



**HAL**  
open science

# Koszul duality and Frobenius structure for restricted enveloping algebras

Simon Riche

► **To cite this version:**

Simon Riche. Koszul duality and Frobenius structure for restricted enveloping algebras. 2010. hal-00522983

**HAL Id: hal-00522983**

**<https://hal.science/hal-00522983>**

Preprint submitted on 4 Oct 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# KOSZUL DUALITY AND FROBENIUS STRUCTURE FOR RESTRICTED ENVELOPING ALGEBRAS

SIMON RICHE

ABSTRACT. Let  $\mathfrak{g}$  be the Lie algebra of a connected, simply connected semisimple algebraic group over an algebraically closed field of sufficiently large positive characteristic. We study the compatibility between the Koszul grading on the restricted enveloping algebra  $(\mathcal{U}\mathfrak{g})_0$  of  $\mathfrak{g}$  constructed in [Ri], and the structure of Frobenius algebra of  $(\mathcal{U}\mathfrak{g})_0$ . This answers a question raised to the author by W. Soergel.

## CONTENTS

Introduction	1
1. Linear Koszul duality and homological duality	4
2. Reminder on localization in positive characteristic	11
3. Koszul duality and ordinary duality: regular case	21
4. Koszul duality and ordinary duality: singular case	26
5. Example: $\mathrm{SL}(2)$	28
References	29

## INTRODUCTION

0.1. Let  $G$  be a connected, simply connected semisimple algebraic group over an algebraically closed field  $\mathbb{k}$  of characteristic  $p > 0$ . Let

$$(\mathcal{U}\mathfrak{g})_0 := \mathcal{U}\mathfrak{g} / \langle X^p - X^{[p]}, X \in \mathfrak{g} \rangle$$

be the associated restricted enveloping algebra. Generalizing a result from [AJS], we have proved in [Ri] that, if  $p$  is sufficiently large, this algebra can be endowed with a Koszul grading, i.e. a grading which makes it a Koszul ring in the sense of [BGS].

On the other hand, it is well-known and easy to prove (see [Be]) that  $(\mathcal{U}\mathfrak{g})_0$  is a Frobenius algebra. More precisely, there is a natural isomorphism of algebra  $\mathcal{U}\mathfrak{g} \rightarrow \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ , induced by the assignment  $X \in \mathfrak{g} \mapsto -X$ , which induces an isomorphism

$$\Phi : (\mathcal{U}\mathfrak{g})_0 \xrightarrow{\sim} (\mathcal{U}\mathfrak{g})_0^{\mathrm{op}}.$$

Using this isomorphism, the standard duality for  $\mathbb{k}$ -vector spaces  $M \mapsto M^* := \mathrm{Hom}_{\mathbb{k}}(M, \mathbb{k})$  induces an anti-equivalence of the category of finite dimensional (left)  $(\mathcal{U}\mathfrak{g})_0$ -modules, denoted  $(-)^{\vee}$ . Then the fact that  $(\mathcal{U}\mathfrak{g})_0$

is a Frobenius algebra amounts to the existence of an isomorphism of left  $(\mathcal{U}\mathfrak{g})_0$ -modules

$$\theta : (\mathcal{U}\mathfrak{g})_0 \xrightarrow{\sim} ((\mathcal{U}\mathfrak{g})_0)^\vee.$$

Our aim in this article is to compare these two structures. We show that there exists a Koszul grading on  $(\mathcal{U}\mathfrak{g})_0$  such that  $\Phi$  becomes an isomorphism of *graded* algebras, and such that  $\theta$  is an isomorphism of *graded*  $(\mathcal{U}\mathfrak{g})_0$ -modules, up to some shifts to be explained below.

This problem was suggested to us by Wolfgang Soergel.

0.2. To state our results precisely, we need to introduce the various blocks of the algebra  $(\mathcal{U}\mathfrak{g})_0$ . For simplicity, we assume from now on that  $p > h$ , where  $h$  is the Coxeter number of  $G$ . Let also  $T \subset B \subset G$  be a maximal torus and a Borel subgroup,  $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$  their Lie algebras, and  $\mathbb{X} := X^*(T)$  the character lattice. Let also  $W$  be the Weyl group,  $W'_{\text{aff}} := W \ltimes \mathbb{X}$  the extended affine Weyl group, and let  $\rho$  be the opposite of the half sum of the roots of  $\mathfrak{b}$ . We consider the following “dot” actions of  $W$  on  $\mathfrak{t}^*$ , respectively of  $W'_{\text{aff}}$  on  $\mathbb{X}$ :

$$w \bullet \lambda = w(\lambda + \rho) - \rho, \quad \text{respectively} \quad (wt_\mu) \bullet \nu = w(\nu + p\mu + \rho) - \rho.$$

The subalgebra  $(\mathcal{U}\mathfrak{g})^G \subset \mathcal{U}\mathfrak{g}$  is central, and isomorphic to  $\mathbb{k}[\mathfrak{t}^*/(W, \bullet)]$ . Its image in  $(\mathcal{U}\mathfrak{g})_0$  is a central subalgebra  $\mathfrak{Z}_0$ , whose set of characters is naturally parametrized by  $\mathbb{X}/(W'_{\text{aff}}, \bullet)$ . For any  $\lambda \in \mathbb{X}$  (or in  $\mathbb{X}/(W'_{\text{aff}}, \bullet)$ ), we denote by  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$  the completion of  $(\mathcal{U}\mathfrak{g})_0$  with respect to the annihilator of  $\lambda$  in  $\mathfrak{Z}_0$ . It is a finite dimensional algebra, whose category of finitely generated modules is equivalent to the category of finitely generated  $(\mathcal{U}\mathfrak{g})_0$ -modules with generalized central character  $\lambda$  (for  $\mathfrak{Z}_0$ ). Moreover, there is a natural isomorphism of algebras

$$(\mathcal{U}\mathfrak{g})_0 \cong \prod_{\lambda \in \mathbb{X}/(W'_{\text{aff}}, \bullet)} (\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}.$$

Hence, constructing a Koszul grading on  $(\mathcal{U}\mathfrak{g})_0$  is equivalent to constructing a Koszul grading on each on the subalgebras  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$ .

For any  $\lambda \in \mathbb{X}$ ,  $\Phi$  induces an isomorphism

$$\Phi_\lambda : (\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}} \xrightarrow{\sim} ((\mathcal{U}\mathfrak{g})_0^{-\widehat{\lambda-2\rho}})^{\text{op}},$$

and  $\theta$  induces an isomorphism of  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$ -modules

$$\theta_\lambda : (\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}} \xrightarrow{\sim} ((\mathcal{U}\mathfrak{g})_0^{-\widehat{\lambda-2\rho}})^\vee.$$

0.3. Our main result is the following.

**Theorem.** *Assume  $p \gg 0$ .*

*There exists a Koszul grading on the algebra  $(\mathcal{U}\mathfrak{g})_0$  such that:*

- (1) *The natural isomorphism  $\Phi : (\mathcal{U}\mathfrak{g})_0 \xrightarrow{\sim} ((\mathcal{U}\mathfrak{g})_0)^{\text{op}}$  is an isomorphism of graded rings.*

- (2) For any  $\lambda \in \mathbb{X}$ , there exists an explicit  $N_\lambda \in \mathbb{Z}$  and an isomorphism of graded left  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$ -modules

$$(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}} \cong ((\mathcal{U}\mathfrak{g})_0^{-\widehat{\lambda-2\rho}})^\vee \langle N_\lambda \rangle.$$

0.4. Let us first explain more precisely what is  $N_\lambda$ . Recall that a weight  $\lambda \in \mathbb{X}$  is called *regular* if  $\langle \lambda + \rho, \alpha^\vee \rangle \notin p\mathbb{Z}$  for any root  $\alpha$ . For such a weight, we have  $N_\lambda = 2 \dim(G/B)$ . In this case, the theorem is proved in Propositions 3.2.2 and 3.4.1.

For a general  $\mu \in \mathbb{X}$ , there exists a standard parabolic subgroup  $P \subset G$ , and some  $w \in W'_{\text{aff}}$ , such that the stabilizer of  $w \bullet \mu$  in  $W'_{\text{aff}}$  (for the dot action) is the Weyl group of the Levi of  $P$  (see [BMR2, Lemma 1.5.2]). In this case, we have  $N_\mu = 2 \dim(G/P)$ , and the theorem is proved in Propositions 4.2.1 and 4.4.1.

0.5. Now, let us explain the condition on  $p$ . It is the same as that of [Ri]. It depends on the weight. For regular weights, the condition is that Lusztig's conjecture on characters of simple  $G$ -modules is true. (Recall that it is conjectured that this character formula holds as soon as  $p > h$ , but it is only known that it is true for  $p$  bigger than an explicit bound which is much larger than  $h$ , see [Fi]. See also [Ri, §0.5] for further references.) For singular weights, we make an extra assumption which is known to be true for  $p$  bigger than an explicit bound depending on  $G$  and the weight under consideration.

0.6. As suggested by the description of  $N_\lambda$  in §0.4, the proof of the theorem (whose statement is of algebraic nature) is based on geometry, and more precisely on the localization theory in positive characteristic developed by Bezrukavnikov, Mirković and Rumynin, see [BMR, BMR2, BM]. We also use linear Koszul duality, a geometric version of the standard Koszul duality between symmetric and exterior algebras due to I. Mirković, see [Ri, Section 2]. (Note that here we use the *covariant* version of this duality, and not the *contravariant* version of [MR1, MR2].)

We first prove, in a general context, that linear Koszul duality commutes with homological duality (see Proposition 1.3.1). Then, as in [Ri], we apply linear Koszul duality to a particular geometric context to construct a “Koszul duality” between certain categories of  $\mathcal{U}\mathfrak{g}$ -modules. The “key result” of [Ri] states that, if  $p \gg 0$ , this duality sends certain (uniquely determined) lifts of simples to certain lifts of indecomposable projectives. It follows from the particular case of Proposition 1.3.1 that a geometric version of the duality  $(-)^{\vee}$  commutes with our Koszul duality. We deduce that this geometric version of  $(-)^{\vee}$  sends our lifts of projectives to lifts of projectives (see Proposition 3.1.2). We deduce the main theorem from this geometric statement.

**0.7. Organization of the paper.** In Section 1 we prove our general results on linear Koszul duality. In fact we reprove the equivalence of [Ri, Theorem 2.3.10] under weaker hypotheses, and with shorter proofs. We also construct homological dualities for the categories under consideration, and prove that linear Koszul duality commutes with homological duality.

In Section 2 we review the results of [BMR, BMR2] and [Ri] that will be needed. We also explain how one can explicitly construct the Koszul grading on  $(\mathcal{U}\mathfrak{g})_0$ , given our preferred lifts of indecomposable projectives (see Theorems 2.4.3 and 2.5.5).

In Section 3, we prove the theorem for regular blocks. We first prove a geometric statement (see Proposition 3.1.2), and show that it implies our algebraic statements.

In Section 4, we prove the theorem for singular blocks. The proof is very similar to that for regular blocks.

Finally, in Section 5 we explain the content of our results in the special case  $G = \mathrm{SL}(2)$ .

**0.8. Acknowledgements.** We warmly thank W. Soergel for suggesting this problem.

The author is supported by the French National Research Agency (ANR-09-JCJC-0102-01).

**0.9. Some notation.** For  $\mathbb{k}$  an algebraically closed field of positive characteristic, and  $Y$  a  $\mathbb{k}$ -scheme, we denote by  $Y^{(1)}$  the Frobenius twist of  $Y$  (see [BMR, §1.1]).

For  $Y$  a scheme, and  $X \subset Y$  a subscheme, we denote by  $\mathrm{Coh}_X(Y)$  the category of coherent sheaves on  $Y$  which are set-theoretically supported on  $X$ .

For any smooth variety  $X$  endowed with an action of an algebraic group  $H$ , we denote by

$$\mathbb{D}_X : \mathcal{D}^b \mathrm{Coh}^H(X) \xrightarrow{\sim} (\mathcal{D}^b \mathrm{Coh}^H(X))^{\mathrm{op}}$$

the equivalence given by  $\mathbb{D}_X := R\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ . When  $X = Y^{(1)}$ , when no confusion can arise we sometimes write  $\mathbb{D}_Y$  instead of  $\mathbb{D}_{Y^{(1)}}$ .

We denote by  $V^*$  the dual of a vector space or more generally a vector bundle  $V$ , and by  $\mathcal{V}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}, \mathcal{O}_X)$  the dual of an  $\mathcal{O}_X$ -module  $\mathcal{V}$ , or a complex of  $\mathcal{O}_X$ -modules.

In any category of  $\mathbb{G}_m$ -equivariant objects (in particular for graded modules over a graded algebra), we denote by  $\langle 1 \rangle$  the tensor product with the 1-dimensional  $\mathbb{k}^\times$ -module given by  $\mathrm{Id}_{\mathbb{k}^\times}$ . We denote by  $\langle j \rangle$  the  $j$ -th power of  $\langle 1 \rangle$ .

## 1. LINEAR KOSZUL DUALITY AND HOMOLOGICAL DUALITY

**1.1. Covariant linear Koszul duality.** In this subsection we reprove the results of [Ri, Section 2] in a more general setting (and with shorter proofs). The main new ideas are taken from [Po]. A similar generalization of the

main result of [MR1] will appear in a forthcoming paper in collaboration with I. Mirković.

Let  $X$  be a scheme, and let  $E \rightarrow X$  be a vector bundle over  $X$ . Let also  $F \subset E$  be a subbundle, and  $F^\perp \subset E^*$  be the orthogonal to  $F$ . Let also  $\mathcal{E}$  and  $\mathcal{F}$  be the (locally free) sheaves of sections of  $E$  and  $F$ , and let  $\mathcal{F}^\perp$  be the orthogonal to  $\mathcal{F}$  inside  $\mathcal{E}^*$ . Our goal is to construct a ‘‘Koszul duality’’ equivalence between certain categories of coherent (dg-)sheaves on the dg-schemes<sup>1</sup>  $F$  and  $F^\perp \cap_{E^*} X$ , where  $X \subset E^*$  is the zero section.

Consider the  $\mathbb{G}_m$ -equivariant dg-algebras

$$\mathcal{S} := S(\mathcal{F}^*), \quad \mathcal{R} := S(\mathcal{F}^*), \quad \mathcal{T} := \Lambda(\mathcal{F})$$

endowed with the trivial differential, where  $\mathcal{F}^*$  is in bidegree  $(2, -2)$ , respectively  $(0, -2)$ , and  $\mathcal{F}$  is in bidegree  $(-1, 2)$ . For a  $\mathbb{G}_m$ -equivariant dg-algebra  $\mathcal{A}$ , we denote by  $\mathcal{C}(\mathcal{A}\text{-Mod}^{\text{qc}})$  the category of  $\mathbb{G}_m$ -equivariant  $\mathcal{A}$ -dg-modules which are  $\mathcal{O}_X$ -quasi-coherent<sup>2</sup>, and by  $\mathcal{C}(\mathcal{A}\text{-Mod}^{\text{bc}})$  the subcategory whose objects are bounded above for the internal degree (uniformly in the cohomological degree).

Consider the functors

$$\mathcal{F} : \begin{cases} \mathcal{C}(\mathcal{S}\text{-Mod}^{\text{qc}}) & \rightarrow \mathcal{C}(\mathcal{T}\text{-Mod}^{\text{qc}}) \\ \mathcal{M} & \mapsto \mathcal{T}^* \otimes_{\mathcal{O}_X} \mathcal{M} \end{cases}$$

and

$$\mathcal{G} : \begin{cases} \mathcal{C}(\mathcal{T}\text{-Mod}^{\text{qc}}) & \rightarrow \mathcal{C}(\mathcal{S}\text{-Mod}^{\text{qc}}) \\ \mathcal{N} & \mapsto \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{N} \end{cases}.$$

Here, the actions and differentials on the dg-modules  $\mathcal{T}^* \otimes_{\mathcal{O}_X} \mathcal{M}$  and  $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{N}$  are defined as in [Ri, §2.1].

