



HAL
open science

Absence of exponentially localized solitons for the Manakov-Novikov-Veselov equation at positive energy

Roman Novikov

► **To cite this version:**

Roman Novikov. Absence of exponentially localized solitons for the Manakov-Novikov-Veselov equation at positive energy. 2010. hal-00523343v2

HAL Id: hal-00523343

<https://hal.science/hal-00523343v2>

Preprint submitted on 6 Oct 2010 (v2), last revised 24 Jan 2011 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**Absence of exponentially localized solitons
for the Manakov-Novikov-Veselov equation at positive energy
R.G. Novikov**

CNRS (UMR 7641), Centre de Mathématiques Appliquées, Ecole Polytechnique,
91128 Palaiseau, France
e-mail: novikov@cmap.polytechnique.fr

Abstract. In this note we show that the Manakov-Novikov-Veselov equation (MNV-equation) at positive energy (an analog of KdV in 2+1 dimensions) has no exponentially localized solitons in the two-dimensional sense.

1. Introduction and Theorem 1. We consider the following 2+1 - dimensional analog of the KdV equation:

$$\begin{aligned} \partial_t v &= 4\operatorname{Re}(4\partial_z^3 v + \partial_z(vw) - E\partial_z w), \\ \partial_z w &= -3\partial_z v, \quad v = \bar{v}, \quad E \in \mathbb{R}, \\ v &= v(x, t), \quad w = w(x, t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \end{aligned} \tag{1}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right). \tag{2}$$

We assume that

$$\begin{aligned} v &\text{ is sufficiently regular and has sufficient decay as } |x| \rightarrow \infty, \\ w &\text{ is decaying as } |x| \rightarrow \infty. \end{aligned} \tag{3}$$

Equation (1) is contained implicitly in the paper of S.V.Manakov [M] as an equation possessing the following representation

$$\frac{\partial(L - E)}{\partial t} = [L - E, A] + B(L - E), \tag{4}$$

where $L = -\Delta + v(x, t)$, $\Delta = 4\partial_z\partial_{\bar{z}}$, A and B are suitable differential operators of the third and zero order respectively, $[\cdot, \cdot]$ denotes the commutator. Equation (1) was written in an explicit form by S.P.Novikov and A.P.Veselov in [NV1], [NV2], where higher analogs of (1) were also constructed.

For the case when

$$v(x_1, x_2, t), \quad w(x_1, x_2, t) \text{ are independent of } x_2 \tag{5}$$

equation (1) is reduced to

$$\partial_t v = 2\partial_x^3 v - 12v\partial_x v + 6E\partial_x v, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{6}$$

In terms of $u(x, t)$ such that

$$v(x, t) = u(-2t, x + 6Et), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (7)$$

equation (6) takes the standard form of the KdV equation (see [NMPZ]):

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \quad (8)$$

It is well-known (see [NMPZ]) that (8) has the soliton solutions

$$u(x, t) = u_{\kappa, \varphi}(x - 4\kappa^2 t) = -\frac{2\kappa^2}{ch^2(\kappa(x - 4\kappa^2 t - \varphi))}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad \kappa \in]0, +\infty[, \quad \varphi \in \mathbb{R}. \quad (9)$$

In addition, one can see that

$$\begin{aligned} u_{\kappa, \varphi} &\in C^\infty(\mathbb{R}), \\ \partial_x^j u_{\kappa, \varphi}(x) &= O(e^{-2\kappa|x|}) \quad \text{as } x \rightarrow \infty, \quad j = 0, 1, 2, 3, \dots \end{aligned} \quad (10)$$

Properties (10) show, in particular, that the solitons of (9) are exponentially localized in x .

