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**Absence of exponentially localized solitons
for the Novikov-Veselov equation at positive energy
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Abstract. In this note we show that the Novikov-Veselov equation (NV-equation) at positive energy (an analog of KdV in 2+1 dimensions) has no exponentially localized solitons in the two-dimensional sense.

1. Introduction and Theorem 1. We consider the following 2+1 - dimensional analog of the KdV equation (Novikov-Veselov equation):

$$\begin{aligned} \partial_t v &= 4\operatorname{Re}(4\partial_z^3 v + \partial_z(vw) - E\partial_z w), \\ \partial_{\bar{z}} w &= -3\partial_z v, \quad v = \bar{v}, \quad E \in \mathbb{R}, \\ v &= v(x, t), \quad w = w(x, t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \end{aligned} \tag{1}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \partial_{\bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right). \tag{2}$$

We assume that

$$\begin{aligned} v &\text{ is sufficiently regular and has sufficient decay as } |x| \rightarrow \infty, \\ w &\text{ is decaying as } |x| \rightarrow \infty. \end{aligned} \tag{3}$$

Equation (1) is contained implicitly in the paper of S.V.Manakov [M] as an equation possessing the following representation

$$\frac{\partial(L - E)}{\partial t} = [L - E, A] + B(L - E), \tag{4}$$

where $L = -\Delta + v(x, t)$, $\Delta = 4\partial_z\partial_{\bar{z}}$, A and B are suitable differential operators of the third and zero order respectively, $[\cdot, \cdot]$ denotes the commutator. Equation (1) was written in an explicit form by S.P.Novikov and A.P.Veselov in [NV1], [NV2], where higher analogs of (1) were also constructed.

For the case when

$$v(x_1, x_2, t), \quad w(x_1, x_2, t) \text{ are independent of } x_2 \tag{5}$$

equation (1) is reduced to

$$\partial_t v = 2\partial_x^3 v - 12v\partial_x v + 6E\partial_x v, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{6}$$

In terms of $u(x, t)$ such that

$$v(x, t) = u(-2t, x + 6Et), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (7)$$

equation (6) takes the standard form of the KdV equation (see [NMPZ]):

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \quad (8)$$

It is well-known (see [NMPZ]) that (8) has the soliton solutions

$$u(x, t) = u_{\kappa, \varphi}(x - 4\kappa^2 t) = -\frac{2\kappa^2}{ch^2(\kappa(x - 4\kappa^2 t - \varphi))}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad \kappa \in]0, +\infty[, \quad \varphi \in \mathbb{R}. \quad (9)$$

In addition, one can see that

$$\begin{aligned} u_{\kappa, \varphi} &\in C^\infty(\mathbb{R}), \\ \partial_x^j u_{\kappa, \varphi}(x) &= O(e^{-2\kappa|x|}) \quad \text{as } x \rightarrow \infty, \quad j = 0, 1, 2, 3, \dots \end{aligned} \quad (10)$$

Properties (10) show, in particular, that the solitons of (9) are exponentially localized in x .

In the present note we obtain, in particular, the following result:

Theorem 1. *Let v, w satisfy (1) for $E = E_{fix} > 0$, where*

$$\begin{aligned} v(x, t) &= V(x - ct), \quad x \in \mathbb{R}^2, \quad c = (c_1, c_2) \in \mathbb{R}^2, \\ V &\in C^3(\mathbb{R}^2), \quad \partial_x^j V(x) = O(e^{-\alpha|x|}) \quad \text{for } |x| \rightarrow \infty, \quad |j| \leq 3 \quad \text{and some } \alpha > 0, \end{aligned} \quad (11a)$$

(where $j = (j_1, j_2) \in (0 \cup \mathbb{N})^2$, $|j| = |j_1| + |j_2|$, $\partial_x^j = \partial^{j_1+j_2} / \partial x_1^{j_1} \partial x_2^{j_2}$),

$$w(\cdot, t) \in C(\mathbb{R}^2), \quad w(x, t) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad t \in \mathbb{R}. \quad (11b)$$

Then $V \equiv 0$, $v \equiv 0$, $w \equiv 0$.

Theorem 1 shows that equation (1) has no nonzero solitons (travel wave solutions) exponentially localized in x in the two-dimensional sense.

The proof of Theorem 1 is based on Proposition 1 and Proposition 2, see Section 4. In turn, Proposition 2 is based, in particular, on Lemma 1 and Lemma 2.

Lemma 1, Lemma 2 and Proposition 1 are recalled in Section 2. Proposition 2 is given in Section 3. It seems that the result of Proposition 2 (that sufficiently localized travel wave solutions for the NV-equation (1) for $E = E_{fix} > 0$ have zero scattering amplitude for the two-dimensional Schrödinger equation (12)) was not yet formulated in the literature.

