdot, 0) > c > 0 & \text{in } [-\alpha, \alpha]. \end{cases}$$

Then, blow-up results from [4] imply the blowing-up in finite time of  $u$ .  $\square$

**Theorem 3.3.** *Let  $p > 2$  and  $\lambda \leq 0$ . For all  $\alpha > 0$ , the positive solution  $u$  of Problem 22 blows up in finite time if the initial data  $\varphi \in \mathcal{C}([-\alpha, \alpha])$  satisfies*

$$\forall x \in [-\alpha, \alpha], \varphi(x) > 0.$$



*Proof.* Since of  $\lambda \leq 0$ , the function  $u$  verifies

$$\begin{cases} \partial_t u \geq \partial_x^2 u - u \partial_x u + u^p & \text{in } [-\alpha, \alpha] \text{ for } t > 0, \\ \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \{-\alpha, \alpha\} \text{ for } t > 0, \\ u(\cdot, 0) > 0 & \text{in } [-\alpha, \alpha]. \end{cases}$$

Thanks to the blow-up results from [4], we know that  $u$  blows up in finite time.  $\square$

**Remark 4.** The Neumann boundary conditions are included here, with the special case  $\sigma \equiv 0$ .

**3.2. Global existence in unbounded domains.** We study the Problem 1 under the Dirichlet, the Neumann and the dynamical boundary conditions when  $\Omega$  is an unbounded domain. Using some explicit super-solutions, we look for global existence in the three types of unbounded domains:  $(-\infty, 0)$ ,  $(0, \infty)$  and  $\mathbb{R}$ . We begin with the case  $\lambda > 0$ :

**Theorem 3.4.** *Let  $p > 1$ ,  $\lambda > 0$ ,  $\varphi \in \mathcal{C}(\overline{\Omega})$  a non-negative function, and let  $\Omega$  be any unbounded domain. Then, the Problem 1 admits a global positive solution if the initial data satisfies*

$$0 \leq \varphi \leq \lambda^{\frac{1}{p-1}},$$

*and when  $\mathcal{B}(u) = 0$  stands for the Dirichlet, the Neumann, the Robin ( $\partial_\nu u + au = 0$  with  $a \geq 0$ ) or the dynamical boundary conditions.*

*Proof.* As in the proof of Theorem 3.2, we consider the constant function  $v(x, t) = \lambda^{\frac{1}{p-1}}$  for all  $(x, t) \in \Omega \times (0, \infty)$ . Then,  $v$  satisfies Burger's Equation, the choice of  $\varphi$  implies  $\varphi \leq v(\cdot, 0)$  in  $\Omega$ . On the boundary, we have:

$$\begin{aligned} v &\geq 0 && \text{(Dirichlet).} \\ \partial_\nu v &= 0 && \text{(Neumann).} \\ \partial_\nu v + av &\geq 0 && \text{(Robin).} \\ \sigma \partial_t v + \partial_\nu v &= 0 && \text{(Dynamical).} \end{aligned}$$

Thus,  $v$  is super-solution of 1 for the four boundary conditions above, and we conclude with the comparison principle [2].  $\square$

If  $\lambda \leq 0$ , we must add some restrictions, and we obtain the following results.

**Theorem 3.5.** *Assume  $\Omega = (0, \infty)$  and let  $p \in (1, 2]$ ,  $\lambda \leq 0$  and  $\varphi \in \mathcal{C}(\overline{\Omega})$  a non-negative function. Then, the Problem 1 admits a global positive solution if the initial data is bounded and when  $\mathcal{B}(u) = 0$  stands for the Dirichlet boundary conditions or the dynamical boundary conditions with  $\sigma > 0$  constant.*

*Proof.* We deal with the comparison principle [2] and the explicit function  $v(x, t) = Ae^{\alpha x + (t+t_0)^2}$  defined in  $\mathbb{R}^+ \times \mathbb{R}^+$ . Computing the partial derivatives, we have

$$\begin{aligned} \partial_t v(x, t) &= 2(t+t_0)v. \\ \partial_x v(x, t) &= \alpha v. \\ \partial_x^2 v(x, t) &= \alpha^2 v. \end{aligned}$$

