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Exactness of the Bogoliubov approximation in random external potentials

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Abstract

We investigate the validity of the Bogoliubov c-number approximation in the case of interacting Bose-gas in a *homogeneous random* media. To take into account the possible occurence of *type* III generalized Bose-Einstein condensation (i.e. the occurrence of condensation in an infinitesimal band of low kinetic energy modes *without* macroscopic occupation of any of them) we generalize the c-number substitution procedure to this band of modes with low momentum. We show that, as in the case of the one-mode condensation for translation-invariant interacting systems, this procedure has *no effect* on the exact value of the pressure in the thermodynamic limit, assuming that the c-numbers are chosen according to a suitable *variational principle*. We then discuss the relation between these c-numbers and the (total) density of the condensate.

Keywords: Generalized Bose-Einstein Condensation, Random Potentials, Bogoliubov c-number Approximation, Bogoliubov Quasi-Averages, Berezin-Lieb Inequalities.

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1 Motivation

In 1947 Bogoliubov [1] proposed an *ansatz* that for large Boson systems the particle creation and annihilation operators: a_0^*, a_0, corresponding to the zero momentum $k = 0$, can be replaced by complex numbers. This procedure is known as the Bogoliubov c-number approximation. It is based on the idea that these creation and annihilation operators, when divided by the square root of the volume, V, of the region Λ containing the system, can be expressed as space averages:

$$\frac{a_0^\#}{\sqrt{V}} := \frac{1}{V}\int_\Lambda \mathrm{d}x\, a^\#(x),$$

where $a^\#(x)$ are the usual local creation and annihilation operators. Therefore for translation-invariant ergodic states these operators should converge to multiples of the identity:

$$\frac{a_0^\#}{\sqrt{V}} \to \alpha^\# \ ,$$

in some *weak sense*, see e.g. [3]. These ideas were exploited to construct a various truncations of the full interacting boson Hamiltonian. We refer the reader to, e.g., [14] for a review of these models and to [22], [23] for some recent applications.

The most spectacular result derived from this ansatz was its application to a homogeneous model of a weakly interacting Bose gas [1], [14], which provided explicitly a spectrum of collective excitations satisfying the Landau criteria of *superfluidity*. Note that this *microscopic* theory of superfluidity is also based on two other Bogoluibov's *ansatse*: the occurrence in the weakly interacting boson system of condensation in the $k = 0$ mode and the truncation of the full Hamiltonian, keeping only the "dominant" terms, that is those that involves at least two particles from the condensate.

Recently, see [17], [18], the Bogoliubov approximation has been used to study interacting bosons systems in homogeneous external random potentials, where the notion of the ground-state as well as the existence of the condensation are quite subtle. The aim of our paper is to study the validity of the Bogoliubov scheme for this kind of models.

The first rigorous result concerning the Bogoliubov c-number approximation was due to Ginibre [4]. For a homogeneous boson gas with a two-body superstable interaction he proved that the Bogoliubov *ansatz*, supplemented by a self-consistency *condensate equation* which maximizes the *approximated pressure* with respect to the c-number α, gives the right pressure in the thermodynamic limit. A transparent and elegant proof of this and other related results has been recently given by Lieb et al [5], using, in particular, the Berezin-Lieb and the Bogoliubov convexity inequalities. The paper [5] also investigates a delicate point: namely, whether the value of the variational parameter $\alpha_{\max}$ maximizing the approximating pressure *coincides* with the *total* condensate density. There it was shown that the maximizer $\alpha_{\max}$ corresponds to the zero-mode condensate density *if* the gauge symmetry breaking term (quasi-average *sources*) of the form: $\sqrt{V}(\eta\, a_0^* + \overline{\eta}\, a_0)$ is added to the full Hamiltonian. The idea of breaking the gauge-symmetry of the quantum Gibbs state for $k = 0$ is due to Bogoliubov, [2]. This forces the *totality* of the condensate to be concentrated in the zero-mode (ground state). It must be switched off ($\eta = |\eta|\, e^{i \arg \eta}$, $|\eta| \to 0$ with a fixed gauge $\phi := \arg \eta$)

after the thermodynamic limit to produce a limiting Gibbs state. The expectation defined by this state is called the Bogoliubov *quasi-average* with respect to this source. The quasi-averages of the operators $a_0^{\#}/\sqrt{V}$ coincide with $|\alpha_{\max}|\, e^{\pm i\phi}$, i.e., $|\alpha_{\max}|^2$ is equal to the *total* condensate density [5]. It has been argued in [2], [5] that this quasi-average is the only physically reliable quantity to describe Bose Einstein condensation.

We emphasize this point, because it has been known since [6] that the Bose-Einstein condensation in the *gauge-invariant* systems does *not* necessarily imply a macroscopic occupation of the ground state only. Indeed, the condensate can be spread over many (possibly *infinitely* many) quantum states, and in some cases, *none* of these states are macroscopically occupied. In all cases however, the *total* amount of condensate in an arbitrary small band of energy in the vicinity of the ground-state is the same, a phenomenon known as *generalized* condensation in the terminology of van den Berg-Lewis-Pulé [6].

After Ginibre [4] it was tempting to conjecture that the limiting value of the solution to the finite volume *condensate equation* yields the correct condensate density under all circumstances. By means of a counter example it was shown in [7] that this is not so. Although the Bogoliubov c-number approximation still gives the right pressure in these systems, it has been shown for the mean-field boson gas [7], that the solution $\alpha_{\max}$ of the condensate equation does *not* provide the ground-state condensate density ρ_0, but the *total* amount of the generalized condensate, i.e., $\rho_0 \leq |\alpha_{\max}|^2$. This was in a striking contrast with a general conviction that the *gauge-invariant*, translation invariant boson systems always manifest the *total* amount condensation in the ground state, and hence that the Bogoliubov c-number approximation in the zero-mode coincides with this amount.

We return to this point at the end of the present paper where we discuss our main result (Theorem 3.1) in relation to the results of [4], [7], [2], and [5].

The aim of our paper is to prove that the Bogoliubov c-number approximation to interacting Bose-gas can be extended to the case of homogeneous random external potentials (random media). Note that the arguments in [5] are valid for inhomogeneous systems and allows to treat condensation in many modes, as long as its number is much smaller than the volume V. In the case of random media this problem is more complicated. First, we have to investigate a random inhomogeneous system, albeit with non-random properties (*self-averaging*) in the thermodynamic limit [8]. Secondly, since the randomness seems to force the generalised condensation to be of type III, as we disucussed in [15], the number of modes occupied by the condensation in the random potentials is *a priori* of the order of V.

Recall that for non-interacting (perfect) bosons systems embedded into a *bona fide* random potential, the generalized condensation occurs even in low-dimensional systems (see [9, 10, 11] and [12], [15] for a review) even though it does not occur for the corresponding translation-invariant systems. This is caused by the fact that the one-particle density of states has *Lifshitz tails*, that is an extremely low density of quantum states near the bottom of the spectrum, a well-known feature of random systems widely believed to be associated with the existence of localised eigenstates. Kac and Luttinger [9, 10] conjectured that in a homogeneous random potential condensation occurs in the state with the lowest energy (ground state). Indeed, one can check this conjecture [13] for the particular case of the Luttinger-Sy model [11]. Recently in [15], it was shown that whenever condensation occurs in the random perfect Bose-gas, then there is a *generalized* condensation in the kinetic-energy momentum states, and the both densities have the same value. This result can be partially

extended to some simple models of interacting Bose gases (mean-field models). It was proved in [16] that under a fairly weak assumption about localization this condensation is of the *type* III, i.e. there is no macroscopic accumulation of particles in any single momentum state. We conjecture that this holds also for general interacting systems in the presence of homogeneous random potentials.

For the above reasons, we would like to follow the general philosophy of the Bogoliubov c-number substitution ansatz and extend it to a *generalized* Bogoliubov approximation, in order to cover the possible case of *type* III generalized condensation. By this, we mean a replacement of all creation/annihilation operators corresponding to momentum states with kinetic energy ε_k in the energy band $0 \leq \varepsilon_k < \delta$ by complex numbers $\{\sqrt{V}\alpha_k^\sharp\}_{\{k:\varepsilon_k<\delta\}}$.
First we show that this extension of the Bogoliubov approximation applied to interacting Bose-gas in homogeneous random potentials is valid as far as the pressure is concerned (Section 3). In this case as in the case one-mode condensation the corresponding trial pressure is maximized with respect to these complex numbers and then one lets the parameter $\delta \to 0$ *after* the thermodynamic limit. Note that for each realization of the random potential, the system is not translation invariant and the minimizer has a random value *before* the thermodynamic limit. For this reason, the proof use the stationarity and ergodicity of the random potential with respect to the space translations.
Finally, we discuss the variational problem established for the pressure (Section 4). In particular, we highlight the fact that the link between the c-numbers that maximize the trial pressure and the structure of the condensate is highly non-trivial. By the mean of a simple example, we show that the Bogoliubov quasi-average technique of adding external sources [2] is not "satisfactory" in this case, since we suspect that the generalised condensate should be of type III, while this procedure is able to drastically alter the fine structure of the generalised condensate. This brings us back to the discussion of the gauge symmetry breaking and the physical reliability of the Bogoliubov quasi-averages in the case of condensation [2], [5].

