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Alexandre Cabot, Pierre Frankel

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ASYMPTOTICS FOR SOME SEMILINEAR HYPERBOLIC EQUATIONS WITH NON-AUTONOMOUS DAMPING

A. CABOT AND P. FRANKEL

ABSTRACT. Let V and H be Hilbert spaces such that $V \subset H \subset V'$ with dense and continuous injections. Consider a linear continuous operator $A : V \to V'$ which is assumed to be symmetric, monotone and semi-coercive. Given a function $f : V \to H$ and a map $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ such that $\lim_{t\to+\infty} \gamma(t) = 0$, our purpose is to study the asymptotic behavior of the following semilinear hyperbolic equation

$$(E) \qquad \frac{d^2u}{dt^2}(t) + \gamma(t)\frac{du}{dt}(t) + Au(t) + f(u(t)) = 0, \quad t \geq 0.$$

The nonlinearity f is assumed to be monotone and conservative. Condition $\int_0^{+\infty} \gamma(t)\, dt = +\infty$ guarantees that some suitable energy function tends toward its minimum. The main contribution of this paper is to provide a general result of convergence for the trajectories of (E): if the quantity $\gamma(t)$ behaves as k/t^α, for some $\alpha \in]0, 1[$, $k > 0$ and t large enough, then $u(t)$ weakly converges in V toward an equilibrium as $t \to +\infty$. Strong convergence in V holds true under compactness or symmetry conditions. We also give estimates for the speed of convergence of the energy under some ellipticity-like conditions. The abstract results are applied to particular semilinear evolution problems at the end of the paper.

1. INTRODUCTION

Throughout this paper, V stands for a real Hilbert space, whose scalar product and norm are respectively denoted by $((\cdot,\cdot))$ and $\|\cdot\|$. Let H be another real Hilbert space with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. Suppose that V is dense in H with continuous injection. By duality, the topological dual space H' of H is identified with a dense subspace of the topological dual V' of V. Identifying H with H', we obtain $V \subset H \subset V'$, where each space is dense in the next one, each injection being continuous. We denote by $\langle\cdot,\cdot\rangle_{V',V}$ the duality pairing between V' and V. Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear form satisfying

$$(h_1) \qquad a(.,.) \text{ is symmetric, positive,}$$

$$(h_2) \qquad \exists \lambda \geq 0, \mu > 0 \quad \text{such that} \quad \forall u \in V, \quad a(u,u) + \lambda|u|^2 \geq \mu\|u\|^2.$$

This last property is known as the semi-coercivity of the form a. We associate with $a(.,.)$ the continuous operator $A : V \to V'$ defined by $\langle Au, v\rangle_{V',V} = a(u,v)$ for all $u, v \in V$. We denote by $D(A)$ the domain of the operator A, *i.e.* $D(A) = \{v \in V;\ Av \in H\}$. Given a function $f : V \to H$ and a map $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$, we

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consider the following semilinear evolution equation of second-order in time

$$(E) \qquad \frac{d^2u}{dt^2}(t) + \gamma(t)\frac{du}{dt}(t) + Au(t) + f(u(t)) = 0, \quad t \geq 0.$$

The nonlinearity f is assumed to be conservative, *i.e.* derives from some potential $F \in \mathcal{C}^1(V, \mathbb{R})$. The main purpose of the paper is to investigate the asymptotic behavior of the trajectories of (E) for a vanishing damping term, *i.e.* $\gamma(t) \to 0$ as $t \to +\infty$. It is clear that the decay properties of the map γ play a central role in the analysis. In particular, if the quantity $\gamma(t)$ tends to 0 too rapidly as $t \to +\infty$, convergence of the trajectories may fail. To motivate our study, let us show how it is connected to other questions of interest.

Case of a constant damping. If $\gamma(t) \equiv \gamma$, existence and uniqueness are well-known in the framework of damped wave equations. More precisely, if the map $f : V \to H$ is Lipschitz continuous on the bounded sets of V and if the map F satisfies suitable growth conditions, then for any $(u_0, v_0) \in D(A) \times V$, there exists a unique solution $u \in W^{1,\infty}_{loc}(\mathbb{R}_+, V) \cap W^{2,\infty}_{loc}(\mathbb{R}_+, H)$ of (E) such that $u(0) = u_0$ and $\frac{du}{dt}(0) = v_0$, see [12, Theorem II.3.2.1] or [20, Ch. IV, Theorem 4.1]. The trajectories of (E) are known to converge toward an equilibrium point $u_\infty \in \{v \in V,\ Av + f(v) = 0\}$ under assumptions like monotonicity or analyticity. In the case of a monotone map f, convergence is obtained for the weak topology of V and the main ingredient of the proof is the Opial lemma, *cf.* [3]. When the nonlinearity is analytic, convergence of the trajectories relies on the Lojasiewicz inequality, see [15, 16] and the pioneering work [19] for parabolic problems.

Averaged heat equation. With the same assumptions as above, consider the abstract heat equation

$$\frac{dv}{ds}(s) + Av(s) = 0, \quad s \geq 0. \tag{1.1}$$

It may be of interest to examine the case where the velocity $\frac{dv}{ds}(s)$ is proportional, not to the instantaneous vector $Av(s)$, but to some average taken over the interval $[0, s]$. The simplest such equation is

$$\frac{dv}{ds}(s) + \frac{1}{s}\int_0^s Av(\sigma)\, d\sigma = 0, \quad s > 0. \tag{1.2}$$

After multiplying this equality by s and differentiating, we obtain the following second-order in time equation

$$s\,\frac{d^2v}{ds^2}(s) + \frac{dv}{ds}(s) + Av(s) = 0, \quad s > 0.$$

The change of variable $s = \frac{t^2}{4}$ allows to rewrite the above equation as

$$\frac{d^2u}{dt^2}(t) + \frac{1}{t}\,\frac{du}{dt}(t) + Au(t) = 0, \quad t > 0,$$

where the map u is defined by $u(t) = v\left(\frac{t^2}{4}\right)$ for every $t \geq 0$. This is exactly equation (E) with $\gamma(t) = \frac{1}{t}$ and $f \equiv 0$. Assuming that the injection $V \hookrightarrow H$ is compact, there exists a nondecreasing sequence $(\lambda_i)_{i\geq 1}$ of eigenvalues of A, along with a complete orthonormal basis of H, $(e_i)_{i\geq 1}$ consisting of the corresponding

eigenvectors. Let $u(t) = \sum_{i=1}^{+\infty} u_i(t)\, e_i$ be the decomposition of the solution $u(t)$ on the basis of eigenfunctions. Every component u_i satisfies the following equation

$$\ddot{u}_i(t) + \frac{1}{t}\, \dot{u}_i(t) + \lambda_i\, u_i(t) = 0, \quad t > 0.$$

It ensues that each kernel component u_i, $i \in \{1, \dots, \dim(\ker A)\}$ verifies $u_i(t) = a_i \ln t + b_i$, for some $a_i, b_i \in \mathbb{R}$. In particular, it cannot converge as $t \to +\infty$, unless it is stationary. When the eigenvalue λ_i is positive, we let the reader check that

$$u_i(t) = a_i'\, J_0\left(\sqrt{\lambda_i}\, t\right) + b_i'\, Y_0\left(\sqrt{\lambda_i}\, t\right), \quad \text{for some } a_i', b_i' \in \mathbb{R},$$

where J_0 and Y_0 denote respectively the zeroth Bessel functions of the first and second kind[1]. Recalling that

$$J_0(t) \sim \sqrt{\frac{2}{\pi\, t}} \cos\left(t - \frac{\pi}{4}\right) \quad \text{and} \quad Y_0(t) \sim \sqrt{\frac{2}{\pi\, t}} \sin\left(t - \frac{\pi}{4}\right) \quad \text{as } t \to +\infty,$$

we deduce that $u_i(t) \sim \frac{c_i}{\sqrt{t}} \cos(\sqrt{\lambda_i}\, t - \varphi_i)$ as $t \to +\infty$, for some c_i, $\varphi_i \in \mathbb{R}$. Coming back to the averaged heat equation (1.2), we then obtain for each component v_i

$$v_i(s) \sim \frac{c_i}{\sqrt{2}}\, s^{-\frac{1}{4}} \cos\left(2\sqrt{\lambda_i\, s} - \varphi_i\right) \quad \text{as } s \to +\infty.$$

It converges toward zero much more slowly than the corresponding component of the "pure" heat equation, equal to $v_i(0)\, e^{-\lambda_i s}$. The above discussion shows that the global behavior of (1.2) -or more generally (E)- differs considerably from the one of equation (1.1).

Heavy ball with asymptotically small friction. Given a continuous map $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ and a potential $\Phi : H \to \mathbb{R}$ of class $\mathcal{C}^1$ with a locally Lipschitz gradient, let us consider the following ordinary differential equation in the Hilbert space H

$$\ddot{x}(t) + \gamma(t)\, \dot{x}(t) + \nabla\Phi(x(t)) = 0, \quad t \geq 0. \tag{1.3}$$

When $\gamma(t) \equiv \gamma > 0$, the above equation is known under the terminology of "Heavy Ball with Friction" system, (HBF) for short. From a mechanical point of view, (HBF) corresponds to the equation describing the motion of a material point subjected to the conservative force $-\nabla\Phi(x)$ and the viscous friction force $-\gamma\, \dot{x}$. The (HBF) system can be studied in the classical framework of the theory of dissipative dynamical systems, *cf.* [11, 13]. The trajectories of (HBF) are known to converge toward a critical point of Φ under various assumptions (see [2, 4] for convex potentials and [14] for analytic ones). In the recent papers [8, 9], it is considered the case of a vanishing damping $\gamma(t) \to 0$ as $t \to +\infty$. The corresponding equation is typically obtained from a first-order gradient system involving some memory aspects, see [7]. If the function Φ is convex and has a unique minimum $\overline{x}$, condition $\int_0^{+\infty} \gamma(t)\, dt = +\infty$ is sufficient to ensure (weak) convergence of the trajectories of (1.3) toward $\overline{x}$. When the function Φ has a continuum of equilibria, the more stringent condition $\displaystyle\int_0^{+\infty} e^{-\int_0^t \gamma(s)\, ds} dt < +\infty$ is necessary to obtain convergence of the trajectories. In the one-dimensional case, the slightly stronger condition $\displaystyle\int_0^{+\infty} e^{-\theta \int_0^t \gamma(s)\, ds} dt < +\infty$, for some $\theta \in]0, 1[$ is shown to be sufficient.

[1]See [1, 5] for standard references on Bessel equations.

In the higher-dimensional case, the general question of convergence is left open in [8, 9]. The new techniques developed in the present paper allow to address this question and to fill partially the gap between necessary and sufficient conditions for convergence, see comments below.

Let us come back to equation (E) and precise now the framework of the paper. The nonlinearity f is assumed to be monotone and conservative, *i.e.* derives from some convex potential $F \in \mathcal{C}^1(V, \mathbb{R})$. The set of equilibria $S = \{v \in V,\ Av + f(v) = 0\}$ is supposed to be nonempty. It is not our purpose to develop the well-posedness of equation (E) for given initial conditions. Throughout the paper, we assume the existence of a solution to equation (E) in the class

$$u \in W^{1,1}_{loc}(\mathbb{R}_+, V) \cap W^{2,1}_{loc}(\mathbb{R}_+, H). \tag{1.4}$$

We define the energy function $\mathcal{E}$ along each trajectory by

$$\mathcal{E}(t) = \frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \frac{1}{2}\,a(u(t), u(t)) + F(u(t)).$$

The major contribution of this paper is to provide a result of (weak) convergence in V for the trajectories of (E): if the quantity $\gamma(t)$ behaves as k/t^α, for some $\alpha \in]0, 1[$, $k > 0$ and t large enough, there exists an equilibrium $u_\infty \in S$ such that $u(t) \rightharpoonup u_\infty$ weakly in V as $t \to +\infty$. The exact statement is in fact slightly more general, see Theorem 3.10. The main ingredients of the proof are the Opial lemma along with accurate estimates of the energy decay, *cf.* Proposition 3.7. Strong convergence in V holds true under compactness or symmetry conditions, see Theorem 3.13. We stress the fact that the result and the technique of the proof are new, and they are also applicable to the differential equation (1.3).

The second contribution of the paper is to give sharp estimates for the speed of convergence of the energy $\mathcal{E}(t)$ as $t \to +\infty$. In the linear case ($f = 0$) and under some ellipticity-like condition, we obtain the following estimate

$$\mathcal{E}(t) \sim K\, e^{-\int_0^t \gamma(s)ds} \quad \text{as} \quad t \to +\infty, \qquad \text{for some } K > 0. \tag{1.5}$$

Notice that this estimate fails to be true if the trajectory is contained in $\ker A$, see Theorem 2.7 for a precise statement. In the nonlinear case, the same kind of estimate is obtained at a slightly lower degree of precision[2], *cf.* Theorem 3.15.