2. *Lemma 1, Lemma 2 and Proposition 1.* Consider the equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^2, \quad E = E_{fix} > 0, \quad (12)$$

where

$$\begin{aligned} v(x) &= \overline{v(x)}, \quad x \in \mathbb{R}^2, \\ (1 + |x|)^{2+\varepsilon} v(x) &\in L^\infty(\mathbb{R}^2) \quad (\text{as a function of } x \in \mathbb{R}^2) \quad \text{for some } \varepsilon > 0. \end{aligned} \quad (13)$$

It is known that for any $k \in \mathbb{R}^2$, such that $k^2 = E$, there exists a unique bounded solution $\psi^+(x, k)$ of equation (12) with the following asymptotics:

$$\psi^+(x, k) = e^{ikx} - i\pi\sqrt{2\pi}e^{-i\pi/4}f(k, |k|\frac{x}{|x|})\frac{e^{i|k||x|}}{\sqrt{|k||x|}} + o\left(\frac{1}{\sqrt{|x|}}\right) \quad \text{as } |x| \rightarrow \infty. \quad (14)$$

This solution describes scattering of incident plane wave e^{ikx} on the potential v . The function f on

$$\mathcal{M}_E = \{k \in \mathbb{R}^2, l \in \mathbb{R}^2 : k^2 = l^2 = E\} \quad (15)$$

arising in (14) is the scattering amplitude for v in the framework of equation (12). Under assumptions (13), it is known, in particular, that

$$f \in C(\mathcal{M}_E). \quad (16)$$

Lemma 1. *Let v satisfy (13) and $v_y, y \in \mathbb{R}^2$, be defined by*

$$v_y(x) = v(x - y), \quad x \in \mathbb{R}^2. \quad (17)$$

Then the scattering amplitude f for v and the scattering amplitude f_y for v_y are related by the formula

$$f_y(k, l) = f(k, l)e^{iy(k-l)}, \quad (k, l) \in \mathcal{M}_E, \quad y = (y_1, y_2) \in \mathbb{R}^2. \quad (18)$$

Lemma 1 follows, for example, from the definition of the scattering amplitude by means of (14) and the fact that $\psi^+(x - y, k)$ solves (12) for v replaced by v_y , where $k^2 = E$.

Lemma 1 was given, for example, in [N3].

Lemma 2. *Let v, w satisfy (1), (3), where $E = E_{fix} > 0$. Then the scattering amplitude $f(\cdot, \cdot, t)$ for $v(\cdot, t)$ and the scattering amplitude $f(\cdot, \cdot, 0)$ for $v(\cdot, 0)$ are related by*

$$f(k, l, t) = f(k, l, 0) \exp[2it(k_1^3 - 3k_1k_2^2 - l_1^3 + 3l_1l_2^2)], \quad (k, l) \in \mathcal{M}_E, \quad t \in \mathbb{R}. \quad (19)$$

Lemma 2 was given for the first time in [N1].

Note that in the framework of Lemma 2 properties (3) can be specified as follows:

$$\begin{aligned} v, w &\in C(\mathbb{R}^2 \times \mathbb{R}) \quad \text{and for each } t \in \mathbb{R} \quad \text{the following properties are fulfilled :} \\ v(\cdot, t) &\in C^3(\mathbb{R}^2), \quad \partial_x^j v(x, t) = O(|x|^{-2-\varepsilon}) \quad \text{for } |x| \rightarrow \infty, \quad |j| \leq 3 \quad \text{and some } \varepsilon > 0, \\ w(x, t) &\rightarrow 0 \quad \text{for } |x| \rightarrow \infty. \end{aligned} \quad (20)$$

Proposition 1. *Let*

$$v(x) = \overline{v(x)}, \quad e^{\alpha|x|}v(x) \in L^\infty(\mathbb{R}^2) \quad (\text{as a function of } x) \quad \text{for some } \alpha > 0 \quad (21)$$

and the scattering amplitude $f \equiv 0$ on \mathcal{M}_E for this potential for some $E = E_{fix} > 0$. Then $v \equiv 0$ in $L^\infty(\mathbb{R}^2)$.

In the general case the result of Proposition 1 was given for the first time in [GN]. Under the additional assumption that v is sufficiently small (in comparison with E) the result of Proposition 1 was given for the first time in [N2]-[N4].

3. *Transparency of solitons.* In this section we show that sufficiently localized solitons (travel wave solutions) for the NV-equation (1) for $E = E_{fix} > 0$ have zero scattering amplitude for the two-dimensional Schrödinger equation (12).