**Lemma 1.1.1.** *The functors  $\mathcal{F}$  and  $\mathcal{G}$  are exact (i.e. they send acyclic dg-modules to acyclic dg-modules, or equivalently quasi-isomorphisms to quasi-isomorphisms).*

*Proof.* This proof is taken from [Po, Theorem A.1.2]. We give the proof for  $\mathcal{G}$ ; the case of  $\mathcal{F}$  is similar and simpler.

Let  $\mathcal{N}$  be an acyclic object of  $\mathcal{C}(\mathcal{S}\text{-Mod}^{\text{qc}})$ . Up to some shift, we can assume  $\mathcal{N}_i = 0$  for  $i > 0$ . Then we have

$$(\mathcal{G}(\mathcal{N}))_n = \bigoplus_{\substack{i \leq 0, j \leq 0, \\ n = i + j}} \mathcal{S}_i \otimes_{\mathcal{O}_X} \mathcal{N}_j.$$

Remark that this sum is finite. Hence the homogeneous components of  $\mathcal{G}(\mathcal{N})$  are obtained from the homogeneous components of  $\mathcal{N}$  by tensoring with the homogeneous components of  $\mathcal{S}$  (which are flat) and taking shifts

<sup>1</sup>See [Ri, §1.8] for generalities on dg-schemes, and in particular on derived intersections.

<sup>2</sup>Here for simplicity we restrict from the beginning to quasi-coherent dg-modules. However, this assumption is used only in the proof of Proposition 1.1.5. Equivalently, one can put the quasi-coherency assumption on the cohomology of the dg-modules instead, as in [Ri].

and cones a finite number of times. Hence they are acyclic, which proves the result.  $\square$

We denote by

$$\overline{\mathcal{F}} : \mathcal{D}(\mathcal{S}\text{-Mod}_{\underline{q}^c}) \rightarrow \mathcal{D}(\mathcal{T}\text{-Mod}_{\underline{q}^c}), \quad \overline{\mathcal{G}} : \mathcal{D}(\mathcal{T}\text{-Mod}_{\underline{q}^c}) \rightarrow \mathcal{D}(\mathcal{S}\text{-Mod}_{\underline{q}^c})$$

the functors induced between the corresponding derived categories.

**Proposition 1.1.2.** *The functors  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{G}}$  are quasi-inverse equivalences of categories.*

*Proof.* This proof is again taken from [Po, Theorem A.1.2]. Recall the “generalized Koszul complex”  $\mathcal{K}^{(1)}$  of [MR1, §2.5]. (Here, we are in the simpler situation where the vector bundle “ $\mathcal{V}$ ” of [MR1] is zero.)

The functors  $\mathcal{F}$  and  $\mathcal{G}$  are clearly adjoint; hence so are  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{G}}$  (see [Ke, Lemma 13.6]). We show that the adjunction morphism  $\overline{\mathcal{G}} \circ \overline{\mathcal{F}} \rightarrow \text{Id}$  is an isomorphism; the proof for the morphism  $\text{Id} \rightarrow \overline{\mathcal{F}} \circ \overline{\mathcal{G}}$  is similar. Let  $\mathcal{M}$  be an object of  $\mathcal{C}(\mathcal{S}\text{-Mod}_{\underline{q}^c})$ . Then the homogeneous internal degree components of the cone of the morphism  $\mathcal{G} \circ \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{M}$  can be obtained from the negative homogeneous internal degree components of  $\mathcal{K}^{(1)}$  by tensoring with the homogeneous internal degree components of  $\mathcal{M}$  and taking shifts and cones a finite number of times. These negative homogeneous internal degree components of  $\mathcal{K}^{(1)}$  are acyclic by [MR1, Lemma 2.5.1]. Hence it suffices to prove that the tensor product of an acyclic, bounded complex of flat  $\mathcal{O}_X$ -modules with any complex of  $\mathcal{O}_X$ -modules is acyclic. This fact is well-known, see e.g. [Sp, Proposition 5.7].  $\square$

Finally we prove that these equivalences restrict to finitely generated objects. From now on, we assume that  $X$  is noetherian. For a  $\mathbb{G}_{\mathbf{m}}$ -equivariant dg-algebra  $\mathcal{A}$ , we denote by  $\mathcal{D}^{\text{fg}}(\mathcal{A}\text{-Mod}_{\underline{q}^c})$  the subcategory of  $\mathcal{D}(\mathcal{A}\text{-Mod}_{\underline{q}^c})$  whose objects have locally finitely generated cohomology (over  $H(\mathcal{A})$ ). We let also  $\mathcal{CFG}^{\text{qc}}(\mathcal{A})$  be the category of  $\mathcal{O}_X$ -quasi-coherent, locally finitely generated  $\mathbb{G}_{\mathbf{m}}$ -equivariant  $\mathcal{A}$ -dg-modules. We denote by  $\mathcal{DFG}^{\text{qc}}(\mathcal{A})$  the corresponding derived category. The following lemma can be proved as in [MR1, Lemma 3.6.1].

**Lemma 1.1.3.** *For  $\mathcal{A} = \mathcal{R}, \mathcal{S}$  or  $\mathcal{T}$ , the natural inclusion  $\mathcal{CFG}^{\text{qc}}(\mathcal{A}) \hookrightarrow \mathcal{C}(\mathcal{A}\text{-Mod}_{\underline{q}^c})$  induces an equivalence of categories*

$$\mathcal{DFG}^{\text{qc}}(\mathcal{A}) \cong \mathcal{D}^{\text{fg}}(\mathcal{A}\text{-Mod}_{\underline{q}^c}).$$

Recall the regrading functor

$$\xi : \mathcal{C}(\mathcal{S}\text{-Mod}_{\underline{q}^c}) \rightarrow \mathcal{C}(\mathcal{R}\text{-Mod}_{\underline{q}^c})$$

of [MR1, §3.5]. It is defined by the condition

$$\xi(\mathcal{M})_j^i := \mathcal{M}_j^{i-j},$$

and it induces equivalences of triangulated categories

$$(1.1.4) \quad \mathcal{DFG}^{\text{qc}}(\mathcal{S}) \cong \mathcal{DFG}^{\text{qc}}(\mathcal{R}), \quad \mathcal{D}^{\text{fg}}(\mathcal{S}\text{-Mod}_{\underline{q}^c}) \cong \mathcal{D}^{\text{fg}}(\mathcal{R}\text{-Mod}_{\underline{q}^c})$$

(because it respects the condition of being bounded above for the internal grading).

**Proposition 1.1.5.** *The equivalences  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{G}}$  restrict to equivalences*

$$\mathcal{D}^{\text{fg}}(\mathcal{S}\text{-Mod}_{-}^{\text{qc}}) \cong \mathcal{D}^{\text{fg}}(\mathcal{T}\text{-Mod}_{-}^{\text{qc}}).$$

*Proof.* Any object of the category  $\mathcal{C}(\mathcal{T}\text{-Mod}_{-}^{\text{qc}})$  has a finite filtration (as a  $\mathcal{T}$ -dg-module) such that  $\mathcal{T}$  acts trivially on the associated graded. Hence, using Lemma 1.1.3, the category  $\mathcal{D}^{\text{fg}}(\mathcal{T}\text{-Mod}_{-}^{\text{qc}})$  is generated, as a triangulated category, by objects of  $\mathcal{D}^b\text{Coh}^{\mathbb{G}_m}(X)$  (endowed with a trivial  $\mathcal{T}$ -action), where  $\mathbb{G}_m$  acts trivially on  $X$ .

On the other hand, we claim that  $\mathcal{D}^{\text{fg}}(\mathcal{S}\text{-Mod}_{-}^{\text{qc}})$  is generated, as a triangulated category, by objects of the form  $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{F}$ , for  $\mathcal{F}$  in  $\mathcal{D}^b\text{Coh}^{\mathbb{G}_m}(X)$ . Indeed, using the regrading functor, it is enough to prove the same result for  $\mathcal{R}$ -dg-modules. Then this follows from the following general result (see [CG, p. 266, last paragraph]), using the fact that  $\mathcal{DFG}(\mathcal{R})$  is equivalent to the bounded derived category of  $\mathbb{G}_m$ -equivariant coherent sheaves on  $F$ .

**Proposition 1.1.6.** *Let  $H$  be an algebraic group, and let  $\pi : V \rightarrow Y$  be an  $H$ -equivariant vector bundle. Then the category  $\mathcal{D}^b\text{Coh}^H(V)$  is generated, as a triangulated category, by objects of the form  $\pi^*\mathcal{F}$  for  $\mathcal{F}$  in  $\text{Coh}^H(Y)$ .*

We have determined a set of generators  $\mathcal{G}_1$ , respectively  $\mathcal{G}_2$ , for the triangulated category  $\mathcal{D}^{\text{fg}}(\mathcal{T}\text{-Mod}_{-}^{\text{qc}})$ , respectively  $\mathcal{D}^{\text{fg}}(\mathcal{S}\text{-Mod}_{-}^{\text{qc}})$ . By definition,  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{G}}$  induce equivalences between the subcategories whose objects are in  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . Hence they induce an equivalence  $\mathcal{D}^{\text{fg}}(\mathcal{T}\text{-Mod}_{-}^{\text{qc}}) \cong \mathcal{D}^{\text{fg}}(\mathcal{S}\text{-Mod}_{-}^{\text{qc}})$ .  $\square$

Let us now introduce the following notation:

$$\text{DGCoh}^{\text{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} X) := \mathcal{D}^{\text{fg}}(\mathcal{T}\text{-Mod}_{-}^{\text{qc}}).$$

Comparing Lemma 1.1.3 and [Ri, Proposition 3.3.4], we see that this category is equivalent to that denoted similarly in [Ri, (2.3.8)]. There is a natural forgetful functor

$$(1.1.7) \quad \text{For} : \text{DGCoh}^{\text{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} X) \rightarrow \text{DGCoh}(F^{\perp} \overset{R}{\cap}_{E^*} X).$$

We define also

$$\text{DGCoh}^{\text{gr}}(F) := \mathcal{D}^{\text{fg}}(\mathcal{S}\text{-Mod}_{-}^{\text{qc}}).$$

Consider the action of  $\mathbb{G}_m$  on  $F$  where  $t \in \mathbb{G}_m$  acts by multiplication by  $t^2$  in the fibers. By the second equivalence in (1.1.4), the category  $\text{DGCoh}^{\text{gr}}(F)$  is equivalent to  $\mathcal{D}^b\text{Coh}^{\mathbb{G}_m}(F)$ . In particular, there is a natural forgetful functor

$$(1.1.8) \quad \text{For} : \text{DGCoh}^{\text{gr}}(F) \rightarrow \mathcal{D}^b\text{Coh}(F).$$

Let us prove that the category  $\text{DGCoh}^{\text{gr}}(F)$  is equivalent to the category denoted similarly in [Ri, (2.3.6)]. Consider the category  $\mathcal{D}_{\mathbb{G}_m}^{+, \text{qc}, \text{fg}}(X, \mathcal{S})$



of [Ri]. By [Ri, (2.3.5)], there is a natural functor  $\phi : \mathcal{D}_{\mathbb{G}_m}^{+, \text{qc}, \text{fg}}(X, \mathcal{S}) \rightarrow \mathcal{D}^{\text{fg}}(\mathcal{S}\text{-Mod}_{-}^{\text{qc}})$ . Moreover, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}_{\mathbb{G}_m}^{+, \text{qc}, \text{fg}}(X, \mathcal{S}) & & \\ \phi \downarrow & \searrow \psi & \\ \mathcal{D}^{\text{fg}}(\mathcal{S}\text{-Mod}_{-}^{\text{qc}}) & \xrightarrow{\overline{\mathcal{F}}} & \text{DGCoh}^{\text{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} X), \end{array}$$

where  $\psi$  denotes the equivalence of [Ri, (2.3.1)]. As both  $\overline{\mathcal{F}}$  and  $\psi$  are equivalences, so is  $\phi$ .

Hence we have obtained the following more general version of [Ri, Theorem 2.3.10]. Note also that the new definition of the category  $\text{DGCoh}^{\text{gr}}(F)$  would allow to simplify (and generalize) the constructions of [Ri, §§2.4, 2.5 and 4.2].

**Theorem 1.1.9.** *There exists an equivalence of triangulated categories*

$$\kappa : \text{DGCoh}^{\text{gr}}(F) \xrightarrow{\sim} \text{DGCoh}^{\text{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} X),$$

called linear Koszul duality.

*Remark 1.1.10.* The functor  $\kappa$  commutes with internal shifts. However, we have rather  $\xi(\mathcal{M}\langle m \rangle) = \xi(\mathcal{M})\langle m \rangle[-m]$  for  $m \in \mathbb{Z}$ . Hence the functor  $\text{For}$  of (1.1.8) satisfies  $\text{For}(\mathcal{M}\langle m \rangle) = \text{For}(\mathcal{M})[-m]$ .

**1.2. Homological duality.** For simplicity, from now on we assume that  $X$  is a smooth variety over an algebraically closed field  $\mathbb{k}$ . We have seen in §1.1 that the category  $\text{DGCoh}^{\text{gr}}(F)$  is canonically equivalent to  $\mathcal{D}^b \text{Coh}^{\mathbb{G}_m}(F)$ . Now it is well-known that the functor

$$\mathbb{D}_F : \begin{cases} \mathcal{D}^b \text{Coh}^{\mathbb{G}_m}(F) & \rightarrow & \mathcal{D}^b \text{Coh}^{\mathbb{G}_m}(F) \\ \mathcal{M} & \mapsto & R\mathcal{H}om_{\mathcal{O}_F}(\mathcal{M}, \mathcal{O}_F) \end{cases}$$

is an equivalence of categories, such that  $\mathbb{D}_F \circ \mathbb{D}_F \cong \text{Id}$ . We denote by

$$\mathbb{D}_{\mathcal{S}} : \text{DGCoh}^{\text{gr}}(F) \xrightarrow{\sim} \text{DGCoh}^{\text{gr}}(F)$$

the induced equivalence.

Now we define a duality functor

$$\mathbb{D}_{\mathcal{T}} : \text{DGCoh}^{\text{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} X) \xrightarrow{\sim} \text{DGCoh}^{\text{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} X).$$

Consider the functor

$$\mathbb{D}_{\mathcal{T}} : \begin{cases} \mathcal{C}\mathcal{F}\mathcal{G}^{\text{qc}}(\mathcal{T}) & \rightarrow & \mathcal{C}\mathcal{F}\mathcal{G}^{\text{qc}}(\mathcal{T})^{\text{op}} \\ \mathcal{M} & \mapsto & \mathcal{H}om_{\mathcal{T}}(\mathcal{M}, \mathcal{T}) \end{cases}.$$

Here, the left  $\mathcal{T}$ -action is induced by the left multiplication of  $\mathcal{T}$  on itself.

**Lemma 1.2.1.** *The functor  $\mathbb{D}_{\mathcal{T}}$  admits a left derived functor*

$$\mathbb{D}_{\mathcal{T}} : \text{DGCoh}^{\text{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} X) \rightarrow \text{DGCoh}^{\text{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} X),$$

which is an equivalence of categories such that  $\mathbb{D}_{\mathcal{T}} \circ \mathbb{D}_{\mathcal{T}} \cong \text{Id}$ .