Outline of the paper. Section 2 is concerned with the linear hyperbolic equation (E_0) obtained by taking $f = 0$ in (E). We analyze the behavior of the trajectories by studying respectively their components with respect to the spaces $\ker A$ and $(\ker A)^\perp$. A sharp estimate of the energy decay is given under some ellipticity-like condition. In section 3, we deal with the general equation (E) by assuming that the nonlinearity f is monotone. It is shown in paragraph 3.1 that the energy $\mathcal{E}(t)$ vanishes as $t \to +\infty$, which allows to prove (weak) convergence of the trajectories in the case of a unique minimum. The general problem of convergence for a continuum of minima is treated in paragraph 3.2, which is the core of the paper. Additional results of strong convergence in V are given under some compactness or symmetry assumptions. Finally, the abstract results are applied to particular semilinear evolution problems in section 4.

[2]In this case, a factor $\frac{2}{3}$ has to be introduced in the exponent of formula (1.5).

2. Linear hyperbolic equation

Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear form satisfying (h_1)-(h_2) and let $A : V \to V'$ be the associate operator. Given a map $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$, we consider the following linear hyperbolic equation

$$(E_0) \qquad \frac{d^2u}{dt^2}(t) + \gamma(t)\frac{du}{dt}(t) + Au(t) = 0, \quad t \geq 0.$$

We assume the existence of a solution to equation (E_0) in the class (1.4). We define the energy function $\mathcal{E}$ along each trajectory by

$$\mathcal{E}(t) = \frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \frac{1}{2}a(u(t), u(t)).$$

We have $\mathcal{E} \in W^{1,1}_{loc}(\mathbb{R}_+)$ and

$$\begin{aligned} \dot{\mathcal{E}}(t) &= \left(\frac{d^2u}{dt^2}(t), \frac{du}{dt}(t)\right) + \left\langle Au(t), \frac{du}{dt}(t)\right\rangle_{V',V} \\ &= -\gamma(t)\left|\frac{du}{dt}(t)\right|^2 \leq 0 \quad \text{a.e. on } \mathbb{R}_+, \end{aligned}$$

hence the function $\mathcal{E}$ is a Lyapunov function for the system (E_0). The purpose of this section is to establish results of convergence for the trajectory u, along with estimates of the energy decay. The key assumptions on the map γ are the following

$$(l_1) \qquad \lim_{t\to+\infty} \gamma(t) = 0$$

$$(l_2) \qquad \dot{\gamma} \in L^1(0, +\infty).$$

For every $t \geq 0$, we set $\widehat{u}(t) = Pu(t)$, where P denotes the orthogonal projection onto the subspace[3] $\ker A$ in the sense of H. Since $\widehat{u}(t) \in \ker A$ for every $t \geq 0$, we have

$$\forall t \geq 0, \quad \frac{d^2\widehat{u}}{dt^2}(t) + \gamma(t)\frac{d\widehat{u}}{dt}(t) = 0.$$

By integrating this equality twice, we find

$$\begin{aligned} \forall t \geq 0, \quad \widehat{u}(t) &= \widehat{u}(0) + \left(\int_0^t e^{-\int_0^s \gamma(\tau)d\tau} ds\right)\frac{d\widehat{u}}{dt}(0) \\ &= Pu_0 + \left(\int_0^t e^{-\int_0^s \gamma(\tau)d\tau} ds\right) Pv_0. \end{aligned} \tag{2.1}$$

If $Pv_0 \neq 0$, the above equality shows that the asymptotic behavior of the component $\widehat{u}$ is strongly related with the convergence of the integral $\int_0^{+\infty} e^{-\int_0^s \gamma(\tau)d\tau} ds$. The next proposition summarizes the different possible cases.

Proposition 2.1. *Let us set* $\overline{\omega} = \int_0^{+\infty} e^{-\int_0^s \gamma(\tau)d\tau} ds \in \mathbb{R}_+ \cup \{+\infty\}$.

- *If* $v_0 \in (\ker A)^\perp$, *then* $\widehat{u}(t) = Pu_0$ *for every* $t \geq 0$.

[3]By using assumptions (h_1)-(h_2), it is easy to check that $\ker A$ is closed in H. See also Remark 3.2.

- *If $v_0 \notin (\ker A)^\perp$, then the solution $\widehat{u}$ converges if and only if $\overline{\omega} < +\infty$. More precisely, we have $\lim_{t\to+\infty} |\widehat{u}(t)| = +\infty$ if $\overline{\omega} = +\infty$ while $\lim_{t\to+\infty} \widehat{u}(t) = P(u_0 + \overline{\omega} v_0)$ if $\overline{\omega} < +\infty$.*

Our purpose is now to evaluate the energy decay along each trajectory $u(.)$. We start with a preliminary result corresponding to the case $\ker A = \{0\}$.

Lemma 2.2. *Assume that the bilinear form $a(.,.)$ satisfies (h_1)-(h_2) and that*

$$\exists \eta > 0, \forall u \in V, \quad a(u,u) \geq \eta |u|^2. \tag{2.2}$$

Let $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ be a function satisfying (l_1)-(l_2). Let u be a solution in the class (1.4) to equation (E_0). Then, either the solution u is stationary, or there exists $K > 0$ such that

$$\mathcal{E}(t) \sim K e^{-\int_0^t \gamma(s)ds} \quad \text{as} \quad t \to +\infty.$$

Proof. The main idea of the proof consists in using the function $\mathcal{F}$ defined by[4]

$$\begin{aligned} \mathcal{F}(t) &= \frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \frac{1}{2}a(u(t),u(t)) + \frac{\gamma(t)}{2}\left(\frac{du}{dt}(t), u(t)\right) \\ &= \mathcal{E}(t) + \frac{\gamma(t)}{2}\left(\frac{du}{dt}(t), u(t)\right). \end{aligned}$$

We have $\mathcal{F} \in W^{1,1}_{loc}(\mathbb{R}_+)$ and by differentiating the function $\mathcal{F}$, we find for almost every $t \geq 0$

$$\begin{aligned} \dot{\mathcal{F}}(t) &= \dot{\mathcal{E}}(t) + \frac{\dot{\gamma}(t)}{2}\left(\frac{du}{dt}(t), u(t)\right) + \frac{\gamma(t)}{2}\left(\frac{d^2u}{dt^2}(t), u(t)\right) + \frac{\gamma(t)}{2}\left|\frac{du}{dt}(t)\right|^2 \\ &= -\frac{\gamma(t)}{2}\left|\frac{du}{dt}(t)\right|^2 - \frac{\gamma(t)}{2}a(u(t),u(t)) + \left(\frac{\dot{\gamma}(t)}{2} - \frac{\gamma(t)^2}{2}\right)\left(\frac{du}{dt}(t), u(t)\right). \end{aligned}$$

Therefore we have

$$\dot{\mathcal{F}}(t) + \gamma(t)\mathcal{F}(t) = \frac{\dot{\gamma}(t)}{2}\left(\frac{du}{dt}(t), u(t)\right) \quad \text{a.e. on } \mathbb{R}_+. \tag{2.3}$$

Since $\left|\left(\frac{du}{dt}(t), u(t)\right)\right| \leq \frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \frac{1}{2}|u(t)|^2$ and $a(u(t),u(t)) \geq \eta\,|u(t)|^2$ by assumption (2.2), we have

$$\left|\left(\frac{du}{dt}(t), u(t)\right)\right| \leq C\,\mathcal{E}(t), \quad \text{for some } C > 0. \tag{2.4}$$

Recalling that $\lim_{t\to+\infty}\gamma(t) = 0$, the expression of $\mathcal{F}$ shows that

$$\mathcal{F}(t) \sim \mathcal{E}(t) \quad \text{as } t \to +\infty. \tag{2.5}$$

We deduce from (2.3), (2.4) and (2.5) the existence of $D > 0$ and $t_0 \geq 0$ such that

$$\left|\dot{\mathcal{F}}(t) + \gamma(t)\mathcal{F}(t)\right| \leq D\,|\dot{\gamma}(t)|\mathcal{F}(t) \quad \text{a.e. on } [t_0, +\infty[.$$

Let us multiply each member of this inequality by $e^{\int_0^t \gamma(s)ds}$ and set $\mathcal{G}(t) = e^{\int_0^t \gamma(s)ds}\mathcal{F}(t)$. We obtain

$$|\dot{\mathcal{G}}(t)| \leq D\,|\dot{\gamma}(t)|\mathcal{G}(t) \quad \text{a.e. on } [t_0, +\infty[. \tag{2.6}$$

[4]The use of such an auxiliary function is classical, see for example [13, Lemma 3.2.6] in the case of an autonomous damping.

Observe that if $\mathcal{G}(t_1) = 0$ for some $t_1 \geq t_0$, then we have $\mathcal{F}(t_1) = 0$ and $\mathcal{E}(t_1) = 0$. Since the map $\mathcal{E}$ is nonincreasing, we conclude that $\mathcal{E}(t) = 0$ for every $t \geq t_1$, *i.e.* the solution u is stationary. Now assume that $\mathcal{G}(t) > 0$ for every $t \geq t_0$ and divide each member of equality (2.6) by $\mathcal{G}(t)$. Since $\dot{\gamma} \in L^1(0, +\infty)$ by assumption, we deduce that

$$\left| \frac{d}{dt} \ln \mathcal{G} \right| (t) = \frac{|\dot{\mathcal{G}}(t)|}{\mathcal{G}(t)} \quad \in L^1(0, +\infty).$$

It ensues that $\lim_{t \to +\infty} \ln \mathcal{G}(t)$ exists in $\mathbb{R}$. We deduce that $\lim_{t \to +\infty} e^{\int_0^t \gamma(s) ds} \mathcal{F}(t) = K > 0$. The conclusion immediately follows from estimate (2.5). □

Remark 2.3. A result similar to Lemma 2.2 can be obtained by eliminating the first order term in (E_0) via the change of variable $v(t) = e^{\frac{1}{2} \int_0^t \gamma(s) ds} u(t)$. The details are left to the reader.

Remark 2.4 (Case γ constant). Assuming that $\gamma(t) \equiv \gamma > 0$ and that $a(u, u) \geq \eta \, |u|^2$ for every $u \in V$, the estimate $\mathcal{E}(t) = O\left(e^{-\gamma t}\right)$ remains true as $t \to +\infty$ if $\gamma < 2 \, \eta^{1/2}$, see [13, Lemma 3.2.6]. However, it fails to be valid if $\gamma \geq 2 \, \eta^{1/2}$, see [13, Proposition 3.2.5].

We now assume the following ellipticity-like condition

$$\forall u \in V, \quad a(u, u) \geq \eta \, |u - Pu|^2, \quad \text{for some } \eta > 0. \tag{2.7}$$

Remark 2.5. Under (h_2), this condition is equivalent to the following one[5]

$$\forall u \in V, \quad a(u, u) \geq \eta' \, \|u - Pu\|^2, \quad \text{for some } \eta' > 0. \tag{2.8}$$

Indeed, assume that condition (2.7) is satisfied. Recalling that $Pu \in \ker A$, we deduce from (h_2) that

$$\forall u \in V, \quad a(u, u) + \lambda \, |u - Pu|^2 \geq \mu \, \|u - Pu\|^2.$$

It ensues that $\left(1 + \frac{\lambda}{\eta}\right) a(u, u) \geq \mu \, \|u - Pu\|^2$ for every $u \in V$ and finally (2.8) is fulfilled with $\eta' = \frac{\eta \, \mu}{\eta + \lambda}$.

Remark 2.6. Suppose that the injection $V \hookrightarrow H$ is compact and that (h_1)-(h_2) hold true. The eigenvalues of A then define an nondecreasing sequence of nonnegative scalars tending to $+\infty$ and there exists an orthonormal basis of H consisting of the corresponding eigenvectors, see for example [17, 20]. If η denotes the smallest eigenvalue of A greater than 0, it is clear that $a(u, u) \geq \eta |u|^2$ for every $u \in (\ker A)^\perp \cap V$ and therefore condition (2.7) holds true.

The next result allows to estimate the energy decay under condition (2.7).

Theorem 2.7. *Assume that the bilinear form $a(.,.)$ satisfies conditions (h_1)-(h_2) and (2.7). Let $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ be a function satisfying (l_1)-(l_2). Let u be a solution in the class (1.4) to equation (E_0). Then, either the trajectory is contained in* $\ker A$*, or there exists $K > 0$ such that*

$$\mathcal{E}(t) \sim K \, e^{-\int_0^t \gamma(s) ds} \quad \textit{as} \quad t \to +\infty. \tag{2.9}$$

[5]Condition (2.8) is used in [21, Section 4], where estimates of the energy decay are provided in the case of an autonomous damping.

