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The Fujita phenomenon in exterior domains under the Robin boundary conditions

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Abstract

The Fujita phenomenon for nonlinear parabolic problems $\partial_t u = \Delta u + u^p$ in an exterior domain of \mathbb{R}^N under the Robin boundary conditions is investigated in the superlinear case. As in the case of Dirichlet boundary conditions (see Trans. Amer. Math. Soc 316 (1989), 595-622 and Isr. J. Math. 98 (1997), 141-156), it turns out that there exists a critical exponent $p = 1 + 2/N$ such that blow-up of positive solutions always occurs for subcritical exponents, whereas in the supercritical case global existence can occur for small non-negative initial data.

Key words: Nonlinear parabolic problems; Robin boundary conditions ; Global solutions; Blow-up.

1 Introduction

Let Ω be an exterior domain of \mathbb{R}^N , that is to say a connected open set Ω such that $\overline{\Omega}^c$ is a bounded domain when $N \geq 2$, and in dimension one, Ω is the complement of a real closed interval. We always suppose that the boundary $\partial\Omega$ is of class \mathcal{C}^2 . The outer normal unit vector field is denoted by $\nu : \partial\Omega \rightarrow \mathbb{R}^N$ and the outer normal derivative by ∂_ν . Let p be a real number with $p > 1$, α a non-negative continuous function on $\partial\Omega \times \mathbb{R}^+$ and φ a continuous function in $\overline{\Omega}$. Consider the following nonlinear parabolic problem

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ \partial_\nu u + \alpha u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}. \end{cases} \quad (1)$$

In this paper, we give a positive answer to Levine & Zhang's question [1]: the Fujita phenomenon, well-known in the case of $\Omega = \mathbb{R}^N$ (see Ref. [2]), remains true for the Robin boundary conditions. The case limiting $\alpha \equiv 0$ and $\alpha = +\infty$

were proved by Levine & Zhang in [1] and by Bandle & Levine in [3], respectively. The real number $1 + \frac{2}{N}$ is still the critical exponent, and we prove the blowing-up of all positive solutions of Problem (1) for subcritical exponents p , whereas in the supercritical case, we show the existence of global positive solutions of Problem (1) for sufficiently small initial data. In the last section, we study the case of a general second order elliptic operator replacing the Laplacian. We also consider a non-linearity including a time and a space dependence. Throughout, we shall assume that α is non-negative

$$\alpha \geq 0 \text{ on } \partial\Omega \times \mathbb{R}^+, \quad (2)$$

and, in order to deal with classical solutions, we need some regularity on α

$$\alpha \in \mathcal{C}(\partial\Omega \times \mathbb{R}^+). \quad (3)$$

To construct solutions with the truncation procedure (see Section 2), we suppose

$$\varphi \in \mathcal{C}(\overline{\Omega}), \quad 0 < \|\varphi\|_\infty < \infty, \quad \varphi \geq 0, \quad \lim_{\|x\|_2 \rightarrow \infty} \varphi(x) = 0. \quad (4)$$

In the case $\Omega = \mathbb{R}^N$, the boundary conditions are dropped, and the result is well-known by the classical paper of Fujita [2]. Thus we suppose $\Omega \neq \mathbb{R}^N$.

2 Preliminaries

First, we give the definition of positive solution which is understood along this paper.

Definition 2.1 *A positive solution of Problem (1) is a positive function $u : (x, t) \mapsto u(x, t)$ of class $\mathcal{C}(\overline{\Omega} \times [0, T]) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0, T))$, satisfying*

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ \partial_\nu u + \alpha u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}, \end{cases}$$

where α and φ are given with (2), (3) and (4). The time $T = T(\alpha, \varphi) \in (0, +\infty]$ denotes the maximal existence time of the solution u . If $T = +\infty$, the solution is called global.