2 Model and definitions

Let $\{\Lambda_l := (-l/2, l/2)^d\}_{l\geqslant 1}$ be a sequence of hypercubes of side l in $\mathbb{R}^d, d \geqslant 1$, centered at the origin of coordinates with volumes $V_l = l^d$. We consider a system of identical bosons, of mass m, contained in Λ_l. For simplicity, we use a system of units such that $\hbar = m = 1$. First we define the self-adjoint one-particle kinetic-energy operator of our system by:

$$h_l^0 := -\tfrac{1}{2}\,\Delta_P, \tag{2.1}$$

acting in the Hilbert space $\mathscr{H}_l := L^2(\Lambda_l)$. The subscript P stands for the *periodic* boundary condition. We denote by $\{\psi_k^l, \varepsilon_k^l\}_{k\in\Lambda^*}$ the set of normalized eigenfunctions and eigenvalues corresponding to operator h_l^0

$$\psi_k^l(x) \,=\, \frac{1}{\sqrt{V_l}}\, e^{ik\cdot x}, \quad \varepsilon_k^l \,=\, \tfrac{1}{2}\, k^2\ , \tag{2.2}$$

and Λ_l^* is the usual dual space to Λ_l, that is $\Lambda_l^* := \{k \in \mathbb{R}^d : k^2 = (2\pi n)^2/l^2, n \in \mathbb{Z}^d\}$. Finally, we denote by ν_l^0 the finite-volume *integrated density of states* (IDS), that is,

$$\nu_l^0(E) \,:=\, \frac{1}{V_l}\,\sharp\{k \in \Lambda_l^* : \varepsilon_k^l \leqslant E\}, \tag{2.3}$$

and we let $\nu^0(E) := \lim_{l\to\infty} \nu_l^0(E)$. Note that the limiting IDS, $\nu^0(E)$, has support on $[0,\infty)$ and that it is known explicitly: $\nu^0(E) = C_d\, E^{d/2}$ (the *Weyl formula*).

Definition 2.1 *We define an external random potential* $v^{(\cdot)}(\cdot) : \Omega\times\mathbb{R}^d \to \mathbb{R},\ x \mapsto v^\omega(x)$ *as a measurable random field on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, satisfying the following conditions:*
(i) $v^\omega, \omega \in \Omega$*, is non-negative;*
(ii) $p := \mathbb{P}\{\omega : v^\omega(0) = 0\} < 1$.
As usual, we assume also (*see e.g. Appendix B in* [15]) *that this field is:*
(iii) *homogeneous (stationary) and ergodic with respect to the group* $\{\tau_x\}_{x\in\mathbb{R}^d}$ *of probability preserving translations on* $(\Omega, \mathcal{F}, \mathbb{P})$ *;*
(iv) φ*-mixing for* Σ_Λ*-measurable functions, where* Σ_Λ *is the* σ*-algebra generated by the field* $\{v^\omega(x)\}_{x\in\Lambda}$ *for* $\Lambda \subset \mathbb{R}^d$.

Then the corresponding self-adjoint random Schrödinger operator acting in $\mathscr{H} := L^2(\mathbb{R}^d)$ is a perturbation of the kinetic-energy operator:

$$h^\omega := -\tfrac{1}{2}\,\Delta \,\dot{+}\, v^\omega, \tag{2.4}$$

defined as a sum in the *quadratic-forms* sense. The restriction to the box Λ_l, is specified by the periodic boundary conditions and for regular potentials one gets the self-adjoint operator:

$$h_l^\omega := \left(-\tfrac{1}{2}\,\Delta + v^\omega\right)_P = h_l^0 \,\dot{+}\, v_l^\omega, \tag{2.5}$$

acting in $\mathscr{H}_l$, where v_l^ω is the restriction of v^ω to Λ_l. We denote by $\{\phi_i^{\omega,l}, E_i^{\omega,l}\}_{i\geqslant 1}$ the set of normalized eigenfunctions and the corresponding eigenvalues of the random operator h_l^ω. We order the eigenvalues (counting the multiplicity) in such a way that $E_1^{\omega,l} \leqslant E_2^{\omega,l} \leqslant E_3^{\omega,l} \ldots$. Note that the *non-negativity* of the random potential implies that $E_1^{\omega,l} > 0$. So, for convenience we *assume* also that in the thermodynamic limit the lowest edge of this random one-particle spectrum $\sigma(h_l^\omega)$ satisfies the fifth condition:
(v) $\lim_{l\to\infty} E_1^{\omega,l} = 0$, *almost surely* (a.s.) with respect to the probability $\mathbb{P}$.

Remark 2.1 *Note that* (v) *is in fact an implicit condition on the random potential saying that a.s., one can find a sequence of regions* (gaps) *with* $v^\omega(x) = 0$*, with volumes tending to infinity in the van Hove sense.*

Now, we turn to the many-body problem. Let $\mathscr{F}_l := \mathscr{F}_l(\mathscr{H}_l)$ be the symmetric Fock space constructed over $\mathscr{H}_l$. Then $H_l^0 := \mathrm{d}\Gamma(h_l^\omega)$ denotes the second quantization of the *one-particle* Schrödinger operator h_l^ω in $\mathscr{F}_l$. For simplicity, we omit the explicit mention of the randomness of the Hamiltonians and all related quantities, unless this is necessary for the sake of clarity. Since for any $\omega \in \Omega$ the one-particle eigenstates $\{\phi_i := \phi_i^{\omega,l}\}_{i\geq 1}$ of h_l^ω form a basis in $\mathscr{H}_l$, the operator H_l^0 acting in $\mathscr{F}_l$ can be expressed as:

$$H_l^0 = \sum_{i\geqslant 1} E_i^{\omega,l}\; a^*(\phi_i)a(\phi_i) \;=\; \sum_{i\geqslant 1} E_i^{\omega,l} N_l(\phi_i)\ . \tag{2.6}$$

Here $a^*(\phi_i), a(\phi_i)$ are the creation and annihilation operators, satisfying the boson *Canonical Commutation Relations*, and $N_l(\phi_i)$ is the particle-number operator in the state $\phi_i^{\omega,l}$. Note

that, since $[h_l^0, h_l^\omega] \neq 0$, the Hamiltonian (2.6) cannot be expressed as a function of the operators $N_l(\psi_k^l)$ in the kinetic-energy eigenstates (2.2).

(vi) Below we assume that the particles interact through a suitable *non-negative* two-body *translation-invariant* potential $\Phi(x,y) := u(|x-y|)$. More precisely, we assume that the function u has a continuous, bounded Fourier transformation $\hat{u}(q)$, and there is $\gamma < \infty$ such that $|\hat{u}(q)| < \gamma$ for all $q \in \Lambda_l^*$ and for all l. For example, one can choose $u \in L^1(\mathbb{R}^d)$.

Then the second quantization of interaction: $U_l := \mathrm{d}\Gamma(\Phi)$ has a simple form in the translation-invariant basis $\{\psi_k := \psi_k^l\}_{k\in\Lambda_l^*}$ of the kinetic-energy operator h_l^0:

$$U_l = \frac{1}{2V_l} \sum_{q,k,k'\in\Lambda_l^*} \hat{u}(q)\ a^*(\psi_{k+q})a^*(\psi_{k'-q})a(\psi_{k'})a(\psi_k)\ . \tag{2.7}$$

The full Hamiltonian with the chemical potential μ included has the form:

$$H_l(\mu) \ := \ H_l^0 - \mu N_l + U_l \tag{2.8}$$

Note that the creation and annihilation operators in the interaction term (2.7) are in the momentum eigenstates $\{\psi_k\}_{k\in\Lambda_l^*}$, although the perfect Bose-gas Hamiltonian (2.6) is not diagonal when expressed in this basis.

By $\langle - \rangle_l$ we denote below the grand-canonical equilibrium state defined by the Hamiltonian $H_l(\mu)$:

$$\langle A \rangle_l(\beta,\mu) := \frac{1}{\Xi_l(\beta,\mu)} \mathrm{Tr}_{\mathscr{F}_l} \exp(-\beta H_l(\mu)),$$

and by $p_l(\beta,\mu)$ its associated grand-canonical pressure

$$p_l(\beta,\mu) \ := \ \frac{1}{\beta V_l} \ln \Xi_l(\beta,\mu), \tag{2.9}$$

where

$$\Xi_l(\beta,\mu) \ := \ \mathrm{Tr}_{\mathscr{F}_l} \exp(-\beta H_l(\mu))$$

is the corresponding grand-canonical partition function.

