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On the stochastic ordering of folded binomials

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Abstract

Folded binomial arises from binomial distributions when the number of successes is considered equivalent to the number of failures or they are indistinguishable. Formally, if $Y \sim \text{Bin}(m, \pi)$ is a binomial random variable, then the random variable $X = \min(Y, m - Y)$ is folded binomial distributed with parameters m and $p = \min(\pi, 1 - \pi)$. In this work, we present results on the stochastic ordering of folded binomial distributions. Providing an equivalence between their cumulative distribution functions (cdf) and a combination of two Beta random variable cdf's, we prove both that folded binomials are stochastically ordered with respect to their parameter p given the number of trials m , and that they are stochastically ordered with respect to their parameter m given p . Furthermore, the reader is offered two corollaries on strict stochastic dominance.

Keywords: Data depth, First-order stochastic dominance, Fisher's sign test, Strict stochastic dominance.

1 Introduction

Folded binomial arises from binomial distributions when the number of successes is considered equivalent to the number of failures or they are indistinguishable (Urbakh, 1967; Gart, 1970; Mantel, 1970). In other words, a folded binomial observation is given by $x = \min(y, m - y)$ when m independent Bernoulli trials with equal probability π have yielded y successes. It arises when two subsets of outcomes for m trials can be identified, but either which are the successes and which are the failures cannot be said, or successes and failures are considered equivalent. Thus, 10 trials yielding 4 successes will give the same pattern as 10 trials yielding 6 successes.

A folded binomial can be formally defined as a transformation of a binomial random variable. Let $Y \sim \text{Bin}(m, \pi)$ be a binomial random variable, where m is the number of trials and π is the success probability, with m a positive integer and $0 \leq \pi \leq 1$. Then, the random variable

$$X = \min(Y, m - Y)$$

is folded binomial distributed, with the following parameters: the number of trials m , and the probability $p = \min(\pi, 1 - \pi)$, $0 \leq p \leq 1/2$. In other words, $p = \pi$ if $0 \leq \pi \leq 1/2$ and $p = 1 - \pi$ if $1/2 < \pi \leq 1$. When X is a folded binomial random variable with parameters m and p , we will write $X \sim f\text{Bin}(m, p)$.

After Gart (1970), the probability mass function (pmf) of $X \sim f\text{Bin}(m, p)$ is given by:

$$P(X = x) = [1 - \frac{1}{2}\delta_{x, m-x}] \binom{m}{x} [p^x(1-p)^{(m-x)} + p^{(m-x)}(1-p)^x], \quad (1)$$

where $\delta_{k,j} = 1$ or 0 as $k = j$ or $k \neq j$, respectively.

The support set \mathcal{X} of the distribution depends on the number of Bernoulli trials, and it is given by $\mathcal{X} = \{0, 1, \dots, \lfloor m/2 \rfloor\}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x (the floor function). That is, if m is even, $\mathcal{X} = \{0, 1, \dots, m/2\}$, while if m is odd, $\mathcal{X} = \{0, 1, \dots, (m-1)/2\}$. We note that in Equation (1) $\delta_{x, m-x} = 1$ only if $x = m/2$, and this latter case occurs only for m even. For m odd, $\delta_{x, m-x} = 0$ for all x .

Unlike binomial distributions, folded binomials degenerate for $m = 1$. In this case $P(X = 0) = 1$. For this reason, it is worth studying the properties of folded binomials for $m > 1$.

Folded binomials have been used within epidemiology (Mantel *et al.*, 1976), genetic studies (Nordheim *et al.*, 1983; Nordheim *et al.*, 1984), in experimental psychology (Coombs and Huang, 1970; Aschenbrenner, 1984), in population growth studies (Pickens and Mode, 1986), and in physics (Toke *et al.*, 1997).

Furthermore, considering the two-sided (or non-directional) distribution free Fisher's sign test (Hollander and Wolfe, 1999; Sec. 3.4), we find that the sample distribution of the test statistic is a folded binomial (with $p = 1/2$ under H_0). In our opinion, this is an undervalued result that can lead to further applications. In particular, it can be exploited to study the sign test power functions.

The aim of this work is to present results on the stochastic ordering of folded binomial distributions. That is, we investigate the relationships between the cumulative distribution functions (cdf) of folded binomials in

order to establish which cdf attaches more probability to larger values than the others. Specifically, we prove both that folded binomials are stochastically ordered with respect to their parameter p given the number of trials m (Theorem 1), and that they are stochastically ordered with respect to their parameter m given p (Theorem 2). Furthermore, the reader is offered two corollaries on strict stochastic dominance.

