# Preconditioning operators and $L^\infty$ attractor for a class of reaction-diffusion systems

Boris Andreianov, Halima Labani

# PRECONDITIONING OPERATORS AND $L^\infty$ ATTRACTOR FOR A CLASS OF REACTION-DIFFUSION SYSTEMS

Boris Andreianov

Laboratoire de Mathématiques CNRS UMR 6623
Université de Franche-Comté, 16 route de Gray
25030 Besançon Cedex, France

Halima Labani

Université Chouaïb Doukkali
Faculté des Sciences, Département de Mathématiques et Informatique
BP20 24000 El Jadida, Morocco

Abstract. We suggest an approach for proving global existence of bounded solutions and existence of a maximal attractor in $L^\infty$ for a class of abstract $3\times3$ reaction-diffusion systems. The motivation comes from the concrete example of "facilitated diffusion" system with different non-homogeneous boundary conditions modelling the blood oxigenation reaction $\mathrm{Hb}+\mathrm{O}_2 \rightleftharpoons \mathrm{HbO}_2$.

The method uses the $L^p$ techniques developed by Martin and Pierre [MP1] and Bénilan and Labani [BL2] and the hint of "preconditioning operators": roughly speaking, the study of solutions of $\big(\partial_t + A_i\big)\, u = f$ is reduced to the study of solutions to

$$\big(\partial_t + B\big)\,(B^{-1}u) = B^{-1}f + (\mathrm{I}-B^{-1}A_i)\, u,$$

with a conveniently chosen operator $B$. In particular, we need the $L^\infty - L^p$ regularity of $B^{-1}A_i$ and the positivity of the operator $(B^{-1}A_i - \mathrm{I})$ on the domain of $A_i$.

The same ideas can be applied to systems of higher dimension. To give an example, we prove the existence of a maximal attractor in $L^\infty$ for the $5\times5$ system of facilitated diffusion modelling the coupled reactions $\mathrm{Hb}+\mathrm{O}_2 \rightleftharpoons \mathrm{HBO}_2$, $\mathrm{Hb}+\mathrm{CO}_2 \rightleftharpoons \mathrm{HbCO}_2$.

Contents



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*Date*: October 1, 2010.

1991 *Mathematics Subject Classification.* Primary: 35B41, 35K57, 35Q92 Secondary: 35K90.

*Key words and phrases.* Reaction-diffusion system, global existence, maximal attractor, analytic semigroup, maximal regularity, preconditioning operator.

The second author is supported by the French-Moroccan project No. MA/164/07.

## 1. Introduction

Consider the following reaction-diffusion systems in $(0,+\infty)\times\Omega$, where $\Omega$ is a bounded domain of $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$:

$$\begin{cases} \partial_t u_1 - d_1\Delta u_1 = u_3 - u_1u_2 \\ \partial_t u_2 - d_2\Delta u_2 = u_3 - u_1u_2 \\ \partial_t u_3 - d_3\Delta u_3 = u_1u_2 - u_3, \end{cases} \tag{1}$$

$$\begin{cases} \partial_t u_1 - d_1\Delta u_1 = K_2u_2 - K_1u_1u_5 \\ \partial_t u_2 - d_2\Delta u_2 = -K_2u_2 + K_1u_1u_5 \\ \partial_t u_3 - d_3\Delta u_3 = K_4u_4 - K_3u_3u_5 \\ \partial_t u_4 - d_4\Delta u_4 = -K_4u_4 + K_3u_3u_5 \\ \partial_t u_5 - d_5\Delta u_5 = K_2u_2 + K_4u_4 - K_1u_1u_5 - K_3u_3u_5 \end{cases} \tag{2}$$

with the boundary conditions (BC, for short) of the following general form:

$$\lambda_i\partial_n u_i + (1-\lambda_i)u_i = \alpha_i \quad \text{on } \Omega, \quad \alpha_i \geq 0, \quad i = 1..3 \ \text{ or } \ i = 1..5. \tag{3}$$

Here $0 \leq \lambda_i \leq 1$.