Proof. For every $t \geq 0$, we set $\widehat{u}(t) = Pu(t)$ and $\widetilde{u}(t) = u(t) - Pu(t)$. Since $\widehat{u}(t) \in \ker A$, $\frac{d\widehat{u}}{dt}(t) \in \ker A$ and $\frac{d\widetilde{u}}{dt}(t) \in (\ker A)^{\perp}$, we have for every $t \geq 0$

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2}\left|\frac{d\widehat{u}}{dt}(t) + \frac{d\widetilde{u}}{dt}(t)\right|^2 + \frac{1}{2}a(\widehat{u}(t)+\widetilde{u}(t), \widehat{u}(t)+\widetilde{u}(t)) \\ &= \frac{1}{2}\left|\frac{d\widehat{u}}{dt}(t)\right|^2 + \frac{1}{2}\left|\frac{d\widetilde{u}}{dt}(t)\right|^2 + \frac{1}{2}a(\widetilde{u}(t), \widetilde{u}(t)). \end{aligned} \tag{2.10}$$

From equality (2.1), we deduce that for every $t \geq 0$

$$\left|\frac{d\widehat{u}}{dt}(t)\right|^2 = e^{-2\int_0^t \gamma(s)ds}\left|\frac{d\widehat{u}}{dt}(0)\right|^2. \tag{2.11}$$

Let us now set $V_1 = (\ker A)^{\perp} \cap V$, $a_1 = a_{|V_1 \times V_1}$ and $A_1 = A_{|V_1}$. It is clear that $\widetilde{u}$ is a solution of

$$\frac{d^2\widetilde{u}}{dt^2}(t) + \gamma(t)\frac{d\widetilde{u}}{dt}(t) + A_1\widetilde{u}(t) = 0.$$

On the other hand, condition (2.7) implies that $a_1(u,u) \geq \eta\,|u|^2$ for every $u \in V_1$. By applying Lemma 2.2 to the solution $\widetilde{u}$, we obtain that either the map $\widetilde{u}$ is stationary or there exists $K_1 > 0$ such that

$$\frac{1}{2}\left|\frac{d\widetilde{u}}{dt}(t)\right|^2 + \frac{1}{2}a(\widetilde{u}(t), \widetilde{u}(t)) \sim K_1\, e^{-\int_0^t \gamma(s)ds} \quad \text{as} \quad t \to +\infty. \tag{2.12}$$

We now combine equalities (2.10), (2.11) with estimate (2.12). If $\int_0^{+\infty}\gamma(s)ds = +\infty$, we immediately obtain (2.9) with $K = K_1$. If $\int_0^{+\infty}\gamma(s)ds < +\infty$, then

$$\lim_{t\to+\infty}\mathcal{E}(t) = \frac{1}{2}\,e^{-2\int_0^{+\infty}\gamma(s)ds}\left|\frac{d\widehat{u}}{dt}(0)\right|^2 + K_1\,e^{-\int_0^{+\infty}\gamma(s)ds},$$

hence (2.9) is satisfied with $K = \frac{1}{2}\,e^{-\int_0^{+\infty}\gamma(s)ds}\left|\frac{d\widehat{u}}{dt}(0)\right|^2 + K_1$. □

Remark 2.8. If the trajectory $u(.)$ is contained in $\ker A$, estimate (2.9) is no more valid. In this case, we infer from equality (2.11) that $\mathcal{E}(t) = \frac{1}{2}\,e^{-2\int_0^t\gamma(s)ds}\left|\frac{d\widehat{u}}{dt}(0)\right|^2$ for every $t \geq 0$.

Corollary 2.9. *Under the hypotheses of Theorem 2.7, assume moreover that*[6]

$$(l_3) \qquad \gamma \notin L^1(0,+\infty).$$

Then we have $\lim_{t\to+\infty}\mathcal{E}(t) = 0$. *If* $\ker A = \{0\}$, *then* $u(t) \to 0$ *strongly in* V *as* $t \to +\infty$.

Proof. The first assertion is an immediate consequence of estimate (2.9), while the second one follows from

$$\forall t \geq 0, \qquad \mathcal{E}(t) \geq \frac{1}{2}\,a(u(t),u(t)) \geq \frac{\eta'}{2}\,\|u(t)\|^2,$$

see inequality (2.8). □

[6]Assumption (l_3) expresses that the quantity $\gamma(t)$ tends rather slowly toward 0 as $t \to +\infty$.

When $\ker A \neq \{0\}$, convergence of the trajectories is obtained under the following stronger assumption

$$\int_0^{+\infty} e^{-\frac{1}{2}\int_0^s \gamma(\tau)d\tau} ds < +\infty. \tag{2.13}$$

Corollary 2.10. *Under the hypotheses of Theorem 2.7, assume moreover that condition (2.13) is satisfied. Then, there exists $u_\infty \in \ker A$ such that $u(t) \to u_\infty$ strongly in V as $t \to +\infty$.*

Proof. First assume that the trajectory is contained in $\ker A$. Observing that $\overline{\omega} = \int_0^{+\infty} e^{-\int_0^s \gamma(\tau)d\tau} ds < +\infty$, we deduce from Proposition 2.1 that $u(t)$ converges strongly in H as $t \to +\infty$. If the trajectory is not contained in $\ker A$, we derive from estimate (2.9) that

$$\left|\frac{du}{dt}(t)\right| = O\left(e^{-\frac{1}{2}\int_0^t \gamma(s)ds}\right) \quad \text{as} \quad t \to +\infty,$$

hence $\frac{du}{dt} \in L^1(\mathbb{R}_+, H)$ in view of condition (2.13). The trajectory u has a finite length, hence strongly converges in H toward some $u_\infty \in \ker A$. Using now the semi-coercivity condition (h_2), we have

$$\mu\|u(t) - u_\infty\|^2 \leq \lambda|u(t) - u_\infty|^2 + a(u(t) - u_\infty, u(t) - u_\infty) = \lambda|u(t) - u_\infty|^2 + a(u(t), u(t)).$$

Since $\lim_{t\to+\infty} |u(t) - u_\infty| = 0$ and $\lim_{t\to+\infty} a(u(t), u(t)) = 0$ in view of Corollary 2.9, we conclude that $\lim_{t\to+\infty} \|u(t) - u_\infty\| = 0$.

□

Example 2.11. Suppose that there exist α, $k > 0$ such that $\gamma(t) = \frac{k}{t^\alpha}$ for t large enough. The assumptions (l_1)-(l_2) are clearly satisfied. If the bilinear form $a(.,.)$ satisfies conditions (h_1)-(h_2) and (2.7), we deduce from Theorem 2.7 and Corollary 2.10 that

- if $\alpha > 1$, then $\lim\limits_{t\to+\infty} \mathcal{E}(t) > 0$;
- if $\alpha = 1$, then $\mathcal{E}(t) \sim \frac{K}{t^k}$ as $t \to +\infty$ and the trajectory $u(.)$ strongly converges in V as soon as $k > 2$;
- if $\alpha \in (0,1)$, then $\mathcal{E}(t) \sim Ke^{-\frac{k}{1-\alpha}t^{1-\alpha}}$ as $t \to +\infty$ and the trajectory $u(.)$ strongly converges in V for every $k > 0$.

Other results of convergence will be provided in the more general framework of semilinear equations.

3. Monotone conservative nonlinearity

The assumptions concerning the spaces V, H, the linear operator $A : V \to V'$ and the map $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are the same as in section 2. We consider the following semilinear hyperbolic equation

$$(E) \qquad \frac{d^2u}{dt^2}(t) + \gamma(t)\frac{du}{dt}(t) + Au(t) + f(u(t)) = 0, \quad t \geq 0.$$

We suppose that the nonlinearity $f : V \to H$ is conservative, *i.e.*

$$(k_1) \qquad \exists F \in \mathcal{C}^1(V, \mathbb{R}) \text{ such that } \forall u, v \in V, \quad \langle F'(u), v\rangle_{V',V} = (f(u), v).$$

Moreover, we assume that the map f is monotone

$$(k_2) \qquad \forall u, v \in V, \quad (f(u) - f(v), u - v) \geq 0,$$

which is equivalent to the convexity of the potential F. Defining $\Phi : V \to \mathbb{R}$ by

$$\Phi(v) = \tfrac{1}{2} a(v, v) + F(v),$$

we obtain a function of class $\mathcal{C}^1$ whose first derivative is given by $\langle \Phi'(u), v \rangle_{V',V} = a(u, v) + (f(u), v)$, or equivalently $\Phi'(u) = Au + f(u)$. Moreover, Φ is convex, which amounts to

$$\forall u, v \in V, \; a(u, v - u) + (f(u), v - u) \leq \Phi(v) - \Phi(u). \tag{3.1}$$

Consequently, minimum and stationary points of Φ coincide, *i.e.*

$$\operatorname{argmin} \Phi = \{v \in V \mid Av + f(v) = 0\}, \tag{3.2}$$

where $\operatorname{argmin} \Phi = \{v \in V \mid \Phi(v) = \inf \Phi\}$. It is clear in view of equation (E) that nothing is changed if some constant is added to the potential Φ. Without loss of generality, we will systematically assume that $\inf \Phi = 0$. Suppose moreover that

$$(k_3) \qquad S = \operatorname{argmin} \Phi \neq \emptyset.$$

Remark 3.1. Assume that a is coercive, *i.e.* (h_2) holds with $\lambda = 0$. Then the map $u \mapsto a(u, u)$ is strongly convex and since the function F is convex, the map Φ is also strongly convex. This implies immediately that the set $\operatorname{argmin} \Phi$ is a singleton, hence the non-vacuity condition (k_3) holds true. Now assume that (h_2) holds with $\lambda > 0$. To overcome the lack of coercivity, suppose that there exist $\varepsilon > 0$ and $C \geq 0$ such that $F(u) \geq \varepsilon |u|^2 - C$ for every $u \in V$. Without loss of generality, we can assume that $\varepsilon \leq \frac{\lambda}{2}$. For every $u \in V$, we have

$$\begin{aligned} \Phi(u) = \frac{1}{2} a(u, u) + F(u) \quad &\geq \quad \frac{\varepsilon}{\lambda} a(u, u) + F(u) \\ &\geq \quad \frac{\varepsilon \mu}{\lambda} \|u\|^2 - \varepsilon |u|^2 + \varepsilon |u|^2 - C \\ &= \quad \frac{\varepsilon \mu}{\lambda} \|u\|^2 - C, \end{aligned}$$

which shows that $\lim_{\|u\| \to +\infty} \Phi(u) = +\infty$. Since the function Φ is convex and continuous, this classically implies condition (k_3).

It is immediate to check that the set S is convex, closed in V and that $S \subset D(A)$.

Remark 3.2. Under assumption (h_2), let us show that S is closed in H. Let (u_n) be a sequence in S such that $\lim_{n \to +\infty} u_n = \overline{u}$ strongly in H, for some $\overline{u} \in H$. Since the function F is convex, there exist $b, c \in \mathbb{R}$ such that, for all $u \in V$, $F(u) \geq -b|u| - c$. Therefore we have for all $u \in V$,

$$\frac{1}{2} a(u, u) \leq \Phi(u) + b|u| + c. \tag{3.3}$$

Recalling that $\Phi(u_n) = 0$ for every $n \in \mathbb{N}$, we deduce that $\frac{1}{2} a(u_n, u_n) \leq b|u_n| + c$, hence the sequence $(a(u_n, u_n))$ is bounded. From hypothesis (h_2), we infer that the sequence (u_n) is bounded in V. It ensues that there exist $\widehat{u} \in V$ and a subsequence (u_{n_k}) such that $\lim_{k \to +\infty} u_{n_k} = \widehat{u}$ weakly in V. We immediately have $\widehat{u} = \overline{u}$ and the weak lower semicontinuity of Φ implies that $\Phi(\overline{u}) \leq \liminf_{k \to +\infty} \Phi(u_{n_k}) = 0$, hence $\overline{u} \in S$.

Remark 3.3 (Case $f(0) = 0$). If $f(0) = 0$ then we have

$$S = \ker A \cap \{v \in V \mid f(v) = 0\} \neq \emptyset.$$

Indeed, if $w \in S$ then in particular $(Aw, w) + (f(w), w) = 0$, and by monotonicity of f we have $(f(w) - f(0), w) \geq 0$, hence $(Aw, w) = (f(w), w) = 0$ and therefore $Aw = 0$.