It is known that the pressure of the corresponding non-random model (i.e. for $v^\omega(x) = 0$) with a *bona fide* interaction exists and is independent of the boundaries condition for a large class of them, including the periodic case, see e.g. [19]. The proof of this statement consists essentially in showing the existence of the Dirichlet pressure using sub-additivity

$$p_\Lambda^D(\beta,\mu) \ \geqslant \ p_{\Lambda'}^D(\beta,\mu) + p_{\tau_x\Lambda''}^D(\beta,\mu)$$

where Λ', Λ'' are disjoints subsets of Λ, and τ_x denotes translation by x. The exact value of x is chosen according to the usual *tempering condition* required of the two-body interaction potential (vi). Then, using translation invariance of the non-random model, one obtains

$$p_\Lambda^D(\beta,\mu) \ \geqslant \ p_{\Lambda'}^D(\beta,\mu) + p_{\Lambda''}^D(\beta,\mu)\ , \tag{2.10}$$

and the boundeness of the pressure, which is provided by the *superstability* of the interaction ($u \geq 0$, (vi)), leads to the existence and finiteness of the limiting pressure for any μ. Then,

one can show using functional integration techniques, see [20], that the others boundary conditions converge to the same limit.

The last part of this prove can be carried through verbatim in the presence of the external random potential. However, because of the lack of translation invariance in the random case, the inequality (2.10) for the Dirichlet pressure is modified as follows:

$$p_{\Lambda}^{D,\omega}(\beta,\mu) \geqslant p_{\Lambda'}^{D,\omega}(\beta,\mu) + p_{\tau_x\Lambda''}^{D,\omega}(\beta,\mu) = p_{\Lambda'}^{D,\omega}(\beta,\mu) + p_{\Lambda''}^{D,\tau_x\omega}(\beta,\mu) \ . \tag{2.11}$$

We used the *stationarity* of the random potential in the last identity. To prove the the existence of the thermodynamic limit one can use the Kingman *sub-additive* ergodic theorem, see [21]:

Proposition 2.1 *Let τ be measure preserving transformation of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{g_n\}_{n\geq 1}$ be a sequence of functions $g_n \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the condition:*

$$g_{n+m}(\omega) \leq g_n(\omega) + g_m(\tau^n\omega) \tag{2.12}$$

Then one gets that

$$a.s. - \lim_{n\to\infty} g_n(\omega)/n = g(\omega) \ , \tag{2.13}$$

where the function $g(\omega)$ is τ-invariant: $g(\tau^s\omega) = g(\omega)$. If in addtion, the functions g_n are ergodic, it follows that the limit $g(\omega)$ is a.s. non random.

3 Generalized Bogoliubov c-numbers approximation

3.1 Existence of the approximating pressure

Following the Bogoliubov approximation philosophy, we want to replace all creation/annihilation operators in the momentum states ψ_k with kinetic energy less than some $\delta > 0$ by c-numbers. To this end let $I_\delta \subset \Lambda_l^*$ be the set of all *replaceable* modes, that is,

$$I_\delta := \big\{k \in \Lambda_l^* : k^2/2 \leqslant \delta\big\},$$

and we denote $n_\delta := \sharp\{k : k \in I_\delta\}$. The number of quantum states n_δ is of the order V_l, since by definition of the IDS (2.3): $n_\delta = V_l\ \nu_l^0(\delta)$. Let $\mathscr{H}_l^\delta$ to be the subspace of $\mathscr{H}_l$ spanned by the set of ψ_k^l with $k \in I_\delta$, and P_δ be orthogonal projector onto this subspace. Hence, we have a natural decomposition of the total space $\mathscr{H}_l$ and the corresponding representation for the associated symmetrised Fock space:

$$\mathscr{H}_l = \mathscr{H}_l^\delta \oplus \mathscr{H}_l^\perp \ , \qquad \mathscr{F}_l \approx \mathscr{F}_l^\delta \otimes \mathscr{F}_l^\perp. \tag{3.1}$$

Then we proceed to the Bogoliubov substitution $a_k \to c_k$ and $a_k^* \to \overline{c}_k$ for all $k \in I_\delta$, which provides an approximating Hamiltonian that we denote by $H_l^{low}(\mu, \{c_k\})$. The meaning of the superscript *low* will become clear in the next section. We postpone to Appendix A a description of the explicit form of this operator. The partition function and the corresponding pressure for this approximating Hamiltonian have the form:

$$\Xi_l^{low}(\mu, \{c_k\}) \ := \ \mathrm{Tr}_{\mathscr{F}_l^\perp}\ e^{-\beta H_l^{low}(\mu,\{c_k\})} \ , \tag{3.2}$$

$$p_{l,\delta}^{low}(\mu, \{c_k\}) \ := \ \frac{1}{V_l} \ln \Xi_l^{low}(\mu, \{c_k\}) \ . \tag{3.3}$$

The principal result of the present paper is the following main theorem:

Theorem 3.1 *The c-numbers substitution for all operators in the energy-band I_δ does not affect the original pressure (2.9) in the following sense:*

$$\text{a.s.}-\lim_{l\to\infty} p_l(\beta,\mu) \;=\; \lim_{\delta\downarrow 0}\liminf_{l\to\infty}\max_{\{c_k\}} p^{low}_{l,\delta}(\mu,\{c_k\}) \;=\; \lim_{\delta\downarrow 0}\limsup_{l\to\infty}\max_{\{c_k\}} p^{low}_{l,\delta}(\mu,\{c_k\}) \;. \quad (3.4)$$

Note that the number of the substituted modes is of order V, since we let $\delta\downarrow 0$ after the thermodynamic limit.

3.2 Proof of the main Theorem

Our method is a generalisation of the one invented in [5]. For the convenience of the reader, we postpone the proof of some technical lemmas to the next section.

First we define a family of *normalized* coherent vectors in the Fock space $\mathscr{F}_l$, with the vacuum state $|0\rangle$:

$$|c\rangle \;=\; \bigotimes_{k\in I_\delta} e^{-|c_k|^2/2+c_k a_k^*}\;|0\rangle \;, \quad (3.5)$$

labeled by the set of complex numbers $\{c_k\}_{k\in I_\delta}$. With the help of vectors (3.5), we define the *lower symbol* A^{low} for any operator A in $\mathscr{F}_l$ by the partial inner product:

$$A^{low}(\{c_k\}) \;:=\; \langle c|A|c\rangle \;,$$

which is an operator in $\mathscr{F}_l^\perp$, see (3.1). Similarly, using the tensor structure of (3.1), we can define on $\mathscr{F}_l^\perp$ the *upper symbol* A^{up} of the operator A in $\mathscr{F}_l$ through the *integral representation*

$$A \;=:\; \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\, A^{up}(\{c_k\})\,|c\rangle\langle c| \;.$$

Here $d^2c_j \;:=\; d\,\mathrm{Re}(c_j)d\,\mathrm{Im}(c_j)/\pi$ and $|c\rangle\langle c| \;:=\; \bigotimes_{k\in I_\delta}|c_k\rangle\langle c_k|$ is the projector on the coherent vectors (3.5). Similar to the one-mode case one has the completeness property $\int_{\mathbb{C}^{n_\delta}}\mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\,|c\rangle\langle c| \;=\; I$. Note that, contrary to the lower symbols, the upper symbol does not necessarily *exists*, and it may not be *unique*. Although, this poses no problem in our case, since the existence (though not the unicity) of the upper symbols follows from the fact that the Hamiltonian (2.8) is polynomial in creation/annihilation operators. We postpone the explicit expressions of these upper symbols to Appendix A.

We then define two approximating Hamiltonians, that we denote by $H_l^{\mathrm{low}}(\mu,\{c_k\})$ and $H_l^{\mathrm{up}}(\mu,\{c_k\})$. They are obtained by replacing all creations and annihilations operators $a_k^\sharp$ by their *lower*, respectively *upper*, symbols. We refer the reader to Appendix A for the details and the explicit expressions.
Note that $H_l^{\mathrm{low}}(\mu,\{c_k\})$ is obtained simply by replacing all operators $\{a_k^\sharp\}_{k\in I_\delta}$ by the corresponding complex numbers $\{c_k^\sharp\}_{k\in I_\delta}$. Formally it corresponds to the Hamiltonian obtained by the standard Bogoliubov approximation, that is why it appears in Theorem 3.1.

In a way similar to (3.2), (3.3), one can define by $\Xi_l^{up}(\mu, \{c_k\})$ the partition function for the Hamiltonian $H_l^{\rm up}(\mu, \{c_k\})$, and by $p_{l,\delta}^{up}(\mu, \{c_k\})$ the corresponding pressure.