Choosing  $t_0 \geq \frac{1}{2}(\alpha^2 - \lambda)$ , we obtain

$$\partial_t v - \partial_x^2 v + v \partial_x v - v^p + \lambda v \geq v^2(\alpha - v^{p-2}).$$

Thanks to  $p \leq 2$  and with  $\alpha x + (t+t_0)^2 \geq 0$  in  $\mathbb{R}^+ \times \mathbb{R}^+$ , we have  $v^{p-2} \leq A^{p-2}$ . Choosing  $A^{p-2} \leq \alpha$ , we obtain  $\partial_t v - \partial_x^2 v + v \partial_x v - v^p + \lambda v \geq 0$ . Since  $v \geq 0$ , the

case of the Dirichlet boundary conditions is trivial. Choosing  $t_0 \geq \frac{\alpha}{2\sigma}$ , the case of the dynamical boundary conditions is verified thanks to

$$\sigma \partial_t v + \partial_n uv = v \left( 2\sigma(t + t_0) - \alpha \right) \geq 0.$$

Finally, we have a super-solution choosing  $A \geq \sup \varphi$ .  $\square$

**Remark 5.** In the previous proof, one can see that the dynamical boundary conditions are satisfied for a more general coefficient  $\sigma$  verifying

$$\sigma(x, t) \geq \frac{\alpha}{2(t + t_0)}.$$

And replacing the function  $v$  by  $w(x, t) = Ae^{\alpha x + (t+t_0)^n}$ , we can consider smaller coefficients  $\sigma > 0$  with  $\sigma(x, t) \underset{t \rightarrow \infty}{\sim} t^{-n+1}$ .

**Corollary 2.** *Suppose  $\Omega = (-\infty, 0)$  or  $\Omega = \mathbb{R}$ . Let  $p = 2$ ,  $\lambda \leq 0$  and  $\varphi \in \mathcal{C}(\Omega)$ . Then the Problem 1 admits a global positive solution if there exists  $C > 0$  and  $\alpha > 0$  such that*

$$0 \leq \varphi(x) \leq Ce^{\alpha x} \text{ in } \Omega$$

and when  $\mathcal{B}(u) = 0$  stands for the Dirichlet, the Neumann or the dynamical boundary conditions with  $\sigma > 0$ .

*Proof.* As in the previous theorem, we consider  $v(x, t) = Ae^{\alpha x + (t+t_0)^2}$ . Thanks to  $p = 2$  and with some appropriate constants  $A$  and  $\alpha$ , we have

$$\begin{cases} \partial_t v - \partial_x^2 v + v \partial_x v - v^p + \lambda v \geq 0 & \text{in } \Omega \times [0, \infty). \\ v(\cdot, 0) \geq \varphi & \text{in } \Omega. \end{cases}$$

The case  $\Omega = \mathbb{R}$  (no boundary) and the case of Dirichlet boundary conditions are trivial. For  $\Omega = (-\infty, 0)$ , we have  $\partial_\nu v = \partial_x v = \alpha v > 0$  on the boundary. Thus, the Neumann boundary conditions and the dynamical boundary conditions with  $\sigma \geq 0$  are verified.  $\square$

When  $\lambda = 0$ ,  $\Omega = (-\infty, 0)$  and  $p > 3$ , the Green's function of the heat equation is a suitable super-solution for the Problem 1.

**Theorem 3.6.** *Assume  $\Omega = (-\infty, 0)$ ,  $p > 3$  and  $\varphi \in \mathcal{C}(\Omega)$ . Then the Problem 1 admits a global positive solution if the initial data  $\varphi$  is sufficiently small and when  $\mathcal{B}(u) = 0$  stands for the Dirichlet, the Neumann or the dynamical boundary conditions with  $\sigma > 0$  constant.*

*Proof.* Consider the function  $v(x, t) = A(t+1)^{-\gamma} e^{\frac{-(x+y)^2}{4t+4}}$  defined in  $\mathbb{R}^- \times \mathbb{R}^+$  with  $A > 0$ ,  $\gamma = \frac{1}{p-1}$  and  $y = -2\sigma\gamma$ . A simple calculus leads to

$$\partial_t v - \partial_x^2 v + v \partial_x v - v^p + \lambda v = \frac{v}{2(t+1)} \left( -2\gamma + 1 - (x+y)v - v^{p-1} \right).$$