Notice that the first theorem proves that the two-sided Fisher's sign test is unbiased. On the other hand, both theorems can be exploited to derive properties of the convex hull probability depth (Porzio and Ragozini, 2009).

2 Stochastic ordering

In this section we offer two theorems on the stochastic ordering of folded binomial random variables. Results on strict stochastic dominance are provided as well. With these aims, we first formally define the stochastic ordering notion we consider. Then we rewrite Gart's expression of the folded binomial pmf and of the corresponding cdf, exploiting it to establish the equivalence between the folded binomial cdf and a combination of two Beta random variable cdf's (Lemma 1). The latter result is used in turn to prove our theorems.

Thus let X_1 and X_2 be two univariate random variables such that

$$P(X_1 > u) \leq P(X_2 > u) \quad \forall u \in \mathbb{R}.$$

Then X_1 is said to be *stochastically smaller than or equal to* X_2 , and this is denoted by $X_1 \leq_{st} X_2$ (Whitt, 1988). Equivalently, we can say that $X_1 \leq_{st} X_2$ if

$$F_{X_1}(u) \geq F_{X_2}(u) \quad \forall u \in \mathbb{R}, \quad (2)$$

where F_{X_1} and F_{X_2} denote the cdf's of X_1 and X_2 , respectively.

When this latter inequality holds, it can also be said that X_2 stochastically dominates X_1 , where this kind of dominance is referred to as *first-order stochastic dominance*.

Finally, if in addition to the inequality (2), we also have that:

$$F_{X_1}(u) > F_{X_2}(u) \quad \text{for some } u \in \mathbb{R}, \quad (3)$$

then X_1 is said to be *stochastically smaller than* X_2 , or equivalently a *strict stochastic dominance* of X_2 on X_1 holds.

For our purposes, let us now rewrite Gart's expression (Equation 1) of

the pmf of $X \sim fBin(m, p)$ as:

$$\begin{aligned}
 P(X = x) &= [1 - \frac{1}{2}I_{\{m/2\}}(x)] \left[\binom{m}{x} p^x (1-p)^{(m-x)} + \binom{m}{m-x} p^{(m-x)} (1-p)^x \right] = \\
 &= I_{[0, m/2[}(x) \left[\binom{m}{x} p^x (1-p)^{(m-x)} + \binom{m}{m-x} p^{(m-x)} (1-p)^x \right] + \\
 &\quad + I_{\{m/2\}}(x) \binom{m}{m/2} p^{m/2} (1-p)^{m/2},
 \end{aligned} \tag{4}$$

with $x \in \mathcal{X}$, and $I_{\mathcal{A}}(x) = 1$ if and only if $x \in \mathcal{A}$.

Our expression of the folded binomial pmf highlights that the $\delta_{x, m-x}$ term in Equation (1) accounts only for the case $x = m/2$. As stated, this latter case occurs only for m even. For m odd, the second term of the sum in Equation (4) simply cancels itself out.

Consequently, the cdf of X , $F_X(x)$ can be written as:

$$\begin{aligned}
 F_X(x) &= \sum_{k=0}^x \left\{ I_{[0, m/2[}(k) \left[\binom{m}{k} p^k (1-p)^{(m-k)} + \binom{m}{m-k} p^{(m-k)} (1-p)^k \right] + \right. \\
 &\quad \left. + I_{\{m/2\}}(k) \binom{m}{m/2} p^{m/2} (1-p)^{m/2} \right\} \quad 0 \leq x \leq m/2.
 \end{aligned} \tag{5}$$

Given that $F_X(m/2) = 1$ for m both even and odd, we eventually have:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^x \left[\binom{m}{k} p^k (1-p)^{(m-k)} + \binom{m}{m-k} p^{(m-k)} (1-p)^k \right] & 0 \leq x < m/2 \\ 1 & x \geq m/2 \end{cases} \tag{6}$$

This latter equation allows us to prove the following Lemma.

Lemma 1. *Let X be a folded binomial random variable, that is $X \sim fBin(m, p)$. Then, the cdf of X can be expressed as:*

$$F_X(x) = \begin{cases} 0 & x < 0 \\ F_{W_1}(1-p) + [1 - F_{W_2}(1-p)] & 0 \leq x < m/2 \\ 1 & x \geq m/2 \end{cases}$$

where $W_1 \sim Beta(m-x, x+1)$, and $W_2 \sim Beta(x+1, m-x)$.