Finally, we denote by $\langle - \rangle_{low}$ and $\langle - \rangle_{up}$ the grand-canonical equilibrium states related to the following (integrated) partition functions:

$$\Xi_l^{low}(\mu) := \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \dots \mathrm{d}^2 c_{n_\delta} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\rm low}(\mu, \{c_k\})}, \tag{3.6}$$

$$\Xi_l^{up}(\mu) := \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \dots \mathrm{d}^2 c_{n_\delta} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\rm up}(\mu, \{c_k\})} , \tag{3.7}$$

and we denote the associated pressures by $p_{l,\delta}^{low}(\mu), p_{l,\delta}^{up}(\mu)$.

Now one can mimic the arguments of [5] and extend them to the *multi-mode* projections $|c\rangle\langle c|$ case to produce the Bogoliubov-Peierls and the Berezin-Lieb inequalities for (3.6) and (3.7). These inequalities yield respectively lower and upper estimates for the grand partition function:

$$\int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \dots \mathrm{d}^2 c_{n_\delta} \, \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\rm low}(\mu, \{c_k\})} \leqslant \Xi_l(\mu) \leqslant \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \dots \mathrm{d}^2 c_{n_\delta} \, \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\rm up}(\mu, \{c_k\})}. \tag{3.8}$$

By a straightforward generalization of the arguments in [5] to the case of multi-mode coherent projection $|c\rangle\langle c|$, we obtain for any $\{\widetilde{c}_k\}_{k \in I_\delta}$ the bound

$$\mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\rm low}(\mu, \{\widetilde{c}_k\})} \leqslant \Xi_l(\mu)$$

on the *integrand* in the left-hand side of (3.8). This in particular implies

$$\max_{\{c_k\} \in \mathbb{C}^{n_\delta}} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\rm low}(\mu, \{c_k\})} \leqslant \Xi_l(\mu) , \tag{3.9}$$

i.e. the estimate of the grand partition function from *below*.

To find a similar bound for the right-hand side of (3.8) from *above*, we note that $H_l^{\rm low}(\mu, \{c_k\})$ and $H_l^{\rm up}(\mu, \{c_k\})$ are related by

$$H_l^{\rm up}(\mu, \{c_k\}) = H_l^{\rm low}(\mu, \{c_k\}) + \kappa(\mu, \{c_k\}) ,$$

see Appendix A, equation (A.22) for an explicit expression of $\kappa(\mu, \{c_k\})$. If this is combined with the Bogoliubov convexity inequality:

$$\begin{aligned} & \ln \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \dots \mathrm{d}^2 c_{n_\delta} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\rm up}(\mu, \{c_k\})} - \ln \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \dots \mathrm{d}^2 c_{n_\delta} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\rm low}(\mu, \{c_k\})} \\ \leqslant \; & \frac{\int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \dots \mathrm{d}^2 c_{n_\delta} \mathrm{Tr}_{\mathscr{F}_l^\perp} \big(- \kappa(\mu, \{c_k\}) e^{-\beta H_l^{\rm up}(\mu, \{c_k\})} \big)}{\int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \dots \mathrm{d}^2 c_{n_\delta} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\rm up}(\mu, \{c_k\})}} , \end{aligned} \tag{3.10}$$

then (3.8) and (3.10) provide the upper bound:

$$\ln \mathrm{Tr}_{\mathscr{F}_l} e^{-\beta H_l(\mu)} \leqslant \ln \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \dots \mathrm{d}^2 c_{n_\delta} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\rm low}(\mu, \{c_k\})} - \langle \kappa(\mu, \{c_k\}) \rangle_{up} . \tag{3.11}$$

Using the orthogonal projection $P_\delta : \mathscr{H}_l \mapsto \mathscr{H}_l^\delta$, and in view of (A.22), one can estimate the last term in (3.11) explicitly:

$$- \kappa(\mu, \{c_k\}) \leqslant \mathrm{Tr}(h_l^\omega - \mu) P_\delta \tag{3.12}$$

$$
\begin{aligned}
&- \gamma\Big(\nu_l^0(\delta) + \frac{V_l}{2}\big(\nu_l^0(\delta)\big)^2 + V_l\nu_l^0(\delta)\nu_l^0(2\delta)\Big) \\
&+ \frac{\gamma}{2}\big(\frac{4}{V_l} + 2\nu_l^0(\delta) + 2\nu_l^0(2\delta)\big)\sum_{k\in I_\delta}|c_k|^2 \\
&+ \frac{\gamma}{2}\big(2\nu_l^0(\delta) + 2\nu_l^0(\delta)\big)\sum_{k\in I_\delta^c} a_k^* a_k \\
&\leqslant \mathrm{Tr}\big((h_l^\omega - \mu)P_\delta\big) - \gamma\nu_l^0(\delta)\Big(1 - 4V_l\nu_l^0(2\delta) + \frac{V_l}{2}\nu_l^0(\delta) + V_l\nu_l^0(2\delta)\Big) \\
&+ 4\gamma\nu_l^0(2\delta)\Big(\sum_{k\in I_\delta}\big(|c_k|^2 - 1\big) + \sum_{k\in I_\delta^c} a_k^* a_k\Big)\ .
\end{aligned}
$$

Here $I_\delta^c := \Lambda_l^* \setminus I_\delta$. Then taking into account the upper symbol of the total number operator, we find that

$$
H_l^{\mathrm{up}}(\mu, \{c_k\}) + a\Big(\sum_{k\in I_\delta}(|c_k|^2 - 1) + \sum_{k\in I_\delta^c} a_k^* a_k\Big) \ = \ H_l^{\mathrm{up}}(\mu - a, \{c_k\})\ , \tag{3.13}
$$

which together with equation (3.11) provides the following estimate

$$
\ln \mathrm{Tr}_{\mathscr{F}_l} e^{-\beta H_l(\mu)} \ \leqslant \ \ln \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2c_1 \ldots \mathrm{d}^2c_{n_\delta}\, \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\mathrm{low}}(\mu,\{c_k\})} \tag{3.14}
$$

$$
\begin{aligned}
&+ \mathrm{Tr}\big((h_l^\omega - \mu)P_\delta\big) - \gamma\nu_l^0(\delta)\Big(1 - 4V_l\nu_l^0(2\delta) + \frac{V_l}{2}\nu_l^0(\delta) + V_l\nu_l^0(2\delta)\Big) \\
&+ 4\gamma\nu_l^0(2\delta)\partial_\mu \ln \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2c_1 \ldots \mathrm{d}^2c_{n_\delta}\, \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\mathrm{low}}(\mu,\{c_k\})}
\end{aligned} \tag{3.15}
$$

To finish the proof, we need three lemmas. We postpone their proofs to the next section.

Lemma 3.1 *Suppose that the system of interacting bosons (2.8) has a bounded limiting particle density $\rho(\mu)$ for any fixed $\mu \in \mathbb{R}$:*

$$
\rho(\mu) := \partial_\mu p(\mu) := \partial_\mu \lim_{l\to\infty} p_l(\beta, \mu) \ < \ \infty \tag{3.16}
$$

see (2.9). Then one gets the estimates:

$$
\limsup_{\delta\downarrow 0}\limsup_{l\to\infty} \frac{1}{\beta V_l}\partial_\mu p_{l,\delta}^{low}(\mu) \ \leqslant \ \rho(\mu), \tag{3.17}
$$

$$
\limsup_{\delta\downarrow 0}\limsup_{l\to\infty} \frac{1}{\beta V_l}\partial_\mu p_{l,\delta}^{up}(\mu) \ \leqslant \ \rho(\mu)\ . \tag{3.18}
$$

Next, we relate the integrated pressure $p_{l,\delta}^{low}(\mu)$ defined by (3.6) to the maximum of the corresponding integrand.

Lemma 3.2 *For any $\alpha > 1$, one has the estimate:*

$$
\begin{aligned}
&\frac{1}{\beta V_l}\ln \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2c_1 \ldots \mathrm{d}^2c_{n_\delta}\, \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{low}(\mu,\{c_k\})} \\
\leqslant\ & \frac{1}{\beta V_l}\ln \max_{\{c_k\}} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{low}(\mu,\{c_k\})} \ - \ \frac{1}{\beta V_l}\ln(1 - 1/\alpha) \ + \ \frac{\nu_l^0(\delta)}{\beta}\ln(\alpha\partial_\mu p_{l,\delta}^{low}(\mu)) \\
+\ & \frac{1}{\beta}\nu_l^0(\delta) - \frac{1}{2\beta}\frac{\ln V_l}{V_l} - \frac{\nu_l^0(\delta)}{\beta}\ln\big(\nu_l^0(\delta)\big) - \frac{1}{2\beta V_l}\ln\big(\nu_l^0(\delta)\big).
\end{aligned} \tag{3.19}
$$

Notice that above statements are independent of the possible presence of random potentials. The next Lemma serves to control (ergodic) random external potentials.