By definition of  $\gamma$  and  $p > 3$ , we have  $-2\gamma + 1 > 0$ . Since  $v^{p-1} \leq A^{p-1}$ , and because  $-(x+y) > 0$  for all  $x \in \Omega$ , we obtain  $\partial_t v - \partial_x^2 v + v \partial_x v - v^p + \lambda v \geq 0$  by choosing  $A$  small enough. The case of the Dirichlet boundary conditions is clear because  $v \geq 0$ . For the dynamical boundary conditions and the Neumann boundary conditions ( $\sigma = 0$ ), we use the definition of  $y$  and we have

$$\sigma \partial_t v + \partial_\nu v \geq \frac{v}{2(t+1)} \left( -2\sigma\gamma - y \right) \geq 0.$$

Thus,  $v$  is a super-solution of the Problem 1 as soon as  $\varphi \leq v(\cdot, 0)$  in  $\Omega$ .  $\square$

**3.3. Blow up in unbounded domains.** Here, using some weighted  $L^1$ -norms, we examine blow-up phenomena for some solutions of Problem 1 in unbounded domains satisfying the Neumann, the Robin, and some nonlinear boundary conditions and this growth order at infinity: for all  $a > 0$  and for all  $t > 0$

$$\lim_{|x| \rightarrow \infty} u(x, t)e^{-a|x|} = 0 \text{ and } \lim_{|x| \rightarrow \infty} |\partial_x u(x, t)|e^{-a|x|} = 0. \quad (23)$$

Unless otherwise stated, we always suppose  $\Omega = (0, \infty)$ . We begin with a lemma which gives a criterion for the blowing-up of the solution.

**Lemma 3.7.** *Let  $u$  be a solution of Problem 1 satisfying the condition 23. If there exists  $\alpha > 0$  such that*

$$N_\alpha(t) := \int_0^\infty u(x, t)e^{-\alpha x} dx$$

*blows-up in finite time, then  $u$  also blows-up in finite time.*

*Proof.* Consider  $\alpha > 0$  such that  $N_\alpha$  blows-up in finite time. Using the following inequality

$$N_\alpha(t) \leq \int_0^\infty e^{-\alpha x/2} dx \cdot \sup_\Omega u(x, t)e^{-\frac{\alpha}{2}x} = \frac{2}{\alpha} \sup_\Omega u(x, t)e^{-\frac{\alpha}{2}x},$$

and because  $N_\alpha$  blows up, we can deduce the blowing up in finite time of the function  $u(x, t)e^{-\frac{\alpha}{2}x}$ . Then, thanks to the growth order condition 23, the solution  $u$  must blow up too.  $\square$

We also need this technical lemma.

**Lemma 3.8.** *Let  $u$  be a solution of Problem 1. Then, for all  $\tau > 0$  there exists  $c > 0$  such that*

$$u(0, t) \geq c \text{ for all } t \geq \tau.$$

*Proof.* Let  $v$  be the positive solution of the following problem

$$\begin{cases} \partial_t v = \partial_x^2 v - v \partial_x v + v^p - \lambda v & \text{in } [0, 1] \times [0, \infty), \\ \mathcal{B}(v) = 0 & \text{on } \{0\} \times [0, \infty), \\ v = 0 & \text{on } \{1\} \times [0, \infty), \\ v(\cdot, 0) = \varphi_1 & \text{in } [0, 1], \end{cases}$$

where  $\mathcal{B}(v) = 0$  is the same boundary condition as in Problem 1, where  $\varphi_1 \in \mathcal{C}^2([0, 1])$  satisfies  $\varphi_1(1) = 0$ ,  $\partial_x^2 \varphi_1 - \varphi_1 \partial_x \varphi_1 + \varphi_1^p - \lambda \varphi_1 \geq 0$  and  $0 \leq \varphi_1 \leq \varphi$  in  $[0, 1]$ . Thanks to  $u(\cdot, 0) \geq v(\cdot, 0)$  in  $[0, 1]$  and  $u(1, t) \geq 0 = v(1, t)$  for all  $t > 0$ , the comparison principle from [2] implies