In the present note we obtain, in particular, the following result:

Theorem 1. *Let v, w satisfy (1) for $E = E_{fix} > 0$, where*

$$\begin{aligned} v(x, t) &= V(x - ct), \quad x \in \mathbb{R}^2, \quad c = (c_1, c_2) \in \mathbb{R}^2, \\ V &\in C^3(\mathbb{R}^2), \quad \partial_x^j V(x) = O(e^{-\alpha|x|}) \quad \text{for } |x| \rightarrow \infty, \quad |j| \leq 3 \quad \text{and some } \alpha > 0, \end{aligned} \quad (11a)$$

(where $j = (j_1, j_2) \in (0 \cup \mathbb{N})^2$, $|j| = |j_1| + |j_2|$, $\partial_x^j = \partial^{j_1+j_2} / \partial x_1^{j_1} \partial x_2^{j_2}$),

$$w(\cdot, t) \in C(\mathbb{R}^2), \quad w(x, t) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad t \in \mathbb{R}. \quad (11b)$$

Then $V \equiv 0$, $v \equiv 0$, $w \equiv 0$.

Theorem 1 shows that equation (1) has no nonzero solitons (travel wave solutions) exponentially localized in x in the two-dimensional sense.

The proof of Theorem 1 is based on Proposition 1 and Proposition 2, see Section 4. In turn, Proposition 2 is based, in particular, on Lemma 1 and Lemma 2.

Lemma 1, Lemma 2 and Proposition 1 are recalled in Section 2. Proposition 2 is given in Section 3. It seems that the result of Proposition 2 (that sufficiently localized travel wave solutions for the MNV-equation (1) for $E = E_{fix} > 0$ have zero scattering amplitude for the two-dimensional Schrödinger equation (12)) was not yet formulated in the literature.

2. *Lemma 1, Lemma 2 and Proposition 1.* Consider the equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^2, \quad E = E_{fix} > 0, \quad (12)$$

where

$$\begin{aligned} v(x) &= \overline{v(x)}, \quad x \in \mathbb{R}^2, \\ (1 + |x|)^{2+\varepsilon} v(x) &\in L^\infty(\mathbb{R}^2) \quad (\text{as a function of } x \in \mathbb{R}^2) \quad \text{for some } \varepsilon > 0. \end{aligned} \quad (13)$$

It is known that for any $k \in \mathbb{R}^2$, such that $k^2 = E$, there exists an unique bounded solution $\psi^+(x, k)$ of equation (12) with the following asymptotics:

$$\psi^+(x, k) = e^{ikx} - i\pi\sqrt{2\pi}e^{-i\pi/4}f(k, |k|\frac{x}{|x|})\frac{e^{i|k||x|}}{\sqrt{|k||x|}} + o\left(\frac{1}{\sqrt{|x|}}\right) \quad \text{as } |x| \rightarrow \infty. \quad (14)$$

This solution describes scattering of incident plane wave e^{ikx} on the potential v . The function f on

$$\mathcal{M}_E = \{k \in \mathbb{R}^2, l \in \mathbb{R}^2 : k^2 = l^2 = E\} \quad (15)$$

arising in (14) is the scattering amplitude for v in the framework of equation (12). Under assumptions (13), it is known, in particular, that

$$f \in C(\mathcal{M}_E). \quad (16)$$

Lemma 1. *Let v satisfy (13) and $v_y, y \in \mathbb{R}^2$, be defined by*

$$v_y(x) = v(x - y), \quad x \in \mathbb{R}^2. \quad (17)$$

Then the scattering amplitude f for v and the scattering amplitude f_y for v_y are related by the formula

$$f_y(k, l) = f(k, l)e^{iy(k-l)}, \quad (k, l) \in \mathcal{M}_E, \quad y = (y_1, y_2) \in \mathbb{R}^2. \quad (18)$$

Lemma 1 follows, for example, from the definition of the scattering amplitude by means of (14) and the fact that $\psi^+(x - y, k)$ solves (12) for v replaced by v_y , where $k^2 = E$.

Lemma 1 was given, for example, in [N3].

Lemma 2. *Let v, w satisfy (1), (3), where $E = E_{fix} > 0$. Then the scattering amplitude $f(\cdot, \cdot, t)$ for $v(\cdot, t)$ and the scattering amplitude $f(\cdot, \cdot, 0)$ for $v(\cdot, 0)$ are related by*

$$f(k, l, t) = f(k, l, 0) \exp[2it(k_1^3 - 3k_1k_2^2 - l_1^3 + 3l_1l_2^2)], \quad (k, l) \in \mathcal{M}_E, \quad t \in \mathbb{R}. \quad (19)$$

Lemma 2 was given for the first time in [N1].