Lemma 3.3 *Taking into account our assumptions on the random potentials, see Section 2, one gets the following inequality:*

$$\limsup_{l\to\infty} \frac{1}{V_l}\mathrm{Tr}(h_l^\omega - \mu)P_\delta \leqslant \nu^0(\delta)\Big((\delta-\mu) + \mathbb{E}\big(v^\omega(0)\big)\Big) .$$

Here $\mathbb{E}(\cdot)$ *denotes the expectations in the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

Returning back to the proof of Theorem 3.1, we get from (3.9) and (3.14) the estimates:

$$\max_{\{c_k\}} p_{l,\delta}^{low}(\mu,\{c_k\}) \leqslant p_l(\mu) \leqslant p_{l,\delta}^{low}(\mu) + \frac{1}{\beta V_l}\Big(\mathrm{Tr}\big((h_l^\omega-\mu)P_\delta\big) - \gamma\nu_l^0(\delta)\big(1 - 4V_l\nu_l^0(2\delta) + \frac{V_l}{2}\nu_l^0(\delta) + V_l\nu_l^0(2\delta)\big)\Big) + 4\gamma\nu_l^0(2\delta)\frac{1}{\beta V_l}\partial_\mu \ln\int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\mathrm{low}}(\mu,\{c_k\})} .$$

By Lemma 3.2, this implies for any configuration ω the estimates:

$$\max_{\{c_k\}} p_{l,\delta}^{low}(\mu,\{c_k\}) \leqslant p_l(\mu) \leqslant \max_{\{c_k\}} p_{l,\delta}^{low}(\mu,\{c_k\}) + K^\omega(l,\delta) , \tag{3.20}$$

where the random parameter $K^\omega(l,\delta)$ is given by

$$\begin{aligned} K^\omega(l,\delta) : &= \frac{1}{\beta V_l}\Big(\mathrm{Tr}\big((h_l^\omega-\mu)P_\delta\big) - \gamma\nu_l^0(\delta)\big(1 - 4V_l\nu_l^0(2\delta) + \frac{V_l}{2}\nu_l^0(\delta) + V_l\nu_l^0(2\delta)\big)\Big) \\ &+ 4\gamma\nu_l^0(2\delta)\frac{1}{\beta V_l}\partial_\mu \ln\int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\mathrm{low}}(\mu,\{c_k\})}. \end{aligned}$$

Note that, by Lemmas 3.1 and 3.3, we can control this error term since for any configuration one gets:

$$\lim_{\delta\downarrow 0}\liminf_{l\to\infty} K^\omega(l,\delta) = \lim_{\delta\downarrow 0}\limsup_{l\to\infty} K^\omega(l,\delta) = 0 .$$

Hence, (3.20) yields

$$\begin{aligned} \limsup_{l\to\infty}\max_{\{c_k\}} p_{l,\delta}^{low}(\mu,\{c_k\}) \leqslant p(\mu) \leqslant \limsup_{l\to\infty}\max_{\{c_k\}} p_{l,\delta}^{low}(\mu,\{c_k\}) + \liminf_{l\to\infty} K^\omega(l,\delta) \\ \limsup_{\delta\downarrow 0}\limsup_{l\to\infty}\max_{\{c_k\}} p_{l,\delta}^{low}(\mu,\{c_k\}) \leqslant p(\mu) \leqslant \liminf_{\delta\downarrow 0}\limsup_{l\to\infty}\max_{\{c_k\}} p_{l,\delta}^{low}(\mu,\{c_k\}) , \end{aligned}$$

which proves the first equality in (3.4) in Theorem 3.1. The second one can be proven in a similar way.

3.3 Proofs of technical results

Recall that the *finite-volume* IDS (2.3): $\nu_l^0(\delta) \geq 0$, converges in the limit $l \to \infty$ to the Weyl formula $\nu^0(\delta) = C_d\delta^{d/2}$, uniformly in δ on any finite interval $0 \leq \delta \leq E$. Below we use this uniformity in a systematic way.

Proof of Lemma 3.1
Since

$$H_l^{\rm low}(\mu,\{c_k\}) + a\Big(\sum_{k\in I_\delta}|c_k|^{\,2} + \sum_{k\in I_\delta^c} a_k^* a_k\Big) \;=\; H_l^{\rm low}(\mu - a,\{c_k\})\;, \tag{3.21}$$

the estimate (3.12) yields:

$$\begin{aligned} H_l^{\rm up}(\mu,\{c_k\}) \;&\geqslant\; H_l^{\rm low}(\mu + 4\gamma\nu_l^0(2\delta),\{c_k\}) \\ &-\; {\rm Tr}\big((h_l^\omega - \mu)P_\delta\big) - \gamma\nu_l^0(\delta)\Big(1 + \frac{V_l}{2}\nu_l^0(\delta) + \nu_l^0(2\delta)\Big). \end{aligned}$$

Applying now the Bogoliubov-Peierls and the Berezin-Lieb inequalities, we obtain:

$$p_{l,\delta}^{low}(\mu) \;\leqslant\; p_l(\mu) \;\leqslant\; p_{l,\delta}^{up}(\mu) \;\leqslant\; p_{l,\delta}^{low}(\mu + 4\gamma\nu_l^0(2\delta)) + \frac{M^\omega(l,\delta)}{V_l}\;,$$

where

$$M^\omega(l,\delta) \;:=\; {\rm Tr}\big((h_l^\omega - \mu)P_\delta\big) + \gamma\nu_l^0(\delta)\Big(1 + \frac{V_l}{2}\nu_l^0(\delta) + \nu_l^0(2\delta)\Big)\;. \tag{3.22}$$

Consequently, for any configuration ω one gets in the limit the estimates:

$$\begin{aligned} \limsup_{l\to\infty} p_{l,\delta}^{low}(\mu) \leqslant p(\mu) \leqslant \liminf_{l\to\infty} p_{l,\delta}^{up}(\mu) \leqslant \limsup_{l\to\infty} p_{l,\delta}^{up}(\mu) \;&\leqslant\; \liminf_{l\to\infty} p_{l,\delta}^{low}(\mu + 4\gamma\nu_l^0(2\delta)) \\ &+\; \liminf_{l\to\infty} \frac{M^\omega(l,\delta)}{V_l}\;. \end{aligned} \tag{3.23}$$

Since $p_{l,\delta}^{up}(\mu)$ is a convex functions of μ, for any $t>0$ we have

$$\partial_\mu p_{l,\delta}^{up}(\mu) \;\leqslant\; \frac{1}{t}\big(p_{l,\delta}^{up}(\mu+t) - p_{l,\delta}^{up}(\mu)\big)\;.$$

Then by virtue of estimates (3.23) we obtain:

$$\begin{aligned} \limsup_{l\to\infty} \partial_\mu p_{l,\delta}^{up}(\mu) \;&\leqslant\; \frac{1}{t}\big(\limsup_{l\to\infty} p_{l,\delta}^{up}(\mu+t) - \liminf_{l\to\infty} p_{l,\delta}^{up}(\mu)\big) \\ &\leqslant\; \frac{1}{t}\big(\liminf_{l\to\infty} p_{l,\delta}^{low}(\mu+t+4\gamma\nu_l^0(2\delta)) + \liminf_{l\to\infty}\frac{1}{V_l}M^\omega(l,\delta,\mu) - p(\mu)\big) \\ &\leqslant\; \frac{1}{t}\big(p(\mu+t+4\gamma\nu^0(2\delta)) + \liminf_{l\to\infty}\frac{1}{V_l}M^\omega(l,\delta,\mu) - p(\mu)\big)\;. \end{aligned} \tag{3.24}$$

Since by (3.22) and Lemma 3.3 for any configuration ω one gets:

$$\lim_{\delta\downarrow 0}\;\liminf_{l\to\infty}\frac{1}{V_l}M^\omega(l,\delta) \;=\; 0\;,$$

we obtain from (3.24)

$$\limsup_{\delta\downarrow 0}\limsup_{l\to\infty}\partial_\mu p_{l,\delta}^{up}(\mu) \;\leqslant\; \frac{1}{t}\big(p(\mu+t) - p(\mu)\big) \tag{3.25}$$

which is valid for any $t>0$. Letting $t\downarrow 0$ leads to the second estimate (3.18).

