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Abstract

Electromagnetic circuits are the electromagnetic analog at fixed frequency of mass-spring networks in elastodynamics. By interchanging the roles of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ in electromagnetic circuits one obtains magnetoelectric circuits. Here we show that by introducing tetrahedral connectors having $\boldsymbol{\varepsilon} = \boldsymbol{\mu} = 0$ one can join electromagnetic and magnetoelectric circuits to obtain hybrid circuits. Their response is governed by a symmetric matrix with negative semidefinite imaginary part. Conversely given any such matrix a recipe is given for constructing a hybrid circuit which has that matrix as its response matrix.

Key words: Electromagnetic circuits, Electromagnetism, Circuits, Metamaterials
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At a given fixed frequency ω Maxwell's equations, which can be written in the form

$$\frac{\partial}{\partial x_i}\left(C_{ijk\ell}\frac{\partial E_\ell}{\partial x_k}\right) + \{i\omega\mathbf{j}\}_j = -\{\omega^2\boldsymbol{\varepsilon}\mathbf{E}\}_j \quad (1)$$

where $C_{ijk\ell} = e_{ijm}e_{k\ell n}\{\boldsymbol{\mu}^{-1}\}_{mn}$ [in which $\mathbf{E}$ is the electric field, $\mathbf{j}$ is the free current density $\boldsymbol{\varepsilon}$ the electric permittivity tensor, $\boldsymbol{\mu}$ the magnetic permeability tensor, and $e_{ijm} = 1$ (-1) if ijm is an even (odd) permutation of 123 and is zero otherwise] bear a close resemblance to the equations of continuum elastodynamics

$$\frac{\partial}{\partial x_i}\left(C_{ijk\ell}\frac{\partial u_\ell}{\partial x_k}\right) + \{\mathbf{f}\}_j = -\{\omega^2\boldsymbol{\rho}\mathbf{u}\}_j \quad (2)$$

[in which $\mathbf{u}$ is the displacement field, $\mathbf{f}$ is the body-force density $\boldsymbol{\rho}$ is the density tensor, and $\mathbf{C}$ is now the elasticity tensor]. It is therefore natural to ask: What is analogous in electrodynamics to a discrete system of springs and masses in elastodynamics? The answer is an electromagnetic circuit,

introduced by us in [1]. The idea of an electromagnetic circuit generalizes the idea of Engheta, Salandrino, and Alú [2] and Engheta [3] who realized that normal linear electrical circuits could be approximated, in the quasistatic limit (which does not imply the frequency is low, but only that the size of the network is small compared to the wavelength) by a connected network of thin cylinders each of material with a suitably scaled value of $\boldsymbol{\varepsilon} = \varepsilon\mathbf{I}$ surrounded by a cladding of zero-dielectric material (with $\boldsymbol{\varepsilon} = 0$): a cylinder with a real positive value of ε approximates a capacitor, while a cylinder with a positive imaginary value of ε approximates a resistor, and a cylinder with an almost negative real value of ε approximates an inductor.

A system of springs and masses can be approximated by massless elastic bars (having $\boldsymbol{\rho} = 0$ and $\mathbf{C}$ appropriately scaled; buckling of the bars is ignored as one working within the framework of linear elasticity) with rigid spherical masses (having $\mathbf{C} = \infty$ and $\boldsymbol{\rho}$ appropriately scaled) at the junction nodes between bars and surrounded by void

(having $\rho = 0$ and $\mathbf{C} = 0$). A subset of nodes are chosen to be terminal nodes at which displacements are prescribed. Similarly an electromagnetic circuit, as illustrated in Figure 1, can be approximated by zero-dielectric diamagnetic thin triangular plates (of width $h \rightarrow 0$ having $\boldsymbol{\varepsilon} = 0$ and $\boldsymbol{\mu}$ appropriately scaled) with zero-magnetic dielectric cylinders (having $\boldsymbol{\mu} = 0$ and $\boldsymbol{\varepsilon}$ appropriately scaled) at the junction edges between triangular plates and surrounded by a cladding (having $\boldsymbol{\varepsilon} = 0$ and $\boldsymbol{\mu} = \infty$). A subset of edges are chosen to be terminal edges along which the electric field is prescribed. As expected from (1)-(2) $\boldsymbol{\varepsilon}$ plays the role of $\boldsymbol{\rho}$ and $\boldsymbol{\mu}^{-1}$ plays the role of $\mathbf{C}$. What is perhaps not quite so expected is the different geometrical structure of the elements: triangular plates instead of bars and cylinders instead of spheres. However this is necessitated by the continuity of the fields. Indeed in the cladding one has $\mathbf{H} = 0$, similarly to the way one has a stress $\boldsymbol{\sigma} = 0$ in the void surrounding the elastodynamic network. A bar in such a medium cannot have a constant non-zero value of $\mathbf{H}$ inside it, by continuity of the tangential component of $\mathbf{H}$ across the boundary, while a triangular plate in the medium can have a constant value of $\mathbf{H}$, directed normal to the surface. When $\mathbf{j} = 0$ Maxwell's equations remain invariant when the roles of $(\mathbf{E}, \mathbf{D}, \varepsilon, \omega)$ are switched with those of $(\mathbf{H}, \mathbf{B}, \boldsymbol{\mu}, -\omega)$. Therefore for every electromagnetic circuit there is a corresponding magnetoelectric circuit, where the roles of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are interchanged.