In the sequel, we assume the existence of a solution to equation (E) in the class (1.4). We define the energy function $\mathcal{E}$ along each trajectory by

$$\mathcal{E}(t) = \frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \Phi(u(t)).$$

We have $\mathcal{E} \in W^{1,1}_{loc}(\mathbb{R}_+)$ and

$$\begin{aligned}\dot{\mathcal{E}}(t) &= \left(\frac{d^2u}{dt^2}(t), \frac{du}{dt}(t)\right) + \left\langle Au(t) + f(u(t)), \frac{du}{dt}(t)\right\rangle_{V',V} \\ &= -\gamma(t)\left|\frac{du}{dt}(t)\right|^2 \leq 0 \quad \text{a.e. on } \mathbb{R}_+,\end{aligned}$$

hence the function $\mathcal{E}$ is a Lyapunov function for the equation (E). We deduce that for every $t \geq 0$

$$\frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 \leq \mathcal{E}(t) \leq \mathcal{E}(0) \quad \text{and} \quad \Phi(u(t)) \leq \mathcal{E}(t) \leq \mathcal{E}(0). \tag{3.4}$$

In particular, we have $\frac{du}{dt} \in L^\infty(\mathbb{R}_+, H)$. In the sequel, we will consider only solutions which are bounded in H, *i.e.* satisfying $u \in L^\infty(\mathbb{R}_+, H)$.

Remark 3.4. Under assumption (h_2), it is easy to see that $u \in L^\infty(\mathbb{R}_+, H)$ implies $u \in L^\infty(\mathbb{R}_+, V)$. Indeed, let us assume that $\{u(t); t \geq 0\}$ is bounded in H. From inequality (3.3), we have $\frac{1}{2}a(u(t), u(t)) \leq \Phi(u(t)) + b|u(t)| + c$ for all $t \in \mathbb{R}_+$. Recalling that $\Phi(u(t)) \leq \mathcal{E}(0)$ in view of (3.4), we infer that $\{a(u(t), u(t)); t \geq 0\}$ is bounded. From hypothesis (h_2), we conclude that $\{u(t); t \geq 0\}$ is bounded in V.

3.1. **Summability of the energy. Case of a unique equilibrium.** We now prove that the map $\gamma\,\mathcal{E}$ is summable over $\mathbb{R}_+$ and that $\lim_{t\to+\infty} \mathcal{E}(t) = 0$.

Proposition 3.5. *Assume that the bilinear form $a(.,.)$ and the function f satisfy respectively hypotheses (h_1)-(h_2) and (k_1)-(k_3). Let $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ be a map satisfying (l_1)-(l_3). Let u be a solution in the class (1.4) to equation (E) and assume that $u \in L^\infty(\mathbb{R}_+, H)$. Then*

(i) $\int_0^{+\infty} \gamma(t)\,\mathcal{E}(t)\,dt < +\infty$.
(ii) $\lim\limits_{t\to+\infty} \mathcal{E}(t) = 0$, *hence*

$$\lim_{t\to+\infty}\left|\frac{du}{dt}(t)\right| = 0 \quad \textit{and} \quad \lim_{t\to+\infty} \Phi(u(t)) = 0. \tag{3.5}$$

Proof. (i) The proof follows the same arguments as those of [8, Prop. 3.1]. Let us take $v \in S$ and define the function $p : \mathbb{R}_+ \to \mathbb{R}_+$ by $p(t) = \frac{1}{2}|u(t) - v|^2$. By differentiating, we find for every $t \geq 0$

$$\dot{p}(t) = \left(\frac{du}{dt}(t), u(t) - v\right).$$

Since $\frac{du}{dt} \in W^{1,1}_{loc}(\mathbb{R}_+, H)$ by assumption, it is immediate to check that $\dot{p} \in W^{1,1}_{loc}(\mathbb{R}_+)$. Hence the map $\dot{p}$ is differentiable almost everywhere on $\mathbb{R}_+$ and we have

$$\ddot{p}(t) = \left(\frac{d^2u}{dt^2}(t), u(t) - v\right) + \left|\frac{du}{dt}(t)\right|^2 \quad \text{a.e. on } \mathbb{R}_+.$$

By combining the expressions of $\dot{p}$, $\ddot{p}$ and by using the convexity of the function Φ, we obtain

$$\begin{aligned} \ddot{p}(t) + \gamma(t)\dot{p}(t) &= a(u(t), v - u(t)) + (f(u(t), v - u(t)) + \left|\frac{du}{dt}(t)\right|^2 \\ &\leq -\Phi(u(t)) + \left|\frac{du}{dt}(t)\right|^2 \quad \text{a.e. on } \mathbb{R}_+. \end{aligned} \tag{3.6}$$

It follows that

$$\ddot{p}(t) + \gamma(t)\dot{p}(t) + \mathcal{E}(t) \leq \frac{3}{2}\left|\frac{du}{dt}(t)\right|^2 \quad \text{a.e. on } \mathbb{R}_+. \tag{3.7}$$

Let us multiply this inequality by $\gamma(t)$ and integrate on $[0, t]$. By using the fact that $\dot{\mathcal{E}}(t) = -\gamma(t)\left|\frac{du}{dt}(t)\right|^2$ almost everywhere on $\mathbb{R}_+$, we derive that

$$\int_0^t \gamma(s)\,\mathcal{E}(s)\,ds \leq \frac{3}{2}\left(\mathcal{E}(0) - \mathcal{E}(t)\right) - \int_0^t \gamma(s)\ddot{p}(s)ds - \int_0^t \gamma(s)^2\dot{p}(s)ds. \tag{3.8}$$

Then, remark that

$$\int_0^t \gamma(s)\ddot{p}(s)ds = \gamma(t)\dot{p}(t) - \gamma(0)\dot{p}(0) - \int_0^t \dot{\gamma}(s)\dot{p}(s)ds.$$

Recall that the map u is bounded in H by assumption. On the other hand, the energy function $\mathcal{E}$ is nonincreasing hence majorized. We deduce that the map $\frac{du}{dt}$ is bounded in H. Hence we infer the existence of $M > 0$ such that $p(t) \leq M$ and $|\dot{p}(t)| \leq M$ for every $t \geq 0$. Therefore

$$\left|\int_0^t \gamma(s)\ddot{p}(s)ds\right| \leq M\gamma(t) + M\gamma(0) + M\int_0^t |\dot{\gamma}(s)|\,ds.$$

Since $\dot{\gamma} \in L^1(0, +\infty)$ by assumption, the right-hand side is majorized by some $M' \geq 0$. In the same way, there exists $M'' \geq 0$ such that $\left|\int_0^t \gamma(s)^2\dot{p}(s)ds\right| \leq M''$ for every $t \geq 0$. The expected estimate is then a consequence of inequality (3.8).
(ii) Let us argue by contradiction and assume that $\lim_{t\to+\infty}\mathcal{E}(t) = l > 0$. The map $\mathcal{E}$ is nonincreasing, hence $\mathcal{E}(t) \geq l$ for every $t \geq 0$. Since $\gamma \notin L^1(0, +\infty)$, we deduce that

$$\int_0^{+\infty} \gamma(t)\,\mathcal{E}(t)\,dt \geq l\int_0^{+\infty} \gamma(t)\,dt = +\infty,$$

a contradiction with the result of (i). The last assertion is immediate. □

In view of the previous result, we can prove weak convergence of the trajectories in the case of a unique equilibrium. The general case of multiple equilibria is more delicate and will be discussed in section 3.2.

Corollary 3.6 (Case of a unique equilibrium)**.** *Under the hypotheses of Proposition 3.5, assume moreover that* $\operatorname{argmin}\Phi = \{\overline{u}\}$ *for some* $\overline{u} \in V$*. Then the solution* $u(t)$ *weakly converges in* V *toward* $\overline{u}$ *as* $t \to +\infty$*. Furthermore, if* $u(t)$ *strongly converges*[7] *in* H *then it strongly converges in* V*.*

Proof. By assumption, the solution u is bounded in H. In view of hypothesis (h_2) and Remark 3.4, it is also bounded in V. Hence there exist $u_\infty \in V$ and a subsequence (t_n) tending to $+\infty$ such that $\lim_{n\to+\infty} u(t_n) = u_\infty$ weakly in V. Since Φ is convex and continuous for the strong topology of V, it is lower semicontinuous for the weak topology of V. Hence, we have $\Phi(u_\infty) \leq \liminf_{n\to+\infty} \Phi(u(t_n))$. From the second part of (3.5) we deduce that $\Phi(u_\infty) \leq 0$, *i.e.* $u_\infty \in \operatorname{argmin}\Phi = \{\overline{u}\}$. Hence $\overline{u}$ is the unique limit point of the map $t \mapsto u(t)$ as $t \to +\infty$ for the weak topology of V. It ensues that $\lim_{t\to+\infty} u(t) = \overline{u}$ weakly in V. Let us now prove the second point. The argument is given in [3, p. 548-549] but we recall it for the sake of completeness. From (h_2), we have

$$\begin{aligned} \mu \, \|u(t) - \overline{u}\|^2 &\leq \lambda \, |u(t) - \overline{u}|^2 + a(u(t) - \overline{u}, u(t) - \overline{u}) \\ &= \lambda \, |u(t) - \overline{u}|^2 + 2\,\Phi(u(t)) - 2\,F(u(t)) - 2\,a(u(t), \overline{u}) + a(\overline{u}, \overline{u}). \end{aligned} \tag{3.9}$$

Since $u(t) \to \overline{u}$ strongly in H and weakly in V, we have $\lim_{t\to+\infty} |u(t) - \overline{u}|^2 = 0$ and $\lim_{t\to+\infty} a(u(t), \overline{u}) = a(\overline{u}, \overline{u})$. On the other hand, by weak lower semicontinuity of the continuous convex function $F : V \to \mathbb{R}$, we infer that $\liminf_{t\to+\infty} F(u(t)) \geq F(\overline{u})$. Recalling finally property (3.5), we deduce from inequality (3.9) that

$$\mu \limsup_{t\to+\infty} \|u(t) - \overline{u}\|^2 \leq -2\,F(\overline{u}) - a(\overline{u}, \overline{u}) = 0.$$

We conclude that $u(t) \to \overline{u}$ strongly in V. □

3.2. **Convergence of the trajectories.** When the damping coefficient $\gamma(t)$ is constant, *i.e.* $\gamma(t) \equiv \gamma$, the solutions of (E) weakly converge in V toward an equilibrium point, see [3]. We are going to show that this property still holds true if the quantity $\gamma(t)$ behaves as k/t^α, for some $\alpha \in]0, 1[$, $k > 0$ and t large enough.

The first step consists in establishing an improved version of Proposition 3.5. The corresponding estimates are obtained by strengthening the assumptions on the map γ.

Proposition 3.7. *Assume that the bilinear form* $a(.,.)$ *and the function* f *satisfy respectively hypotheses* (h_1)*-*(h_2) *and* (k_1)*-*(k_3)*. Let* $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ *be a function satisfying* (l_1)*. Assume that there exists* $t_0 > 0$ *such that* $\gamma(t) \geq \frac{3}{t}$ *for every* $t \geq t_0$ *and that* $\displaystyle\int_0^{+\infty} t^{1-\left(\frac{1}{2}\right)^n} |\dot{\gamma}(t)| \, dt < +\infty$ *for some* $n \in \mathbb{N}$*. Let* u *be a solution in the class (1.4) to equation* (E) *and assume that* $u \in L^\infty(\mathbb{R}_+, H)$*. Then we have*

(i) $\displaystyle\int_0^{+\infty} t^{1-\left(\frac{1}{2}\right)^n} \mathcal{E}(t) \, dt < +\infty.$

(ii) $\displaystyle\lim_{t\to+\infty} t^{2-\left(\frac{1}{2}\right)^n} \mathcal{E}(t) = 0.$

(iii) $\displaystyle\int_0^{+\infty} t^{2-\left(\frac{1}{2}\right)^n} \gamma(t) \left|\frac{du}{dt}(t)\right|^2 dt < +\infty.$

[7] This assumption is satisfied if the injection $V \hookrightarrow H$ is compact.