Proof. By Equation (6), for $0 \leq x < m/2$ we have:

$$\begin{aligned}
 F_X(x) &= \sum_{k=0}^x \binom{m}{k} p^k (1-p)^{(m-k)} + \sum_{k=0}^x \binom{m}{m-k} p^{(m-k)} (1-p)^k = \\
 &= P(Y \leq x) + P(Y \geq m-x) = \\
 &= F_Y(x) + [1 - F_Y(m-x-1)],
 \end{aligned}$$

where $Y \sim \text{Bin}(m, p)$.

Then, the Lemma is established given the relationship between the cdf's of binomial and Beta random variables. \square

We now use Lemma 1 to present some results on stochastic ordering of folded binomials.

Theorem 1 (Stochastic ordering w.r.t. p). *Let X_1 and X_2 be two folded binomial distributions with m independent Bernoulli trials ($m > 1$) and probability of success p_1 and p_2 , respectively. That is, $X_1 \sim f\text{Bin}(m, p_1)$ and $X_2 \sim f\text{Bin}(m, p_2)$. If $p_1 < p_2$, then $X_1 \leq_{st} X_2$. That is, X_1 is stochastically smaller than or equal to X_2 .*

Proof. By definition of stochastic ordering (Equation 2), and by Equation (6), it is enough to evaluate for $0 \leq x < m/2$ the inequality:

$$F_{X_2}(x) \leq F_{X_1}(x) \quad p_1 < p_2,$$

or equivalently $F_{X_1}(x) - F_{X_2}(x) \geq 0$.

Let $W_1 \sim \text{Beta}(m-x, x+1)$ and $W_2 \sim \text{Beta}(x+1, m-x)$. By Lemma 1, the latter inequality is equivalent to:

$$F_{W_1}(1-p_1) - F_{W_1}(1-p_2) - [F_{W_2}(1-p_1) - F_{W_2}(1-p_2)] \geq 0.$$

That is,

$$\begin{aligned}
 &B(m-x, x+1) \int_{1-p_2}^{1-p_1} t^{m-x-1} (1-t)^x dt + \\
 &-B(x+1, m-x) \int_{1-p_2}^{1-p_1} t^x (1-t)^{m-x-1} dt \geq 0,
 \end{aligned} \tag{7}$$

where $B(\cdot, \cdot)$ denotes the Beta function.

As $B(\alpha, \beta) = B(\beta, \alpha)$ and the integrals in Equation (7) are defined over the same interval, the theorem holds if the inequality of the integrand functions

$$t^{m-x-1}(1-t)^x \geq t^x(1-t)^{m-x-1} \quad (8)$$

holds over all the possible values of $(1-p_2) < t < (1-p_1)$. As $0 \leq p_1 < p_2 \leq 1/2$, Equation (8) must be evaluated for $1/2 < t < 1$. It becomes $[t/(1-t)]^{m-2x-1} \geq 1$.

Given that $t/(1-t) > 1$, the inequality (8) is then verified when $m - 2x - 1 \geq 0$, i.e.:

$$x \leq (m-1)/2. \quad (9)$$

Hence, $F_{X_2}(x) \leq F_{X_1}(x)$ for $0 \leq x \leq (m-1)/2$.

Furthermore, we note that for m odd $F_{X_1}(\frac{m-1}{2}) = F_{X_2}(\frac{m-1}{2}) = 1$, and thus the ordering holds for any x .

For m even, folded binomials share the following cdf properties:

$$F_X\left(\frac{m-2}{2}\right) = F_X\left(\frac{m-1}{2}\right) = P(X < m/2). \quad (10)$$

Consequently, as by Equation (9) $F_{X_1}(\frac{m-2}{2}) \geq F_{X_2}(\frac{m-2}{2})$, X_1 and X_2 are stochastically ordered also for $(m-1)/2 \leq x < m/2$.

Finally, given that $F_X(m/2) = 1$, for $p_1 < p_2$, we have:

$$\begin{cases} F_{X_2}(x) = F_{X_1}(x) = 0 & x < 0 \\ F_{X_2}(x) \leq F_{X_1}(x) & 0 \leq x < m/2 \\ F_{X_2}(x) = F_{X_1}(x) = 1 & x \geq m/2 \end{cases}$$

□

Corollary 1 (Strict stochastic dominance w.r.t. p). *Let X_1 and X_2 be two folded binomial distributions with m independent Bernoulli trials ($m > 1$) and probability of success p_1 and p_2 , respectively. That is, $X_1 \sim fBin(m, p_1)$ and $X_2 \sim fBin(m, p_2)$. If $p_1 < p_2$, then X_2 strictly stochastically dominates X_1 .*