Of course such extreme values of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are difficult to achieve, even with metamaterials over a narrow frequency range. So why introduce electromagnetic circuits when they are so difficult to realize? One answer is that they provide new ways of manipulating electromagnetic fields, that are relatively easy to analyze. Indeed, the possible responses of electromagnetic circuits in which no two terminal edges are connected have been completely characterized [1]. If a desired manipulation is possible with an electromagnetic circuit this should motivate the search to achieve a similar manipulation with more realistic materials. Another answer is more fundamental. Shin, Shen and Fan [4] have shown that metamaterials can exhibit macroscopic electromagnetic behavior which is non-Maxwellian, even though they are governed by Maxwell's equations at the microscale. [See also Dubovik, Martsenyuk and Saha [5] where other non-Maxwellian macroscopic equations are proposed]. So what sort of macroscopic electromagnetic equations can
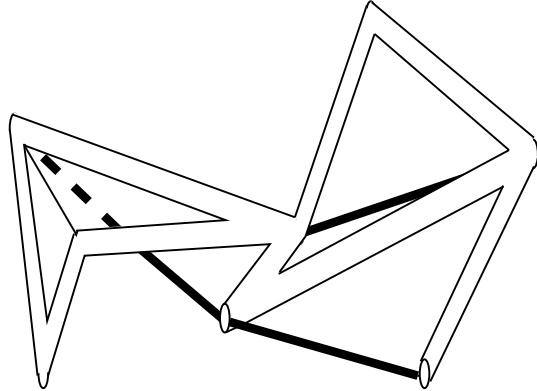

Figure 1: An example of an electromagnetic circuit (EM-circuit) with six zero-dielectric diamagnetic thin triangular plates (having $\boldsymbol{\varepsilon} = 0$ and $\boldsymbol{\mu} \neq 0$). It has small diameter zero-magnetic dielectric cylinders with $\varepsilon \neq 0$ and $\mu = 0$ along a selection of edges including possibly the terminal edges. Here there are three terminal edges marked by thicker solid black lines. One internal edge, marked by the thin line, has a cylinder with $\varepsilon = \mu = 0$ attached to it.

one obtain? The success of Camar-Eddine and Seppecher [6, 7] in addressing such questions in the context of three-dimensional (static) conductivity and elasticity, suggests one should try to characterize the continuum macroscopic behaviors of electromagnetic circuits and then try to prove that this encompasses all possible macroscopic behaviors. The continuum limits of electrical circuits can have interesting macroscopic behaviors as discussed in the book [8] and references therein. The continuum limits of electromagnetic circuits should have an even richer span of macroscopic behaviors.

When discussing elastic networks it is quite usual to talk about applying a force to a terminal node. This could be a concentrated body force, or could be provided by say a medium external to the network which we do not need to precisely specify when talking about the applied force. In a similar way in an electromagnetic network it is convenient to talk about applying a free electrical current to a terminal edge. This could be a concentrated current caused by an electrochemical potential, or could be an intense $\mathbf{H}$ field provided by an external medium, acting across the terminal edge. To be more precise, at the interface between the external medium and the terminal edge we require that $\mathbf{n} \cdot \mathbf{C} \partial \mathbf{E} / \partial \mathbf{x}$ be continuous, which is nothing more than requiring the tangential component of $\mathbf{H}$ to be continuous. In elastodynamics the quantity $\mathbf{n} \cdot \mathbf{C} \partial \mathbf{u} / \partial \mathbf{x}$ is called