If moreover there exists $k > 0$ such that $\gamma(t) \geq \frac{k}{t^{1-\left(\frac{1}{2}\right)^{n+1}}}$ for t large enough, then

(iv) $\displaystyle\int_0^{+\infty} t^{1-\left(\frac{1}{2}\right)^{n+1}} \left|\frac{du}{dt}(t)\right|^2 dt < +\infty.$

Proof. It is divided into two steps.
Step A. First we establish a preliminary result that will be used recursively in the second step. We assume that there exist $\theta \in [0,1[$ and $k > 0$ such that $\left|\frac{du}{dt}(t)\right| \leq \frac{k}{t^\theta}$ for every $t > 0$. Suppose moreover that $\int_0^{+\infty} t^\theta \left|\dot{\gamma}(t)\right| dt < +\infty$. Let us consider the map p defined by $p(t) = \frac{1}{2}\left|u(t) - v\right|^2$ for some $v \in S$, see the proof of Proposition 3.5. Recall that we have from inequality (3.7)

$$\mathcal{E}(t) \leq \frac{3}{2}\left|\frac{du}{dt}(t)\right|^2 - \ddot{p}(t) - \gamma(t)\,\dot{p}(t) \quad \text{a.e. on } \mathbb{R}_+. \tag{3.10}$$

Now define the map $\mathcal{E}_\theta : \mathbb{R}_+ \to \mathbb{R}_+$ by $\mathcal{E}_\theta(t) = t^{1+\theta}\,\mathcal{E}(t)$. It is clear that $\mathcal{E}_\theta \in W^{1,1}_{loc}(\mathbb{R}_+)$. Since $\dot{\mathcal{E}}(t) = -\gamma(t)\left|\frac{du}{dt}(t)\right|^2$ for almost every $t \geq 0$, we have

$$\dot{\mathcal{E}}_\theta(t) = (1+\theta)\,t^\theta\,\mathcal{E}(t) - t^{1+\theta}\,\gamma(t)\left|\frac{du}{dt}(t)\right|^2 \quad \text{a.e. on } \mathbb{R}_+. \tag{3.11}$$

From the assumption $\gamma(t) \geq \frac{3}{t}$ for every $t \geq t_0$, we deduce that

$$t^\theta\left|\frac{du}{dt}(t)\right|^2 \leq \frac{1}{3}(1+\theta)\,t^\theta\,\mathcal{E}(t) - \frac{1}{3}\dot{\mathcal{E}}_\theta(t) \quad \text{a.e. on } [t_0, +\infty[. \tag{3.12}$$

By combining inequalities (3.10) and (3.12), we infer that

$$\frac{1}{2}(1-\theta)\,t^\theta\,\mathcal{E}(t) \leq -\frac{1}{2}\dot{\mathcal{E}}_\theta(t) - t^\theta\,\ddot{p}(t) - t^\theta\,\gamma(t)\,\dot{p}(t) \quad \text{a.e. on } [t_0, +\infty[.$$

Let us integrate this inequality on $[t_0, t]$; we find

$$\frac{1}{2}(1-\theta)\int_{t_0}^t s^\theta\,\mathcal{E}(s)ds \leq \frac{1}{2}\mathcal{E}_\theta(t_0) - \int_{t_0}^t s^\theta\,\ddot{p}(s)\,ds - \int_{t_0}^t s^\theta\,\gamma(s)\,\dot{p}(s)\,ds. \tag{3.13}$$

For the last two integrals, let us use a technique of integration by parts.

$$\begin{aligned} -\int_{t_0}^t s^\theta\,\ddot{p}(s)\,ds &= -t^\theta\,\dot{p}(t) + t_0^\theta\,\dot{p}(t_0) + \theta\int_{t_0}^t s^{\theta-1}\,\dot{p}(s)\,ds \\ &= -t^\theta\,\dot{p}(t) + t_0^\theta\,\dot{p}(t_0) + \theta\,t^{\theta-1}\,p(t) - \theta\,t_0^{\theta-1}\,p(t_0) - \theta\,(\theta-1)\int_{t_0}^t s^{\theta-2}\,p(s)\,ds. \end{aligned}$$

The map u is bounded in H by assumption, hence there exist M, $M' > 0$ such that $p(t) \leq M$ and $|\dot{p}(t)| \leq M'\left|\frac{du}{dt}(t)\right|$ for every $t \geq 0$. Therefore we deduce from the above equality that

$$-\int_{t_0}^t s^\theta\,\ddot{p}(s)\,ds \leq M'\,t^\theta\left|\frac{du}{dt}(t)\right| + M'\,t_0^\theta\left|\frac{du}{dt}(t_0)\right| + \theta\,M\,t^{\theta-1} + \theta\,(1-\theta)\,M\int_{t_0}^t s^{\theta-2}\,ds.$$

By using the assumption $\left|\frac{du}{dt}(t)\right| \leq \frac{k}{t^\theta}$ for every $t > 0$, we obtain

$$-\int_{t_0}^t s^\theta\,\ddot{p}(s)\,ds \leq 2\,k\,M' + \theta\,M\,t^{\theta-1} + \theta\,M\,(t_0^{\theta-1} - t^{\theta-1}) = 2\,k\,M' + \theta\,M\,t_0^{\theta-1}. \tag{3.14}$$

On the other hand, we have

$$\begin{aligned} -\int_{t_0}^{t} s^\theta\,\gamma(s)\,\dot p(s)\,ds &= -t^\theta\,\gamma(t)\,p(t) + t_0^\theta\,\gamma(t_0)\,p(t_0) + \theta\int_{t_0}^{t} s^{\theta-1}\,\gamma(s)\,p(s)\,ds + \int_{t_0}^{t} s^\theta\,\dot\gamma(s)\,p(s)\,ds \\ &\le M\,t_0^\theta\,\gamma(t_0) + \theta\,M\int_{t_0}^{t} s^{\theta-1}\,\gamma(s)\,ds + M\int_{t_0}^{t} s^\theta\,|\dot\gamma(s)|\,ds. \end{aligned}$$

From the assumption $\int_0^{+\infty} s^\theta\,|\dot\gamma(s)|\,ds < +\infty$ and Lemma 3.8 (i) below[8], we deduce that

$$-\int_{t_0}^{t} s^\theta\,\gamma(s)\,\dot p(s)\,ds \le M\,t_0^\theta\,\gamma(t_0) + \theta\,M\int_{t_0}^{+\infty} s^{\theta-1}\,\gamma(s)\,ds + M\int_{t_0}^{+\infty} s^\theta\,|\dot\gamma(s)|\,ds < +\infty. \tag{3.15}$$

By combining inequalities (3.13), (3.14) and (3.15), we conclude that the quantity $\int_{t_0}^{t} s^\theta\,\mathcal{E}(s)ds$ is uniformly majorized with respect to t, hence

$$\int_0^{+\infty} s^\theta\,\mathcal{E}(s)\,ds < +\infty. \tag{3.16}$$

Lemma 3.8. *Let $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+,\mathbb{R}_+)$ be a map satisfying (l_1). For every $\theta > 0$, we have*

(i) *If $\int_0^{+\infty} t^\theta\,|\dot\gamma(t)|\,dt < +\infty$, then $\int_0^{+\infty} t^{\theta-1}\,\gamma(t)\,dt < +\infty$.*

(ii) *If the map γ is nonincreasing and if $\int_0^{+\infty} t^{\theta-1}\,\gamma(t)\,dt < +\infty$, then $\int_0^{+\infty} t^\theta\,|\dot\gamma(t)|\,dt < +\infty$.*

The proof of Lemma 3.8 is postponed to the appendix. Let us now come back to equation (3.11). By taking the positive part of each member, we find $(\dot{\mathcal{E}}_\theta)_+(t) \le (1+\theta)\,t^\theta\,\mathcal{E}(t)$. This implies that $(\dot{\mathcal{E}}_\theta)_+ \in L^1(0,+\infty)$ and therefore $l = \lim_{t\to+\infty} t^{1+\theta}\,\mathcal{E}(t)$ exists in $\mathbb{R}_+$. If $l > 0$, this implies that $t^\theta\,\mathcal{E}(t) \sim l/t$ as $t \to +\infty$, a contradiction with estimate (3.16). Hence we have

$$\lim_{t\to+\infty} t^{1+\theta}\,\mathcal{E}(t) = 0. \tag{3.17}$$

Finally, by integrating equality (3.11) on $[0,t]$, we obtain

$$\begin{aligned} \int_0^t s^{1+\theta}\,\gamma(s)\left|\frac{du}{ds}(s)\right|^2 ds &= (1+\theta)\int_0^t s^\theta\,\mathcal{E}(s)\,ds + \mathcal{E}_\theta(0) - \mathcal{E}_\theta(t) \\ &\le (1+\theta)\int_0^{+\infty} s^\theta\,\mathcal{E}(s)\,ds + \mathcal{E}_\theta(0) < +\infty, \end{aligned}$$

hence

$$\int_0^{+\infty} s^{1+\theta}\,\gamma(s)\left|\frac{du}{ds}(s)\right|^2 ds < +\infty. \tag{3.18}$$

<u>*Step B.*</u> The function $\mathcal{E}$ is nonincreasing, hence majorized. We deduce the existence of $k_1 > 0$ such that $\left|\frac{du}{dt}(t)\right| \le k_1$ for every $t \ge 0$. Since $\int_0^{+\infty} |\dot\gamma(t)|\,dt < +\infty$, we can apply the result of Step A with $\theta = 0$. We then obtain from assertion (3.17) that $\lim\limits_{t\to+\infty} t\,\mathcal{E}(t) = 0$. Hence there exists $k_2 > 0$ such that $\left|\frac{du}{dt}(t)\right| \le \frac{k_2}{t^{1/2}}$ for every $t > 0$. Since $\int_0^{+\infty} t^{1/2}\,|\dot\gamma(t)|\,dt < +\infty$ as soon as $n \ge 1$, we can apply the result of Step A with $\theta = \frac{1}{2}$. We obtain in particular that $\lim\limits_{t\to+\infty} t^{3/2}\,\mathcal{E}(t) = 0$. By iterating

[8] Lemma 3.8 applies only for $\theta > 0$. When $\theta = 0$, the corresponding term to be majorized in the above inequality is equal to 0.

the above arguments, we let the reader check that $\lim_{t\to+\infty} t^{2-\left(\frac{1}{2}\right)^{n-1}}\mathcal{E}(t) = 0$. This implies the existence of $k_{n+1} > 0$ such that $\left|\frac{du}{dt}(t)\right| \leq \frac{k_{n+1}}{t^{1-\left(\frac{1}{2}\right)^n}}$ for every $t > 0$. Since $\int_0^{+\infty} t^{1-\left(\frac{1}{2}\right)^n}|\dot{\gamma}(t)|\,dt < +\infty$ by assumption, assertions (3.16), (3.17) and (3.18) applied with $\theta = 1-\left(\frac{1}{2}\right)^n$ respectively yield conclusions (i), (ii) and (iii). Finally, by combining (iii) with the additional assumption $\gamma(t) \geq \frac{k}{t^{1-\left(\frac{1}{2}\right)^{n+1}}}$, we immediately find (iv). □

In the sequel, we denote by (l_4) the following condition

$$(l_4) \qquad \int_0^{+\infty} t^{1-\left(\frac{1}{2}\right)^n}|\dot{\gamma}(t)|\,dt < +\infty \qquad \text{and} \qquad \forall t \geq t_0, \quad \gamma(t) \geq \frac{k}{t^{1-\left(\frac{1}{2}\right)^{n+1}}},$$

for some $n \in \mathbb{N}$, $k > 0$ and $t_0 > 0$. Hypothesis (l_4) automatically implies (l_2) together with (l_3).

Remark 3.9. Assume that the map $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing. If the integer n arising in (l_4) is equal to 0, condition $\int_0^{+\infty} t^{1-\left(\frac{1}{2}\right)^n}|\dot{\gamma}(t)|\,dt < +\infty$ is automatically satisfied. If $n \geq 1$, we deduce from Lemma 3.8 that $\int_0^{+\infty} t^{1-\left(\frac{1}{2}\right)^n}|\dot{\gamma}(t)|\,dt$ converges if and only if $\int_0^{+\infty} \frac{\gamma(t)}{t^{\left(\frac{1}{2}\right)^n}}\,dt$ is finite. This last condition is realized if there exist $\theta > 1-\left(\frac{1}{2}\right)^n$ and $k' > 0$ such that $\gamma(t) \leq \frac{k'}{t^\theta}$ for t large enough. It can be easily seen that condition (l_4) is satisfied if the following assertion holds true

$$\forall t \geq t_0, \quad \frac{k}{t^\alpha} \leq \gamma(t) \leq \frac{k'}{t^\alpha}, \qquad \text{for some } \alpha \in]0,1[\text{ and } k, k' > 0.$$

The integer n is then uniquely[9] defined by the inclusion $\alpha \in \left]1-\left(\frac{1}{2}\right)^n, 1-\left(\frac{1}{2}\right)^{n+1}\right]$. Notice that if $\alpha \in \left]0, \frac{1}{2}\right]$, we have $n = 0$ and one may take $k' = +\infty$ (no required upper bound).

Let us now state the main result of this section.