Proof. By definition of strict stochastic dominance (Equation 3), and considering Equation (8), we note that the strict inequality holds for $0 \leq x < (m-1)/2$. Hence:

$$F_{X_2}(x) < F_{X_1}(x) \quad 0 \leq x < (m-1)/2.$$

□

Theorem 2 (Stochastic ordering w.r.t. m). *Let X_1 and X_2 be two folded binomial distributions with m_1 and m_2 independent Bernoulli trials ($m_1 > 1, m_2 > 1$), respectively, and both with the same probability of success p , $0 \leq p \leq 1/2$. That is, $X_1 \sim fBin(m_1, p)$ and $X_2 \sim fBin(m_2, p)$. If $m_1 < m_2$, then $X_1 \leq_{st} X_2$. That is, X_1 is stochastically smaller than or equal to X_2 .*

Proof. By definition of stochastic ordering (Equation 2), by Equation (6), and as $m_1 < m_2$, it is enough to evaluate for $0 \leq x < m_2/2$ the inequality:

$$F_{X_2}(x) \leq F_{X_1}(x), \quad m_1 < m_2. \quad (11)$$

Given that $F_{X_1}(m_1/2) = 1$ and $F_{X_2}(m_1/2) \leq 1$, the ordering in Equation (11) holds for $m_1/2 \leq x < m_2/2$.

Hence, it is left to evaluate the inequality (11) for $0 \leq x < m_1/2$. Let $W_{11} \sim Beta(m_1 - x, x + 1)$, $W_{21} \sim Beta(x + 1, m_1 - x)$, $W_{12} \sim Beta(m_2 - x, x + 1)$, and $W_{22} \sim Beta(x + 1, m_2 - x)$. By Lemma 1, the inequality (11) is equivalent to:

$$F_{W_{11}}(1 - p) + [1 - F_{W_{21}}(1 - p)] - \{F_{W_{12}}(1 - p) + [1 - F_{W_{22}}(1 - p)]\} \geq 0$$

For $p = 0$, the equality holds. This corresponds to the fact that if $p = 0$, then $P(X = 0) = 1$, whatever the number of Bernoulli trials, and in this case we have $F_{X_1}(x) = F_{X_2}(x)$ for any x . For $0 < p \leq 1/2$, we have:

$$\begin{aligned} & B(m_1 - x, x + 1) \int_0^{1-p} t^{m_1-x-1} (1-t)^x dt - B(x + 1, m_1 - x) \int_0^{1-p} t^x (1-t)^{m_1-x-1} dt + \\ & - \left[B(m_2 - x, x + 1) \int_0^{1-p} t^{m_2-x-1} (1-t)^x dt - B(x + 1, m_2 - x) \int_0^{1-p} t^x (1-t)^{m_2-x-1} dt \right] \geq 0 \end{aligned}$$

Given that $B(\alpha, \beta) = B(\beta, \alpha)$, and that the integrals are defined over the same integrating interval, we may equivalently write:

$$\begin{aligned} & B(m_1 - x, x + 1) \int_0^{1-p} [t^{m_1-x-1} (1-t)^x - t^x (1-t)^{m_1-x-1}] dt + \\ & - B(m_2 - x, x + 1) \int_0^{1-p} [t^{m_2-x-1} (1-t)^x - t^x (1-t)^{m_2-x-1}] dt \geq 0 \end{aligned}$$

For the sake of simplicity, let us for a while set:

$$\Delta(m_j) = \int_0^{1-p} [t^{m_j-x-1}(1-t)^x - t^x(1-t)^{m_j-x-1}]dt \quad j = 1, 2.$$

The inequality (11) thus holds if:

$$B(m_1 - x, x + 1)\Delta(m_1) \geq B(m_2 - x, x + 1)\Delta(m_2), \quad (12)$$

for any $m_2 > m_1$, for any $0 \leq x < m_1/2$, and for any $0 < p \leq 1/2$. This is always true, as both $B(m_1 - x, x + 1) > B(m_2 - x, x + 1)$ and $\Delta(m_1) > \Delta(m_2)$.