Before giving the proof of this lemma, we give an alternative definition of the functor  $D_{\mathcal{T}}$ . Let  $n = \text{rk}(\mathcal{F})$ , and  $\mathcal{L} := \Lambda^n(\mathcal{F})$  (considered as a line bundle on  $X$ , in internal degree 0). For any  $i = 0, \dots, n$ , consider the morphism

$$\begin{cases} \Lambda^i(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} & \rightarrow & \mathcal{H}om_{\mathcal{O}_X}(\Lambda^{n-i}(\mathcal{F}), \mathcal{O}_X) \\ (x \otimes a) & \mapsto & (y \mapsto (-1)^{n \cdot |y|} (x \wedge y) \otimes a) \end{cases} .$$

This collection of morphisms induces an isomorphism of  $\mathbb{G}_{\mathbf{m}}$ -equivariant  $\mathcal{T}$ -dg-modules

$$(1.2.2) \quad \mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \xrightarrow{\sim} \text{Coind}_{\mathcal{T}}(\mathcal{O}_X)[n]\langle 2n \rangle$$

(see [Ri, §1.2] for the definition of the coinduction functor). Using this isomorphism and [Ri, §1.2], we obtain isomorphisms of  $\mathcal{T}$ -dg-modules, for any  $\mathcal{M}$  in  $\mathcal{C}\mathcal{F}\mathcal{G}^{\text{gr}}(\mathcal{T})$ :

$$(1.2.3) \quad \begin{aligned} D_{\mathcal{T}}(\mathcal{M}) &\cong \mathcal{H}om_{\mathcal{T}}(\mathcal{M}, \text{Coind}_{\mathcal{T}}(\mathcal{O}_X)) \otimes_{\mathcal{O}_X} \mathcal{L}[n]\langle 2n \rangle \\ &\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L}[n]\langle 2n \rangle. \end{aligned}$$

Here, the  $\mathcal{T}$ -module structure on  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$  is given by  $(t \cdot \phi)(m) = (-1)^{|t| \cdot |\phi|} (t \cdot m)$ .

The following lemma can be proved exactly as in [MR1, Proposition 3.1.1].

**Lemma 1.2.4.** *For every  $\mathcal{M}$  in  $\mathcal{C}\mathcal{F}\mathcal{G}^{\text{gr}}(\mathcal{T})$ , there exists an object  $\mathcal{P}$  of  $\mathcal{C}\mathcal{F}\mathcal{G}^{\text{gr}}(\mathcal{T})$  such that for every  $i$  and  $j$  the  $\mathcal{O}_X$ -module  $\mathcal{P}_j^i$  is locally free of finite rank, and a (surjective) quasi-isomorphism  $\mathcal{P} \xrightarrow{\text{qis}} \mathcal{M}$ .*

Finally we can prove Lemma 1.2.1.

*Proof of Lemma 1.2.1.* Using isomorphism (1.2.3), it is clear that any object  $\mathcal{P}$  as in Lemma 1.2.4 is split on the left for the functor  $D_{\mathcal{T}}$ . This lemma asserts that there are enough such objects in  $\mathcal{C}\mathcal{F}\mathcal{G}^{\text{gr}}(\mathcal{T})$ , which implies the existence of the derived functor.

It also follows from (1.2.3) that  $D_{\mathcal{T}} \circ D_{\mathcal{T}}(\mathcal{P}) \cong \mathcal{P}$  naturally for any object  $\mathcal{P}$  as in Lemma 1.2.4. Hence  $\mathbb{D}_{\mathcal{T}} \circ \mathbb{D}_{\mathcal{T}} \cong \text{Id}$ .  $\square$

**1.3. Compatibility.** In this subsection we prove that the Koszul duality  $\kappa$  is compatible with the duality equivalences  $\mathbb{D}_{\mathcal{T}}$  and  $\mathbb{D}_{\mathcal{S}}$ . More precisely, we prove the following. Recall the notation  $n, \mathcal{L}$  introduced in §1.2

**Proposition 1.3.1.** *Consider the following diagram of equivalences*

$$\begin{array}{ccc} \text{DGCoh}^{\text{gr}}(F) & \xrightarrow[\sim]{\kappa} & \text{DGCoh}^{\text{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} X) \\ \mathbb{D}_{\mathcal{S}} \downarrow \wr & & \wr \downarrow \mathbb{D}_{\mathcal{T}} \\ \text{DGCoh}^{\text{gr}}(F) & \xrightarrow[\sim]{\kappa} & \text{DGCoh}^{\text{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} X). \end{array}$$

For any  $\mathcal{M}$  in  $\text{DGCoh}^{\text{gr}}(F)$  there is a functorial isomorphism

$$\mathbb{D}_{\mathcal{T}} \circ \kappa(\mathcal{M}) \cong \kappa \circ \mathbb{D}_{\mathcal{S}}(\mathcal{M}) \otimes_{\mathcal{O}_X} \mathcal{L}\langle 2n \rangle[n].$$

*Proof.* In fact we will rather work with the equivalence  $\kappa^{-1}$ . Let  $\mathcal{P}$  be an object of  $\mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} X)$  which satisfies the assumptions of Lemma 1.2.4. Then we have, using (1.2.3),

$$\kappa^{-1} \circ \mathbb{D}_{\mathcal{T}}(\mathcal{P}) \cong \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L}[n]\langle 2n \rangle.$$

The differential on the right hand side is a Koszul differential. On the other hand, we have

$$\mathbb{D}_{\mathcal{S}} \circ \kappa^{-1}(\mathcal{P}) \cong \mathbb{D}_{\mathcal{S}}(\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{P}) \cong \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{O}_X).$$

Again, the differential on the right hand side is a Koszul differential. The result follows.  $\square$

**1.4. Duality on  $F^\perp \overset{R}{\cap}_{E^*} X$  and  $F^\perp$ .** To finish this section, we study the relation between our equivalence  $\mathbb{D}_{\mathcal{T}}$  and the standard duality on  $F^\perp$ .

Let us consider the functor  $\mathbb{D}_{F^\perp} : \mathcal{D}^b \mathrm{Coh}^{\mathbb{G}_m}(F^\perp) \xrightarrow{\sim} \mathcal{D}^b \mathrm{Coh}^{\mathbb{G}_m}(F^\perp)$ . Let also  $p : F^\perp \overset{R}{\cap}_{E^*} X \rightarrow F^\perp$  be the natural morphism of dg-schemes, and let

$$Rp_* : \mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} X) \rightarrow \mathcal{D}^b \mathrm{Coh}^{\mathbb{G}_m}(F^\perp)$$

be the associated direct image functor (see [Ri, §1.8]). To make this functor more explicit, it is better to use a different realization of the category  $\mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} X)$ . Namely, using the Koszul resolution

$$\mathrm{S}(\mathcal{E}) \otimes_{\mathcal{O}_X} \Lambda(\mathcal{E}) \xrightarrow{\mathrm{qis}} \mathcal{O}_X,$$

one can realize the dg-sheaf of functions on  $F^\perp \overset{R}{\cap}_{E^*} X$  as the dg-algebra

$$\mathcal{Q} := \Lambda_{\mathcal{O}_{F^\perp}}(p_{F^\perp}^* \mathcal{E}),$$

where  $p_{F^\perp} : F^\perp \rightarrow X$  is the projection. (Here the differential is *not* trivial.) Then the category  $\mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} X)$  is naturally equivalent to the subcategory of the derived category of  $\mathbb{G}_m$ -equivariant quasi-coherent  $\mathcal{Q}$ -dg-modules whose cohomology is locally finitely generated. We denote the latter category by  $\mathcal{D}^{\mathrm{fg}}(\mathcal{Q}\text{-Mod}^{\mathrm{qc}})$ . One can easily check that there is well-defined equivalence

$$\mathbb{D}_{\mathcal{Q}} : \mathcal{D}^{\mathrm{fg}}(\mathcal{Q}\text{-Mod}^{\mathrm{qc}}) \rightarrow \mathcal{D}^{\mathrm{fg}}(\mathcal{Q}\text{-Mod}^{\mathrm{qc}})$$

which is obtained as the derived functor of the functor  $\mathcal{M} \mapsto \mathcal{H}om_{\mathcal{Q}}(\mathcal{M}, \mathcal{Q})$ , and which corresponds to  $\mathbb{D}_{\mathcal{T}}$  under the equivalence mentioned above.

Under the equivalence  $\mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} X) \cong \mathcal{D}^{\mathrm{fg}}(\mathcal{Q}\text{-Mod}^{\mathrm{qc}})$ , the functor  $Rp_*$  is simply the restriction functor from  $\mathcal{Q}$ -dg-modules to complexes of  $\mathcal{O}_{F^\perp}$ -modules.

Let  $m$  be the rank of  $\mathcal{E}$ , and let  $\mathcal{K} := \Lambda^m(\mathcal{E})$ , considered as a line bundle on  $X$  in internal degree 0.

**Lemma 1.4.1.** *For any  $\mathcal{M}$  in  $\mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} X)$ , there is a functorial isomorphism*

$$Rp_* \circ \mathbb{D}_{\mathcal{T}}(\mathcal{M}) \cong (\mathbb{D}_{F^\perp} \circ Rp_*(\mathcal{M})) \otimes_{\mathcal{O}_{F^\perp}} p_{F^\perp}^* \mathcal{K}[m]\langle 2m \rangle.$$

*Proof.* Using the remarks above, one can work with  $\mathcal{Q}$ -dg-modules instead of  $\mathcal{T}$ -dg-modules. For  $\mathcal{M}$  split on the left for  $\mathbb{D}_{\mathcal{Q}}$ , we have

$$Rp_*\mathbb{D}_{\mathcal{Q}}(\mathcal{M}) \cong \mathcal{H}om_{\mathcal{Q}}(\mathcal{M}, \mathcal{Q}).$$

Now, one easily checks (as for isomorphism (1.2.2)) that there is an isomorphism of  $\mathcal{Q}$ -dg-modules

$$\mathcal{Q} \cong \text{Coind}_{\mathcal{Q}}(\mathcal{O}_{F^\perp}) \otimes_{\mathcal{O}_{F^\perp}} p_{F^\perp}^* \mathcal{K}[m](2m).$$

Hence we obtain

$$Rp_*\mathbb{D}_{\mathcal{Q}}(\mathcal{M}) \cong \mathcal{H}om_{\mathcal{O}_{F^\perp}}(\mathcal{M}, \mathcal{O}_{F^\perp}) \otimes_{\mathcal{O}_{F^\perp}} p_{F^\perp}^* \mathcal{K}[m](2m),$$

which gives the result.  $\square$

*Remark 1.4.2.* It is natural to consider that the “dimension” of the dg-scheme  $F^\perp \overset{R}{\frown}_{E^*} X$  is  $\dim(F^\perp) + \dim(X) - \dim(E^*)$ . Hence the “difference of dimensions” between  $F^\perp$  and  $F^\perp \overset{R}{\frown}_{E^*} X$  is  $m = \text{rk}(E)$ , which makes Lemma 1.4.1 consistent with the general fact that proper direct image commutes with Grothendieck-Serre duality.

**1.5. Equivariant analogues.** Let us note that if an algebraic group acts on the scheme  $X$ , and acts linearly on the vector bundle  $E$  preserving  $F$ , then there are obvious equivariant analogues of all the constructions and results of this section. As we will not use these equivariant versions, we do not state them.

## 2. REMINDER ON LOCALIZATION IN POSITIVE CHARACTERISTIC

**2.1. Notation.** Let  $\mathbb{k}$  be an algebraically closed field of characteristic  $p > 0$ . Let  $R$  be a root system, and  $G$  the corresponding connected, semi-simple, simply-connected algebraic group over  $\mathbb{k}$ . In the whole paper we assume that

$$p > h,$$

where  $h$  is the Coxeter number of  $G$ . Let  $B$  be a Borel subgroup of  $G$ ,  $T \subset B$  a maximal torus,  $U$  the unipotent radical of  $B$ . Let  $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{n}$ , be their respective Lie algebras. Let  $R^+ \subset R$  be the positive roots, chosen as the roots in  $\mathfrak{g}/\mathfrak{b}$ . Let  $\mathcal{B} := G/B$  be the flag variety of  $G$ , and  $\tilde{\mathcal{N}} := T^*\mathcal{B}$  its cotangent bundle. We have the geometric description

$$\tilde{\mathcal{N}} = \{(X, gB) \in \mathfrak{g}^* \times \mathcal{B} \mid X|_{\mathfrak{g}/\mathfrak{b}} = 0\}.$$

We also introduce the “extended cotangent bundle”

$$\tilde{\mathfrak{g}} := \{(X, gB) \in \mathfrak{g}^* \times \mathcal{B} \mid X|_{\mathfrak{g}/\mathfrak{n}} = 0\}.$$

Let  $\mathfrak{h}$  denote the “abstract” Cartan subalgebra of  $\mathfrak{g}$ , isomorphic to  $\mathfrak{b}_0/[\mathfrak{b}_0, \mathfrak{b}_0]$  for any Borel subalgebra  $\mathfrak{b}_0$  of  $\mathfrak{g}$ .

We denote by  $\mathbb{Y} := \mathbb{Z}R$  the root lattice of  $R$ , and by  $\mathbb{X} := X^*(T)$  the weight lattice. Let  $W$  be the Weyl group of  $(G, T)$ ,  $W_{\text{aff}} := W \ltimes \mathbb{Y}$  the affine Weyl group, and  $W'_{\text{aff}} := W \ltimes \mathbb{X}$  the extended affine Weyl group. Let  $\rho \in \mathbb{X}$

be the half sum of the positive roots. The “dot-action” of  $W$  on  $\mathfrak{t}^*$  is defined by

$$w \bullet \lambda = w(\lambda + \rho) - \rho$$

(where we identify  $\rho$  and its differential). Similarly, there is a dot-action of  $W$  on  $\mathfrak{t}$  obtained by duality. There is also a dot-action of  $W'_{\text{aff}}$  on  $\mathbb{X}$ , defined by

$$(wt_\lambda) \bullet \mu = w(\mu + p\lambda + \rho) - \rho.$$

We set

$$C_0 := \{\lambda \in \mathbb{X} \mid \forall \alpha \in R^+, 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p\},$$

the set of integral weights in the fundamental alcove. We will also consider its “closure”

$$\overline{C_0} := \{\lambda \in \mathbb{X} \mid \forall \alpha \in R^+, 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq p\}.$$

If  $P \subseteq G$  is a parabolic subgroup containing  $B$ ,  $\mathfrak{p}$  its Lie algebra,  $\mathfrak{p}^u$  the nilpotent radical of  $\mathfrak{p}$ , and  $\mathcal{P} = G/P$  the corresponding flag variety, we consider the following analogue of the variety  $\tilde{\mathfrak{g}}$ :

$$\tilde{\mathfrak{g}}_{\mathcal{P}} := \{(X, gP) \in \mathfrak{g}^* \times \mathcal{P} \mid X|_{\mathfrak{g}, \mathfrak{p}^u} = 0\}.$$

In particular,  $\tilde{\mathfrak{g}}_{\mathcal{B}} = \tilde{\mathfrak{g}}$ . The quotient morphism  $\pi_{\mathcal{P}} : \mathcal{B} \rightarrow \mathcal{P}$  induces a morphism

$$(2.1.1) \quad \tilde{\pi}_{\mathcal{P}} : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}_{\mathcal{P}}.$$

We also denote by  $W_P \subseteq W$  the Weyl group of the Levi of  $P$ .

For any dominant weight  $\lambda$ , we denote by  $L(\lambda)$  the simple  $G$ -module with highest weight  $\lambda$ .

Finally, we set  $N := \dim(G/B)$ ,  $N_P := \dim(G/P)$ ,  $d = \dim(\mathfrak{g})$ .

**2.2. Localization.** In this subsection we review the localization theory in positive characteristic developed in [BMR, BMR2, BM].