$$u(x, t) \geq v(x, t) \text{ for all } x \in [0, 1] \text{ and } t > 0.$$

Then, the comparison principle and the maximum principle from [2] imply

$$\partial_t v(x, t) \geq 0 \text{ and } v(x, t) > 0.$$

for all  $x \in [0, 1]$  and  $t > 0$ , see Lemma 2.1 in [3]. Thus, for all  $\tau > 0$ , we obtain

$$u(0, t) \geq v(0, t) \geq v(0, \tau) > 0 \text{ for all } t \geq \tau.$$

$\square$

**Theorem 3.9.** *Let  $\lambda < 0$  and  $p \geq 2$ . Then the Problem 1 admits no global positive solution when  $\mathcal{B}(u) = 0$  stands for the Neumann boundary conditions.*

*Proof.* We aim to prove the existence of  $\alpha > 0$  and  $\beta > 0$  such that  $N'_\alpha \geq \beta N_\alpha^p$  where

$$N_\alpha(t) := \int_0^\infty u(x,t)e^{-\alpha x} dx$$

Derivating the function  $N_\alpha$ , we obtain

$$\begin{aligned} N'_\alpha(t) &= \int_0^\infty \partial_t u(x,t)e^{-\alpha x} dx \\ &= \int_0^\infty \left( \partial_x^2 u(x,t) \right) e^{-\alpha x} dx - \int_0^\infty \left( u(x,t) \partial_x u(x,t) \right) e^{-\alpha x} dx \\ &\quad + \int_0^\infty u^p(x,t)e^{-\alpha x} dx - \lambda \int_0^\infty u(x,t)e^{-\alpha x} dx. \end{aligned}$$

Using the growth order condition 23 and integrating by parts, we obtain

$$\int_0^\infty \left( \partial_x^2 u(x,t) \right) e^{-\alpha x} dx = \alpha^2 \int_0^\infty u(x,t)e^{-\alpha x} dx + \partial_\nu u(0,t) - \alpha u(0,t)$$

and

$$\int_0^\infty \left( u(x,t) \partial_x u(x,t) \right) e^{-\alpha x} dx = \frac{\alpha}{2} \int_0^\infty u^2(x,t)e^{-\alpha x} dx - \frac{u^2(0,t)}{2}.$$

Thus, we have

$$\begin{aligned} N'_\alpha(t) &= \int_0^\infty u(x,t)e^{-\alpha x} \left( \alpha^2 - \frac{\alpha}{2} u(x,t) - \lambda + u^{p-1}(x,t) \right) dx \\ &\quad + \partial_\nu u(0,t) - \alpha u(0,t) + \frac{u^2(0,t)}{2}. \end{aligned} \tag{24}$$

Thanks to Lemma 3.8, and considering  $u$  from a time  $\tau > 0$ , we can assume that

$$c := \min_{t>0} u(0,t) > 0.$$

Then, if  $\alpha$  is small enough ( $\alpha \leq c/2$ ), we have  $-\alpha u(0,t) + \frac{u^2(0,t)}{2} \geq 0$ . Then, the Neumann boundary conditions imply

$$N'_\alpha(t) \geq \int_0^\infty u(x,t)e^{-\alpha x} \left( \alpha^2 - \frac{\alpha}{2} u(x,t) - \lambda + u^{p-1}(x,t) \right) dx. \tag{25}$$