the surface force $\mathbf{F}$. We adopt a similar terminology for electrodynamics and call $(i\omega)^{-1}\mathbf{n}\cdot\mathbf{C}\partial\mathbf{E}/\partial\mathbf{x}$ the surface free current $\mathbf{J}$. (The additional factor of $(i\omega)^{-1}$ is introduced because $i\omega\mathbf{j}$ in (1) plays the role of $\mathbf{f}$ in (2).) In a domain Ω which is divided in two subdomains Ω_1, Ω_2 we can say that Ω_2 exerts on Ω_1 a surface free current $\mathbf{J}$ while Ω_1 exerts on Ω_2 the opposite surface free current $-\mathbf{J}$. This formulation does not mean, in any way, that there exist actual free currents in the material, just like the Newtonian action-reaction law does not imply the existence of actual surface forces inside the domain. Similarly in magnetoelectric circuits it is convenient to talk about applying a free magnetic monopole current to a terminal edge. Again this in no way implies that there exist actual magnetic monopole currents, but rather that the equivalent effect is provided by intense $\mathbf{E}$ fields in an external medium acting across the terminal edge. But mathematically there is no barrier to thinking of magnetic monopole currents: in the presence of such currents the equation analogous to (1) is

$$\frac{\partial}{\partial x_i}\left(L_{ijk\ell}\frac{\partial H_\ell}{\partial x_k}\right) - \{i\omega\mathbf{g}\}_j = -\{\omega^2\boldsymbol{\mu}\mathbf{H}\}_j \quad (3)$$

where $L_{ijk\ell} = e_{ijm}e_{k\ell n}\{\boldsymbol{\varepsilon}^{-1}\}_{mn}$ and $\mathbf{g}$ is the free magnetic monopole current density.

The response of a electromagnetic circuit with n terminal edges is governed by a linear relation $i\omega\mathbf{J} = \mathbf{W}\mathbf{V}$ between the variables $\mathbf{J} = (J_1, J_2, \ldots, J_n)$ (not to be confused with the $\mathbf{J}$ in the previous paragraph) whose components now represent surface free currents acting along the terminal edges (the surface free current along terminal edge i is constant along the edge and directed along the edge, and the complex scalar J_i represents the total current flow in that direction) and the variables $\mathbf{V} = (V_1, V_2, \ldots, V_n)$ which are the line integrals of the electric field $\mathbf{E}$ along these edges. As shown in [1], the matrix $\mathbf{W}$ is symmetric, with a negative semidefinite imaginary part. Furthermore if the terminal edges are disjoint, given any matrix $\mathbf{W}$ with these properties there exists a electromagnetic circuit (specifically an electromagnetic ladder network, as described in [1]) which has $\mathbf{W}$ as its response matrix. Similarly, the response of a magnetoelectric circuit with m terminal edges is governed by a linear relation $-i\omega\mathbf{G} = \mathbf{Y}\mathbf{U}$, or equivalently $\omega\mathbf{G} = \mathbf{Y}(i\mathbf{U})$, between the variables $\mathbf{G} = (G_1, G_2, \ldots, G_m)$ which represent surface free magnetic monopole currents acting on the terminal edges, and the variables $\mathbf{U} = (U_1, U_2, \ldots, U_m)$
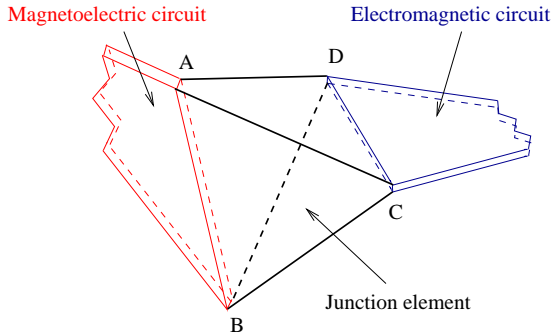

Figure 2: The tetrahedral connector, which allows us to connect a terminal edge in an electromagnetic circuit (E-terminal) with a terminal edge in a magnetoelectric circuit (H-terminal).

which are the line integrals of the magnetic field $\mathbf{H}$ along these edges. The matrix $\mathbf{Y}$ is symmetric, with a negative semidefinite imaginary part.