Let  $\mathfrak{Z}$  be the center of  $\mathcal{U}\mathfrak{g}$ , the enveloping algebra of  $\mathfrak{g}$ . The subalgebra of  $G$ -invariants,  $\mathfrak{Z}_{\text{HC}} := (\mathcal{U}\mathfrak{g})^G$  is central in  $\mathcal{U}\mathfrak{g}$ . This is the “Harish-Chandra part” of the center, which is isomorphic to  $S(\mathfrak{t})^{(W, \bullet)}$ , the algebra of  $W$ -invariants in the symmetric algebra of  $\mathfrak{t}$ , for the dot-action. The center  $\mathfrak{Z}$  also has an other part, the “Frobenius part”  $\mathfrak{Z}_{\text{Fr}}$ , which is generated, as an algebra, by the elements  $X^p - X^{[p]}$  for  $X \in \mathfrak{g}$ . It is isomorphic to  $S(\mathfrak{g}^{(1)})$ , the functions on the Frobenius twist of  $\mathfrak{g}^*$ . Under our assumption  $p > h$ , there is an isomorphism

$$\mathfrak{Z}_{\text{HC}} \otimes_{\mathfrak{Z}_{\text{Fr}} \cap \mathfrak{Z}_{\text{HC}}} \mathfrak{Z}_{\text{Fr}} \xrightarrow{\sim} \mathfrak{Z}.$$

Hence, a character of  $\mathfrak{Z}$  is given by a compatible pair  $(\nu, \chi) \in \mathfrak{t}^* \times \mathfrak{g}^{*(1)}$ . In this paper we will only consider the case when  $\chi = 0$ , and  $\nu \in \mathfrak{t}^*$  is *integral*,

i.e. in the image of the natural map  $\mathbb{X} \rightarrow \mathfrak{t}^*$ . If  $\lambda \in \mathbb{X}$ , we still denote by  $\lambda$  its image in  $\mathfrak{t}^*$ . We denote the corresponding specializations by

$$\begin{aligned} (\mathcal{U}\mathfrak{g})^\lambda &:= (\mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{Z}_{\text{HC}}} \mathbb{k}_\lambda, \\ (\mathcal{U}\mathfrak{g})_0 &:= (\mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{Z}_{\text{Fr}}} \mathbb{k}_0. \end{aligned}$$

Let  $\text{Mod}^{\text{fg}}(\mathcal{U}\mathfrak{g})$  be the abelian category of finitely generated  $\mathcal{U}\mathfrak{g}$ -modules. If  $\lambda \in \mathbb{X}$ , we denote by  $\text{Mod}_{(\lambda,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g})$  the category of finitely generated  $\mathcal{U}\mathfrak{g}$ -modules on which  $\mathfrak{Z}$  acts with generalized character  $(\lambda, 0)$ . We define similarly the categories  $\text{Mod}_0^{\text{fg}}((\mathcal{U}\mathfrak{g})^\lambda)$ ,  $\text{Mod}_\lambda^{\text{fg}}((\mathcal{U}\mathfrak{g})_0)$ . Hence we have inclusions

$$\begin{array}{ccc} \text{Mod}_0^{\text{fg}}((\mathcal{U}\mathfrak{g})^\lambda) & \hookrightarrow & \text{Mod}_{(\lambda,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g}) \longrightarrow \text{Mod}^{\text{fg}}(\mathcal{U}\mathfrak{g}). \\ & \nearrow & \\ \text{Mod}_\lambda^{\text{fg}}((\mathcal{U}\mathfrak{g})_0) & & \end{array}$$

Note the category  $\text{Mod}_\lambda^{\text{fg}}((\mathcal{U}\mathfrak{g})_0)$  is equivalent to the category of finitely generated modules over  $(\mathcal{U}\mathfrak{g})_0^\lambda$ , the completion of  $(\mathcal{U}\mathfrak{g})_0$  with the respect to the image of the ideal of  $\mathfrak{Z}_{\text{HC}}$  defined by  $\lambda$  (see e.g. [Ri, §4.4]).

Recall that a weight  $\lambda \in \mathbb{X}$  is called *regular* if, for any root  $\alpha$ ,  $\langle \lambda + \rho, \alpha^\vee \rangle \notin p\mathbb{Z}$ , i.e. if  $\lambda$  is not on any reflection hyperplane of  $W_{\text{aff}}$  (for the dot-action). If  $\mu \in \mathbb{X}$ , we denote by  $\text{Stab}_{(W_{\text{aff}}, \bullet)}(\mu)$  the stabilizer of  $\mu$  for the dot-action of  $W_{\text{aff}}$  on  $\mathbb{X}$ . Under our hypothesis  $p > h$ , we have  $(p\mathbb{X}) \cap \mathbb{Y} = p\mathbb{Y}$ . It follows that  $\text{Stab}_{(W_{\text{aff}}, \bullet)}(\mu)$  is also the stabilizer of  $\mu$  for the action of  $W'_{\text{aff}}$  on  $\mathbb{X}$ .

The localization theory in positive characteristic (due to Bezrukavnikov, Mirković and Rumynin) provides a geometric description of the categories of  $\mathcal{U}\mathfrak{g}$ -modules considered above. More precisely we have (see [BMR, Theorem 5.3.1] for (i), and [BMR2, Theorem 1.5.1.c, Lemma 1.5.2.b] for (ii)):

**Theorem 2.2.1.** (i) *Let  $\lambda \in \mathbb{X}$  be regular. There exist equivalences of triangulated categories*

$$(2.2.2) \quad \mathcal{D}^b \text{Coh}_{\mathcal{B}(1)}(\tilde{\mathfrak{g}}^{(1)}) \cong \mathcal{D}^b \text{Mod}_{(\lambda,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g}),$$

$$(2.2.3) \quad \mathcal{D}^b \text{Coh}_{\mathcal{B}(1)}(\tilde{\mathcal{N}}^{(1)}) \cong \mathcal{D}^b \text{Mod}_0^{\text{fg}}((\mathcal{U}\mathfrak{g})^\lambda).$$

(ii) *More generally, let  $\mu \in \mathbb{X}$ , and let  $P$  be a parabolic subgroup of  $G$  containing  $B$  such that  $\text{Stab}_{(W_{\text{aff}}, \bullet)}(\mu) = W_P$ . Let  $\mathcal{P} = G/P$  be the corresponding flag variety. Then there exists an equivalence of triangulated categories*

$$(2.2.4) \quad \mathcal{D}^b \text{Coh}_{\mathcal{P}(1)}(\tilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}) \cong \mathcal{D}^b \text{Mod}_{(\mu,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g}).$$

As in [BMR], let us consider  $\tilde{\mathcal{D}} := (q_* \mathcal{D}_{G/U})^T$ , where  $q : G/N \rightarrow \mathcal{B}$  is the projection, and  $T$  acts on  $G/U$  via right multiplication. This algebra is an Azumaya algebra over  $\tilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^*$ . (Here, the morphism  $\mathfrak{h}^* \rightarrow \mathfrak{h}^*(1)$  is the Artin-Schreier map, see [BMR].) This Azumaya algebra splits on the formal

neighborhood of  $\mathcal{B}^{(1)} \times \{\lambda\}$ , for any  $\lambda \in \mathbb{X}$ . Moreover, there are natural choices of splitting bundles. For any regular  $\lambda \in \mathbb{X}$ , we denote by  $\mathcal{M}^\lambda$  the splitting bundle constructed as in [BMR2, §1.3.5], and by

$$\begin{aligned}\gamma_\lambda^{\mathcal{B}} &: \mathcal{D}^b \text{Coh}_{\mathcal{B}^{(1)}}(\tilde{\mathfrak{g}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b \text{Mod}_{(\lambda,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g}) \\ \epsilon_\lambda^{\mathcal{B}} &: \mathcal{D}^b \text{Coh}_{\mathcal{B}^{(1)}}(\tilde{\mathcal{N}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b \text{Mod}_0^{\text{fg}}((\mathcal{U}\mathfrak{g})^\lambda),\end{aligned}$$

the associated equivalences (2.2.2) and (2.2.3). Note that these equivalences depend on  $\lambda$ , and not only on its image in  $\mathbb{X}/(W'_{\text{aff}}, \bullet)$ . Note also that the projection  $\tilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^* \rightarrow \tilde{\mathfrak{g}}^{(1)}$  induces an isomorphism<sup>3</sup> between the formal neighborhood of  $\mathcal{B}^{(1)} \times \{\lambda\}$  and that of  $\mathcal{B}^{(1)}$ , hence one can consider  $\mathcal{M}^\lambda$  as a vector bundle on the formal neighborhood of the zero section in  $\tilde{\mathfrak{g}}^{(1)}$ .

Similarly, for  $\mu, \mathcal{P}$  as in Theorem 2.2.1, we denote by  $\mathcal{M}_\mu^{\mathcal{P}}$  the splitting bundle on the formal neighborhood of  $\mathcal{P}^{(1)} \times \{\mu\}$  in  $\tilde{\mathfrak{g}}_{\mathcal{P}}^{(1)} \times_{\mathfrak{h}^{*(1)}/W_{\mathcal{P}}} \mathfrak{h}^*/(W_{\mathcal{P}}, \bullet)$  (or equivalently on the formal neighborhood of the zero section in  $\tilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}$ ) constructed as in [BMR2, §1.3.5], and by

$$\gamma_\mu^{\mathcal{P}} : \mathcal{D}^b \text{Coh}_{\mathcal{P}^{(1)}}(\tilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b \text{Mod}_{(\mu,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g})$$

the associated equivalence (2.2.4).

Finally, the following theorem is proved in [Ri, Theorem 3.4.1, Proposition 3.4.13, Theorem 3.4.14]. As in §1.4, we denote by  $p : (\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)} \rightarrow \tilde{\mathfrak{g}}^{(1)}$  and  $p_{\mathcal{P}} : (\tilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)} \rightarrow \tilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}$  the natural morphisms of dg-schemes.

**Theorem 2.2.5.** (i) *Let  $\lambda \in \mathbb{X}$  be regular. There exists an equivalence of triangulated categories*

$$\hat{\gamma}_\lambda^{\mathcal{B}} : \text{DGCoh}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \xrightarrow{\sim} \mathcal{D}^b \text{Mod}_\lambda^{\text{fg}}((\mathcal{U}\mathfrak{g})_0)$$

such that the following diagram commutes, where the functor  $\text{Incl}$  is induced by the inclusion  $\text{Mod}_\lambda^{\text{fg}}((\mathcal{U}\mathfrak{g})_0) \hookrightarrow \text{Mod}_{(\lambda,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g})$ :

$$\begin{array}{ccc} \text{DGCoh}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) & \xrightarrow[\sim]{\hat{\gamma}_\lambda^{\mathcal{B}}} & \mathcal{D}^b \text{Mod}_\lambda^{\text{fg}}((\mathcal{U}\mathfrak{g})_0) \\ \downarrow Rp_* & & \downarrow \text{Incl} \\ \mathcal{D}^b \text{Coh}_{\mathcal{B}^{(1)}}(\tilde{\mathfrak{g}}^{(1)}) & \xrightarrow[\sim]{\gamma_\lambda^{\mathcal{B}}} & \mathcal{D}^b \text{Mod}_{(\lambda,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g}). \end{array}$$

(ii) *Let  $\mu, \mathcal{P}$  be as in Theorem 2.2.1(ii). There exists an equivalence of triangulated categories*

$$\hat{\gamma}_\mu^{\mathcal{P}} : \text{DGCoh}((\tilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \xrightarrow{\sim} \mathcal{D}^b \text{Mod}_\mu^{\text{fg}}((\mathcal{U}\mathfrak{g})_0)$$

<sup>3</sup>In fact, this property is already used to deduce equivalence (2.2.2) from [BMR, Theorem 5.3.1].

such that the following diagram commutes, where  $\text{Incl}$  is induced by the inclusion  $\text{Mod}_\mu^{\text{fg}}((\mathcal{U}\mathfrak{g})_0) \hookrightarrow \text{Mod}_{(\mu,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g})$ :

$$\begin{array}{ccc} \text{DGCoh}((\tilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) & \xrightarrow[\sim]{\hat{\gamma}_\mu^{\mathcal{P}}} & \mathcal{D}^b \text{Mod}_\mu^{\text{fg}}((\mathcal{U}\mathfrak{g})_0) \\ \downarrow R(p_{\mathcal{P}})_* & & \downarrow \text{Incl} \\ \mathcal{D}^b \text{Coh}_{\mathcal{P}^{(1)}}(\tilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}) & \xrightarrow[\sim]{\gamma_\mu^{\mathcal{P}}} & \mathcal{D}^b \text{Mod}_{(\mu,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g}). \end{array}$$

*Remark 2.2.6.* Equivalently, the condition of Theorem 2.2.1(ii) says that  $\mu$  is on the reflection hyperplane corresponding to any simple root of  $W_P$ , but not on any reflection hyperplane of a reflection (simple or not) in  $W_{\text{aff}} - W_P$ . With this description, it is clear that if  $\mu$  satisfies this condition, then  $-\mu - 2\rho$  also satisfies it (for the same parabolic subgroup).

**2.3. Geometric description of duality.** There exists a natural isomorphism of algebras  $\mathcal{U}\mathfrak{g} \xrightarrow{\sim} \mathcal{U}\mathfrak{g}^{\text{op}}$ , induced by the assignment  $X \in \mathfrak{g} \mapsto -X$ . Hence the duality  $M \mapsto M^*$  induces a duality operation  $M \mapsto M^\vee$  on the category of finite dimensional (left)  $\mathcal{U}\mathfrak{g}$ -modules. This duality induces a duality between the categories  $\text{Mod}_{(\lambda,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g})$  and  $\text{Mod}_{(-\lambda-2\rho,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g})^4$ , between  $\text{Mod}_0^{\text{fg}}((\mathcal{U}\mathfrak{g})^\lambda)$  and  $\text{Mod}_0^{\text{fg}}((\mathcal{U}\mathfrak{g})^{-\lambda-2\rho})$ , and between  $\text{Mod}_\lambda^{\text{fg}}((\mathcal{U}\mathfrak{g})_0)$  and  $\text{Mod}_{-\lambda-2\rho}^{\text{fg}}((\mathcal{U}\mathfrak{g})_0)$ . We denote all these dualities similarly.

We denote by  $\Phi : (\mathcal{U}\mathfrak{g})_0 \xrightarrow{\sim} ((\mathcal{U}\mathfrak{g})_0)^{\text{op}}$  the induced isomorphism. For any  $\lambda \in \mathbb{X}$ , it induces an isomorphism

$$(2.3.1) \quad \Phi_\lambda : (\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}} \xrightarrow{\sim} ((\mathcal{U}\mathfrak{g})_0^{-\lambda-2\rho})^{\text{op}}.$$

In [BMR2, Section 3], the authors give a geometric description of these dualities. More precisely, they prove part (i) of the following proposition (see [BMR2, Proposition 3.0.9]). Part (ii) can be proved similarly (see Remark 2.2.6). Let  $\sigma : \tilde{\mathfrak{g}}^{(1)} \rightarrow \tilde{\mathfrak{g}}^{(1)}$  be the automorphism given by multiplication by  $-1$  in the fibers of the vector bundle. We use the same notation for the similarly defined automorphism of  $\tilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}$ .

**Proposition 2.3.2.** (i) *Let  $\lambda \in \mathbb{X}$  be regular. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{D}^b \text{Coh}_{\mathcal{B}^{(1)}}(\tilde{\mathfrak{g}}^{(1)}) & \xrightarrow{\sigma^* \mathbb{D}_{\tilde{\mathfrak{g}}}[d]} & \mathcal{D}^b \text{Coh}_{\mathcal{B}^{(1)}}(\tilde{\mathfrak{g}}^{(1)}) \\ \gamma_\lambda^{\mathcal{B}} \downarrow \wr & & \wr \downarrow \gamma_{-\lambda-2\rho}^{\mathcal{B}} \\ \mathcal{D}^b \text{Mod}_{(\lambda,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g}) & \xrightarrow{(-)^\vee} & \mathcal{D}^b \text{Mod}_{(-\lambda-2\rho,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g}). \end{array}$$

<sup>4</sup>It is more usual to replace  $-\lambda - 2\rho$  by  $-w_0\lambda$ . However we have  $-w_0\lambda = w_0 \bullet (-\lambda - 2\rho)$ , hence these weights induce the same character of the Harish-Chandra center.



(ii) More generally, let  $\mu, P$  be as in Theorem 2.2.1(ii). Then the following diagram commutes.

$$\begin{array}{ccc} \mathcal{D}^b \text{Coh}_{\mathcal{P}(1)}(\tilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}) & \xrightarrow{\sigma^* \mathbb{D}_{\tilde{\mathfrak{g}}_{\mathcal{P}}} [d]} & \mathcal{D}^b \text{Coh}_{\mathcal{P}(1)}(\tilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}) \\ \gamma_{\mu}^{\mathcal{P}} \downarrow \wr & & \wr \downarrow \gamma_{-\mu-2\rho}^{\mathcal{P}} \\ \mathcal{D}^b \text{Mod}_{(\mu,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g}) & \xrightarrow{(-)^{\vee}} & \mathcal{D}^b \text{Mod}_{(-\mu-2\rho,0)}^{\text{fg}}(\mathcal{U}\mathfrak{g}). \end{array}$$

The same proof works also for the other categories of  $\mathcal{U}\mathfrak{g}$ -modules. We obtain the following results, where again  $\sigma : \tilde{\mathcal{N}}^{(1)} \rightarrow \tilde{\mathcal{N}}^{(1)}$ ,  $\sigma : (\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)} \rightarrow (\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}$  and  $\sigma : (\tilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)} \rightarrow (\tilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}$  denote multiplication by  $-1$  in the fibers.