For bounded nonnegative initial conditions $u_i(0) = u_i^0$, global existence of solutions, attractor in $L^\infty$ and asymptotic behaviour for (1),(3) and for (2),(3) have been studied in many works (see e.g. [Rot, E, MP1, L, AL]), under different restrictions on the boundary conditions and data. The aim of the present work is to show global existence and to construct the attractor in $L^\infty$ in some of the cases which, to the authors knowledge, are not covered by the existing literature. In particular, we treat the case of non-homogeneous Robin boundary conditions for (1),(3) with $\lambda_1, \lambda_2, \lambda_3$ which do not coincide (see § 5.1). The case where some components may have Neumann boundary conditions is more subtle (see § 5.2), and we only prove the global existence in this case.

Let us briefly recall the known results and the methods used to obtain them.

Rothe [Rot] showed the global existence for (1), under the homogeneous Neumann boundary conditions and for $n \leq 5$, using feedback or bootstrap arguments ant Sobolev embeddings. He also studied the asymptotic behaviour under the same assumptions, showing that the solution converges, as $t \to +\infty$, to the unique equilibrium point. This result was achieved thanks an entropy production functional. In the same direction as Rothe, Ebel [E] considered system (1) with a nonhomogeneous Dirichlet condition on $u_1$ and the homogeneous Neumann conditon on $u_2, u_3$. The most general results are obtained for $n = 1$; namely, Ebel showed that the solution exists globally in time; using a Lyapunov functional, she proved that the solution converges to the asymptotically stable steady-state solution uniquely determined by the value $C = \frac{1}{|\Omega|}\int_\Omega (u_2 + u_3)$. For $n \leq 5$, same results are obtained for a special choice of the Dirichlet BC on $u_1$. Further, system (1) was studied by Martin and Pierre in [MP1]; they developed the $L^p$ technique in order to prove the global existence under the Neumann boundary condition for any space dimension $n$. Using these $L^p$ techniques, Labani in [L] has shown, in collaboration with Ph. Bénilan, the result of global existence and convergence to zero of solution of (1), for all $n$, under the homogeneous condition (3) with

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda \tag{4}$$

with $\lambda \in [0,1)$ (either the Dirichlet condition, or the same Robin condition is imposed on all the three components). In the same way, Amraoui and Labani [AL] gave global existence of solutions and existence of a maximal attractor in $((L^\infty(\Omega))^+)^3$ with non-homogeneous BC under the restriction (4).

System (2) was studied by many authors (see Morgan [Mo], W. Feng [Fe1, Fe2] and references therein). In particular, using upper and lower solutions and following the result of Rothe, Feng proved global existence for $n \leq 5$ under the homogeneous boundary condition (or when some of the five components of the solution have the same boundary condition, like in (4); see [Fe2] for the precise statements). Under the same conditions, the convergence of the solutions to a steady-state solution is shown, using a Lyapunov functional and the idea of $\omega$-limit set.

The motivation of this paper is to study system (1) under non-homogeneous boundary conditions in all space dimension, and to relax considerably the assumption (4) used by Amraoui and Labani. Namely, we assume that one of the following three situations occurs:

$$\begin{array}{l} \text{either } \lambda_i \in (0,1),\ i = 1..3, \text{ or } \lambda_1 = \lambda_2 = \lambda_3 = 0, \\ \text{or } \lambda_i \in [0,1) \text{ with } \alpha_i = 0 \text{ for } i \text{ such that } \lambda_i = 0. \end{array} \tag{5}$$

These assumptions allow to introduce an operator $B$ as a “preconditioner” for the system; the sense of this term will become clear from the use we make of this preconditioning operator. Let us briefly describe our approach and give the main assumptions. Following Bénilan and Labani [BL2], we recast problem (1),(3) under the abstract form :

$$(S) \quad \begin{cases} \dfrac{d}{dt} u_i + A_i(u_i - \bar{\alpha}_i) = f_i(u_1, u_2, u_3), \\ u_i(0) = u_i^0, \quad i = 1..3, \end{cases}$$

where for $i = 1..3$, $(-A_i)$ is the infinitesimal generator of an analytic exponentially stable semigroup of positive linear operators $e^{-tA_i}$ on $L^2(\Omega)$; we assume that these semigroups are $L^p$-nonexpansive and hypercontractive. We refer to Section 2 for the exact assumptions on $A_i$ and for the definition of a solution.