From [3], if $T < +\infty$, u blows up in finite time, that is to say:

$$\limsup_{t \nearrow T} \limsup_{x \in \overline{\Omega}} u(x, t) = +\infty.$$

Then, let us recall a standard procedure to construct solutions of Problem (1) in outer domains for uniformly bounded and continuous initial data φ . For more details, we refer to [4], [5] and references therein. Let $(D_n)_{n \in \mathbb{N}}$ be a sequence of nested bounded domains such that

$$\overline{\Omega}^c \subseteq D_0 \subseteq D_1 \subseteq \dots \subseteq \bigcup_{n \in \mathbb{N}} D_n = \mathbb{R}^N.$$

Let u_n be the solution of

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \overline{\Omega} \cap D_n \times (0, +\infty), \\ \partial_\nu u + \alpha u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial D_n \times (0, +\infty), \\ u(\cdot, 0) = \varphi_n & \text{in } \overline{\Omega} \cap D_n, \end{cases} \quad (5)$$

where $(\varphi_n)_{n \in \mathbb{N}}$ denotes a sequence of functions in $C_0(\overline{\Omega} \cap D_n)$ such that

$$0 \leq \varphi_n \leq \varphi \text{ in } \overline{\Omega} \cap D_n$$

and $\varphi_n \rightarrow \varphi$ uniformly in any compact of $\overline{\Omega} \cap D_n$ as $n \rightarrow +\infty$. Let z denote the solution of the ODE

$$\begin{cases} \dot{z} = z^p, \\ z(0) = \|\varphi\|_\infty, \end{cases}$$

with maximal existence time $S = \frac{1}{(p-1)\|\varphi\|_\infty^{p-1}}$. By the comparison principle (see [6]), we have

$$0 \leq u_n(x, t) \leq u_{n+1}(x, t) \leq z(x, t) \text{ in } \overline{\Omega} \cap D_n \times [0, S].$$

Standard arguments based on a priori estimates for the heat equation imply $u_n \rightarrow u$ in the sense of $C_{loc}^{2,1}(\overline{\Omega} \times (0, S))$ as $n \rightarrow +\infty$, where u is a positive solution of Problem (1). Moreover, since u_n vanishes on ∂D_n for each $n \in \mathbb{N}^*$, the solution u vanishes at infinity:

$$\lim_{\|x\|_2 \rightarrow \infty} u(x, t) = 0, \forall t \in (0, T).$$

3 Blow up case

In this section, we compare the solution of Problem (1) with an appropriate Dirichlet solution. We prove the following theorem:

Theorem 3.1 *Suppose that conditions (2), (3) and (4) are fulfilled. Then all non-trivial positive solutions of Problem (1) blow up in finite time for $p \in (1, 1 + 2/N)$. Moreover, if $N \geq 3$, blow up also occurs for $p = 1 + 2/N$.*

Proof: Ab absurdo, suppose that there exists α and a non-trivial φ satisfying the hypotheses above, and such that the solution u of Problem (1) with these parameters is global. Then, consider u_n the solution of the truncated Problem (5). By the comparison principle from [6], we obtain

$$0 \leq u_n(x, t) \leq u(x, t) \text{ in } \overline{\Omega} \cap D_n \text{ for } t > 0.$$

Thus, u_n can not blow up in finite time, and u_n must be global. Next, define v_n the solution of the following problem

$$\begin{cases} \partial_t v_n = \Delta v_n + v_n^p & \text{in } \overline{\Omega} \cap D_n \times (0, +\infty), \\ v_n = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v_n = 0 & \text{on } \partial D_n \times (0, +\infty), \\ v_n(\cdot, 0) = \varphi_n & \text{in } \overline{\Omega} \cap D_n. \end{cases}$$

Again, the comparison principle from [6] implies $0 \leq v_n(x, t) \leq u_n(x, t)$ in $\overline{\Omega} \cap D_n$ for $t > 0$. Then, we consider v the solution of the Dirichlet problem

$$\begin{cases} \partial_t v = \Delta v + v^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ v = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v(\cdot, 0) = \varphi & \text{in } \overline{\Omega}, \end{cases}$$

obtained as the limit of the v_n by the truncation procedure described in Section 2. Thus, $v \leq u$ in $\overline{\Omega} \times (0, +\infty)$ and v is a global positive solution. A contradiction with Bandle & Levine results [3] (see [7] for the one-dimensional case). If $N \geq 3$ and $p = 1 + 2/N$, the contradiction holds with Mochizuki & Suzuki's results [8] and [9]. Hence, our solution u must blow up in finite time. ■