**Proposition 2.3.3.** *Let  $\lambda \in \mathbb{X}$  be regular. The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{D}^b \text{Coh}_{\mathcal{B}(1)}(\tilde{\mathcal{N}}^{(1)}) & \xrightarrow{\sigma^* \mathbb{D}_{\tilde{\mathcal{N}}} [2N]} & \mathcal{D}^b \text{Coh}_{\mathcal{B}(1)}(\tilde{\mathcal{N}}^{(1)}) \\ \epsilon_{\lambda}^{\mathcal{B}} \downarrow & & \downarrow \epsilon_{-\lambda-2\rho}^{\mathcal{B}} \\ \mathcal{D}^b \text{Mod}_0^{\text{fg}}((\mathcal{U}\mathfrak{g})^{\lambda}) & \xrightarrow{(-)^{\vee}} & \mathcal{D}^b \text{Mod}_0^{\text{fg}}((\mathcal{U}\mathfrak{g})^{-\lambda-2\rho}). \end{array}$$

Recall the construction of the functor  $\mathbb{D}_{\mathcal{T}}$  in §1.2. Similar arguments allow to construct duality functors

$$\begin{aligned} \mathbb{D}_{\mathcal{T}}^0 : \text{DGCoh}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) &\xrightarrow{\sim} \text{DGCoh}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}), \\ \mathbb{D}_{\mathcal{T}, \mathcal{P}}^0 : \text{DGCoh}((\tilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) &\xrightarrow{\sim} \text{DGCoh}((\tilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}). \end{aligned}$$

We have:

**Proposition 2.3.4.** (i) *Let  $\lambda \in \mathbb{X}$  be regular. The following diagram commutes.*

$$\begin{array}{ccc} \text{DGCoh}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) & \xrightarrow{\sigma^* \mathbb{D}_{\mathcal{T}}^0} & \text{DGCoh}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \\ \hat{\gamma}_{\lambda}^{\mathcal{B}} \downarrow & & \downarrow \hat{\gamma}_{-\lambda-2\rho}^{\mathcal{B}} \\ \mathcal{D}^b \text{Mod}_{\lambda}^{\text{fg}}((\mathcal{U}\mathfrak{g})_0) & \xrightarrow{(-)^{\vee}} & \mathcal{D}^b \text{Mod}_{-\lambda-2\rho}^{\text{fg}}((\mathcal{U}\mathfrak{g})_0). \end{array}$$

(ii) *More generally, let  $\mu, P$  be as in Theorem 2.2.1(ii). The following diagram commutes.*

$$\begin{array}{ccc} \text{DGCoh}((\tilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) & \xrightarrow{\sigma^* \mathbb{D}_{\mathcal{T}, \mathcal{P}}^0} & \text{DGCoh}((\tilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \\ \hat{\gamma}_{\mu}^{\mathcal{P}} \downarrow & & \downarrow \hat{\gamma}_{-\mu-2\rho}^{\mathcal{P}} \\ \mathcal{D}^b \text{Mod}_{\mu}^{\text{fg}}((\mathcal{U}\mathfrak{g})_0) & \xrightarrow{(-)^{\vee}} & \mathcal{D}^b \text{Mod}_{-\mu-2\rho}^{\text{fg}}((\mathcal{U}\mathfrak{g})_0). \end{array}$$

**2.4. Localization and Koszul duality: regular case.** Let us apply the constructions of §1.1 to the case  $X = \mathcal{B}^{(1)}$ ,  $E = (\mathfrak{g}^* \times \mathcal{B})^{(1)}$ ,  $F = \tilde{\mathcal{N}}^{(1)}$ . Under our assumptions on  $p$ , there exists an isomorphism of  $G$ -modules  $\mathfrak{g} \cong \mathfrak{g}^*$ . We fix such an isomorphism, and use it to identify the vector bundles  $E$  and  $E^*$ . Under this identification, the orthogonal of  $\tilde{\mathcal{N}}$  is  $\tilde{\mathfrak{g}}$ . Hence Theorem 1.1.9 yields an equivalence

$$\kappa_{\mathcal{B}} : \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathcal{N}}^{(1)}) \xrightarrow{\sim} \mathrm{DGCoh}^{\mathrm{gr}}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}).$$

Now, fix a regular weight  $\lambda \in \mathbb{X}$ . Using Theorem 2.2.1(i) and Theorem 2.2.5(i), we have the following situation:

$$\begin{array}{ccc} \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathcal{N}}^{(1)}) & \xrightarrow[\kappa_{\mathcal{B}}]{\sim} & \mathrm{DGCoh}^{\mathrm{gr}}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \\ \mathrm{For} \downarrow (1.1.8) & & (1.1.7) \downarrow \mathrm{For} \\ \mathcal{D}^b \mathrm{Coh}_{\mathcal{B}^{(1)}}(\tilde{\mathcal{N}}^{(1)}) & \xrightarrow{\quad} & \mathrm{DGCoh}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \\ \wr \downarrow \epsilon_{\lambda}^{\mathcal{B}} & & \hat{\gamma}_{\lambda}^{\mathcal{B}} \downarrow \wr \\ \mathcal{D}^b \mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^{\lambda}) & & \mathcal{D}^b \mathrm{Mod}_{\lambda}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0). \end{array}$$

Let  $y \in W_{\mathrm{aff}}$  be the unique element such that  $y^{-1} \bullet \lambda \in C_0$ . Let also

$$W^0 := \{w \in W'_{\mathrm{aff}} \mid w \bullet C_0 \text{ contains a restricted dominant weight}\}.$$

It is well-known (see [Ri, §4.4] and references therein) that the simple objects in the categories  $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^{\lambda})$  and  $\mathrm{Mod}_{\lambda}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$  are parametrized by  $W^0$ : they are the  $\mathcal{U}\mathfrak{g}$ -modules induced by the  $G$ -modules  $L(wy^{-1} \bullet \lambda)$ . For any  $w \in W^0$ , we set

$$\mathfrak{L}_w^y := (\epsilon_{\lambda}^{\mathcal{B}})^{-1} L(wy^{-1} \bullet \lambda) \in \mathcal{D}^b \mathrm{Coh}_{\mathcal{B}^{(1)}}(\tilde{\mathcal{N}}^{(1)}).$$

(This object only depends on  $y$ , and not on  $\lambda$ .) Similarly, for  $w \in W^0$ , we denote by  $P(wy^{-1} \bullet \lambda)$  the projective cover of  $L(wy^{-1} \bullet \lambda)$  in  $\mathrm{Mod}_{\lambda}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$ , and we set

$$\mathfrak{P}_w^y := (\hat{\gamma}_{\lambda}^{\mathcal{B}})^{-1} P(wy^{-1} \bullet \lambda) \in \mathrm{DGCoh}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}).$$

(Again, this object only depends on  $y$ , and not on  $\lambda$ .) For simplicity, here and below, when  $y = 1$  we omit it from the notation.

The “key-result” of [Ri] is the following (see [Ri, Theorems 4.4.3 and 8.5.2]). Let  $\tau_0 = t_{\rho} \cdot w_0 \in W'_{\mathrm{aff}}$ , and consider the natural functor

$$\zeta : \mathcal{D}^b \mathrm{Coh}_{\mathcal{B}^{(1)}}^{\mathbb{G}_m}(\tilde{\mathcal{N}}^{(1)}) \hookrightarrow \mathcal{D}^b \mathrm{Coh}^{\mathbb{G}_m}(\tilde{\mathcal{N}}^{(1)}) \xrightarrow{\xi^{-1}} \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathcal{N}}^{(1)}).$$

**Theorem 2.4.1.** *Assume  $p > h$  is such that Lusztig’s conjecture is true.*

*There is a unique choice of lifts  $\mathfrak{P}_v^{y, \mathrm{gr}} \in \mathrm{DGCoh}^{\mathrm{gr}}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$  of  $\mathfrak{P}_v^y$ , respectively  $\mathfrak{L}_v^{y, \mathrm{gr}} \in \mathcal{D}^b \mathrm{Coh}_{\mathcal{B}^{(1)}}^{\mathbb{G}_m}(\tilde{\mathcal{N}}^{(1)})$  of  $\mathfrak{L}_v^y$  ( $v \in W^0$ ), such that for all  $w \in$*

$W^0$ ,

$$\kappa_{\mathcal{B}}^{-1} \mathfrak{P}_{\tau_0 w}^{y, \text{gr}} \cong \zeta(\mathcal{L}_w^{y, \text{gr}}) \otimes_{\mathcal{O}_{\mathcal{B}(1)}} \mathcal{O}_{\mathcal{B}(1)}(-\rho) \quad \text{in } \text{DGCoh}^{\text{gr}}(\tilde{\mathcal{N}}^{(1)}).$$

In [Ri, Theorem 9.5.1], we have deduced from Theorem 2.4.1 the Koszulity of regular blocks of  $(\mathcal{U}\mathfrak{g})_0$ . Here we give a more “concrete” construction of this grading. As above, fix a regular weight  $\lambda \in \mathbb{X}$ , and let  $y \in W_{\text{aff}}$  be such that  $y^{-1} \bullet \lambda \in C_0$ . We have an isomorphism of left  $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$ -modules

$$(2.4.2) \quad (\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}} \cong \bigoplus_{w \in W^0} P(wy^{-1} \bullet \lambda)^{\oplus n_w^{\lambda}},$$

where  $n_w^{\lambda} = \dim(L(wy^{-1} \bullet \lambda))$ . We fix such an isomorphism. (It can be easily checked that nothing below really depends on this choice.)

The right multiplication of  $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$  on itself induces an isomorphism

$$(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}} \xrightarrow{\sim} \text{End}_{(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}}((\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}})^{\text{op}}.$$

We obtain algebra isomorphisms

$$\begin{aligned} (\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}} &\xrightarrow{\sim} \text{End}_{(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}} \left( \bigoplus_{w \in W^0} P(wy^{-1} \bullet \lambda)^{\oplus n_w^{\lambda}} \right)^{\text{op}} \\ &\xrightarrow[\sim]{(\hat{\gamma}_{\lambda}^R)^{-1}} \text{End}_{\text{DGCoh}((\tilde{\mathfrak{g}} \cap_{\mathfrak{g}^*} \times_{\mathcal{B}} \mathcal{B})^{(1)})} \left( \bigoplus_{w \in W^0} (\mathfrak{P}_w^y)^{\oplus n_w^{\lambda}} \right)^{\text{op}}. \end{aligned}$$

Now the right hand side has a natural grading (as an algebra), given by

$$\begin{aligned} &\text{End}_{\text{DGCoh}((\tilde{\mathfrak{g}} \cap_{\mathfrak{g}^*} \times_{\mathcal{B}} \mathcal{B})^{(1)})} \left( \bigoplus_{w \in W^0} (\mathfrak{P}_w^y)^{\oplus n_w^{\lambda}} \right) \cong \\ &\bigoplus_{m \in \mathbb{Z}} \text{Hom}_{\text{DGCoh}^{\text{gr}}((\tilde{\mathfrak{g}} \cap_{\mathfrak{g}^*} \times_{\mathcal{B}} \mathcal{B})^{(1)})} \left( \bigoplus_{w \in W^0} (\mathfrak{P}_w^{y, \text{gr}})^{\oplus n_w^{\lambda}}, \bigoplus_{w \in W^0} (\mathfrak{P}_w^{y, \text{gr}})^{\oplus n_w^{\lambda}} \langle -m \rangle \right) \end{aligned}$$

(i.e. we consider the grading given by the natural structure of  $\mathbb{G}_{\mathbf{m}}$ -equivariant algebra on a point). Hence we obtain a grading on the algebra  $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$ .

**Theorem 2.4.3.** *Assume  $p > h$  is such that Lusztig’s conjecture is true.*

(i) *This grading only depends on the image of (the differential of)  $\lambda$  in  $\mathfrak{h}^*/(W, \bullet)$ .*

(ii) *This grading makes  $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$  a Koszul ring.*

*Proof.* (i) This follows easily from the construction of the objects  $\mathfrak{P}_w^{y, \text{gr}}$  in terms of the objects  $\mathfrak{P}_w^{\text{gr}}$ , see [Ri, §8.5].

(ii) Thanks to (i), we can assume  $y = 1$ , i.e.  $\lambda \in C_0$ . Using the isomorphisms of [Ri, §9.3], for any  $v, w$  and  $m$  we have

$$\text{Hom}_{\text{DGCoh}^{\text{gr}}((\tilde{\mathfrak{g}} \cap_{\mathfrak{g}^*} \times_{\mathcal{B}} \mathcal{B})^{(1)})} (\mathfrak{P}_w, \mathfrak{P}_v \langle -m \rangle) \cong \text{Ext}_{(\mathcal{U}\mathfrak{g})_0}^m(L(\tau_0 w \bullet \lambda), L(\tau_0 v \bullet \lambda)).$$

Hence this grading is non-negative, and its 0-part is isomorphic to

$$\prod_{w \in W^0} \text{Mat}_{n_w^\lambda}(\mathbb{k}),$$

hence is semisimple. Then, by [BGS, Proposition 2.1.3], it is enough to prove that for any simple graded modules  $L_1$  and  $L_2$  concentrated in (internal) degree 0,

$$(2.4.4) \quad \text{Ext}_{\text{Mod}^{\text{gr}}((\mathcal{U}\mathfrak{g})_0^\wedge)}^i(L_1, L_2\langle j \rangle) = 0 \quad \text{unless } i = j.$$

However,  $(\mathcal{U}\mathfrak{g})_0^\wedge$  is clearly Morita equivalent, as a graded ring, to the graded ring

$$\begin{aligned} & \text{End}_{\text{DGCoh}((\tilde{\mathfrak{g}} \hat{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})} \left( \bigoplus_{w \in W^0} \mathfrak{P}_w \right)^{\text{op}} \\ & \cong \left( \bigoplus_{m \in \mathbb{Z}} \text{Ext}_{(\mathcal{U}\mathfrak{g})_0^0}^m \left( \bigoplus_{w \in W^0} L(w \bullet \lambda), \bigoplus_{w \in W^0} L(w \bullet \lambda) \right) \right)^{\text{op}}. \end{aligned}$$

(Here the grading on the left hand side is defined as above.) The latter graded ring is Koszul by [Ri, Theorem 9.5.1 and its proof]. Hence it satisfies condition (2.4.4). It follows that the same is true for  $(\mathcal{U}\mathfrak{g})_0^\wedge$ . (Note that the Morita equivalence under consideration preserves modules concentrated in internal degree 0.)  $\square$

**2.5. Localization and Koszul duality: singular case.** Let  $\mu, P$  be as in Theorem 2.2.1(ii). Let also  $\lambda \in \mathbb{X}$  be a regular weight such that  $\langle \lambda, \alpha^\vee \rangle = 0$  for any root  $\alpha$  of the Levi of  $P$ , and such that  $\mu$  is in the closure of the alcove of  $\lambda$ . Let  $y \in W_{\text{aff}}$  be such that  $y^{-1} \bullet \lambda \in C_0$ . Then  $\mu_0 := y^{-1} \bullet \mu \in \overline{C_0}$ .

As in [BMR2, §1.10] and [Ri, §10.1], we denote by  $\mathfrak{D}_{\mathcal{P}}^\lambda$  the sheaf of  $\lambda$ -twisted differential operators on  $\mathcal{P}$ , and we define  $U_{\mathcal{P}}^\lambda := \Gamma(\mathcal{P}, \mathfrak{D}_{\mathcal{P}}^\lambda)$ . The action of  $G$  on  $G/P$  induces an algebra morphism

$$\phi_{\mathcal{P}}^\lambda : (\mathcal{U}\mathfrak{g})^\lambda \rightarrow U_{\mathcal{P}}^\lambda$$

(see [BMR2, §1.10.7]). We consider the following two conditions:

$$(2.5.1) \quad R^i \Gamma(\mathfrak{D}_{\mathcal{P}}^\lambda) = 0 \quad \text{for } i > 0,$$

$$(2.5.2) \quad \phi_{\mathcal{P}}^\lambda \text{ is surjective.}$$

It is known that both of these conditions are satisfied for  $p \gg 0$  (see [Ri, Footnotes 21 and 22] for details).