Further, in $(S)$ we assume

$$\bar{\alpha}_i \in (L^\infty(\Omega))^+ \ \text{ with } \ e^{-tA_i}\bar{\alpha}_i \leq \bar{\alpha}_i, \quad i = 1..3 \tag{6}$$

(to get from (1),(3) to $(S)$ one takes for $\bar{\alpha}_i$ the solution of the appropriately defined elliptic problem with BC (3); in particular, $\bar{\alpha}_i = 0$ if $\alpha_i = 0$).

Finally, the source terms $f_i$, $i = 1..3$ in $(S)$ are assumed to be locally Lipschitz continuous on $(\mathbb{R}^+)^3$ and verify for all $u_1, u_2, u_3 \in \mathbb{R}^+$, the properties

$$f_1(0, u_2, u_3) \geq 0, \quad f_2(u_1, 0, u_3) \geq 0, \quad f_3(u_1, u_2, 0) \geq 0; \tag{7}$$

$$f_1(u_1, u_2, u_3) + f_3(u_1, u_2, u_3) \leq 0; \tag{8}$$

$$\exists a \geq 0 \quad f_1(u_1, u_2, u_3) \leq a(1 + u_3), \quad f_2(u_1, u_2, u_3) \leq a(1 + u_3); \tag{9}$$

$$\exists b \geq 0, \beta \geq 0, \gamma \geq 0 \quad f_3(u_1, u_2, u_3) \leq b(1 + u_1^\beta + u_2^\gamma). \tag{10}$$

Local existence and uniqueness of a non-negative $L^\infty$ mild solution for nonnegative $L^\infty$ initial data is obtained by the standard fixed-point technique based upon the Duhamel formula. Under the assumptions we will impose on the operators $A_i$, this solution is a strong solution, in the sense that $u_i \in W^{1,2}_{loc}((0,T); L^2(\Omega)) \cap C([0,T], L^2(\Omega))$ and $(S)$ is verified in $L^2(\Omega)$, for a.e. $t > 0$.

Time-dependent *a priori* $L^\infty$ bounds on the solutions ensure the existence of a nonlinear semigroup $\{S(t)\}_{t\geq 0}$ on $(L^\infty(\Omega))^3$ solving $(S)$. Then, $L^\infty$ estimates of attractor type ((E.A.T.), for short; see (13) for the definition) and the asymptotic compactness of $\{S(t)\}_{t\geq 0}$ are needed in order to construct the maximal attractor for $(S)$ in $L^\infty(\Omega)$ (see [BL2]). The technique to obtain the (E.A.T.) estimates is the main contribution of this paper; let us describe briefly this technique.

We proceed in four steps in order to obtain $L^\infty$ estimates (E.A.T.) for the solution $(u_1, u_2, u_3)$. Firstly, we assume that there exists a "preconditioning operator" $B$ on $L^2(\Omega)$ satisfying the same requirements as those imposed on $A_i$, $i = 1..3$ (see Definition 2.1 below); and such that, for $A = A_i$, $i = 1..3$, the two properties hold:

$$\begin{cases} (I - B^{-1}A) \leq 0 \quad \text{in the sense that} \\ \text{for all } u \in D(A) \cap L^\infty(\Omega),\ u \geq 0,\ \text{one has } u \leq B^{-1}A\, u \end{cases} \tag{11}$$

and (for $i$ such that $\bar{\alpha}_i \not\equiv 0$)

$$\begin{cases} \text{for all } p < +\infty \text{ there exists } C_p > 0 \text{ such that} \\ \text{for all } u \in D(A) \cap L^\infty(\Omega),\ u \geq 0,\ \text{one has } \|B^{-1}A\, u\|_{L^p(\Omega)} \leq C_p \|u\|_{L^\infty(\Omega)}. \end{cases} \tag{12}$$