Similarly one proves the first estimate (3.17). Again using the convexity and (3.23) we obtain

$$\begin{aligned}\limsup_{l\to\infty}\partial_\mu p^{low}_{l,\delta}(\mu) &\leqslant \frac{1}{t}\big(\limsup_{l\to\infty} p^{low}_{l,\delta}(\mu+t)-\liminf_{l\to\infty} p^{low}_{l,\delta}(\mu)\big)\\ &\leqslant \frac{1}{t}\big(\limsup_{l\to\infty} p_l(\mu+t)\\ &- \limsup_{l\to\infty} p^{up}_{l,\delta}(\mu-4\gamma\nu^0_l(2\delta))-\liminf_{l\to\infty}\frac{1}{V_l}M^\omega(l,\delta,\mu-4\gamma\nu^0_l(2\delta))\big)\ .\end{aligned}$$

Applying the estimate (3.23) to the last inequality once more, we get

$$\limsup_{l\to\infty}\partial_\mu p^{low}_{l,\delta}(\mu) \leqslant \frac{1}{t}\big(p(\mu+t)-p(\mu-4\gamma\nu^0(2\delta))-\liminf_{l\to\infty}\frac{1}{V_l}M^\omega(l,\delta,\mu-4\gamma\nu^0_l(2\delta))\big)$$

and in view of the limit (3.22) and Lemma 3.3, the result follows by letting $\delta\downarrow 0$ and then, letting $t\downarrow 0$. □

Proof of Lemma 3.2
Let

$$\mathbb{C}^{n_\delta}_\xi := \{z\in\mathbb{C}^{n_\delta} : |z|^2\leqslant\xi\}.$$

and we denote the volume of this ball by $\mathrm{Vol}(\mathbb{C}^{n_\delta}_\xi)=\pi^{n_\delta}\xi^{n_\delta}/n_\delta\Gamma(n_\delta)$.

We then obtain the following bound

$$\begin{aligned}&\int_{\mathbb{C}^{n_\delta}}\mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}\\ =&\int_{\mathbb{C}^{n_\delta}_\xi}\mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}+\int_{\mathbb{C}^{n_\delta}\setminus\mathbb{C}^{n_\delta}_\xi}\mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}\\ \leqslant&\frac{\mathrm{Vol}(\mathbb{C}^{n_\delta}_\xi)}{\pi^{n_\delta}}\max_{\{c_k\}}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}+\frac{1}{\xi\pi^{n_\delta}}\int_{\mathbb{C}^{n_\delta}}\mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\big(\sum_{k\in I_\delta}|c_k|^2\big)\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}\\ \leqslant&\frac{\mathrm{Vol}(\mathbb{C}^{n_\delta}_\xi)}{\pi^{n_\delta}}\max_{\{c_k\}}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}+\frac{1}{\xi\pi^{n_\delta}}\langle\sum_{k\in I_\delta}|c_k|^2\rangle_{low}\int_{\mathbb{C}^{n_\delta}}\mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}.\end{aligned}$$

Here we used the expectation $\langle\,\cdot\,\rangle_{low}$ defined by the integrated partition function $\Xi^{low}_l(\mu)$, see (3.6). Notice that, using the explicit form of the lower symbol for the total particle number operator, see (3.21), we can represent the last integral in the form:

$$\begin{aligned}&\int_{\mathbb{C}^{n_\delta}}\mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}\\ \leqslant&\frac{\mathrm{Vol}(\mathbb{C}^{n_\delta}_\xi)}{\pi^{n_\delta}}\max_{\{c_k\}}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}+\frac{1}{\xi\pi^{n_\delta}}\big(V_l\partial_\mu p^{low}_{l,\delta}(\mu)\big)\int_{\mathbb{C}^{n_\delta}}\mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}\\ =&\big(\frac{\xi^{n_\delta}}{n_\delta\Gamma(n_\delta)}\big)\max_{\{c_k\}}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}+\frac{1}{\xi\pi^{n_\delta}}\big(V_l\partial_\mu p^{low}_{l,\delta}(\mu)\big)\int_{\mathbb{C}^{n_\delta}}\mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}\ .\end{aligned}$$

Thus, we get that

$$\big(1-\frac{V_l}{\xi}\partial_\mu p^{low}_{l,\delta}(\mu)\big)\int_{\mathbb{C}^{n_\delta}}\mathrm{d}^2c_1\ldots\mathrm{d}^2c_{n_\delta}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}\leqslant\big(\frac{\xi^{n_\delta}}{n_\delta\Gamma(n_\delta)}\big)\max_{\{c_k\}}\mathrm{Tr}_{\mathscr{F}_l^\perp}e^{-\beta H^{\mathrm{low}}_l(\mu,\{c_k\})}\ .$$

If we put $\xi := \alpha V_l \partial_\mu p_{l,\delta}^{low}(\mu)$ for some $\alpha > 1$, and use the Stirling formula for large n_δ, then the coefficient in the right-hand side can be estimated as:

$$\frac{\xi^{n_\delta}}{n_\delta \Gamma(n_\delta)} \leqslant \frac{\big(\alpha V_l \partial_\mu p_{l,\delta}^{low}(\mu)\big)^{n_\delta}}{n_\delta n_\delta^{n_\delta - 1/2} e^{-n_\delta}} \leqslant \frac{\Big(\big(\alpha \partial_\mu p_{l,\delta}^{low}(\mu)\big)^{\nu_l^0(\delta)}\Big)^{V_l}}{\big(V_l \nu_l^0(\delta)\big)^{V_l \nu_l^0(\delta) + 1/2}} e^{-n_\delta} \ .$$

Hence, one finally obtains the estimate:

$$\begin{aligned} &\int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \ldots \mathrm{d}^2 c_{n_\delta} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\mathrm{low}}(\mu, \{c_k\})} \\ \leqslant\ & \frac{1}{1 - 1/\alpha} \Big(\big(\alpha \partial_\mu p_{l,\delta}^{low}(\mu)\big)^{\nu_l^0(\delta)}\Big)^{V_l} V^{-1/2} \big(\nu_l^0(\delta)\big)^{-(\nu_l^0(\delta) V_l + 1/2)} e^{\nu_l^0(\delta) V_l} \max_{\{c_k\}} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\mathrm{low}}(\mu, \{c_k\})}, \end{aligned}$$

which leads to the desired result

$$\begin{aligned} & \frac{1}{\beta V_l} \ln \int_{\mathbb{C}^{n_\delta}} \mathrm{d}^2 c_1 \ldots \mathrm{d}^2 c_{n_\delta} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\mathrm{low}}(\mu, \{c_k\})} \\ \leqslant\ & \frac{1}{\beta V_l} \ln \max_{\{c_k\}} \mathrm{Tr}_{\mathscr{F}_l^\perp} e^{-\beta H_l^{\mathrm{low}}(\mu, \{c_k\})} - \frac{1}{\beta V_l} \ln(1 - 1/\alpha) + \frac{\nu_l^0(\delta)}{\beta} \ln(\alpha \partial_\mu p_{l,\delta}^{low}(\mu)) \\ +\ & \frac{1}{\beta} \nu_l^0(\delta) - \frac{1}{2\beta} \frac{\ln V_l}{V_l} - \frac{\nu_l^0(\delta)}{\beta} \ln\big(\nu_l^0(\delta)\big) - \frac{1}{2\beta V_l} \ln\big(\nu_l^0(\delta)\big). \end{aligned} \tag{3.26}$$

□

Proof of Lemma 3.3

First we note that since the projection P_δ is constructed with respect to the basis of eigenvectors of h_l^0, one gets:

$$\begin{aligned} \frac{1}{V_l} \mathrm{Tr}(h_l^\omega - \mu) P_\delta &= \frac{1}{V_l} \mathrm{Tr}(h_l^0 - \mu) P_\delta + \frac{1}{V_l} \mathrm{Tr}(v^\omega \restriction_{\Lambda_l}) P_\delta \\ &\leqslant (\delta - \mu) \nu_l^0(\delta) + \frac{1}{V_l} \sum_{k \in I_\delta} (\psi_k^l, v^\omega \psi_k^l) \ , \end{aligned}$$

and consequently:

$$\begin{aligned} \frac{1}{V_l} \mathrm{Tr}(h_l^\omega - \mu) P_\delta &\leqslant (\delta - \mu) \nu_l^0(\delta) + \int_{[0,\delta)} (\psi_k^l, v^\omega \psi_k^l)\, \nu_l^0(\mathrm{d}k) \\ &= (\delta - \mu) \nu_l^0(\delta) + \int_{[0,\delta)} \nu_l^0(\mathrm{d}k) \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x\ v^\omega(x) \\ &= \nu_l^0(\delta) \Big((\delta - \mu) + \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x\ v^\omega(x)\Big) \ . \end{aligned}$$

Thus, the ergodic theorem yields:

$$\limsup_{l \to \infty} \frac{1}{V_l} \mathrm{Tr}(h_l^\omega - \mu) P_\delta \leqslant \nu^0(\delta) \Big((\delta - \mu) + \mathbb{E}_\omega\big(v^\omega(0)\big)\Big) \ .$$

□

4 Concluding remarks

We discuss here the meaning of our main result and in particular the way to interpret the solutions of the variational problem established in Theorem 3.1. Note that the theorem is established for a random system in the thermodynamic limit. Its aim is to take into account the possibility of type III generalized condensation, which we believe is favored by the randomness even in interacting systems.