Proposition 2. *Let v, w satisfy (1) for $E = E_{fix} > 0$, where*

$$\begin{aligned} v(x, t) &= V(x - ct), \quad x \in \mathbb{R}^2, \quad c = (c_1, c_2) \in \mathbb{R}^2, \\ V &\in C^3(\mathbb{R}^2), \quad \partial_x^j V(x) = O(|x|^{-2-\varepsilon}) \quad \text{for } |x| \rightarrow \infty, \quad |j| \leq 3 \quad \text{and some } \varepsilon > 0, \end{aligned} \quad (22a)$$

$$w(\cdot, t) \in C(\mathbb{R}^2), \quad w(x, t) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad t \in \mathbb{R}. \quad (22b)$$

Then

$$f \equiv 0 \quad \text{on } \mathcal{M}_E, \quad (23)$$

where f is the scattering amplitude for $v(x) = V(x)$ in the framework of the Schrödinger equation (12).

The proof of Proposition 2 consists in the following.

We consider

$$T = \{\lambda \in \mathbb{C} : |\lambda| = 1\}. \quad (24)$$

We use that

$$\mathcal{M}_E \approx T \times T, \quad E = E_{fix} > 0, \quad (25)$$

where diffeomorphism (25) is given by the formulas:

$$\lambda = \frac{k_1 + ik_2}{\sqrt{E}}, \quad \lambda' = \frac{l_1 + il_2}{\sqrt{E}}, \quad (k, l) \in \mathcal{M}_E, \quad (26)$$

$$\begin{aligned} k_1 &= \frac{\sqrt{E}}{2} \left(\lambda + \frac{1}{\lambda} \right), \quad k_2 = \frac{i\sqrt{E}}{2} \left(\frac{1}{\lambda} - \lambda \right), \\ l_1 &= \frac{\sqrt{E}}{2} \left(\lambda' + \frac{1}{\lambda'} \right), \quad l_2 = \frac{i\sqrt{E}}{2} \left(\frac{1}{\lambda'} - \lambda' \right), \quad (\lambda, \lambda') \in T \times T. \end{aligned} \quad (27)$$

We use that in the variables λ, λ' formulas (18), (19) take the form

$$f_y(\lambda, \lambda', E) = f(\lambda, \lambda', E) \exp\left[\frac{i}{2}\sqrt{E}\left(\lambda\bar{y} + \frac{y}{\lambda} - \lambda'\bar{y} - \frac{y}{\lambda'}\right)\right], \quad (28)$$

where $(\lambda, \lambda') \in T \times T$, y is considered as $y = y_1 + iy_2$,

$$f(\lambda, \lambda', E, t) = f(\lambda, \lambda', E, 0) \exp\left[iE^{3/2}t\left(\lambda^3 + \frac{1}{\lambda^3} - (\lambda')^3 - \left(\frac{1}{\lambda'}\right)^3\right)\right], \quad (29)$$

where $(\lambda, \lambda') \in T \times T$, $t \in \mathbb{R}$.

The assumptions of Proposition 2 and Lemmas 1 and 2 (with (18), (19) written as (28), (29)) imply that

$$\begin{aligned} f(\lambda, \lambda', E) \exp\left[\frac{i}{2}\sqrt{E}t\left(\lambda\bar{c} + \frac{c}{\lambda} - \lambda'\bar{c} - \frac{c}{\lambda'}\right)\right] = \\ f(\lambda, \lambda', E) \exp\left[iE^{3/2}t\left(\lambda^3 + \frac{1}{\lambda^3} - (\lambda')^3 - \left(\frac{1}{\lambda'}\right)^3\right)\right] \end{aligned} \quad (30)$$

for $(\lambda, \lambda') \in T \times T$, $t \in \mathbb{R}$, where f is the scattering amplitude for $v(x, 0) = V(x)$, c is considered as $c = c_1 + ic_2$.

Property (16), identity (30) and the fact that $\lambda^3, \lambda^{-3}, \lambda, \lambda^{-1}, 1$ are linear independent on each nonempty open subset of T imply (23).

4. *Proof of Theorem 1 and final remark.* Theorem 1 follows from Proposition 1 and Proposition 2.

Finally, note that the result of Theorem 1 does not hold, in general, without the assumption that $V(x) = O(e^{-\alpha|x|})$ as $|x| \rightarrow \infty$ for some $\alpha > 0$: "counter examples" to Theorem 1 with rational bounded V decaying at infinity as $O(|x|^{-2})$ are implicitly contained in [G].

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