Theorem 3.10. *Assume that the bilinear form $a(.,.)$ and the function f satisfy respectively (h_1)-(h_2) and (k_1)-(k_3). Let $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ be a map satisfying (l_1) and (l_4). Let u be a solution in the class (1.4) to equation (E) and assume that $u \in L^\infty(\mathbb{R}_+, H)$. Then, there exists $u_\infty \in S$ such that $u(t) \rightharpoonup u_\infty$ weakly in V as $t \to +\infty$. Furthermore, if $u(t)$ strongly converges[10] in H then it strongly converges in V.*

Proof. Let $v \in S$ and define the map $p : \mathbb{R}_+ \to \mathbb{R}_+$ by $p(t) = \frac{1}{2}|u(t) - v|^2$ as in the proof of Proposition 3.5. Inequality (3.6) implies that

$$\ddot{p}(t) + \gamma(t)\dot{p}(t) \leq \left|\frac{du}{dt}(t)\right|^2 \qquad \text{a.e. on } \mathbb{R}_+.$$

[9] Its explicit expression is given by $n = -\left[\frac{\ln(1-\alpha)}{\ln 2}\right] - 1$, where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

[10] This assumption is satisfied if the injection $V \hookrightarrow H$ is compact.

Let us multiply each member of this inequality by $e^{\int_0^t \gamma(\tau)\,d\tau}$ and integrate on $[0,t]$. Recalling that $\dot{p} \in W^{1,1}_{loc}(\mathbb{R}_+)$, we obtain

$$\dot{p}(t) \leq e^{-\int_0^t \gamma(\tau)\,d\tau}\dot{p}(0) + e^{-\int_0^t \gamma(\tau)\,d\tau}\int_0^t e^{\int_0^s \gamma(\tau)\,d\tau}\left|\frac{du}{ds}(s)\right|^2 ds. \tag{3.19}$$

We now show that the right member of the above inequality is a summable function. From Lemma 3.11 (i) below applied with $\theta = 1 - \left(\frac{1}{2}\right)^{n+1}$, we have

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau)\,d\tau}\,dt < +\infty. \tag{3.20}$$

Lemma 3.11. *Let us assume that there exist $\theta \in]0,1[$, $k > 0$ and $t_0 > 0$ such that $\gamma(t) \geq \frac{k}{t^\theta}$ for every $t \geq t_0$. Then we have*

(i) $\displaystyle\int_0^{+\infty} e^{-\int_0^t \gamma(\tau)\,d\tau}\,dt < +\infty$;

(ii) $\displaystyle\int_s^{+\infty} e^{-\int_0^t \gamma(\tau)\,d\tau}\,dt \leq \frac{2}{k}\,s^\theta\,e^{-\int_0^s \gamma(\tau)\,d\tau}$ *for s large enough.*

The proof of Lemma 3.11 is postponed to the appendix. On the other hand, by applying Fubini theorem, we find

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau)\,d\tau}\int_0^t e^{\int_0^s \gamma(\tau)\,d\tau}\left|\frac{du}{ds}(s)\right|^2 ds\,dt = \int_0^{+\infty}\left|\frac{du}{ds}(s)\right|^2 e^{\int_0^s \gamma(\tau)\,d\tau}\int_s^{+\infty} e^{-\int_0^t \gamma(\tau)\,d\tau}\,dt\,ds. \tag{3.21}$$

In view of Lemma 3.11 (ii) applied with $\theta = 1 - \left(\frac{1}{2}\right)^{n+1}$, this implies that

$$\left|\frac{du}{ds}(s)\right|^2 e^{\int_0^s \gamma(\tau)\,d\tau}\int_s^{+\infty} e^{-\int_0^t \gamma(\tau)\,d\tau}\,dt \leq \frac{2}{k}\,s^{1-\left(\frac{1}{2}\right)^{n+1}}\left|\frac{du}{ds}(s)\right|^2.$$

Since $\displaystyle\int_0^{+\infty} s^{1-\left(\frac{1}{2}\right)^{n+1}}\left|\frac{du}{ds}(s)\right|^2 ds < +\infty$ in view of Proposition 3.7 (iv), we deduce from equality (3.21) that

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau)\,d\tau}\int_0^t e^{\int_0^s \gamma(\tau)\,d\tau}\left|\frac{du}{ds}(s)\right|^2 ds\,dt < +\infty. \tag{3.22}$$

By combining inequality (3.19) with estimates (3.20) and (3.22), we infer that $[\dot{p}]_+ \in L^1(0,+\infty)$ and hence $\lim\limits_{t\to+\infty} p(t)$ exists. The end of the proof is the same as in [3, Theorem 3.1] but the arguments are given for the sake of completeness. Since $u \in L^\infty(\mathbb{R}_+, H)$ by assumption, we deduce from hypothesis (h_2) and Remark 3.4 that $u \in L^\infty(\mathbb{R}_+, V)$. Let $\overline{u} \in V$ be a weak cluster point of $\{u(t); t \to +\infty\}$ for the weak topology of V. There exists a sequence $t_n \to +\infty$ such that $u(t_n) \rightharpoonup \overline{u}$ weakly in V as $n \to +\infty$. Since the function Φ is lower semicontinuous for the weak topology of V, we have in view of assertion (3.5)

$$\Phi(\overline{u}) \leq \liminf_{n\to+\infty}\Phi(u(t_n)) = \lim_{t\to+\infty}\Phi(u(t)) = 0,$$

which implies that $\overline{u} \in S$. Let us prove that $\{u(t); t \to +\infty\}$ has a unique cluster point for the weak topology in V. We apply the following argument due to Opial [18]. Let $\overline{u}_1, \overline{u}_2 \in S$ be two cluster points of $\{u(t); t \to +\infty\}$ for the weak topology of V. According to the first part of the proof, we can assert that $\lim_{t\to+\infty}|u(t) -$

$\overline{u}_i|^2 = l_i$ exists for each $i = 1, 2$. Moreover there exists a sequence $t_n \to +\infty$ such that $u(t_n) \rightharpoonup \overline{u}_1$ weakly in V as $n \to +\infty$. Since the injection $V \hookrightarrow H$ is continuous, $u(t_n) \rightharpoonup \overline{u}_1$ weakly in H as $n \to +\infty$. From the equality

$$|u(t) - \overline{u}_1|^2 - |u(t) - \overline{u}_2|^2 = |\overline{u}_1 - \overline{u}_2|^2 + 2(\overline{u}_1 - \overline{u}_2, \overline{u}_2 - u(t)),$$

we infer that $l_1 - l_2 = -|\overline{u}_1 - \overline{u}_2|^2$. On the other hand, if we take $t_m \to +\infty$ such that $u(t_m) \rightharpoonup \overline{u}_2$ weakly in V as $m \to +\infty$, we find $l_1 - l_2 = |\overline{u}_1 - \overline{u}_2|^2$. As a consequence, $|\overline{u}_1 - \overline{u}_2|^2 = 0$. This establishes the uniqueness of the cluster points of $\{u(t); t \to +\infty\}$ for the weak topology of V. Hence $u(t) \rightharpoonup u_\infty$ weakly in V as $t \to +\infty$ for some $u_\infty \in V$.

For the second point, the reader is referred to the corresponding argument in the proof of Corollary 3.6. □

In view of Remark 3.9, we obtain directly the following corollary of Theorem 3.10.

Corollary 3.12. *Assume that the bilinear form $a(.,.)$ and the function f satisfy the same hypotheses as in Theorem 3.10. Let $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ be a nonincreasing map and suppose that there exist $\alpha \in]0, 1[$, $k, k' > 0$ and $t_0 > 0$ such that*[11]

$$\forall t \geq t_0, \quad \frac{k}{t^\alpha} \leq \gamma(t) \leq \frac{k'}{t^\alpha}.$$

Then we have the same conclusions as in Theorem 3.10.

An interesting situation ensuring strong convergence in V is the case where the non-linearity satisfies the symmetry property $F(-u) = F(u)$ for all $u \in V$.

Theorem 3.13. *Under the hypotheses of Theorem 3.10, assume moreover that the function F is even, i.e. $F(-u) = F(u)$ for all $u \in V$. Then there exists $u_\infty \in S$ such that $u(t) \to u_\infty$ strongly in V.*

Proof. The argument was originated by Bruck, see [6, Theorem 5]. It has been adapted to the framework of second-order in time equations, see for example [2, Theorem 2.4 (i)] or [3, Remark 3.2] in the case of a constant damping parameter γ. Let us fix $t_0 > 0$ and define the map $q : [0, t_0] \to \mathbb{R}$ by

$$q(t) = |u(t)|^2 - |u(t_0)|^2 - \frac{1}{2}|u(t) - u(t_0)|^2.$$

A first differentiation gives for all $t \in [0, t_0]$

$$\dot{q}(t) = \left(\frac{du}{dt}(t), u(t) + u(t_0)\right).$$

Since $\frac{du}{dt} \in W^{1,1}_{loc}(\mathbb{R}_+, H)$ by assumption, it is immediate to check that the map $\dot{q}$ is absolutely continuous, hence differentiable almost everywhere on $[0, t_0]$ and we have

$$\ddot{q}(t) = \left(\frac{d^2u}{dt^2}(t), u(t) + u(t_0)\right) + \left|\frac{du}{dt}(t)\right|^2 \quad \text{a.e. on } [0, t_0].$$

[11] This condition is satisfied if there exists $k'' > 0$ such that $\gamma(t) \sim \frac{k''}{t^\alpha}$ as $t \to +\infty$. On the other hand, one can take $k' = +\infty$ if $\alpha \in \left]0, \frac{1}{2}\right]$, see Remark 3.9.

By combining the expressions of $\dot{q}$, $\ddot{q}$, we obtain for almost every $t \in [0, t_0]$

$$\begin{aligned} \ddot{q}(t) + \gamma(t)\dot{q}(t) &= -a(u(t), u(t) + u(t_0)) - (f(u(t)), u(t) + u(t_0)) + \left|\frac{du}{dt}(t)\right|^2 \\ &= -\langle \Phi'(u(t)), u(t) + u(t_0)\rangle_{V',V} + \left|\frac{du}{dt}(t)\right|^2. \end{aligned} \tag{3.23}$$

Since the function Φ is convex and even, we have for all $u, v \in V$

$$\Phi(v) - \Phi(u) = \Phi(-v) - \Phi(u) \geq -\langle \Phi'(u), v + u\rangle_{V',V}.$$

Hence inequality (3.23) gives

$$\ddot{q}(t) + \gamma(t)\dot{q}(t) \leq \Phi(u(t_0)) - \Phi(u(t)) + \left|\frac{du}{dt}(t)\right|^2 \quad \text{a.e. on } [0, t_0]. \tag{3.24}$$

Recalling that the energy function $\mathcal{E}(t)$ is nonincreasing, we have $\frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \Phi(u(t)) \geq \left|\frac{du}{dt}(t_0)\right|^2 + \Phi(u(t_0))$ for every $t \in [0, t_0]$. Therefore

$$\forall t \in [0, t_0], \quad \Phi(u(t_0)) - \Phi(u(t)) \leq \frac{1}{2}\left|\frac{du}{dt}(t)\right|^2.$$

Using inequality (3.24), we deduce that

$$\ddot{q}(t) + \gamma(t)\dot{q}(t) \leq \frac{3}{2}\left|\frac{du}{dt}(t)\right|^2 \quad \text{a.e. on } [0, t_0].$$

Let us multiply each member of this inequality by $e^{\int_0^t \gamma(\tau)\,d\tau}$ and integrate on $[0, t]$. Since the map $\dot{q}$ is absolutely continuous, we find

$$\dot{q}(t) \leq e^{-\int_0^t \gamma(\tau)\,d\tau}\dot{q}(0) + \frac{3}{2}e^{-\int_0^t \gamma(\tau)\,d\tau}\int_0^t e^{\int_0^s \gamma(\tau)\,d\tau}\left|\frac{du}{ds}(s)\right|^2 ds.$$

Let us integrate this inequality on $[t, t_0]$, we obtain

$$-q(t) \leq \dot{q}(0)\int_t^{t_0} e^{-\int_0^s \gamma(\tau)\,d\tau}\,ds + \frac{3}{2}(h(t_0) - h(t)),$$

where we have set

$$h(t) = \int_0^t e^{-\int_0^s \gamma(\tau)\,d\tau}\int_0^s e^{\int_0^\sigma \gamma(\tau)\,d\tau}\left|\frac{du}{dt}(\sigma)\right|^2 d\sigma\,ds.$$

We deduce from the previous inequality that

$$\frac{1}{2}|u(t) - u(t_0)|^2 \leq |u(t)|^2 - |u(t_0)|^2 + \dot{q}(0)\int_t^{t_0} e^{-\int_0^s \gamma(\tau)\,d\tau}\,ds + \frac{3}{2}(h(t_0) - h(t)). \tag{3.25}$$