First we recall a result established in [5], cf. Section 1.1. For a homogeneous system and the single-mode substitution at $k = 0$, the solution of the variational problem gives the *total* condensate density in the mode $k = 0$, *if* one adds to the Hamiltonian the *zero-mode* gauge-breaking term (quasi-average sources):

$$H_l(\mu;\eta) \;:=\; H_l(\mu) \;+\; \sqrt{V_l}\,\left(\overline{\eta}\,a_0 + \eta\,a_0^*\right)\;.$$

This means that after the Bogoliubov c-number substitution the solution $\alpha_{\max,l}(\beta,\mu;\eta)$ of the (finite-volume) variational problem not only provides the right pressure in the thermodynamic limit, but it also coincides with quasi-averages that give the *total* amount of the condensate in the *zero* mode:

$$\lim_{|\eta|\downarrow 0}\,\lim_{l\to\infty}\alpha_{\max,l}(\beta,\mu;\eta) \;=\; \lim_{|\eta|\downarrow 0}\,\lim_{l\to\infty}\langle a_0^* a_0/V_l\rangle_l(\beta,\mu;\eta)\;.$$

Here $\langle -\rangle_l(\beta,\mu;\eta)$ is the equilibrium state defined by $H_l(\mu;\eta)$.

Using a simple example, we discuss the relevance of this quasi-average approach to more subtle cases of the condensation of type II and III. We show that the Bogoliubov quasi-average sources breaking the gauge invariance [2] are able to alter the fine structure of the condensate reducing it to one-mode (type I).

To see this, consider the perfect Bose-gas in a cubic three-dimensional anisotropic parallelepiped $\Lambda_l := V_l^{\alpha_x}\times V_l^{\alpha_y}\times V_l^{\alpha_z}$, with periodic boundary condition and $\alpha_x \geq \alpha_y \geq \alpha_z$, $\alpha_x+\alpha_y+\alpha_z = 1$. The Hamiltonian is:

$$H_l^0(\mu) \;:=\; \sum_{k\in\Lambda_l^*}(\varepsilon_k-\mu)a_k^* a_k\;, \tag{4.27}$$

with quadratic spectrum and $\varepsilon_{k=0} = 0$.

It is well known that this system exhibits a generalised condensation of type II for $\alpha_x = 1/2$ and of type III for $\alpha_x > 1/2$ for a standard critical density ρ_c, whereas for $\alpha_x < 1/2$, the whole condensate is sitting in the mode $k = 0$, i.e, in the ground state (type I) [6]. Consider the system (4.27) with the quasi-average source in a single mode $\tilde{k}$:

$$H_l^0(\mu;\eta) \;:=\; H_l^0(\mu) \;+\; \sqrt{V_l}\,\left(\overline{\eta}\;a_{\tilde{k}} + \eta\;a_{\tilde{k}}^*\right)\;. \tag{4.28}$$

Then for a fixed density $\overline{\rho}$, the finite-volume equation which defines the corresponding chemical potential $\mu = \overline{\mu}_l(\overline{\rho},\eta)$ takes the form:

$$\begin{aligned}\overline{\rho} \;&=\; \rho_l(\beta,\mu,\eta) \;:=\; \frac{1}{V_l}\sum_{k\in\Lambda_l^*}\langle a_k^* a_k\rangle_l^0(\beta,\mu,\eta) \qquad (4.29)\\ &=\; \frac{1}{V_l}(e^{\beta(\varepsilon_{\tilde{k}}-\mu)}-1)^{-1} \;+\; \frac{1}{V_l}\sum_{k\neq\tilde{k}}\frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} \;+\; \frac{|\eta|^{\,2}}{(\varepsilon_{\tilde{k}}-\mu)^{\,2}}\;.\end{aligned}$$

To investigate a possible condensation, we must take the thermodynamic limit in the right-hand side of (4.29), and then switch off the source, that is let $|\eta| \to 0$. Let us denote by $I(\beta,\mu)$ the limit of $\rho_l(\beta,\mu,\eta=0)$, that is the limiting density function of the *gauge-invariant* system,

$$I(\beta,\mu) \;=\; \lim_{l\to\infty} \frac{1}{V_l} \sum_{k\neq\tilde{k}} \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} \;=\; \int_{\mathbb{R}} \nu^0(\mathrm{d}\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)}-1} \;.$$

with critical density $\rho_c := \sup_{\mu<0} I(\mu)$.

Now we have to distinguish two cases:
(i) For any $\tilde{k}$ such that $\lim_{l\to\infty} \varepsilon_{\tilde{k}} > 0$, we obtain from (4.29)

$$\overline{\rho} \;=\; \lim_{|\eta|\to 0} \lim_{l\to\infty} \rho_l(\beta,\mu,\eta) \;=\; I(\beta,\mu)$$

i.e. the quasi-average coincides with the average and we return to the analysis of the condensate equation (4.29) for $\eta = 0$. This gives again all possible types of condensation as a function of α_x.
(ii) On the other hand, if $\tilde{k}$ is such that $\lim_{l\to\infty} \varepsilon_{\tilde{k}} = 0$, then the condensate equation (4.29) yields for the quasi-average of the total particle density:

$$\overline{\rho} \;=\; \lim_{|\eta|\to 0} \lim_{l\to\infty} \rho_l(\beta,\mu,\eta) \;=\; I(\beta,\mu) + \lim_{\eta\to 0} \frac{|\eta|^2}{\mu^2} \;. \tag{4.30}$$

If $\overline{\rho} \leq \rho_c$, then the asymptotic solution of (4.30) is $\overline{\mu}_\infty(\overline{\rho}) = \lim_{\eta\to 0} \lim_{l\to\infty} \overline{\mu}_l(\overline{\rho},\eta) < 0$ and there is no condensation in any mode.
If $\overline{\rho} > \rho_c$, then $\lim_{\eta\to 0} |\eta|^2/\overline{\mu}_\infty(\overline{\rho},\eta)^2 = \overline{\rho} - \rho_c$. By explicit calculation one also gets that only $\tilde{k}$-mode quasi-average is non-zero:

$$\lim_{\eta\to 0} \lim_{l\to\infty} \frac{1}{V_l} \langle a^*_{\tilde{k}} a_{\tilde{k}} \rangle^0_l(\beta,\overline{\mu}_l,\eta) = \lim_{\eta\to 0} \lim_{l\to\infty} \left\{ \frac{1}{V_l} \frac{1}{e^{\beta(\varepsilon_{\tilde{k}}-\overline{\mu}_l(\overline{\rho},\eta))}-1} + \frac{|\eta|^2}{\overline{\mu}_l^{\,2}} \right\} = \overline{\rho} - \rho_c \;, \tag{4.31}$$

i.e. for any α_x the condensation is type I. Recall that the only condition on $\tilde{k}$ is that the corresponding eigenvalue $\varepsilon_{\tilde{k}}$ vanishes in the infinite volume limit. Since

$$\lim_{\eta\to 0} \lim_{l\to\infty} \frac{1}{V_l} \langle a^*_0 a_0 \rangle^0_l(\beta,\overline{\mu}_l,\eta) = \lim_{\eta\to 0} \lim_{l\to\infty} \frac{1}{V_l} \frac{1}{e^{\beta(-\overline{\mu}_l(\overline{\rho},\eta))}-1} = 0 \;, \tag{4.32}$$

by virtue of (4.31) and (4.32) we see that the Bogoliubov quasi-average procedure not only transforms the generalised condensates of type II or III into a one-mode condensate (i.e., type I), but this mode does not even need to be the ground-state. Therefore, using the quasi-average approach [2], one can force the condensate to be in any given mode $\tilde{k}$, as long as its energy $\varepsilon_{\tilde{k}}$ vanishes in the limit $l \to \infty$.