Note that in the framework of Lemma 2 properties (3) can be specified as follows:

$$\begin{aligned} v, w &\in C(\mathbb{R}^2 \times \mathbb{R}) \quad \text{and for each } t \in \mathbb{R} \quad \text{the following properties are fulfilled :} \\ v(\cdot, t) &\in C^3(\mathbb{R}^2), \quad \partial_x^j v(x, t) = O(|x|^{-2-\varepsilon}) \quad \text{for } |x| \rightarrow \infty, \quad |j| \leq 3 \quad \text{and some } \varepsilon > 0, \\ w(x, t) &\rightarrow 0 \quad \text{for } |x| \rightarrow \infty. \end{aligned} \quad (20)$$

Proposition 1. *Let*

$$v(x) = \overline{v(x)}, \quad e^{\alpha|x|}v(x) \in L^\infty(\mathbb{R}^2) \quad (\text{as a function of } x) \quad \text{for some } \alpha > 0 \quad (21)$$

and the scattering amplitude $f \equiv 0$ on \mathcal{M}_E for this potential for some $E = E_{fix} > 0$. Then $v \equiv 0$ in $L^\infty(\mathbb{R}^2)$.

In the general case the result of Proposition 1 was given for the first time in [GN]. Under the additional assumption that v is sufficiently small (in comparison with E) the result of Proposition 1 was given for the first time in [N2]-[N4].

3. *Transparency of solitons.* In this section we show that sufficiently localized solitons (travel wave solutions) for the MNV-equation (1) for $E = E_{fix} > 0$ have zero scattering amplitude for the two-dimensional Schrödinger equation (12).

Proposition 2. *Let v, w satisfy (1) for $E = E_{fix} > 0$, where*

$$\begin{aligned} v(x, t) &= V(x - ct), \quad x \in \mathbb{R}^2, \quad c = (c_1, c_2) \in \mathbb{R}^2, \\ V &\in C^3(\mathbb{R}^2), \quad \partial_x^j V(x) = O(|x|^{-2-\varepsilon}) \quad \text{for } |x| \rightarrow \infty, \quad |j| \leq 3 \quad \text{and some } \varepsilon > 0, \end{aligned} \quad (22a)$$

$$w(\cdot, t) \in C(\mathbb{R}^2), \quad w(x, t) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad t \in \mathbb{R}. \quad (22b)$$

Then

$$f \equiv 0 \quad \text{on } \mathcal{M}_E, \quad (23)$$

where f is the scattering amplitude for $v(x) = V(x)$ in the framework of the Schrödinger equation (12).

The proof of Proposition 2 consists in the following.

We consider

$$T = \{\lambda \in \mathbb{C} : |\lambda| = 1\}. \quad (24)$$

We use that

$$\mathcal{M}_E \approx T \times T, \quad E = E_{fix} > 0, \quad (25)$$

where diffeomorphism (25) is given by the formulas:

$$\lambda = \frac{k_1 + ik_2}{\sqrt{E}}, \quad \lambda' = \frac{l_1 + il_2}{\sqrt{E}}, \quad (k, l) \in \mathcal{M}_E, \quad (26)$$

$$\begin{aligned} k_1 &= \frac{\sqrt{E}}{2} \left(\lambda + \frac{1}{\lambda} \right), \quad k_2 = \frac{i\sqrt{E}}{2} \left(\frac{1}{\lambda} - \lambda \right), \\ l_1 &= \frac{\sqrt{E}}{2} \left(\lambda' + \frac{1}{\lambda'} \right), \quad l_2 = \frac{i\sqrt{E}}{2} \left(\frac{1}{\lambda'} - \lambda' \right), \quad (\lambda, \lambda') \in T \times T. \end{aligned} \quad (27)$$

We use that in the variables λ, λ' formulas (18), (19) take the form

$$f_y(\lambda, \lambda', E) = f(\lambda, \lambda', E) \exp\left[\frac{i}{2}\sqrt{E}\left(\lambda\bar{y} + \frac{y}{\lambda} - \lambda'\bar{y} - \frac{y}{\lambda'}\right)\right], \quad (28)$$

where $(\lambda, \lambda') \in T \times T$, y is considered as $y = y_1 + iy_2$,

$$f(\lambda, \lambda', E, t) = f(\lambda, \lambda', E, 0) \exp\left[iE^{3/2}t\left(\lambda^3 + \frac{1}{\lambda^3} - (\lambda')^3 - \left(\frac{1}{\lambda'}\right)^3\right)\right], \quad (29)$$

where $(\lambda, \lambda') \in T \times T$, $t \in \mathbb{R}$.