In the proof of Theorem 3.10, we showed that $\lim_{t\to+\infty}|u(t) - v|^2$ exists for all $v \in \operatorname{argmin}\Phi$. Since Φ is convex and even, we have $0 \in \operatorname{argmin}\Phi$, hence $\lim_{t\to+\infty}|u(t)|^2$ exists. On the other hand, from Lemma 3.11 (i) applied with $\theta = 1 - \left(\frac{1}{2}\right)^{n+1}$, we have $\int_0^{+\infty} e^{-\int_0^s \gamma(\tau)\,d\tau}\,ds < +\infty$. Finally, in view of estimate (3.22), we can assert that $\lim_{t\to+\infty} h(t)$ exists. We then deduce from inequality (3.25) that $\{u(t); t \to +\infty\}$

is a Cauchy net in H hence strongly converges in H. It suffices to use the second part of Theorem 3.10 to obtain the strong convergence in V. □

Corollary 3.14 (Linear case)**.** *Assume that the bilinear form $a(.,.)$ satisfies (h_1)-(h_2) and take $f = 0$. Let $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ be a map satisfying (l_1) and (l_4). Let u be a solution in the class (1.4) to equation (E) and assume that $u \in L^\infty(\mathbb{R}_+, H)$. Then, there exists $u_\infty \in \ker A$ such that $u(t) \to u_\infty$ strongly in V as $t \to +\infty$.*

Proof. Use Theorem 3.13 with $F = 0$. □

3.3. **Decay estimates for a strong set of minima.** Recall that the set $S = \operatorname{argmin} \Phi$ is convex and closed in H, see Remark 3.2. Let us denote by P_S the projection operator onto the set S in the sense of H. In this paragraph, we assume that the function $\Phi : V \to \mathbb{R}$ satisfies[12]

$$\exists \eta > 0 \quad \text{such that} \quad \forall u \in V, \quad \Phi(u) \geq \frac{\eta}{2}|u - P_S(u)|^2. \tag{3.26}$$

If $\gamma \notin L^1(0, +\infty)$, we know from Proposition 3.5 (ii) that $\lim_{t\to+\infty} \mathcal{E}(t) = 0$. Under assumption (3.26), we are able to evaluate the speed of convergence of $\mathcal{E}(t)$ as $t \to +\infty$.

Theorem 3.15. *Assume that the bilinear form $a(.,.)$ and the function f satisfy respectively (h_1)-(h_2) and (k_1)-(k_3). Let $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ be a function satisfying (l_1) and $\dot\gamma(t) = o(\gamma(t))$ as $t \to +\infty$. We suppose that the function $\Phi : V \to \mathbb{R}$ defined by $\Phi(u) = \frac{1}{2}a(u,u) + F(u)$ satisfies condition (3.26). Let u be a solution in the class (1.4) to equation (E). Then, for all $m \in]0, \frac{2}{3}[$, there exist $C > 0$ and $t_0 \geq 0$ such that:*

$$\forall t \geq t_0, \quad \mathcal{E}(t) \leq Ce^{-m\int_0^t \gamma(s)ds}.$$

Proof. Define the map $\varphi : \mathbb{R}_+ \to \mathbb{R}$ by $\varphi(t) = \frac{1}{2}d_H^2(u(t), S)$, where $d_H(., S)$ stands for the distance function from the set S in the sense of H. By differentiating, we find for every $t \geq 0$

$$\dot\varphi(t) = \left(\frac{du}{dt}(t), u(t) - P_S(u(t))\right). \tag{3.27}$$

Since $\frac{du}{dt} \in W^{1,1}_{loc}(\mathbb{R}_+, H)$ by assumption, it is immediate to check that $\dot\varphi \in W^{1,1}_{loc}(\mathbb{R}_+)$, hence the map $\dot\varphi$ is differentiable almost everywhere on $\mathbb{R}_+$. Consider now some $t > 0$ where the maps $\dot\varphi$ and $\frac{du}{dt}$ are both differentiable, and let us majorize the quantity $\ddot\varphi(t)$. For that purpose, we use a technique of differential quotient. For all $h \neq 0$, we have

$$\begin{aligned}\frac{1}{h}(\dot\varphi(t+h) - \dot\varphi(t)) \quad &= \quad \frac{1}{h}\left(\frac{du}{dt}(t), u(t+h) - P_S(u(t+h)) - u(t) + P_S(u(t))\right) \\ &\quad + \frac{1}{h}\left(\frac{du}{dt}(t+h) - \frac{du}{dt}(t), u(t+h) - P_S(u(t+h))\right).\end{aligned}$$

The monotonicity of P_S implies that

$$-\frac{1}{h}\left(\frac{du}{dt}(t), P_S(u(t+h)) - P_S(u(t))\right) \leq \frac{1}{h^2}\left(u(t+h) - u(t) - h\frac{du}{dt}(t), P_S(u(t+h)) - P_S(u(t))\right).$$

[12] If $f = 0$, the set S coincides with $\ker A$ and we recover condition (2.7) of section 2.

Hence we obtain

$$\begin{aligned}\frac{1}{h}(\dot\varphi(t+h)-\dot\varphi(t)) \quad &\leq \quad \frac{1}{h}\left(\frac{du}{dt}(t), u(t+h)-u(t)\right)\\ &\quad +\frac{1}{h^2}\left(u(t+h)-u(t)-h\frac{du}{dt}(t), P_S(u(t+h))-P_S(u(t))\right)\\ &\quad +\frac{1}{h}\left(\frac{du}{dt}(t+h)-\frac{du}{dt}(t), u(t+h)-P_S(u(t+h))\right).\end{aligned}$$

Taking the limit as $h \to 0$, we derive that

$$\ddot\varphi(t) \leq \left|\frac{du}{dt}(t)\right|^2 + \left(\frac{d^2u}{dt^2}(t), u(t)-P_S(u(t))\right). \tag{3.28}$$

By combining formulae (3.27) and (3.28), and using the convexity of the function Φ, we deduce that for almost every $t \in \mathbb{R}_+$

$$\begin{aligned}\ddot\varphi(t)+\gamma(t)\dot\varphi(t) \quad &\leq \quad \left|\frac{du}{dt}(t)\right|^2 + \left(\frac{d^2u}{dt^2}(t)+\gamma(t)\,\frac{du}{dt}(t), u(t)-P_S(u(t))\right)\\ &= \quad \left|\frac{du}{dt}(t)\right|^2 - a\,(u(t), u(t)-P_S(u(t))) - (f(u(t)), u(t)-P_S(u(t)))\\ &\leq \quad \left|\frac{du}{dt}(t)\right|^2 - \Phi(u(t)) + \Phi(P_S(u(t))) = \left|\frac{du}{dt}(t)\right|^2 - \Phi(u(t)).\end{aligned}$$

It follows that

$$\ddot\varphi(t)+\gamma(t)\dot\varphi(t)+\mathcal{E}(t) \leq \frac{3}{2}\left|\frac{du}{dt}(t)\right|^2 \qquad \text{a.e. on } \mathbb{R}_+.$$

Multiplying this formula by $\frac{2}{3}\gamma(t)$ and recalling that $\dot{\mathcal{E}}(t) = -\gamma(t)\left|\frac{du}{dt}(t)\right|^2$ for almost every $t \in \mathbb{R}_+$, we obtain

$$\frac{2}{3}\,\gamma(t)\,(\ddot\varphi(t)+\gamma(t)\dot\varphi(t)) + \dot{\mathcal{E}}(t) + \frac{2}{3}\,\gamma(t)\,\mathcal{E}(t) \leq 0 \qquad \text{a.e. on } \mathbb{R}_+. \tag{3.29}$$

This suggests to define the function $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{F}(t) \quad &= \quad \Phi(u(t)) + \frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \frac{2}{3}\gamma(t)\left(\frac{du}{dt}(t), u(t)-P_S(u(t))\right) \qquad (3.30)\\ &= \quad \mathcal{E}(t) + \frac{2}{3}\gamma(t)\,\dot\varphi(t).\end{aligned}$$

In view of inequality (3.29), we immediately find

$$\dot{\mathcal{F}}(t) + \frac{2}{3}\gamma(t)\mathcal{F}(t) \leq \frac{2}{3}\left(\dot\gamma(t) - \frac{1}{3}\gamma(t)^2\right)\left(\frac{du}{dt}(t), u(t)-P_S(u(t))\right) \qquad \text{a.e. on } \mathbb{R}_+. \tag{3.31}$$

Since $\left|\left(\frac{du}{dt}(t), u(t)-P_S(u(t))\right)\right| \leq \frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \frac{1}{2}\,|u(t)-P_S(u(t))|^2$ and $\Phi(u(t)) \geq \frac{\eta}{2}|u(t)-P_S(u(t))|^2$ by assumption, we have

$$\left|\left(\frac{du}{dt}(t), u(t)-P_S(u(t))\right)\right| \leq C\,\mathcal{E}(t), \qquad \text{for some } C > 0. \tag{3.32}$$

Recalling that $\lim_{t\to+\infty}\gamma(t)=0$, the expression of $\mathcal{F}$ shows that

$$\mathcal{F}(t)\sim\mathcal{E}(t)\quad \text{as } t\to+\infty. \tag{3.33}$$

Let us fix some $m\in]0,\frac{2}{3}[$. Using the fact that $\dot\gamma(t)=o(\gamma(t))$ and $\gamma(t)^2=o(\gamma(t))$ as $t\to+\infty$, we deduce from (3.31), (3.32) and (3.33) the existence of $t_0\geq 0$ such that,

$$\dot{\mathcal{F}}(t)+\frac{2}{3}\gamma(t)\mathcal{F}(t)\leq\left(\frac{2}{3}-m\right)\gamma(t)\mathcal{F}(t)\qquad \text{a.e. on } [t_0,+\infty[,$$

hence $\dot{\mathcal{F}}(t)+m\gamma(t)\mathcal{F}(t)\leq 0$ for almost every $t\geq t_0$. Let us multiply by $e^{m\int_0^t\gamma(s)ds}$ and integrate on $[t_0,t]$. Since the function $\mathcal{F}$ is absolutely continuous, we find $\mathcal{F}(t)\leq D\,e^{-m\int_0^t\gamma(s)ds}$, with $D=e^{m\int_0^{t_0}\gamma(s)ds}\,F(t_0)$. Conclusion follows from estimate (3.33). □

Remark 3.16. Under the hypotheses of Theorem 3.15, assume that there exists $k>3$ such that $\gamma(t)\geq\frac{k}{t}$ for t large enough. Fix $m\in\left]\frac{2}{k},\frac{2}{3}\right[$. From Theorem 3.15, there exist $C>0$ and $t_0\geq 0$ such that

$$\forall t\geq t_0,\qquad \frac{1}{2}\left|\frac{du}{dt}(t)\right|^2\leq\mathcal{E}(t)\leq\frac{C}{t^{mk}}.$$

Hence we have $\left|\frac{du}{dt}(t)\right|\leq\frac{(2C)^{1/2}}{t^{mk/2}}$ and since $mk>2$, we deduce that $\left|\frac{du}{dt}\right|\in L^1(0,+\infty)$. The trajectory u has a finite length, therefore it strongly converges in H toward some $u_\infty\in S$.

4. Application to particular semilinear evolution problems

We suppose that Ω is a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ sufficiently regular.

4.1. **Hyperbolic problems of order two in space.**

Example 4.1. Given a map $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ and a function $f\in\mathcal{C}^1(\mathbb{R})$, let us consider the following damped wave equation

$$\frac{d^2u}{dt^2}+\gamma(t)\,\frac{du}{dt}-\Delta u+f(u)=0\quad\text{on}\quad\Omega\times]0,+\infty[, \tag{4.1}$$

with Dirichlet boundary condition:

$$u=0\quad\text{on}\quad\partial\Omega\times]0,+\infty[. \tag{4.2}$$

The functional setting of the evolution problem (4.1)-(4.2) is given by

$$H=L^2(\Omega),\quad V=H_0^1(\Omega)\quad\text{and}\quad a(u,v)=\int_\Omega\nabla u(x)\nabla v(x)dx.$$

Hypothesis (h_1) is trivially verified while hypothesis (h_2) is satisfied with $\lambda=0$, since the bilinear form a is coercive. On the other hand, we assume that the function f satisfies the following properties:

(i) There exist $C,\alpha\geq 0$ such that $(n-2)\alpha\leq 2$ and $|f'(r)|\leq C\,(1+|r|^\alpha)\quad\forall r\in\mathbb{R}$.

(ii) f is nondecreasing.