4 Global existence case

Now, we consider supercritical exponents:

$$p > 1 + \frac{2}{N}.$$

We look for a global positive super-solution of Problem (1), we mean a function U satisfying

$$\begin{cases} \partial_t u \geq \Delta u + u^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ \partial_\nu u + \alpha u \geq 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) \geq \varphi & \text{in } \overline{\Omega}. \end{cases}$$

With this global super-solution and using the comparison principle, we construct the sequence $(u_n)_{n \in \mathbb{N}}$ of global positive solutions of Problems (5). Thus, using the truncation procedure of Section 2, we construct a global positive solution of Problem (1). We use two different super-solutions, and we obtain two results on the global existence with some restrictions on the dimension N or on the coefficient α . First, we only suppose that the dimension

$$N \geq 3.$$

Theorem 4.1 *Under hypotheses (2), (3) and (4), for $N \geq 3$ and*

$$p > 1 + \frac{2}{N},$$

Problem (1) admits global non-trivial positive solutions for sufficiently small initial data φ .

Proof: Consider φ satisfying (4) and v the non-trivial positive solution v of the Neumann problem

$$\begin{cases} \partial_t v = \Delta v + v^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v(\cdot, 0) = \varphi & \text{in } \overline{\Omega}, \end{cases}$$

where the initial data φ is sufficiently small such that the solution v is global. This choice can be achieved because $N \geq 3$ and $p > 1 + 2/N$, see Levine & Zhang [1]. For all $\alpha \geq 0$ on $\partial\Omega \times (0, +\infty)$, we obtain

$$\partial_\nu v + \alpha v \geq 0 \text{ on } \partial\Omega \times (0, +\infty).$$

Thus, v is a super-solution of Problem (1), and we can deduce the statement of the theorem. ■

Now, we suppose that there exists a positive constant $c > 0$ such that

$$\alpha \geq c \text{ on } \partial\Omega \times \mathbb{R}^+. \quad (6)$$

We do not impose any condition on the dimension.

Theorem 4.2 *Let α be a coefficient satisfying (3) and (6), φ an initial data with (4). For*

$$p > 1 + \frac{2}{N},$$

Problem (1) admits global positive solutions for sufficiently small initial data φ .

Proof: We consider the function $U : \overline{\Omega} \times [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$U(x, t) = A(t + t_0)^{-\mu} \exp\left(-\frac{\|x\|_2^2}{4(t + t_0)}\right),$$

where $\mu = 1/(p-1)$, $t_0 > 0$ and $A > 0$ will be chosen below. All the calculus will be detailed in the proof of the general Theorem 5.3. If $A > 0$ is small enough, we have

$$\partial_t U \geq \Delta U + U^p \text{ in } \overline{\Omega} \times (0, +\infty).$$

On the boundary $\partial\Omega$, hypothesis (6) gives

$$\begin{aligned} \partial_\nu U(x, t) + \alpha U(x, t) &\geq \left(\frac{-x \cdot \nu(x)}{2(t + t_0)} + \alpha(x, t)\right) U(x, t) \\ &\geq \left(\frac{-x \cdot \nu(x)}{2(t + t_0)} + c\right) U(x, t) \end{aligned}$$

Since the boundary $\partial\Omega$ is compact, the function $(\partial\Omega \ni x \mapsto -x \cdot \nu(x) \in \mathbb{R})$ is bounded. We choose t_0 sufficiently big such that $-x \cdot \nu(x)/(2t_0) + c \geq 0$. Then we obtain

$$\partial_\nu U + \alpha U \geq 0 \text{ on } \partial\Omega \times (0, +\infty).$$

Finally, if we choose $\varphi \leq U(\cdot, 0)$ in $\overline{\Omega}$, the function U is a super-solution of Problem (1). ■

Remark 4.3 *In the previous proof, one can note that the hypothesis (6) can be relaxed into*

$$\alpha(x, t) \geq \frac{x \cdot \nu(x)}{2(t + t_0)} \text{ for all } (x, t) \in \partial\Omega \times (0, +\infty). \quad (7)$$

This condition gives us an optimal bound on α only if we know the geometry of the domain Ω . For instance, if

$$\Omega = \{\|x\|_2 > R\},$$

we obtain

$$x \cdot \nu(x) = -R \text{ for all } x \in \partial\Omega.$$

Then, the equation (7) is equivalent to

$$\alpha(x, t) \geq \frac{-R}{2(t + t_0)} \text{ for all } (x, t) \in \partial\Omega \times (0, +\infty).$$

In particular, the previous theorem holds for all non-negative α .