Shrinking  $\alpha$ , we can suppose  $\alpha \leq -2\lambda$  and  $\alpha \leq 1$ . When  $u(x,t) \leq 1$ , we have  $-\lambda - \alpha u(x,t)/2 > 0$ . On the other hand, if  $u(x,t) \geq 1$ , we have  $u^{p-1}(x,t) - \alpha u(x,t)/2 \geq u^{p-1}(x,t)/2$ . Hence, we obtain:

$$N'_\alpha(t) \geq \frac{1}{2} \int_0^\infty u^p(x,t)e^{-\alpha x} dx.$$

Hölder inequality

$$\int_0^\infty u(x,t)e^{-\alpha x} dx \leq \left( \int_0^\infty u^p(x,t)e^{-\alpha x} dx \right)^{\frac{1}{p}} \left( \int_0^\infty e^{-\alpha x} dx \right)^{\frac{p-1}{p}}$$

leads to  $N'_\alpha(t) \geq \beta N_\alpha^p(t)$  with

$$\beta = \frac{1}{2} \left( \int_0^\infty e^{-\alpha x} dx \right)^{1-p}.$$

Finally, we prove the blowing-up of  $N_\alpha$  in finite time. Integrating the differential inequality  $N'_\alpha(t) \geq \beta N_\alpha^p(t)$  between 0 and  $t > 0$ , we obtain

$$\frac{1}{1-p} \left( N_\alpha^{1-p}(t) - N_\alpha^{1-p}(0) \right) = \int_{N_\alpha(0)}^{N_\alpha(t)} s^{-p} ds = \int_0^t \frac{N'_\alpha(t)}{N_\alpha^p(t)} dt \geq \beta t,$$

and

$$N_\alpha(t) \geq \left( N_\alpha^{1-p}(0) - (p-1)\beta t \right)^{\frac{-1}{p-1}}.$$

Since of  $\frac{-1}{p-1} < 0$ , the right hand side term blows up at  $t = \frac{N_\alpha^{1-p}(0)}{(p-1)\beta} > 0$ . We conclude with Lemma 3.7.  $\square$

**Corollary 3.** *Let  $\lambda < 0$  and  $p \geq 2$ . Then the Problem 1 admits no global positive solution when  $\mathcal{B}(u) = 0$  stands for the nonlinear boundary conditions  $\partial_\nu u = g(u)$ , where  $g$  is a function such that there exists  $\delta > 0$  and  $\varepsilon \leq 1/2$  satisfying*

$$g(\eta) \geq \delta\eta - \varepsilon\eta^2.$$

*Proof.* We follow the proof of Theorem 3.9. We just change the choice of  $\alpha$ : let  $\alpha > 0$  such that  $\alpha \leq \delta$ , and use the following minoration in Equation 24:

$$\begin{aligned} \partial_\nu u(0, t) - \alpha u(0, t) + \frac{1}{2}u^2(0, t) &= g(u) - \alpha u(0, t) + \frac{1}{2}u^2(0, t) \\ &\geq (\delta - \alpha)u(0, t) + \left(\frac{1}{2} - \varepsilon\right)u^2(0, t) \geq 0. \end{aligned}$$

Then, we return to Equation 25.  $\square$

When  $\lambda = 0$ , the choice of  $\alpha$  is too strict. Meanwhile, we obtain some blow-up results imposing some restrictions on the exponent  $p$  and on the initial data.

**Theorem 3.10.** *Let  $\lambda = 0$  and  $1 < p \leq 3$ . Then the Problem 1 admits no global positive solution when  $\mathcal{B}(u) = 0$  stands for the Neumann boundary conditions.*

*Proof.* Return to the proof of Theorem 3.9. Under the Neumann boundary conditions and with  $\lambda = 0$ , Equation 24 becomes

$$N'_\alpha(t) = \int_0^\infty u(x, t)e^{-\alpha x} \left( \alpha^2 - \frac{\alpha}{2}u(x, t) + u^{p-1}(x, t) \right) dx - \alpha u(0, t) + \frac{u^2(0, t)}{2}.$$

Let  $\beta \in (0, 1)$  and put it into the previous equation:

$$\begin{aligned} N'_\alpha(t) &= \int_0^\infty u(x, t)e^{-\alpha x} \left( \alpha^2 - \frac{\alpha}{2}u(x, t) + \beta u^{p-1}(x, t) \right) dx \\ &\quad - \alpha u(0, t) + \frac{u^2(0, t)}{2} + (1 - \beta) \int_0^\infty u^p(x, t)e^{-\alpha x} dx. \end{aligned}$$