The morphism  $\tilde{\pi}_{\mathcal{P}} : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}_{\mathcal{P}}$  induces a morphism of dg-schemes

$$\hat{\pi}_{\mathcal{P}} : (\tilde{\mathfrak{g}} \hat{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)} \rightarrow (\tilde{\mathfrak{g}}_{\mathcal{P}} \hat{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}.$$

Consider the associated inverse image functors

$$\begin{aligned} L(\hat{\pi}_{\mathcal{P}})^* & : \text{DGCoh}((\tilde{\mathfrak{g}}_{\mathcal{P}} \hat{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \rightarrow \text{DGCoh}((\tilde{\mathfrak{g}} \hat{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}), \\ L(\hat{\pi}_{\mathcal{P}, \mathbb{G}_m})^* & : \text{DGCoh}^{\text{gr}}((\tilde{\mathfrak{g}}_{\mathcal{P}} \hat{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \rightarrow \text{DGCoh}^{\text{gr}}((\tilde{\mathfrak{g}} \hat{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}), \end{aligned}$$

see [Ri, §§1.7 and 5.4].

Recall the equivalence

$$\widehat{\gamma}_\mu^{\mathcal{P}} : \mathrm{DGCoh}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \xrightarrow{\sim} \mathcal{D}^b \mathrm{Mod}_\mu^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$$

of Theorem 2.2.5(ii). Let  $W_\mu^0 \subset W^0$  by the subset of elements  $w$  such that  $w \bullet \mu_0$  is in the upper closure of  $w \bullet C_0$ . The simple objects in the category  $\mathrm{Mod}_\mu^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$  are the images of the simple  $G$ -modules  $L(w \bullet \mu_0)$  for  $w \in W_\mu^0$ . We denote by  $P(w \bullet \mu_0)$  the projective cover of  $L(w \bullet \mu_0)$ . For  $w \in W_\mu^0$ , we set

$$\mathfrak{P}_{\mathcal{P},w}^y := (\widehat{\gamma}_\mu^{\mathcal{P}})^{-1} P(w \bullet \mu_0) \in \mathrm{DGCoh}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}).$$

Assume that  $p > h$  is such that Lusztig's conjecture is true. Then we can consider the objects  $\mathfrak{P}_w^{y,\mathrm{gr}}$  for  $w \in W_\mu^0$ . It is proved in [Ri, (10.2.8)] that for  $w \in W_\mu^0$  there is a unique lift  $\mathfrak{P}_{\mathcal{P},w}^{y,\mathrm{gr}} \in \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)})$  of  $\mathfrak{P}_{\mathcal{P},w}^y$  such that

$$(2.5.3) \quad \mathfrak{P}_w^{y,\mathrm{gr}} \langle N - N_{\mathcal{P}} \rangle \cong L(\widehat{\pi}_{\mathcal{P},\mathbb{G}_m})^* \mathfrak{P}_{\mathcal{P},w}^{y,\mathrm{gr}}.$$

It is proved also in [Ri, Theorem 10.2.4] that, if moreover  $p$  is such that (2.5.1) and (2.5.2) are satisfied, these lifts have properties similar to those of Theorem 2.4.1. In this paper we will rather use the following characterization.

**Lemma 2.5.4.** *Assume that  $p$  is such that Lusztig's conjecture is true.*

*For  $w \in W_\mu^0$ ,  $\mathfrak{P}_{\mathcal{P},w}^{y,\mathrm{gr}}$  is the only object of  $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)})$  (up to isomorphism) such that (2.5.3) is satisfied.*

*Proof.* We have already explained that  $\mathfrak{P}_{\mathcal{P},w}^{y,\mathrm{gr}}$  satisfies this condition. Now it is enough to prove that any object  $\mathfrak{P}$  of  $\mathrm{DGCoh}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)})$  such that  $\mathfrak{P}_w^y \cong L(\widehat{\pi}_{\mathcal{P}})^* \mathfrak{P}$  is isomorphic to  $\mathfrak{P}_{\mathcal{P},w}^y$ . Consider  $P := \widehat{\pi}_\mu^{\mathcal{P}}(\mathfrak{P})$ . Then by assumption and [Ri, Equations (10.2.6) and (10.2.7)] we have  $T_\mu^\lambda(P) \cong P(w \bullet \lambda_0)$  (where  $T_\mu^\lambda$  is the translation functor, see [Ri, §4.3]). Hence  $P$  is projective, i.e. a direct sum of the  $P(v \bullet \mu_0)$ 's ( $v \in W_\mu^0$ ). As [Ri, Equation (10.2.7)] is satisfied by every such  $v$ , we conclude that  $P = P(w \bullet \mu_0)$ . This finishes the proof.  $\square$

As in §2.4, the choice of these lifts provides a grading on the algebra  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\mu}}$ . And the same proof as that of Theorem 2.4.3 gives the following.

**Theorem 2.5.5.** *Assume  $p > h$  is such that Lusztig's conjecture is true, and conditions (2.5.1) and (2.5.2) are satisfied.*

*This grading makes  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\mu}}$  a Koszul ring.*

## 3. KOSZUL DUALITY AND ORDINARY DUALITY: REGULAR CASE

3.1. **Geometry.** Let us define the various geometric duality functors we are going to use. Recall the definition of  $\sigma$  in §2.3. We define

$$\begin{aligned} \mathbf{D}_{\mathcal{T}}^{\text{gr}} &:= \sigma^* \mathbb{D}_{\mathcal{T}} \langle -3N \rangle : \text{DGCoh}^{\text{gr}}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \xrightarrow{\sim} \text{DGCoh}^{\text{gr}}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}), \\ \mathbf{D}_{\mathcal{S}}^{\text{gr}} &:= \sigma^* \mathbb{D}_{\mathcal{S}}[N] \langle -N \rangle : \text{DGCoh}^{\text{gr}}(\tilde{\mathcal{N}}^{(1)}) \xrightarrow{\sim} \text{DGCoh}^{\text{gr}}(\tilde{\mathcal{N}}^{(1)}), \\ \mathbf{D}_{\tilde{\mathcal{N}}}^{\text{gr}} &:= \sigma^* \mathbb{D}_{\tilde{\mathcal{N}}^{(1)}}[2N] \langle -N \rangle : \mathcal{D}^b \text{Coh}^{\mathbb{G}_m}(\tilde{\mathcal{N}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b \text{Coh}^{\mathbb{G}_m}(\tilde{\mathcal{N}}^{(1)}), \\ \mathbf{D}_{\tilde{\mathfrak{g}}}^{\text{gr}} &:= \sigma^* \mathbb{D}_{\tilde{\mathfrak{g}}^{(1)}}[d] \langle 2d - 3N \rangle : \mathcal{D}^b \text{Coh}^{\mathbb{G}_m}(\tilde{\mathfrak{g}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b \text{Coh}^{\mathbb{G}_m}(\tilde{\mathfrak{g}}^{(1)}). \end{aligned}$$

Here, the dualities  $\mathbb{D}_{\mathcal{T}}$  and  $\mathbb{D}_{\mathcal{S}}$  are defined as in §1.2, for our choice  $X = \mathcal{B}^{(1)}$ ,  $E = (\mathfrak{g}^* \times \mathcal{B})^{(1)}$ ,  $F = \tilde{\mathcal{N}}^{(1)}$ . With these definitions, in the following diagram the vertical lines commute:

$$\begin{array}{ccc} \begin{array}{c} \mathbf{D}_{\mathcal{S}}^{\text{gr}} \\ \curvearrowright \\ \text{DGCoh}^{\text{gr}}(\tilde{\mathcal{N}}^{(1)}) \end{array} & \xrightarrow{\sim \kappa_{\mathcal{B}}} & \begin{array}{c} \mathbf{D}_{\mathcal{T}}^{\text{gr}} \\ \curvearrowright \\ \text{DGCoh}^{\text{gr}}((\tilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \end{array} \\ \downarrow \xi & & \downarrow Rp_* \\ \begin{array}{c} \mathcal{D}^b \text{Coh}^{\mathbb{G}_m}(\tilde{\mathcal{N}}^{(1)}) \\ \curvearrowright \\ \mathbf{D}_{\tilde{\mathcal{N}}}^{\text{gr}} \end{array} & & \begin{array}{c} \mathcal{D}^b \text{Coh}^{\mathbb{G}_m}(\tilde{\mathfrak{g}}^{(1)}) \\ \curvearrowright \\ \mathbf{D}_{\tilde{\mathfrak{g}}}^{\text{gr}} \end{array} \end{array}$$

(Use Lemma 1.2.1 and Remark 1.1.10.) Moreover, by Proposition 1.3.1 we have an isomorphism

$$(3.1.1) \quad \mathbf{D}_{\mathcal{T}}^{\text{gr}} \circ \kappa_{\mathcal{B}} \cong (\kappa \circ \mathbf{D}_{\mathcal{S}}^{\text{gr}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-2\rho).$$

Our main geometric result is the following. Fix a regular  $\lambda \in \mathbb{X}$ , and let  $y, z \in W_{\text{aff}}$  be such that  $\lambda \in y \bullet C_0$ ,  $-\lambda - 2\rho \in z \bullet C_0$ . We denote by

$$\iota_{\lambda} : W^0 \xrightarrow{\sim} W^0$$

the bijection such that  $-w_0(wy^{-1} \bullet \lambda) = \iota_{\lambda}(w)z^{-1} \bullet (-\lambda - 2\rho)$ .

**Proposition 3.1.2.** *Assume  $p > h$  is such that Lusztig's conjecture is true.*

*For any  $w \in W^0$ , there are isomorphisms*

$$\mathbf{D}_{\tilde{\mathcal{N}}}^{\text{gr}}(\mathfrak{L}_w^{y, \text{gr}}) \cong \mathfrak{L}_{\iota_{\lambda}(w)}^{z, \text{gr}}, \quad \mathbf{D}_{\mathcal{T}}^{\text{gr}}(\mathfrak{P}_{\tau_0 w}^{y, \text{gr}}) \cong \mathfrak{P}_{\tau_0 \iota_{\lambda}(w)}^{z, \text{gr}}.$$

*Proof.* By definition of  $\iota_{\lambda}$ , for  $w \in W^0$  there is an isomorphism

$$L(wy^{-1} \bullet \lambda)^{\vee} \cong L(\iota_{\lambda}(w)z^{-1} \bullet (-\lambda - 2\rho)).$$

Recall that, given a finite dimensional graded algebra  $A$  and an indecomposable non-graded  $A$ -module  $M$ , there is at most one lift of  $M$  as a graded  $M$ -module, up to isomorphism and shift in the grading (see [Ri, §5.6] and references therein). Hence, comparing Proposition 2.3.3 and the definition of  $\mathbf{D}_{\tilde{\mathcal{N}}}^{\text{gr}}$ , there exists  $n \in \mathbb{Z}$  such that

$$\mathbf{D}_{\tilde{\mathcal{N}}}^{\text{gr}}(\mathfrak{L}_w^{y, \text{gr}}) \cong \mathfrak{L}_{\iota_{\lambda}(w)}^{z, \text{gr}} \langle n \rangle.$$

Then we have

$$\zeta \circ \mathbf{D}_{\mathcal{N}}^{\text{gr}}(\mathfrak{L}_w^{y,\text{gr}}) \cong \zeta(\mathfrak{L}_{\iota_\lambda(w)}^{z,\text{gr}})\langle n \rangle[n].$$

Using Theorem 2.4.1, we deduce

$$\zeta \circ \mathbf{D}_{\mathcal{N}}^{\text{gr}}(\mathfrak{L}_w^{y,\text{gr}}) \otimes_{\mathcal{B}^{(1)}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho) \cong \kappa_{\mathcal{B}}^{-1} \mathfrak{P}_{\tau_0 \iota_\lambda(w)}^{z,\text{gr}}\langle n \rangle[n].$$

Then, using isomorphism (3.1.1), we obtain

$$\mathbf{D}_{\mathcal{T}}^{\text{gr}}(\mathfrak{P}_{\tau_0 w}^{y,\text{gr}}) \cong \mathfrak{P}_{\tau_0 \iota_\lambda(w)}^{z,\text{gr}}\langle n \rangle[n].$$

However, by Proposition 2.3.4, the object

$$\widehat{\gamma}_{\lambda-2\rho}^{\mathcal{B}} \circ \text{For} \circ \mathbf{D}_{\mathcal{T}}^{\text{gr}}(\mathfrak{P}_{\tau_0 w}^{y,\text{gr}}) \cong (\widehat{\gamma}_{\lambda}^{\mathcal{B}}(\mathfrak{P}_{\tau_0 w}^y))^{\vee}$$

is in degree 0. (Here, For is the functor of (1.1.7).) Hence  $n = 0$ , which finishes the proof.  $\square$

*Remark 3.1.3.* (1) We will not use the first isomorphism (for simple objects). We only include it for completeness.

- (2) This proposition, together with Proposition 2.3.4, allows to prove the Frobenius property of regular blocks of  $(\mathcal{U}\mathfrak{g})_0$  without referring to the general result in [Be]. Moreover, it describes explicitly the duals of indecomposable projectives. For example, for  $\lambda$  in  $C_0$ ,  $\iota_\lambda$  does not depend on  $\lambda$ , and is given by  $\iota_0(t_\mu \cdot v) = t_{-w_0\mu} \cdot w_0 v w_0$ . Hence for  $w \in W^0$  we obtain an isomorphism

$$P(w \bullet \lambda)^{\vee} \cong P(\iota_0(w) \bullet (-w_0\lambda)) = P(-w_0(w \bullet \lambda)).$$

In particular, it follows that

$$\text{soc } P(w \bullet \lambda) \cong L(w \bullet \lambda).$$

This is of course well-known, see e.g. [Ja, Proposition I.8.13].

In the rest of this section we deduce from the geometric Proposition 3.1.2 algebraic statements about Koszul gradings on regular blocks of  $(\mathcal{U}\mathfrak{g})_0$ .

**3.2. Grading and the natural anti-isomorphism.** Let again  $\lambda \in \mathbb{X}$  be regular, and let  $y \in W_{\text{aff}}$  be such that  $\lambda \in y \bullet C_0$ .

Consider the restriction  $\widetilde{\mathcal{D}}^\lambda$  of the sheaf of algebras  $\widetilde{\mathcal{D}}$  to the formal neighborhood of  $\mathcal{B}^{(1)} \times \{\lambda\}$  in  $\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^*$ , which we identify with the formal neighborhood of  $\mathcal{B}^{(1)}$  in  $\widetilde{\mathfrak{g}}^{(1)}$ . Consider a decomposition

$$(3.2.1) \quad \mathcal{M}^\lambda = \bigoplus_{i \in I} \mathcal{M}_i^\lambda$$

of the vector bundle  $\mathcal{M}^\lambda$  into indecomposable subbundles. Then it follows from the definitions (see e.g. [BM]) that the collection  $\{(\mathcal{M}_i^\lambda)^* \otimes_{\mathbb{k}} \Lambda(\mathfrak{g}^{(1)}), i \in I\}$  (where the objects are endowed with a Koszul differential) coincides with the collection of the  $\mathfrak{P}_w^y$ 's.