In the one-dimensional case, using symmetry and translation, we can suppose that $\Omega = (-\infty, -1) \cup (1, +\infty)$. Then, without any additional hypothesis on the parameters of Problem (1), we obtain:

Theorem 4.4 *Assume the conditions (2), (3) and (4). For dimension $N = 1$ and*

$$p > 3,$$

Problem (1) admits global positive solutions for sufficiently small initial data φ .

5 Generalization

In the manner of Bandle & Levine's results [7], we generalize our results. We consider the following problem

$$\begin{cases} \partial_t u = Lu + t^q \|x\|_2^s u^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ \partial_\nu u + \alpha u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}, \end{cases} \quad (8)$$

where q and s are two positive real numbers, $p > 1$ is a real number, and L stands for the second order elliptic operator

$$L = \sum_{i,j=1}^N \partial_{x_i} (a_{ij}(x) \partial_{x_j}) + \sum_{i=1}^N b_i(x) \partial_{x_i}.$$

To deal with classical solutions, the coefficients are assumed to be in $\mathcal{C}^2(\overline{\Omega})$. We keep the hypotheses (2), (3) and (4) on the parameters α and φ . In order to state our principal results, we shall introduce some notations.

$$\rho(x) = \sum_{i,j=1}^N a_{ij}(x) \frac{x_i x_j}{\|x\|_2^2}.$$

Throughout, we assume that the matrix $A = (a_{ij})_{1 \leq i,j \leq N}$ is normalized, so that for some $\nu_0 \in (0, 1]$

$$0 < \nu_0 \leq \rho \leq 1 \text{ in } \overline{\Omega}.$$

Denote $b = (b_1, \dots, b_N)$ and let

$$l(x) = \sum_{i,j=1}^N \left(\partial_{x_j} a_{ij}(x) - b_i(x) \right) x_i,$$

$$l^*(x) = \sum_{i,j=1}^N \left(\partial_{x_j} a_{ij}(x) + b_i(x) \right) x_i.$$

We can state the following theorem concerning the blow-up case.

Theorem 5.1 *Assume that $N \geq 2$,*

$$\operatorname{div} b(x) \leq 0 \text{ in } \overline{\Omega},$$

and

$$\rho(x) \leq \frac{\operatorname{trace} A(x) + l(x)}{2} \text{ in } \overline{\Omega}. \quad (9)$$

Then, all non-trivial positive solutions of Problem (8) blow up in finite time for

$$1 < p < 1 + \frac{2 + 2q + s}{N}.$$

Proof: Ab absurdo, we suppose that there exists a non-trivial positive solution v of Problem (8). As in the proof of Theorem 3.1, we deduce that there exists a non-trivial positive solution u of the following Dirichlet problem

$$\begin{cases} \partial_t u = Lu + t^q \|x\|_2^s u^p & \text{in } \overline{\Omega} \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}. \end{cases}$$

According to Bandle & Levine's results from [7], the solution u blows up in finite time under the above hypotheses. Thus, v must blow up too. ■

For the one-dimensional case, Bandle & Levine weaken the hypothesis (9). Then, we obtain:

Theorem 5.2 *Assume that $N = 1$,*

$$\operatorname{div} b(x) \leq 0 \text{ in } \overline{\Omega},$$

and

$$\left(\frac{2 + 2q + s}{p - 1} - 2 \right) a_{11} + l > 0 \text{ in } \overline{\Omega}.$$

If $1 < p < 3 + 2q + s$, then all non-trivial positive solutions of Problem (8) blow up in finite time.

Now, we consider the global existence case.