If  $u \leq 2\alpha$ , we have  $\alpha^2 - \alpha u/2 \geq 0$ , whereas  $u > 2\alpha$  implies

$$-\frac{\alpha}{2}u + \beta u^{p-1} \geq u \left( -\frac{\alpha}{2} + \beta(2\alpha)^{p-2} \right).$$

It is non negative if

$$\beta\alpha^{p-3} \geq 2^{1-p}. \tag{26}$$

Thanks to  $1 < p \leq 3$ , Equation 26 is achieved by choosing  $\alpha > 0$  sufficiently small and  $\beta \in (0, 1)$  depending on  $p$ . Thus, we obtain

$$N'_\alpha(t) \geq -\alpha u(0, t) + \frac{u^2(0, t)}{2} + (1 - \beta) \int_0^\infty u^p(x, t)e^{-\alpha x} dx.$$

Then, we can suppose that  $u(0, t) > c > 0$  for all  $t > 0$  (see Lemma 3.8), and with  $\alpha < c/2$  we have  $-\alpha u(0, t) + \frac{u^2(0, t)}{2} > 0$ . Hence

$$N'_\alpha(t) \geq (1 - \beta) \int_0^\infty u^p(x, t) e^{-\alpha x} dx.$$

As in the proof of Theorem 3.9, we use Hölder inequality and we are led to  $N'_\alpha \geq \delta N_\alpha^p$  with  $\delta > 0$  depending on  $\alpha$ ,  $\beta$  and  $p$ . Hence,  $N_\alpha$  blows-up in finite time, so does the solution  $u$ , see Lemma 3.7.  $\square$

**Theorem 3.11.** *Let  $\lambda = 0$  and  $p > 3$ . Then the Problem 1 admits no global positive solution when  $\mathcal{B}(u) = 0$  stands for the Neumann boundary conditions and if the initial data satisfies  $\varphi(0) > 2^{\frac{1-p}{p-3}}$ .*

*Proof.* The proof is similar to the previous one. Go back to Equation 26: since  $p > 3$ , we must choose  $\alpha$  such that

$$\alpha \geq 2^{\frac{1-p}{p-3}} \beta^{\frac{-1}{p-3}}.$$

Under this condition,  $N_\alpha$  satisfies the differential inequality

$$N'_\alpha(t) \geq -\alpha u(0, t) + \frac{u^2(0, t)}{2} + (1 - \beta) \int_0^\infty u^p(x, t) e^{-\alpha x} dx.$$

Because  $\alpha$  can not be too small, we must use the assumption  $\varphi(0) > 2^{\frac{1-p}{p-3}}$ . Using Lemma 3.8, we have

$$u(0, t) \geq \varphi(0) > 2^{\frac{1-p}{p-3}}, \text{ for all } t > 0.$$

Thus, with  $\beta$  very close to 1 and with  $\alpha = 2^{\frac{1-p}{p-3}} \beta^{\frac{-1}{p-3}}$ , we obtain  $-\alpha u(0, t) + \frac{u^2(0, t)}{2} \geq 0$ . Hence, we have

$$N'_\alpha(t) \geq (1 - \beta) \int_0^\infty u^p(x, t) e^{-\alpha x} dx.$$

We conclude with Hölder inequality and the blowing up of  $N_\alpha$ .  $\square$

**Corollary 4.** *Let  $\lambda = 0$  and  $p > 3$ . Then the Problem 1 admits no global positive solution when  $\mathcal{B}(u) = 0$  stands for the Neumann boundary conditions and if the initial data satisfies*

$$\int_0^\infty \varphi(x) e^{-x} dx > \frac{3p-7}{p-3} \cdot 2^{\frac{5-3p}{p-3}} \cdot \left( \frac{2p-4}{3p-7} \right)^{\frac{4-2p}{p-3}}. \quad (27)$$

*Proof.* Return to the proof of Theorem 3.9. Under the Neumann boundary conditions and introducing  $\beta$  and  $\delta \in (0, 1)$  in Equation 24, we obtain

$$\begin{aligned} N'_\alpha(t) &= \int_0^\infty u(x, t) e^{-\alpha x} \left( \delta \alpha^2 - \frac{\alpha}{2} u(x, t) + \beta u^{p-1}(x, t) \right) dx \\ &\quad - \alpha u(0, t) + \frac{u^2(0, t)}{2} + (1 - \delta) N_\alpha(t) + (1 - \beta) \int_0^\infty u^p(x, t) e^{-\alpha x} dx. \end{aligned}$$