Electromagnetic circuits and magnetoelectric circuits seem like completely different animals. Indeed, an electromagnetic circuit has a cladding with $\boldsymbol{\varepsilon} = 0$ and $\boldsymbol{\mu} = \infty$, whereas a magnetoelectric circuit has a cladding with $\boldsymbol{\varepsilon} = \infty$ and $\boldsymbol{\mu} = 0$. However they can be joined to create hybrid electromagnetic circuits at the cost of introducing an additional circuit element: a connector which joins a terminal edge of an electromagnetic circuit to a terminal edge of a magnetoelectric circuit. As illustrated in figure 2 the connector is comprised of a material with $\boldsymbol{\varepsilon} = \boldsymbol{\mu} = 0$ in the shape of a tetrahedron with vertices $ABCD$, clad on the top and bottom faces ACD and BCD with a material having $\boldsymbol{\varepsilon} = 0$ and $\boldsymbol{\mu} = \infty$ (in which $\mathbf{H} = 0$ and $\mathbf{D} = 0$) and on the two side faces ABC and ABD with a material having $\boldsymbol{\varepsilon} = \infty$ and $\boldsymbol{\mu} = 0$ (in which $\mathbf{E} = 0$ and $\mathbf{B} = 0$). The edge CD and the edge AB are clipped to expose a surface of width $h \to 0$ with $\boldsymbol{\varepsilon} = \boldsymbol{\mu} = 0$. The edge CD may either be a terminal edge (what we will call an E-terminal), or may be connected to the terminal edge, of width h, of an electromagnetic circuit. The edge AB may either be a terminal edge (what we will call an H-terminal), or may be connected to a terminal edge, of width h, of a magnetoelectric circuit.

Inside the tetrahedron $\boldsymbol{\varepsilon} = \boldsymbol{\mu} = 0$, and so both $\mathbf{E}$ and $\mathbf{H}$ are curl-free. Since $\mathbf{H} = 0$ in the cladding on the top and bottom faces, and since the tangential component of $\mathbf{H}$ is continuous across these

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interfaces, it follows that the line integral U_v of $\mathbf{H}$ upwards along the edge BA must equal the line integral of $\mathbf{H}$ upwards across the edge CD of width h. Similarly, since $\mathbf{E} = 0$ in the cladding on the sides, it follows that the line integral V_h of $\mathbf{E}$ forwards along the edge DC must equal the line integral of $\mathbf{E}$ forwards across the edge AB of width h. If J_h denotes the surface free electrical current acting along the edge CD in the forward direction and G_v denotes the surface free magnetic monopole current acting along the edge AB in the upwards direction, then $J_h = U_v$ and $G_v = V_h$, or equivalently

$$\begin{pmatrix} i\omega J_h \\ \omega G_v \end{pmatrix} = \mathbf{M} \begin{pmatrix} V_h \\ iU_v \end{pmatrix}, \tag{4}$$

where

$$\mathbf{M} = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}. \tag{5}$$

The terminal edges in hybrid electromagnetic circuits come in two varieties: those for which we can apply a free surface electrical current directed along the edge, which we call E-terminals, and those for which we can apply (in theory) a free surface magnetic monopole current directed along the edge, which we call H-terminals. The response of general hybrid electromagnetic circuit with n E-terminals and m H-terminals is governed by a linear relation $\mathbf{A} = \mathbf{MX}$ between the variables $\mathbf{A} = (i\omega J_1, i\omega J_2, \ldots, i\omega J_n, \omega G_1, \omega G_2, \ldots, \omega G_m)$ and $\mathbf{X} = (V_1, V_2, \ldots, V_n, iU_1, iU_2, \ldots, iU_m)$, where the matrix $\mathbf{M}$ is symmetric, with a negative semidefinite imaginary part. These two facts are most easily verified if both opposing edges of all tetrahedral connectors are included among the terminal edges. Then the matrix $\mathbf{M}$ takes the form

$$\mathbf{M} = \begin{pmatrix} \mathbf{W} & \mathbf{Z} \\ \mathbf{Z}^T & \mathbf{Y} \end{pmatrix} \tag{6}$$

where $\mathbf{W}$ is the response matrix of the (possibly disconnected) part which is an electromagnetic circuit, $\mathbf{Y}$ is the response matrix of the (possibly disconnected) part which is a magnetoelectric circuit and Z_{ij} is ω if there is a tetrahedral connector which connects E-terminal i with H-terminal j, and is zero otherwise. It then follows, for example from the arguments in section 5 of [1], that these properties of $\mathbf{M}$ extend to hybrid electromagnetic circuits in which some or all of the tetrahedral connector edges are not terminal edges.