The assumptions of Proposition 2 and Lemmas 1 and 2 (with (18), (19) written as (28), (29)) imply that

$$\begin{aligned} f(\lambda, \lambda', E) \exp\left[\frac{i}{2}\sqrt{E}t\left(\lambda\bar{c} + \frac{c}{\lambda} - \lambda'\bar{c} - \frac{c}{\lambda'}\right)\right] = \\ f(\lambda, \lambda', E) \exp\left[iE^{3/2}t\left(\lambda^3 + \frac{1}{\lambda^3} - (\lambda')^3 - \left(\frac{1}{\lambda'}\right)^3\right)\right] \end{aligned} \quad (30)$$

for $(\lambda, \lambda') \in T \times T$, $t \in \mathbb{R}$, where f is the scattering amplitude for $v(x, 0) = V(x)$, c is considered as $c = c_1 + ic_2$.

Property (16), identity (30) and the fact that $\lambda^3, \lambda^{-3}, \lambda, \lambda^{-1}, 1$ are linear independent on each nonempty open subset of T imply (23).

4. *Proof of Theorem 1 and final remark.* Theorem 1 follows from Proposition 1 and Proposition 2.

Finally, note that the result of Theorem 1 does not hold, in general, without the assumption that $V(x) = O(e^{-\alpha|x|})$ as $|x| \rightarrow \infty$ for some $\alpha > 0$: "counter examples" to Theorem 1 with rational bounded V decaying at infinity as $O(|x|^{-2})$ are implicitly contained in [G].

References

- [G] P.G.Grinevich, Rational solitons of the Veselov-Novikov equation - Two-dimensional potentials that are reflectionless for fixed energy, Teoret. i Mat. Fiz. **69**(2), 307-310 (1986) (in Russian); English translation: Theoret. and Math.Phys. **69**, 1170-1172 (1986).
- [GN] P.G.Grinevich, R.G.Novikov, Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials. Commun.Math.Phys. **174**, 400-446 (1995).
- [M] S.V.Manakov, The inverse scattering method and two-dimensional evolution equations, Uspekhi Mat.Nauk **31**(5), 245-246 (1976) (in Russian)
- [N1] R.G.Novikov, Construction of a two-dimensional Schrödinger operator with a given scattering amplitude at fixed energy, Teoret. i Mat Fiz **66**(2), 234-240 (1986) (in Russian); English translation: Theoret. and Math.Phys. **66**, 154-158 (1986).
- [N2] R.G.Novikov, Reconstruction of a two-dimensional Schrödinger operator from the scattering amplitude at fixed energy, Funkt.Anal. i Pril. **20**(3), 90-91 (1986) (in Russian); English translation: Funct.Anal. and Appl. **20**, 246-248 (1986).
- [N3] R.G.Novikov, Inverse scattering problem for the two-dimensional Schrödinger equation at fixed energy and nonlinear equations, PhD Thesis, Moscow State University 1989 (in Russian)
- [N4] R.G.Novikov, The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator, J.Funct.Anal. **103**, 409-463 (1992).

- [NMPZ] S.Novikov, S.V.Manakov, L.P.Pitaevskii, V.Z.Zakharov, Theory of solitons: the inverse scattering method, Springer, 1984.
- [NV1] S.P.Novikov, A.P.Veselov, Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formula and evolution equations. Dokl.Akad.Nauk. SSSR **279**, 20-24 (1984) (in Russian); English translation: Sov.Math.Dokl. **30**, 588-591 (1984).
- [NV2] S.P.Novikov, A.P.Veselov, Finite-zone, two-dimensional Schrödinger operators. Potential operators. Dokl.Akad.Nauk. SSSR **279**, 784-788 (1984) (in Russian); English translation: Sov.Math.Dokl. **30**, 705-708 (1984).