Let us for a while focus on

$$B(m_1 - x, x + 1) > B(m_2 - x, x + 1).$$

Given that $Beta(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, where $\Gamma(\cdot)$ is the *Gamma* function, we have:

$$\frac{\Gamma(m_1 - x)\Gamma(x + 1)}{\Gamma(m_1 + 1)} > \frac{\Gamma(m_2 - x)\Gamma(x + 1)}{\Gamma(m_2 + 1)}.$$

That is:

$$\frac{\Gamma(m_2 + 1)}{\Gamma(m_1 + 1)} > \frac{\Gamma(m_2 - x)}{\Gamma(m_1 - x)}.$$

Given that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, this inequality holds for any $0 \leq x < m_1/2$ and for $m_2 > m_1$. First, let us consider the case of $m_2 = m_1 + 1$. We have:

$$\frac{\Gamma(m_1 + 2)}{\Gamma(m_1 + 1)} > \frac{\Gamma(m_1 - x + 1)}{\Gamma(m_1 - x)},$$

i.e.

$$(m_1 + 1)\frac{\Gamma(m_1 + 1)}{\Gamma(m_1 + 1)} > (m_1 - x)\frac{\Gamma(m_1 - x)}{\Gamma(m_1 - x)},$$

and hence $(m_1 + 1) > (m_1 - x)$. Analogously, it can be shown that the inequality holds even more so for any $m_2 > m_1 + 1$.

Consider now the inequality $\Delta(m_1) > \Delta(m_2)$. As these integrals are defined on the same integrating interval, we can equivalently compare the integrand function for $0 < t < (1 - p) < 1$. We will have $\Delta(m_1) > \Delta(m_2)$ if:

$$[t^{m_1-x-1}(1-t)^x - t^x(1-t)^{m_1-x-1}] - [t^{m_2-x-1}(1-t)^x - t^x(1-t)^{m_2-x-1}] > 0. \quad (13)$$

As $0 < t < 1$, we have:

$$\begin{aligned} & t^{m_1} \frac{(1-t)^x}{t^{x+1}} - (1-t)^{m_1} \frac{t^x}{(1-t)^{x+1}} - \left[t^{m_2} \frac{(1-t)^x}{t^{x+1}} - (1-t)^{m_2} \frac{t^x}{(1-t)^{x+1}} \right] = \\ &= \frac{(1-t)^x}{t^{x+1}} (t^{m_1} - t^{m_2}) - \left\{ \frac{t^x}{(1-t)^{x+1}} [(1-t)^{m_1} - (1-t)^{m_2}] \right\} = \\ &= \frac{(1-t)^x (1-t)^{x+1} (t^{m_1} - t^{m_2}) - t^x t^{x+1} [(1-t)^{m_1} - (1-t)^{m_2}]}{[(1-t)t]^{x+1}}. \end{aligned}$$

Hence, Equation (13) is equivalent to:

$$(1-t)^{2x+1} (t^{m_1} - t^{m_2}) - t^{2x+1} [(1-t)^{m_1} - (1-t)^{m_2}] > 0.$$

That is,

$$(1-t)^{2x+1} t^{m_1} (1 - t^{m_2-m_1}) > t^{2x+1} (1-t)^{m_1} [1 - (1-t)^{m_2-m_1}].$$

In other words, the theorem holds if the following inequality holds:

$$\left(\frac{1-t}{t} \right)^{2x+1} \left(\frac{t}{1-t} \right)^{m_1} \frac{(1-t^{m_2-m_1})}{[1 - (1-t)^{m_2-m_1}]} > 0$$

For $0 < t < 1$, the quantities $\left(\frac{1-t}{t} \right)^{2x+1}$ and $\left(\frac{t}{1-t} \right)^{m_1}$ are both positive, and it is enough to evaluate only if:

$$\frac{(1-t^{m_2-m_1})}{[1 - (1-t)^{m_2-m_1}]} > 0$$

As $m_2 - m_1 > 0$, this holds. We have both $0 < t^{m_2-m_1} < 1$ and $0 < (1-t)^{m_2-m_1} < 1$, since $0 < t < 1$.

□

Corollary 2 (Strict stochastic dominance w.r.t. m). *Let X_1 and X_2 be two folded binomial distributions with m_1 and m_2 independent Bernoulli trials ($m_1 > 1, m_2 > 1$), respectively, and both with the same probability of success p , $0 < p \leq 1/2$. That is, $X_1 \sim fBin(m_1, p)$ and $X_2 \sim fBin(m_2, p)$. If $m_1 < m_2$, then X_2 strictly stochastically dominates X_1 .*

Proof. By definition of strict stochastic dominance (Equation 3), and considering Equation (12), it has already been proved that the strict inequality holds for $0 \leq x < m_1/2$. Hence:

$$F_{X_2}(x) < F_{X_1}(x) \quad 0 \leq x < m_1/2.$$

□

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