Appendix

A The approximating Hamiltonians

In this section, we provide an explicit form of the upper and lower symbols for the Hamiltonian (2.8). For a short-hand we put $E_i^{\omega,l} = E_i$. We note that, as the coherent vector

(3.5) is defined as a tensor product of one-mode coherent states, its effect on each creation/annihilation operator $a_k^\sharp$ is independent of all the others modes operators. First, we give an explicit form of the lower symbol of the full Hamiltonian (2.8):

$$
\begin{aligned}
H_l^{low}(\mu,\{c_k\}) &= \sum_{k\in I_\delta}\big(\sum_{i\geqslant 1}|(\phi_i,\psi_k)|^2(E_i-\mu)\big)|c_k|^2 && \text{(A.1)}\\
&+ \sum_{k\in I_\delta^c}\big(\sum_{i\geqslant 1}|(\phi_i,\psi_k)|^2(E_i-\mu)\big)a_k^*a_k && \text{(A.2)}\\
&+ \sum_{k\in I_\delta,k'\in I_\delta^c}\big(\sum_{i\geqslant 1}(\phi_i,\psi_k)(\psi_{k'},\phi_i)(E_i-\mu)\big)\overline{c_k}a_{k'} && \text{(A.3)}\\
&+ \sum_{k\in I_\delta^c,k'\in I_\delta}\big(\sum_{i\geqslant 1}(\phi_i,\psi_k)(\psi_{k'},\phi_i)(E_i-\mu)\big)c_ka_k^* && \text{(A.4)}\\
&+ \sum_{k,k'\in I_\delta^c,k\neq k'}\big(\sum_{i\geqslant 1}(\phi_i,\psi_k)(\psi_{k'},\phi_i)(E_i-\mu)\big)a_k^*a_{k'} && \text{(A.5)}\\
&+ \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta}\sum_{\substack{q:\,k+q\in I_\delta^c\\ k'-q\in I_\delta^c}}\hat{u}(q)\ c_kc_{k'}\ a_{k+q}^*a_{k'-q}^* && \text{(A.6)}\\
&+ \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta}\sum_{\substack{q:\,k+q\in I_\delta\\ k'-q\in I_\delta^c}}\hat{u}(q)\ \overline{c}_{k+q}c_kc_{k'}\ a_{k'-q}^* && \text{(A.7)}\\
&+ \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta}\sum_{\substack{q:\,k+q\in I_\delta^c\\ k'-q\in I_\delta}}\hat{u}(q)\ \overline{c}_{k'-q}c_kc_{k'}\ a_{k+q}^* && \text{(A.8)}\\
&+ \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta}\sum_{\substack{q:\,k+q\in I_\delta\\ k'-q\in I_\delta}}\hat{u}(q)\ \overline{c}_{k+q}\overline{c}_{k'-q}c_kc_{k'} && \text{(A.9)}\\
&+ \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta^c}\sum_{\substack{q:\,k+q\in I_\delta^c\\ k'-q\in I_\delta^c}}\hat{u}(q)\ c_{k'}\ a_{k+q}^*a_{k'-q}^*a_k && \text{(A.10)}\\
&+ \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta^c}\sum_{\substack{q:\,k+q\in I_\delta\\ k'-q\in I_\delta^c}}\hat{u}(q)\ \overline{c}_{k+q}c_k\ a_{k'-q}^*a_{k'} && \text{(A.11)}\\
&+ \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta^c}\sum_{\substack{q:\,k+q\in I_\delta^c\\ k'-q\in I_\delta}}\hat{u}(q)\ \overline{c}_{k'-q}c_k\ a_{k+q}^*a_{k'} && \text{(A.12)}\\
&+ \frac{1}{2V_l}\sum_{k\in I_\delta}\sum_{k'\in I_\delta^c}\sum_{\substack{q:\,k+q\in I_\delta\\ k'-q\in I_\delta}}\hat{u}(q)\ \overline{c}_{k+q}\overline{c}_{k'-q}c_k\ a_{k'} && \text{(A.13)}\\
&+ \frac{1}{2V_l}\sum_{k\in I_\delta^c}\sum_{k'\in I_\delta}\sum_{\substack{q:\,k+q\in I_\delta^c\\ k'-q\in I_\delta^c}}\hat{u}(q)\ c_{k'}\ a_{k+q}^*a_{k'-q}^*a_k && \text{(A.14)}\\
&+ \frac{1}{2V_l}\sum_{k\in I_\delta^c}\sum_{k'\in I_\delta}\sum_{\substack{q:\,k+q\in I_\delta\\ k'-q\in I_\delta^c}}\hat{u}(q)\ \overline{c}_{k+q}c_{k'}\ a_{k'-q}^*a_k && \text{(A.15)}
\end{aligned}
$$

$$+ \quad \frac{1}{2V_l} \sum_{k\in I_\delta^c} \sum_{k'\in I_\delta} \sum_{\substack{q:\, k+q\in I_\delta^c \\ k'-q\in I_\delta}} \hat{u}(q)\ \overline{c}_{k'-q} c_{k'}\ a^*_{k+q} a_k \tag{A.16}$$

$$+ \quad \frac{1}{2V_l} \sum_{k\in I_\delta^c} \sum_{k'\in I_\delta} \sum_{\substack{q:\, k+q\in I_\delta \\ k'-q\in I_\delta}} \hat{u}(q)\ \overline{c}_{k+q} \overline{c}_{k'-q} c_{k'}\ a_k \tag{A.17}$$

$$+ \quad \frac{1}{2V_l} \sum_{k\in I_\delta^c} \sum_{k'\in I_\delta^c} \sum_{\substack{q:\, k+q\in I_\delta^c \\ k'-q\in I_\delta^c}} \hat{u}(q)\ a^*_{k+q} a^*_{k'-q} a_{k'} a_k \tag{A.18}$$

$$+ \quad \frac{1}{2V_l} \sum_{k\in I_\delta^c} \sum_{k'\in I_\delta^c} \sum_{\substack{q:\, k+q\in I_\delta \\ k'-q\in I_\delta^c}} \hat{u}(q)\ \overline{c}_{k+q}\ a^*_{k'-q} a_{k'} a_k \tag{A.19}$$

$$+ \quad \frac{1}{2V_l} \sum_{k\in I_\delta^c} \sum_{k'\in I_\delta^c} \sum_{\substack{q:\, k+q\in I_\delta^c \\ k'-q\in I_\delta}} \hat{u}(q)\ \overline{c}_{k'-q}\ a^*_{k+q} a_{k'} a_k \tag{A.20}$$

$$+ \quad \frac{1}{2V_l} \sum_{k\in I_\delta^c} \sum_{k'\in I_\delta^c} \sum_{\substack{q:\, k+q\in I_\delta \\ k'-q\in I_\delta}} \hat{u}(q)\ \overline{c}_{k+q} \overline{c}_{k'-q}\ a_{k'} a_k \tag{A.21}$$

Now, we give an explicit form of the upper symbols. We recall the general form of this symbols for polynomials in the creation/annihilation operators of some mode $k \in I_\delta$:

$$(a_k)^{up} = c_k, \quad (a_k^*)^{up} = \overline{c}_k, \quad (a_k a_k)^{up} = (c_k)^2, \quad (a_k^* a_k)^{up} = (\overline{c}_k)^2$$
$$(a_k^* a_k)^{up} = |c_k|^2 - 1, \qquad (a_k^* a_k^* a_k a_k)^{up} = |c_k|^4 - 4|c_k|^2 + 2$$

Note that, since the interaction term of the Hamiltonian term considered on its own does have a momentum conservation law, it is not possible to get exactly three out of four operators in the same mode k. In view of this, it can be seen that the lower and upper symbols of the Hamiltonian will differ only when two or four operators in the same mode appears, that is only terms (A.1), (A.9), (A.12), (A.13), (A.15), (A.16) and (A.17) differs in both approximating Hamiltonians.
Splitting further the sums in these terms leads finally to the final upper symbol of the Hamiltonian

$$H_l^{\mathrm{up}}(\mu, \{c_k\}) \ = \ H_l^{\mathrm{low}}(\mu, \{c_k\}) + \kappa(\mu, \{c_k\}),$$

where

$$\begin{aligned}
\kappa(\mu, \{c_k\}) \ &= \ \sum_{k\in I_\delta} \Big(\sum_{i\leqslant 1} |(\phi_i, \psi_k)|^2 (E_i - \mu) \Big) \qquad (A.22)\\
&+ \ -\frac{1}{2V_l} \Big(2 \sum_{k\in I_\delta} \hat{u}(0) + \sum_{\substack{k,k'\in I_\delta \\ k\neq k'}} \hat{u}(0) + \sum_{q\neq 0} \sum_{k\in I_\delta \cap I-q} \hat{u}(q) \Big) \\
&+ \ \frac{1}{2V_l} \Big(4 \sum_{k\in I_\delta} \hat{u}(0) |c_k|^2 + \sum_{\substack{k,k'\in I_\delta \\ k\neq k'}} \hat{u}(0)(|c_k|^2 + |c_{k'}|^2) + \sum_{q\neq 0} \sum_{\substack{k\in I_\delta\cap I-q \\ k'\in I_\delta\cap I-q \\ k'-k=q}} \hat{u}(q)(|c_k|^2 + |c_{k'}|^2) \Big) \\
&+ \ \frac{1}{2V_l} \Big(2 \sum_{k\in I_\delta} \hat{u}(0) \sum_{k'\in I_\delta^c} a^*_{k'} a_{k'} + \sum_{q\neq 0} \sum_{\substack{k\in I_\delta \\ k'\in I_\delta^c\cap I+q \\ k'-k=q}} \hat{u}(q) a_k^* a_k + \sum_{q\neq 0} \sum_{\substack{k\in I_\delta^c\cap I-q \\ k'\in I_\delta \\ k'-k=q}} \hat{u}(q) a_k^* a_k \Big) \ .
\end{aligned}$$

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