Define the function $F \in \mathcal{C}^2(\mathbb{R})$ by $F(r) = \int_0^r f(s)\,ds$ for every $r \in \mathbb{R}$. For simplicity of notation, we write $F(u)$ for $\int_\Omega F(u(x))\,dx$. Hypothesis (k_1) is a consequence of assumption (i) above, see for example [10, pp. 73-75]. The monotonicity hypothesis (k_2) is ensured by point (ii). Finally the coercivity of the bilinear form a implies that the equilibrium set is a singleton $\{\overline{u}\}$, see Remark 3.1. In particular, the non-vacuity condition (k_3) is satisfied. If the map $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (l_1)-(l_3), we derive from Corollary 3.6 that $u(t) \rightharpoonup \overline{u}$ weakly in $H^1_0(\Omega)$ as $t \to +\infty$. Since the injection $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the second part of Corollary 3.6 shows that the convergence is strong in $H^1_0(\Omega)$. On the other hand, the coercivity of a implies that condition (3.26) is fulfilled. If the map γ satisfies (l_1) and $\dot{\gamma}(t) = o(\gamma(t))$ as $t \to +\infty$, Theorem 3.15 then shows that for every $m \in \left]0, \frac{2}{3}\right[$,

$$\frac{1}{2}\int_\Omega \left\{ \left|\frac{\partial u}{\partial t}(t,x)\right|^2 + |\nabla u(t,x)|^2 \right\} dx + \int_\Omega F(u(t,x))\,dx = O\left(e^{-m\int_0^t \gamma(s)\,ds}\right) \quad \text{as } t \to +\infty.$$

Example 4.2. Let us consider the damped wave equation (4.1) with Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega \times]0, +\infty[$. The functional setting of the evolution problem is given by:

$$H = L^2(\Omega), \quad V = H^1(\Omega) \quad \text{and} \quad a(u,v) = \int_\Omega \nabla u(x).\nabla v(x) dx.$$

The bilinear form a is semi-coercive, hypothesis (h_2) is satisfied with $\lambda = \mu = 1$. To overcome the lack of coercivity, assumptions (i)-(ii) above are supplemented with the following one

(iii) There exist $\varepsilon > 0$ and $D \geq 0$ such that $F(r) \geq \varepsilon r^2 - D$ for every $r \in \mathbb{R}$.

Assumption (iii) implies that condition (k_3) is verified, see Remark 3.1. Hypotheses (k_1)-(k_2) are fulfilled as in the previous example. If the map $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (l_1) and (l_4), we derive from Theorem 3.10 that there exists a solution u_∞ of

$$\begin{cases} -\Delta u + f(u) = 0 & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

such that $u(t) \rightharpoonup u_\infty$ weakly in $H^1(\Omega)$ as $t \to +\infty$. Since the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the second part of Theorem 3.10 shows that the convergence is strong in $H^1(\Omega)$.

Example 4.3. Let us consider the following equation

$$\frac{d^2u}{dt^2} + \gamma(t)\,\frac{du}{dt} - \Delta u - \lambda_1 u + f(u) = 0 \quad \text{on} \quad \Omega \times]0, +\infty[, \tag{4.3}$$

with Dirichlet boundary condition. Here λ_1 stands for the smallest eigenvalue of the Laplacian-Dirichlet operator. The functional setting of the evolution problem is given by:

$$H = L^2(\Omega), \quad V = H^1_0(\Omega) \quad \text{and} \quad a(u,v) = \int_\Omega \left[\nabla u(x).\nabla v(x) - \lambda_1 u(x) v(x)\right] dx.$$

It is immediate to check that (h_1)-(h_2) are satisfied. Under the above assumptions (i), (ii) and (iii), we obtain as previously that conditions (k_1)-(k_3) hold true. If the

map $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (l_1) and (l_4), we derive from Theorem 3.10 that there exists a solution u_∞ of

$$\begin{cases} -\Delta u - \lambda_1 u + f(u) = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

such that $u(t) \to u_\infty$ strongly in $H^1_0(\Omega)$ as $t \to +\infty$.

Example 4.4. The equation arising in the previous example can be generalized as follows

$$\frac{d^2u}{dt^2} + \gamma(t)\,\frac{du}{dt} - \Delta u - \sum_{i=1}^{+\infty} \eta_i P_i u + f(u) = 0 \quad \text{on} \quad \Omega\times]0,+\infty[,$$

see [21, Example 4.5]. We still assume Dirichlet boundary conditions. Let us explicit the notations: $(\lambda_i)_{i\geq 1}$ (respectively $(e_i)_{i\geq 1}$) is the sequence of eigenvalues (respectively eigenfunctions normalized in $L^2(\Omega)$) of $(-\Delta)$ in $H^1_0(\Omega)$. For each $i \geq 1$, P_i denotes the orthogonal projection on span$\{e_i\}$ in the sense of $L^2(\Omega)$. We assume that the nonnegative sequence $(\eta_i)_{i\geq 1}$ is bounded and that $\eta_i \leq \lambda_i$ for every $i \geq 1$. The functional setting of the evolution problem is given by

$$H = L^2(\Omega), \quad V = H^1_0(\Omega) \quad \text{and} \quad a(u,v) = \int_\Omega \nabla u(x).\nabla v(x)dx - \sum_{i=1}^{+\infty} \eta_i \int_\Omega P_i u(x).P_i v(x)dx.$$

It is easy to check that hypotheses (h_1)-(h_2) hold true. Under the additional assumptions (i), (ii) and (iii), we then obtain (k_1)-(k_3). If the map $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (l_1) and (l_4), we obtain as in the previous example the existence of an equilibrium u_∞ such that $u(t) \to u_\infty$ strongly in $H^1_0(\Omega)$ as $t \to +\infty$.

4.2. **A higher-order example.**

Example 4.5. Let us consider the following equation

$$\frac{d^2u}{dt^2} + \gamma(t)\,\frac{du}{dt} + \Delta^2 u + f(u) = 0 \quad \text{on} \quad \Omega\times]0,+\infty[, \tag{4.4}$$

with the boundary condition:

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\times]0,+\infty[. \tag{4.5}$$

The functional setting of the evolution problem (4.4)-(4.5) is given by:

$$H = L^2(\Omega), \quad V = \left\{ u \in H^2(\Omega), \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\} \quad \text{and} \quad a(u,v) = \int_\Omega \Delta u(x).\Delta v(x)dx.$$

Hypothesis (h_1) is trivially verified. Moreover, from the regularity results relative to the Laplacian-Dirichlet problem, there exists $\kappa > 0$ such that $\|u\|_{H^2(\Omega)} \leq \kappa\,|\Delta u|_{L^2(\Omega)}$. Hence condition (h_2) is satisfied with $\lambda = 0$, *i.e.* the bilinear form a is coercive. We assume that the function f satisfies assumption (ii) along with the following variant of (i)

(i′) There exist $C, \alpha \geq 0$ such that $(n-4)\alpha \leq 4$ and $|f'(r)| \leq C\ (1 + |r|^\alpha) \quad \forall r \in \mathbb{R}$.

By using Sobolev's imbedding theorem, we let the reader check that hypothesis (k_1) is a consequence of assumption (i′) above. The monotonicity hypothesis (k_2) is ensured by (ii). Finally in view of Remark 3.1, the coercivity of the bilinear form a

implies that the equilibrium set is a singleton $\{\overline{u}\}$ and in particular (k_3) holds true. If the map $\gamma \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (l_1)-(l_3), we derive from Corollary 3.6 that $u(t) \to \overline{u}$ strongly in $H^2(\Omega)$ as $t \to +\infty$. On the other hand, the coercivity of a implies that condition (3.26) is fulfilled. If the map γ satisfies (l_1) and $\dot{\gamma}(t) = o(\gamma(t))$ as $t \to +\infty$, Theorem 3.15 then shows that for every $m \in \left]0, \frac{2}{3}\right[$,

$$\frac{1}{2}\int_\Omega \left\{ \left| \frac{\partial u}{\partial t}(t,x) \right|^2 + |\Delta u(t,x)|^2 \right\} dx + \int_\Omega F(u(t,x))\, dx = O\left(e^{-m \int_0^t \gamma(s)\, ds} \right) \quad \text{as } t \to +\infty.$$

APPENDIX

Proof of Lemma 3.8. (i) Let us first prove that $\lim_{t\to+\infty} t^\theta \gamma(t) = 0$. Since the map $s \mapsto s^\theta$ is nondecreasing on $\mathbb{R}_+$, we have for every $t \geq 0$

$$\int_t^{+\infty} s^\theta\, |\dot{\gamma}(s)|\, ds \geq t^\theta \int_t^{+\infty} |\dot{\gamma}(s)|\, ds \geq t^\theta \int_t^{+\infty} -\dot{\gamma}(s)\, ds = t^\theta \gamma(t),$$

the last equality being a consequence of the fact that $\lim_{t\to+\infty} \gamma(t) = 0$. In view of assumption $\int_0^{+\infty} s^\theta\, |\dot{\gamma}(s)|\, ds < +\infty$, we infer from the above inequality that $\lim_{t\to+\infty} t^\theta \gamma(t) = 0$. On the other hand, the absolute continuity of the map γ allows to write that

$$\begin{aligned} \theta \int_0^t s^{\theta-1}\, \gamma(s)\, ds &= t^\theta\, \gamma(t) - \int_0^t s^\theta\, \dot{\gamma}(s)\, ds \\ &\leq t^\theta\, \gamma(t) + \int_0^t s^\theta\, |\dot{\gamma}(s)|\, ds. \end{aligned} \tag{4.6}$$

Taking the limit as $t \to +\infty$, we obtain $\theta \int_0^{+\infty} s^{\theta-1}\, \gamma(s)\, ds \leq \int_0^{+\infty} s^\theta\, |\dot{\gamma}(s)|\, ds < +\infty$.

(ii) Since $\dot{\gamma}(s) \leq 0$ for almost every $s \geq 0$, we derive from (4.6) that

$$\int_0^t s^\theta\, |\dot{\gamma}(s)|\, ds = -\int_0^t s^\theta\, \dot{\gamma}(s)\, ds \leq \theta \int_0^t s^{\theta-1}\, \gamma(s)\, ds,$$

and the conclusion immediately follows. □

Proof of Lemma 3.11. (i) From the assumption $\gamma(t) \geq \frac{k}{t^\theta}$, we deduce the existence of $c \in \mathbb{R}$ such that $\int_0^t \gamma(\tau)\, d\tau \geq \frac{k}{1-\theta} t^{1-\theta} + c$ for every $t \geq t_0$. Therefore, we have

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau)\, d\tau}\, dt \leq e^{-c} \int_0^{+\infty} e^{-\frac{k}{1-\theta} t^{1-\theta}}\, dt < +\infty.$$

(ii) By using the assumption $\gamma(t) \geq \frac{k}{t^\theta}$, we find

$$\int_s^{+\infty} e^{-\int_0^t \gamma(\tau)\, d\tau}\, dt \leq \frac{1}{k} \int_s^{+\infty} t^\theta\, \gamma(t)\, e^{-\int_0^t \gamma(\tau)\, d\tau}\, dt. \tag{4.7}$$

An integration by parts in the right-hand side then yields

$$\int_s^{+\infty} t^\theta\, \gamma(t)\, e^{-\int_0^t \gamma(\tau)\, d\tau}\, dt = \left[-t^\theta\, e^{-\int_0^t \gamma(\tau)\, d\tau} \right]_s^{+\infty} + \theta \int_s^{+\infty} t^{\theta-1}\, e^{-\int_0^t \gamma(\tau)\, d\tau}\, dt. \tag{4.8}$$

Remark that $t^\theta\, e^{-\int_0^t \gamma(\tau)\, d\tau} \leq e^{-c} t^\theta\, e^{-\frac{k}{1-\theta} t^{1-\theta}}$, hence $\lim_{t\to+\infty} t^\theta\, e^{-\int_0^t \gamma(\tau)\, d\tau} = 0$. Therefore, we deduce from (4.7) and (4.8) that

$$\int_s^{+\infty} e^{-\int_0^t \gamma(\tau)\, d\tau}\, dt \leq \frac{1}{k}\, s^\theta\, e^{-\int_0^s \gamma(\tau)\, d\tau} + \frac{\theta}{k} \int_s^{+\infty} t^{\theta-1}\, e^{-\int_0^t \gamma(\tau)\, d\tau}\, dt.$$

The right term is clearly negligible with respect to the left one, hence $\frac{\theta}{k}\int_s^{+\infty} t^{\theta-1}\, e^{-\int_0^t \gamma(\tau)\, d\tau}\, dt \leq \frac{1}{2}\int_s^{+\infty} e^{-\int_0^t \gamma(\tau)\, d\tau}\, dt$ for s large enough. The conclusion follows immediately. □

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E-mail address: `acabot@math.univ-montp2.fr, p.frankel30@orange.fr`