Studying both cases  $u \geq 2\alpha\delta$  and  $u \leq 2\alpha\delta$ , we obtain  $\delta\alpha^2 - \alpha u/2 + \beta u^{p-1} \geq 0$  if

$$\alpha = 2^{\frac{1-p}{p-3}} \beta^{\frac{-1}{p-3}} \delta^{\frac{2-p}{p-3}}.$$

Since of  $u^2/2 - \alpha u \geq -\alpha^2/2$  and using Hölder inequality we have

$$N'_\alpha(t) \geq (1 - \delta) N_\alpha(t) + \gamma N_\alpha^p(t) - \frac{\alpha^2}{2}, \quad (28)$$

where  $\gamma = (1 - \beta) \left( \int_0^\infty e^{-\alpha x} dx \right)^{1-p} > 0$ . First, consider this minoration

$$N'_\alpha(t) \geq (1 - \delta)N_\alpha(t) - \frac{\alpha^2}{2}.$$

Thus,  $N_\alpha$  satisfies

$$N_\alpha(t) \geq \frac{\alpha^2}{2(1 - \delta)} + Ae^{(1-\delta)t}, \quad A \in \mathbb{R}.$$

In particular,  $N_\alpha(0) \geq \alpha^2(2 - 2\delta)^{-1} + A$ . With the optimal choice of  $\delta = (2p - 4)/(3p - 7)$  and with  $\beta \in (0, 1)$  close to 1, Hypothesis 27 implies  $N_\alpha(0) > \alpha^2(2 - 2\delta)^{-1}$ . Thus,  $A$  is positive and we obtain

$$(1 - \delta)N_\alpha(t) - \frac{\alpha^2}{2} \geq 0.$$

From Equation 28, we deduce

$$N'_\alpha(t) \geq \gamma N_\alpha^p(t).$$

Hence  $N_\alpha$  blows-up, and the solution  $u$  blows up too, see Lemma 3.7.  $\square$

Finally, if  $\Omega = (-\infty, 0)$ , we must change the weight in  $N_\alpha$  and we obtain this results concerning the nonlinear boundary conditions.

**Theorem 3.12.** *Let  $\lambda \leq 0$  and  $p \geq 2$ . Then the Problem 1 admits no global positive solution when  $\mathcal{B}(u) = 0$  stands for the nonlinear boundary conditions  $\partial_\nu u = g(u)$ , where  $g$  is a function such that there exists  $c > 0$  and  $d > 0$  satisfying*

$$g(\eta) \geq c\eta^2 + d\eta.$$

*Proof.* As in the case of  $\Omega = (0, \infty)$ , we use a weighted  $L^1$ -norm:

$$N_\alpha(t) = \int_{-\infty}^0 u(x, t)e^{\alpha x} dx, \quad \text{with } \alpha > 0.$$

We compute  $N'_\alpha(t) = \int_{-\infty}^0 \partial_t u(x, t)e^{\alpha x} dx$ , and using the equations of Problem 1, integration by parts leads to

$$N'_\alpha(t) = \int_{-\infty}^0 (\alpha^2 u + \alpha u^2 + u^p)e^{\alpha x} dx + \partial_x u(0, t) - \alpha u(0, t) - \frac{\alpha}{2} u^2(0, t).$$

Thanks to  $\partial_\nu u(0, t) = \partial_x u(0, t)$  in  $(-\infty, 0)$ , choosing  $\alpha = \min\{2c, d\}$ , we obtain

$$N'_\alpha(t) \geq \int_{-\infty}^0 (\alpha^2 u + \alpha u^2 + u^p)e^{\alpha x} dx \geq \int_{-\infty}^0 u^p e^{\alpha x} dx.$$

Hölder inequality leads to the differential equation  $N'_\alpha(t) \geq \gamma N_\alpha^p(t)$  with  $\gamma > 0$ . Hence  $N_\alpha$  and the solution  $u$  blow up in finite time.  $\square$

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