Now the response matrix $\mathbf{M}$ can always be expressed in the form (6) if we allow more general (

possibly complex) matrices $\mathbf{Z}$. Then by manipulating the relation $\mathbf{A} = \mathbf{MX}$ we obtain the equivalent relation

$$\begin{pmatrix} i\omega\mathbf{J} \\ i\omega\mathbf{U} \end{pmatrix} = \mathbf{S} \begin{pmatrix} \mathbf{V} \\ -\mathbf{G} \end{pmatrix} \tag{7}$$

where

$$\mathbf{S} = \begin{pmatrix} \mathbf{W} - \mathbf{ZY}^{-1}\mathbf{Z}^T & -\omega\mathbf{ZY}^{-1} \\ -\omega\mathbf{Y}^{-1}\mathbf{Z}^T & -\omega^2\mathbf{Y}^{-1} \end{pmatrix} \tag{8}$$

is symmetric. Conversely it is easy to check that if $\mathbf{S}$ is symmetric so too is $\mathbf{M}$. Also the fact that

$$\begin{aligned}
-\mathbf{X}' &\cdot \mathbf{M}''\mathbf{X}' - \mathbf{X}'' \cdot \mathbf{M}''\mathbf{X}'' \\
&= \mathbf{A}' \cdot \mathbf{X}'' - \mathbf{A}'' \cdot \mathbf{X}' \\
&= (i\omega\mathbf{J})' \cdot \mathbf{V}'' - (i\omega\mathbf{J})'' \cdot \mathbf{V}' \\
&\quad -(i\omega\mathbf{U})' \cdot \mathbf{G}'' + (i\omega\mathbf{U})'' \cdot \mathbf{G}' \\
&= -\begin{pmatrix} \mathbf{V}' \\ -\mathbf{G}' \end{pmatrix} \cdot \mathbf{S}'' \begin{pmatrix} \mathbf{V}' \\ -\mathbf{G}' \end{pmatrix} \\
&\quad -\begin{pmatrix} \mathbf{V}'' \\ -\mathbf{G}'' \end{pmatrix} \cdot \mathbf{S}'' \begin{pmatrix} \mathbf{V}'' \\ -\mathbf{G}'' \end{pmatrix}
\end{aligned} \tag{9}$$

implies $\mathbf{M}''$ is negative semidefinite if and only if $\mathbf{S}''$ is negative semidefinite.

Given our hybrid circuit we can attach m tetrahedral connectors to the m H-terminals, and allow these H-terminals to be internal edges. At the $(n + j)$th E-terminal, which is connected to the former jth H-terminal we have $J_{n+j} = U_j$ and $V_{n+j} = -G_j$ where the minus sign arises, because while G_j is the surface free magnetic monopole current acting on the hybrid circuit at the former jth H-terminal, $-G_j$ is the surface free magnetic monopole current acting on the tetrahedral connector. According to these relations and (7), $\mathbf{S}$ will be the response matrix of this new circuit, i.e. $i\omega\mathbf{J} = \mathbf{SV}$. Thus from the hybrid circuit we have obtained a circuit which responds exactly like a pure electromagnetic circuit.

We can now establish that any given $(n + m) \times (n + m)$ symmetric matrix $\mathbf{S}$ with negative semidefinite imaginary part can be realized by a hybrid circuit with n E-terminals and m H-terminals which have no vertex in common. We first construct the $n + m$ E-terminal electromagnetic ladder network which has $\mathbf{S}$ as its response matrix. Then to the edges $n + 1, n + 2, \ldots, n + m$ we attach tetrahedral connectors to convert these E-terminals into H-terminals (taking these former E-terminals to be internal edges in the new circuit). This leaves the response matrix $\mathbf{S}$ unchanged, which now governs

the response (7) of our new hybrid circuit. The proof is complete. Note that in the ladder network (and more generally in other electromagnetic circuits), the internal zero-magnetic dielectric cylinders which join the terminal edge will generally carry some displacement current $\mathbf{D}$. When we join a tetrahedral connector to the terminal edge this displacement current will flow into the side cladding which has $\boldsymbol{\varepsilon} = \infty$ and $\boldsymbol{\mu} = 0$ and can support a non-zero value of $\mathbf{D}$.

In summary, although hybrid electromagnetic circuits seem to be vastly more general than pure electromagnetic or magnetoelectric circuits, they are in a sense (modulo the addition of tetrahedral connectors to the terminal edges) all equivalent.

Acknowledgements

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