As in §1.1,  $\widetilde{\mathfrak{g}}^{(1)}$  is endowed with an action of  $\mathbb{G}_m$ , where  $t \in \mathbb{k}^\times$  acts by multiplication by  $t^{-2}$  along the fibers of the projection  $\widetilde{\mathfrak{g}}^{(1)} \rightarrow \mathcal{B}^{(1)}$ . It is explained in [Ri, §6.3] how, starting from a  $\mathbb{G}_m$ -equivariant structure on  $\mathcal{M}^\lambda$

(as a sheaf on  $\tilde{\mathfrak{g}}^{(1)}$ ), one can produce a grading on the algebra  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$ , and an equivalence of categories

$$\tilde{\gamma}_\lambda^{\mathcal{B}} : \mathrm{DGCoh}^{\mathrm{gr}}((\tilde{\mathfrak{g}} \cap_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \xrightarrow{\sim} \mathcal{D}^b \mathrm{Mod}_\lambda^{\mathrm{fg}, \mathrm{gr}}((\mathcal{U}\mathfrak{g})_0)$$

which is a “graded version” of  $\tilde{\gamma}_\lambda^{\mathcal{B}}$  (see [Ri, Theorem 6.3.4 and Remark 6.3.5]). (Here,  $\mathrm{Mod}_\lambda^{\mathrm{fg}, \mathrm{gr}}((\mathcal{U}\mathfrak{g})_0)$  is the category of finitely generated graded modules over the graded algebra  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$ .) To choose a  $\mathbb{G}_m$ -equivariant structure on  $\mathcal{M}^\lambda$ , it is enough to choose a  $\mathbb{G}_m$ -equivariant structure on each of the  $\mathcal{M}_i^\lambda$ 's. Such a structure is unique up to an internal shift. (Existence is proved in [Ri, Lemma 6.3.3] or in [BM], and unicity is obvious.) We choose it in such a way that the collection of  $\mathbb{G}_m$ -equivariant objects  $\{(\mathcal{M}_i^\lambda)^* \otimes_{\mathbb{k}} \Lambda(\mathfrak{g}^{(1)}), i \in I\}$  coincides with the collection of the  $\mathfrak{P}_w^{y, \mathrm{gr}}$ 's. We denote by  $\mathcal{M}_{\mathrm{gr}}^\lambda$  the resulting  $\mathbb{G}_m$ -equivariant vector bundle. With this choice, by construction the grading on  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$  is the Koszul grading of Theorem 2.4.3.

Using this one can prove the regular case of point (1) of the main theorem. Recall the isomorphism  $\Phi_\lambda$  of (2.3.1).

**Proposition 3.2.2.** *Assume  $p > h$  is such that Lusztig's conjecture is true.*

*The isomorphism  $\Phi_\lambda : (\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}} \xrightarrow{\sim} ((\mathcal{U}\mathfrak{g})_0^{-\lambda-2\rho})^{\mathrm{op}}$  is an isomorphism of graded algebras, where both algebras are endowed with the Koszul grading given by Theorem 2.4.3.*

*Proof.* Consider the automorphism

$$\tilde{\sigma} : \begin{cases} \tilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^* & \rightarrow & \tilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^*(1)} \mathfrak{h}^* \\ (gB, X, \lambda) & \mapsto & (gB, -X, -\lambda - 2\rho) \end{cases} .$$

As explained in [BMR2, Lemma 3.0.6(a)], the natural isomorphism  $\mathcal{D}_{G/U} \cong \mathcal{D}_{G/U}^{\mathrm{op}}$  (due to triviality of the canonical line bundle on  $G/U$ ) induces an isomorphism of algebras  $\tilde{\mathcal{D}}^{\mathrm{op}} \cong \tilde{\sigma}^* \tilde{\mathcal{D}}$ . Moreover, taking global sections this isomorphism gives the natural isomorphism  $\mathcal{U}\mathfrak{g} \otimes_{\mathfrak{z}_{\mathrm{HC}}} \mathcal{S}(\mathfrak{h}) \cong (\mathcal{U}\mathfrak{g} \otimes_{\mathfrak{z}_{\mathrm{HC}}} \mathcal{S}(\mathfrak{h}))^{\mathrm{op}}$ . In particular, we obtain an isomorphism of algebras on the formal neighborhood of  $\mathcal{B}^{(1)}$  in  $\tilde{\mathfrak{g}}^{(1)}$ :

$$(3.2.3) \quad \tilde{\mathcal{D}}^{\widehat{\lambda}} \cong \sigma^*(\tilde{\mathcal{D}}^{-\lambda-2\rho})^{\mathrm{op}}.$$

Now by [BMR2, Lemma 3.0.6(b)], there exists an isomorphism of  $\tilde{\mathcal{D}}^{\widehat{\lambda}}$ -modules

$$(3.2.4) \quad \mathcal{M}^\lambda \cong \sigma^*(\mathcal{M}^{-\lambda-2\rho})^*,$$

where the  $\tilde{\mathcal{D}}^{\widehat{\lambda}}$ -module structure on the right hand side is induced by isomorphism (3.2.3). It can be easily checked that for any choice of such an



isomorphism  $\phi$ , the induced isomorphism

$$\begin{aligned} \widehat{\mathcal{D}}^\lambda &\cong \mathcal{E}nd(\mathcal{M}^\lambda) \xrightarrow{\phi} \sigma^* \mathcal{E}nd((\mathcal{M}^{-\lambda-2\rho})^*) \\ &\stackrel{(-)^*}{\cong} \sigma^* \mathcal{E}nd(\mathcal{M}^{-\lambda-2\rho})^{\text{op}} \cong \sigma^*(\widehat{\mathcal{D}}^{-\lambda-2\rho})^{\text{op}} \end{aligned}$$

coincides with (3.2.3).

One can choose the decompositions (3.2.1) for  $\lambda$  and for  $-\lambda - 2\rho$  to be compatible with isomorphism (3.2.4). Then by definition and Proposition 3.1.2, isomorphism (3.2.4) becomes an isomorphism of  $\mathbb{G}_{\mathbf{m}}$ -equivariant vector bundles

$$(3.2.5) \quad \mathcal{M}_{\text{gr}}^\lambda \cong \sigma^*(\mathcal{M}_{\text{gr}}^{-\lambda-2\rho})^* \langle 3N \rangle.$$

The result follows, by construction of the grading on  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$  and  $(\mathcal{U}\mathfrak{g})_0^{-\widehat{\lambda}-2\rho}$  (see [Ri, §6.3]).  $\square$

**3.3. Grading and duality.** Using Proposition 3.2.2, one can define a duality functor

$$(-)^\vee : \text{Mod}_\lambda^{\text{fg,gr}}((\mathcal{U}\mathfrak{g})_0) \xrightarrow{\sim} \text{Mod}_{-\lambda-2\rho}^{\text{fg,gr}}((\mathcal{U}\mathfrak{g})_0)^{\text{op}}.$$

Let us give a geometric description of this functor. (The proof is an adaptation of that of [BMR2, Proposition 3.0.9].)

**Proposition 3.3.1.** *Assume  $p > h$  is such that Lusztig's conjecture is true. Let  $\lambda \in \mathbb{X}$  be regular. Then the following diagram commutes:*

$$\begin{array}{ccc} \text{DGCoh}^{\text{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap} \mathfrak{g}^* \times_{\mathcal{B}} \mathcal{B})^{(1)}) & \xrightarrow{\mathbf{D}_T^{\text{gr}}} & \text{DGCoh}^{\text{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap} \mathfrak{g}^* \times_{\mathcal{B}} \mathcal{B})^{(1)}) \\ \widetilde{\gamma}_\lambda^{\mathcal{B}} \downarrow \wr & & \downarrow \wr \widetilde{\gamma}_{-\lambda-2\rho}^{\mathcal{B}} \\ \mathcal{D}^b \text{Mod}_\lambda^{\text{fg,gr}}((\mathcal{U}\mathfrak{g})_0) & \xrightarrow{(-)^\vee \langle 2N \rangle} & \mathcal{D}^b \text{Mod}_{-\lambda-2\rho}^{\text{fg,gr}}((\mathcal{U}\mathfrak{g})_0). \end{array}$$

*Proof.* First we begin with an easy lemma, whose proof is similar to that of [BMR2, Lemma 3.0.1].

**Lemma 3.3.2.** *Let  $M$  be a finite complex of finite dimensional  $S(\mathfrak{g})$ -modules (or equivalently quasi-coherent  $\mathcal{O}_{\mathfrak{g}^*}$ -modules). Then there exists a functorial isomorphism, in the derived category of  $S(\mathfrak{g})$ -modules*

$$\mathbb{D}_{\mathfrak{g}^*}(M) \cong M^*[-d] \langle -2d \rangle.$$

Now, let  $\pi : \widetilde{\mathfrak{g}}^{(1)} \rightarrow \mathfrak{g}^{*(1)}$  be the natural morphism. Then, as Grothendieck-Serre duality commutes with proper direct images, we have an isomorphism

$$(3.3.3) \quad R\pi_* \circ \mathbb{D}_{\widetilde{\mathfrak{g}}} \cong \mathbb{D}_{\mathfrak{g}^*} \circ R\pi_* \langle 2N \rangle.$$

Let  $\mathcal{F} \in \mathrm{DGCoh}^{\mathrm{gr}}((\tilde{\mathfrak{g}} \hat{\cap}_{\mathfrak{g}^* \times \mathcal{B}}^R \mathcal{B})^{(1)})$ . By definition and Lemma 3.3.2 we have an isomorphism

$$\begin{aligned} \tilde{\gamma}_\lambda^{\mathcal{B}}(\mathcal{F})^\vee &= R\Gamma(\mathcal{M}_{\mathrm{gr}}^\lambda \otimes_{\mathcal{O}_{\tilde{\mathfrak{g}}(1)}} Rp_*\mathcal{F})^* \\ &\cong \mathbb{D}_{\mathfrak{g}^*} \circ R\pi_*(\mathcal{M}_{\mathrm{gr}}^\lambda \otimes_{\mathcal{O}_{\tilde{\mathfrak{g}}(1)}} Rp_*\mathcal{F})[d]\langle 2d \rangle. \end{aligned}$$

Using (3.3.3) we deduce

$$\tilde{\gamma}_\lambda^{\mathcal{B}}(\mathcal{F})^\vee \cong R\pi_* \circ \mathbb{D}_{\tilde{\mathfrak{g}}}(\mathcal{M}_{\mathrm{gr}}^\lambda \otimes_{\mathcal{O}_{\tilde{\mathfrak{g}}(1)}} Rp_*\mathcal{F})[d]\langle 2d - 2N \rangle.$$

Now there is a natural isomorphism

$$\mathbb{D}_{\tilde{\mathfrak{g}}}(\mathcal{M}_{\mathrm{gr}}^\lambda \otimes_{\mathcal{O}_{\tilde{\mathfrak{g}}(1)}} \mathcal{F}) \cong (\mathcal{M}_{\mathrm{gr}}^\lambda)^* \otimes_{\mathcal{O}_{\tilde{\mathfrak{g}}(1)}} \mathbb{D}_{\tilde{\mathfrak{g}}}(Rp_*\mathcal{F}).$$

Hence, using isomorphism (3.2.5) we obtain

$$\tilde{\gamma}_\lambda^{\mathcal{B}}(\mathcal{F})^\vee \cong R\Gamma(\mathcal{M}_{\mathrm{gr}}^{-\lambda-2\rho} \otimes_{\mathcal{O}_{\tilde{\mathfrak{g}}(1)}} \sigma^* \mathbb{D}_{\tilde{\mathfrak{g}}}(Rp_*\mathcal{F}))[d]\langle 2d - 5N \rangle.$$

Using finally the isomorphism  $\mathbf{D}_{\tilde{\mathfrak{g}}}^{\mathrm{gr}} \circ Rp_* \cong Rp_* \circ \mathbf{D}_{\mathcal{T}}^{\mathrm{gr}}$ , we obtain

$$\begin{aligned} \tilde{\gamma}_\lambda^{\mathcal{B}}(\mathcal{F})^\vee &\cong R\Gamma(\mathcal{M}_{\mathrm{gr}}^{-\lambda-2\rho} \otimes_{\mathcal{O}_{\tilde{\mathfrak{g}}(1)}} Rp_*(\mathbf{D}_{\mathcal{T}}^{\mathrm{gr}}\mathcal{F}))\langle -2N \rangle \\ &\cong \tilde{\gamma}_{-\lambda-2\rho}^{\mathcal{B}} \circ \mathbf{D}_{\mathcal{T}}^{\mathrm{gr}}(\mathcal{F})\langle -2N \rangle. \end{aligned}$$

One can check that all these isomorphisms are isomorphisms in the derived category of  $(\mathcal{U}\mathfrak{g})_0$ -modules, which concludes the proof.  $\square$

**3.4. Duality and the regular representation.** Now we can prove point (2) of the regular case of the main Theorem.

**Proposition 3.4.1.** *Assume  $p > h$  is such that Lusztig's conjecture is true.*

*Let  $\lambda \in \mathbb{X}$  be regular. There exists an isomorphism of graded  $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$ -modules*

$$(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}} \cong ((\mathcal{U}\mathfrak{g})_0^{-\lambda-2\rho})^\vee \langle 2N \rangle.$$

*Proof.* This follows from Proposition 3.3.1, using the fact that

$$(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}} \cong \tilde{\gamma}_\lambda^{\mathcal{B}}((\mathcal{M}_{\mathrm{gr}}^\lambda)^* \otimes_{\mathbb{k}} \Lambda(\mathfrak{g}^{(1)})),$$

and the natural isomorphism

$$(\mathcal{M}_{\mathrm{gr}}^\lambda)^* \otimes_{\mathbb{k}} \Lambda(\mathfrak{g}^{(1)}) \cong \mathbf{D}_{\mathcal{T}}^{\mathrm{gr}}((\mathcal{M}_{\mathrm{gr}}^{-\lambda-2\rho})^* \otimes_{\mathbb{k}} \Lambda(\mathfrak{g}^{(1)}))$$

which follows from Proposition 3.1.2 (see the proof of Proposition 3.2.2).  $\square$

*Remark 3.4.2.* (1) It follows in particular from this proposition that the maximal degree in the Koszul grading of  $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$  is  $2N$ . It follows also that the Poincaré polynomial  $P_\lambda$  of this grading satisfies  $P_\lambda(t^{-1}) = t^{-2N}P_\lambda(t)$ . (The former fact can also be derived directly from [Ri], using the property that the homological dimension of  $(\mathcal{U}\mathfrak{g})^\lambda$  is  $2N$ .)

- (2) In fact the algebra  $(\mathcal{U}\mathfrak{g})_0$  is a *symmetric* algebra, see [Sc, FP], hence the same is true for  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$ . In other words, there exists an isomorphism of  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$ -bimodules  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}} \cong ((\mathcal{U}\mathfrak{g})_0^{-\widehat{\lambda}-2\rho})^\vee$ . One can easily check that the isomorphism of Proposition 3.4.1 is an isomorphism of graded  $(\mathcal{U}\mathfrak{g})_0^{\widehat{\lambda}}$ -bimodules.

#### 4. KOSZUL DUALITY AND ORDINARY DUALITY: SINGULAR CASE

In this section we prove analogues of the results of Section 3 for singular blocks. Most of the proofs are similar to those for the regular case, hence we omit them.

**4.1. Geometry.** Recall the constructions of §1.2. With the data  $X = \mathcal{P}^{(1)}$ ,  $E = (\mathfrak{g}^* \times \mathcal{P})^{(1)}$ ,  $F = \widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}$ , we obtain a duality functor

$$\mathbb{D}_{\mathcal{T}, \mathcal{P}} : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \xrightarrow{\sim} \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}),$$

which is a “graded version” of the functor  $\mathbb{D}_{\mathcal{T}, \mathcal{P}}^0$  of §2.3. We define

$$\mathbf{D}_{\mathcal{T}, \mathcal{P}}^{\mathrm{gr}} := \sigma^* \mathbb{D}_{\mathcal{T}, \mathcal{P}} \langle -3N \rangle.$$

Then, one easily checks that we have an isomorphism

$$(4.1.1) \quad \mathbf{D}_{\mathcal{T}, \mathcal{P}}^{\mathrm{gr}} \circ L(\widehat{\pi}_{\mathcal{P}, \mathbb{G}_m})^* \langle 2N - 2N_{\mathcal{P}} \rangle \cong L(\widehat{\pi}_{\mathcal{P}, \mathbb{G}_m})^* \circ \mathbf{D}_{\mathcal{T}, \mathcal{P}}^{\mathrm{gr}}.$$

Let us fix a weight  $\mu \in \mathbb{X}$  and a standard parabolic  $P$  as in Theorem 2.2.1(ii). Then  $\mu - 2\rho$  satisfies the same assumption, for the same parabolic subgroup (see Remark 2.2.6). There exist a unique  $y \in W_{\mathrm{aff}}$  such that the alcove  $y \bullet C_0$  contains  $\mu$  in its closure, and contains also a weight  $\lambda$  orthogonal to all the roots of the Levi of  $P$ . Let also  $z \in W_{\mathrm{aff}}$  be defined similarly, for  $-\mu - 2\rho$  instead of  $\mu$ . Then we have the following.