Theorem 5.3 *Assume that condition (6) is satisfied,*

$$\rho(x) \leq 1 \text{ in } \overline{\Omega},$$

and

$$2\gamma_0 := \inf_{\bar{\Omega}} \left(\text{trace } A + l^* \right) > 0.$$

Then, for any

$$p > 1 + \frac{2 + 2q + s}{2\gamma_0},$$

Problem (8) admits global non-trivial positive solutions if the initial data φ is sufficiently small.

Proof: We consider the function $U : \bar{\Omega} \times [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$U(x, t) = A(t + t_0)^{-\mu} \exp \left(- \frac{\|x\|_2^2}{4(t + t_0)} \right),$$

where $\mu = (2 + 2q + s)/(2p - 2)$, $t_0 > 0$ and $A > 0$ will be chosen below. We have

$$\begin{aligned} \partial_t U(x, t) &= \left(\frac{-\mu}{t + t_0} + \frac{\|x\|_2^2}{4(t + t_0)^2} \right) U(x, t), \\ LU(x, t) &= \left(\rho(x) \frac{\|x\|_2^2}{4(t + t_0)^2} - \frac{\text{trace } A + l^*}{2(t + t_0)} \right) U(x, t), \end{aligned}$$

and

$$\partial_\nu U(x, t) = \left(\frac{-x \cdot \nu(x)}{2(t + t_0)} \right) U(x, t).$$

On the boundary $\partial\Omega$, we obtain:

$$\partial_\nu U(x, t) + \alpha U(x, t) = \left(\frac{-x \cdot \nu(x)}{2(t + t_0)} + \alpha \right) U(x, t).$$

Thanks to hypothesis (6), and because the boundary $\partial\Omega$ is compact, we can choose t_0 sufficiently big such that

$$\frac{-x \cdot \nu(x)}{2t_0} + c \geq 0 \text{ on } \partial\Omega.$$

Thus, $\partial_\nu U(x, t) + \alpha U(x, t) \geq 0$ is achieved on $\partial\Omega \times (0, +\infty)$. Then, in $\bar{\Omega}$, we have

$$\partial_t U(x, t) - LU(x, t) = \left((1 - \rho(x)) \frac{\|x\|_2^2}{4(t + t_0)^2} + \frac{\text{trace } A + l^* - 2\mu}{2(t + t_0)} \right) U(x, t).$$

With $\rho \leq 1$, we ignore the t -quadratic term, and by definition of γ_0 , we obtain

$$\partial_t U(x, t) - LU(x, t) \geq \left(\frac{\gamma_0 - \mu}{t + t_0} \right) U(x, t), \quad (10)$$

with $\gamma_0 - \mu > 0$. On the other hand, $t < t + t_0$ implies

$$t^q \|x\|_2^s U^p(x, t) \leq A^{p-1} \|x\|_2^s (t + t_0)^{q - \mu(p-1)} \exp \left(\frac{-(p-1)\|x\|_2^2}{4(t + t_0)} \right) U(x, t).$$

Using the overestimation

$$\left(\frac{2s}{p-1} \right)^{\frac{s}{2}} \exp \left(\frac{-s}{2} \right) (t + t_0)^{\frac{s}{2}} \geq \|x\|_2^s \exp \left(- \frac{\|x\|_2^2 (p-1)}{4(t + t_0)} \right),$$

we obtain

$$t^q \|x\|_2^s U^p(x, t) \leq A^{p-1} \left(\frac{2s}{p-1}\right)^{\frac{s}{2}} \exp\left(\frac{-s}{2}\right) (t+t_0)^{\frac{s}{2}+q-\mu(p-1)} U(x, t). \quad (11)$$

By definition of μ , we have $s/2 + q - \mu(p-1) = -1$. Thus, we just have to choose A sufficiently small, equations (10) and (11) give

$$\partial_t U(x, t) - LU(x, t) \geq t^q \|x\|_2^s U^p(x, t).$$

Finally, if the initial data $\varphi \leq U(\cdot, 0)$ in $\bar{\Omega}$, U is a super-solution of Problem (8), and we can deduce the existence of a solution using the truncation procedure of Section 2. ■

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