**Proposition 4.1.2.** *For  $w \in W_{\mu}^0$ , there is an isomorphism*

$$\mathbf{D}_{\mathcal{T}, \mathcal{P}}^{\mathrm{gr}}(\mathfrak{P}_{\mathcal{P}, w}^{y, \mathrm{gr}}) \cong \mathfrak{P}_{\mathcal{P}, \tau_0 \iota_{\lambda}(\tau_0 w)}^{z, \mathrm{gr}}.$$

*Proof.* By Proposition 3.1.2, there is an isomorphism

$$\mathbf{D}_{\mathcal{T}}^{\mathrm{gr}}(\mathfrak{P}_w^{y, \mathrm{gr}}) \cong \mathfrak{P}_{\tau_0 \iota_{\lambda}(\tau_0 w)}^{z, \mathrm{gr}}.$$

Using equation (2.5.3), we deduce

$$\mathbf{D}_{\mathcal{T}}^{\mathrm{gr}} \circ L(\widehat{\pi}_{\mathcal{P}, \mathbb{G}_m})^* \mathfrak{P}_{\mathcal{P}, w}^{y, \mathrm{gr}} \cong L(\widehat{\pi}_{\mathcal{P}, \mathbb{G}_m})^* \mathfrak{P}_{\mathcal{P}, \tau_0 \iota_{\lambda}(\tau_0 w)}^{z, \mathrm{gr}} \langle N_{\mathcal{P}} - N \rangle.$$

Then, using (4.1.1), we obtain

$$L(\widehat{\pi}_{\mathcal{P}, \mathbb{G}_m})^* \circ \mathbf{D}_{\mathcal{T}, \mathcal{P}}^{\mathrm{gr}}(\mathfrak{P}_{\mathcal{P}, w}^{y, \mathrm{gr}}) \cong \mathfrak{P}_{\mathcal{P}, \tau_0 \iota_{\lambda}(\tau_0 w)}^{z, \mathrm{gr}} \langle N - N_{\mathcal{P}} \rangle.$$

The result follows, by Lemma 2.5.4.  $\square$

*Remark 4.1.3.* (1) One can prove a similar statement for the objects “ $\mathcal{L}_{\mathcal{P}, w}^{y, \mathrm{gr}}$ ” of [Ri, Theorem 10.2.4]. As we do not need this statement (not even for the proof of Proposition 4.1.2), we omit it.

- (2) Again, this proposition allows to prove the Frobenius property of singular blocks of  $(\mathcal{U}\mathfrak{g})_0$  without referring to the general result of [Be]. It also gives a description of duals of indecomposable projectives.

**4.2. Grading and the natural anti-isomorphism.** As in §3.2 we choose a  $\mathbb{G}_m$ -equivariant structure on the vector bundle  $\mathcal{M}_{\mathcal{P}}^{\mu}$  (as a sheaf on the formal neighborhood of  $\mathcal{P}^{(1)}$  in  $\tilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}$ ) which is compatible with the  $\mathbb{G}_m$ -equivariant structures on the  $\mathfrak{P}_{\mathcal{P},w}^{y,\text{gr}}$ 's. Then we have an equivalence of categories

$$\tilde{\gamma}_{\mu}^{\mathcal{P}} : \text{DGCoh}^{\text{gr}}((\tilde{\mathfrak{g}}_{\mathcal{P}}^R \cap_{\mathfrak{g}^*} \times_{\mathcal{B}} \mathcal{B})^{(1)}) \xrightarrow{\sim} \mathcal{D}^b \text{Mod}_{\mu}^{\text{fg,gr}}((\mathcal{U}\mathfrak{g})_0),$$

where the grading on the algebra  $(\mathcal{U}\mathfrak{g})_0^{\hat{\mu}}$  is the Koszul grading provided by Theorem 2.5.5 (see [Ri, §10.2]).

One can make the same constructions for the weight  $-\mu - 2\rho$ . Then, one can prove the following singular analogue of Proposition 3.2.2.

**Proposition 4.2.1.** *Assume  $p > h$  is such that Lusztig's conjecture is true, and conditions (2.5.1) and (2.5.2) are satisfied.*

*The isomorphism  $\Phi_{\mu} : (\mathcal{U}\mathfrak{g})_0^{\hat{\mu}} \rightarrow ((\mathcal{U}\mathfrak{g})_0^{-\mu-2\rho})^{\text{op}}$  is an isomorphism of graded algebras, where both algebras are endowed with the Koszul grading given by Theorem 2.4.3.*

**4.3. Grading and duality.** Using Proposition 4.2.1, one can define a duality functor

$$(-)^{\vee} : \text{Mod}_{\mu}^{\text{fg,gr}}((\mathcal{U}\mathfrak{g})_0) \xrightarrow{\sim} \text{Mod}_{-\mu-2\rho}^{\text{fg,gr}}((\mathcal{U}\mathfrak{g})_0)^{\text{op}}.$$

As in Proposition 3.3.1, one can give a geometric description of this functor.

**Proposition 4.3.1.** *Assume  $p > h$  is such that Lusztig's conjecture is true, and conditions (2.5.1) and (2.5.2) are satisfied.*

*The following diagram commutes:*

$$\begin{array}{ccc} \text{DGCoh}^{\text{gr}}((\tilde{\mathfrak{g}}_{\mathcal{P}}^R \cap_{\mathfrak{g}^*} \times_{\mathcal{P}} \mathcal{P})^{(1)}) & \xrightarrow{\mathbf{D}_{\tilde{\gamma}, \mathcal{P}}^{\text{gr}}} & \text{DGCoh}^{\text{gr}}((\tilde{\mathfrak{g}}_{\mathcal{P}}^R \cap_{\mathfrak{g}^*} \times_{\mathcal{P}} \mathcal{P})^{(1)}) \\ \tilde{\gamma}_{\mu}^{\mathcal{P}} \downarrow \wr & & \downarrow \wr \tilde{\gamma}_{-\mu-2\rho}^{\mathcal{P}} \\ \mathcal{D}^b \text{Mod}_{\mu}^{\text{fg,gr}}((\mathcal{U}\mathfrak{g})_0) & \xrightarrow{(-)^{\vee} \langle 2N_{\mathcal{P}} \rangle} & \mathcal{D}^b \text{Mod}_{-\mu-2\rho}^{\text{fg,gr}}((\mathcal{U}\mathfrak{g})_0). \end{array}$$

**4.4. Duality and the regular representation.** Finally, one can deduce the following.

**Proposition 4.4.1.** *Assume  $p > h$  is such that Lusztig's conjecture is true, and conditions (2.5.1) and (2.5.2) are satisfied.*

*There exists an isomorphism of graded  $(\mathcal{U}\mathfrak{g})_0^{\hat{\mu}}$ -modules*

$$(\mathcal{U}\mathfrak{g})_0^{\hat{\mu}} \cong ((\mathcal{U}\mathfrak{g})_0^{-\mu-2\rho})^{\vee} \langle 2N_{\mathcal{P}} \rangle.$$

*Remark 4.4.2.* (1) It follows in particular from this proposition that the maximal degree in the Koszul grading of  $(\mathcal{U}\mathfrak{g})_0^\mu$  is  $2N_{\mathcal{P}}$ . It follows also that the Poincaré polynomial  $P_\mu$  of this grading satisfies  $P_\mu(t^{-1}) = t^{-2N_{\mathcal{P}}}P_\mu(t)$ .

(2) One can easily check that the isomorphism of Proposition 4.4.1 is an isomorphism of graded  $(\mathcal{U}\mathfrak{g})_0^\mu$ -bimodules.

## 5. EXAMPLE: $\mathrm{SL}(2)$

In this section we set  $G = \mathrm{SL}(2)$ , and we assume  $p > 2$ . There is a natural isomorphism  $\mathbb{X} \cong \mathbb{Z}$ , such that  $\rho = 1$ . Also, we have  $\mathcal{B} \cong \mathbb{P}^1$ . For simplicity, we omit Frobenius twists in this section. Such a twist should appear on every variety we consider.

**5.1. Regular blocks.** In this case the regular blocks are those of the weights  $0, \dots, \frac{p-3}{2}$ . Let us fix  $\lambda \in \{0, \dots, \frac{p-3}{2}\}$ . The simple objects in  $\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$  are  $L(\lambda)$  (of dimension  $\lambda + 1$ ) and  $L(p - 2 - \lambda)$  (of dimension  $p - 1 - \lambda$ ). It is easy to check (see [Ri, Corollary 7.2.6]) that

$$\mathfrak{L}_1 = \mathcal{O}_{\mathbb{P}^1}(-1), \quad \mathfrak{L}_{\tau_0} = \mathcal{O}_{\mathbb{P}^1}(-2)[1]$$

(considered as sheaves on the zero section of  $\tilde{\mathcal{N}}$ ). Hence the Koszul grading on  $(\mathcal{U}\mathfrak{g})_0^\lambda$  is given by the isomorphism

$$(\mathcal{U}\mathfrak{g})_0^\lambda \cong \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}_{\tilde{\mathcal{N}}}^n(\mathfrak{L}_1^{\oplus \lambda+1} \oplus \mathfrak{L}_{\tau_0}^{\oplus p-1-\lambda}, \mathfrak{L}_1^{\oplus \lambda+1} \oplus \mathfrak{L}_{\tau_0}^{\oplus p-1-\lambda}).$$

One can easily check, using the Koszul resolution

$$\mathcal{O}_{\tilde{\mathcal{N}}}(2) \hookrightarrow \mathcal{O}_{\tilde{\mathcal{N}}} \rightarrow \mathcal{O}_{\mathbb{P}^1},$$

that we have

$$\begin{aligned} \mathrm{Ext}_{\tilde{\mathcal{N}}}^*(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) &\cong \mathbb{k} \oplus \mathbb{k}[-2], \\ \mathrm{Ext}_{\tilde{\mathcal{N}}}^*(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}) &\cong V^*, \\ \mathrm{Ext}_{\tilde{\mathcal{N}}}^*(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-1)) &\cong V[-2], \end{aligned}$$

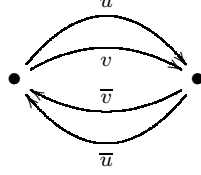
where  $V$  is a 2-dimensional  $\mathbb{k}$ -vector space. Hence we obtain an isomorphism

$$\begin{aligned} (\mathcal{U}\mathfrak{g})_0^\lambda &\cong (\mathrm{Mat}_{\lambda+1}(\mathbb{k}) \otimes_{\mathbb{k}} (\mathbb{k} \oplus \mathbb{k}[-2])) \oplus (\mathrm{Mat}_{p-1-\lambda}(\mathbb{k}) \otimes_{\mathbb{k}} (\mathbb{k} \oplus \mathbb{k}[-2])) \\ &\quad \oplus (V \otimes_{\mathbb{k}} \mathrm{Mat}_{\lambda+1, p-1-\lambda}(\mathbb{k})[-1]) \oplus (V^* \otimes_{\mathbb{k}} \mathrm{Mat}_{p-1-\lambda, \lambda+1}(\mathbb{k})[-1]). \end{aligned}$$

The product is given by the matrix multiplication, together with the natural paring  $V[-1] \times V^*[-1] \rightarrow \mathbb{k}[-2]$ .

In this case  $w_0 = -1$ , hence all the blocks are self-dual. The identification of  $(\mathcal{U}\mathfrak{g})_0^\lambda$  with (the shift of) its dual is given by the natural identification of the dual of  $\mathrm{Mat}_{n,m}(\mathbb{k})$  with  $\mathrm{Mat}_{m,n}(\mathbb{k})$ , via the trace.

In particular, all regular blocks are of dimension  $2p^2$ , and they are Morita equivalent to the path algebra of the quiver



with relations

$$\begin{aligned} \bar{u}v &= \bar{v}u = 0, & \bar{u}u &= \bar{v}v, \\ u\bar{v} &= v\bar{u} = 0, & u\bar{u} &= v\bar{v}. \end{aligned}$$

The grading is obtained by assigning each edge the degree 1.

**5.2. Singular blocks.** The only singular block is that of  $-1$ . And the only simple object in the category  $\text{Mod}_{-1}^{\text{fg}}((\mathcal{U}\mathfrak{g})_0)$  is  $L(p-1)$ , of dimension  $p$ . Moreover, this category is semisimple. Hence we have an isomorphism of graded rings

$$(\mathcal{U}\mathfrak{g})_0^{\widehat{-1}} \cong \text{Mat}_p(\mathbb{k}).$$

Again, the identification of  $(\mathcal{U}\mathfrak{g})_0^{\widehat{-1}}$  with its dual is given by the natural identification of  $\text{Mat}_p(\mathbb{k})$  with its dual.

#### REFERENCES

- [AJS] H. H. Andersen, J. C. Jantzen, W. Soergel, *Representations of quantum groups at a  $p$ -th root of unity and of semisimple groups in characteristic  $p$ : independence of  $p$* , Astérisque **220** (1994), 1–321.
- [BGS] A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473–527.
- [Be] A. Berkson, *The  $u$ -algebra of a restricted Lie algebra is Frobenius*, Proc. Amer. Math. Soc. **15** (1964), 14–15.
- [BM] R. Bezrukavnikov, I. Mirković, *Representations of semisimple Lie algebras in prime characteristic and noncommutative Springer resolution*, arXiv preprint 1001.2562.
- [BMR] R. Bezrukavnikov, I. Mirković, D. Rumynin, *Localization of modules for a semisimple Lie algebra in prime characteristic*, with an appendix by R. Bezrukavnikov and S. Riche, Ann. of Math. **167** (2008), 945–991.
- [BMR2] R. Bezrukavnikov, I. Mirković, D. Rumynin, *Singular localization and intertwining functors for reductive Lie algebras in prime characteristic*, Nagoya Math. J. **184** (2006), 1–55.
- [CG] N. Chriss, V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, 1997.
- [Fi] P. Fiebig, *An upper bound for the exceptional characteristics for Lusztig’s character formula*, arXiv preprint 0811.1674.
- [FP] E. Friedlander, B. Parshall, *Modular representation theory of Lie algebras*, Amer. J. Math. **110** (1988), 1055–1093.
- [Ja] J. C. Jantzen, *Representations of algebraic groups, second edition*, Mathematical surveys and monographs 107, Amer. Math. Soc., 2003.

- [Ke] B. Keller, *Derived categories and their uses*, in *Handbook of algebra, Vol. 1*, edited by M. Hazewinkel, Elsevier, 1996.
- [MR1] I. Mirković, S. Riche, *Linear Koszul duality*, *Compos. Math.* **146** (2010), 233–258.
- [MR2] I. Mirković, S. Riche, *Linear Koszul duality and affine Hecke algebras*, arXiv preprint 0903.0678.
- [Po] L. Positselski, *Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence*, arXiv preprint 0905.2621.
- [Ri] S. Riche, *Koszul duality and modular representations of semisimple Lie algebras*, *Duke Math. J.* **154** (2010), 31–134.
- [Sc] J. Schue, *Symmetry for the enveloping algebra of a restricted Lie algebra*, *Proc. Amer. Math. Soc.* **16** (1965), 1123–1124.
- [Sp] N. Spaltenstein, *Resolutions of unbounded complexes*, *Compos. Math.* **65** (1988), 121–154.

CLERMONT UNIVERSITÉ, UNIVERSITÉ BLAISE PASCAL, LABORATOIRE DE MATHÉMATIQUES, BP 10448, F-63000 CLERMONT-FERRAND.

CNRS, UMR 6620, LABORATOIRE DE MATHÉMATIQUES, F-63177 AUBIERE.

*E-mail address:* `simon.riche@math.univ-bpclermont.fr`