We consider the operator $(\frac{d}{dt} + B)$ applied to the function $B^{-1}\big(u_1(t) - \bar{\alpha}_1 + u_3(t) - \bar{\alpha}_3\big)$. Using (8), we deduce (E.A.T.) estimates on $\|B^{-1}u_i(t)\|_{L^p(\Omega)}$ for $i = 1$ and $i = 3$, for all $p < +\infty$. Then analogous estimates on $\|A_j^{-1}u_i(t)\|_{L^p}$, $j = 1..3$, $i = 1, 3$, follow. Applying $(\frac{d}{dt} + B)$ to the function $B^{-1}(u_2(t) - \bar{\alpha}_2)$, we deduce an (E.A.T.) estimate on $\|A_j^{-1}u_2\|_{L^p}$, $j = 1..3$.

Secondly, we exploit the idea of the proofs in [BL2] and [AL]. Using the first two equations in $(S)$ and property (9), we deduce (E.A.T.) estimates on $\|u_i(t + \cdot)\|_{L^p((0,\delta)\times\Omega)}$ for $i = 1$ and for $i = 2$, for all $p < +\infty$. Here $\delta > 0$ is a fixed real number. At this stage we exploit the $L^p$ maximal regularity property (see [La] and Theorem 3.2 in Section 3) for the operators $A_i$, $i = 1, 2$.

Thirdly, choosing $p$ sufficiently large in the preceding arguments, with the help of the $L^p$ techniques of Martin and Pierre [MP1], from (10) and the third equation of $(S)$ we deduce an (E.A.T.) estimate on $\|u_3(t+\delta)\|_{L^\infty}$. In the two latter steps, we use the exponential decay of $e^{-tA_3}$ and the $L^q - L^p$ regularizing effect of the semigroup, $1 \leq q < p \leq +\infty$.

Finally, using the exponential decay of $e^{-tA_i}$, $i = 1$ and $i = 2$ in $L^p(\Omega)$, from (9) and the first two equations in $(S)$ we deduce (E.A.T.) estimates of $\|u_i\|_{L^\infty}$, $i = 1, 2$.

We refer to Definition 2.1 and Lemma 3.1 for the exact assumptions on the operators and the properties of the semigroups used in our arguments.

The outline of the paper is as follows. In Section 2, we describe our abstract framework, give the definitions, and state the main results (one more result is given in Section 5.3). In Section 3, we give the proofs. Section 4 is devoted to the study of concrete examples of preconditioning operators. In Section 5.1, we apply the abstract results of Section 2 to problem (1) with boundary conditions (3) of Dirichlet or Robin type. Then we give two extensions. Section 5.2 deals with problem (1) with Neumann BC on some of the components; we only prove that the solutions are global in time. In Section 5.3, we apply the technique of Section 3 to the $5 \times 5$ system (2). For Dirichlet or Robin boundary conditions (3), we deduce that solutions exist globally, that there exists a maximal attractor in $(L^\infty(\Omega))^5$ for system (2). If also Neumann boundary conditions are allowed, one can get the global in time existence of solutions, as for the case of system (1).

## 2. The abstract framework and the main result

2.1. **Some notation.** In the sequel, $C$ denotes a generic constant that only depends on the problem, excluding $U^0$; i.e., $C$ depends on $\Omega$, $\bar{\alpha}_i$, $A_i$, $f_i$ (via, in particular, $\|\bar{\alpha}_i\|_{L^\infty(\Omega)}$, meas$(\Omega)$, the regularity of $\partial\Omega$, the constants $a, b$, $\beta, \gamma$, $\omega, \sigma$ appearing in our assumptions). By $\Psi$ we will denote a generic non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ depending on the same parameters.

Similarly, by $\Phi$ we will denote a generic function

$$\begin{aligned} &\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+ \quad \text{satisfying} \\ &\Phi \text{ is bounded, } \ \Phi(\cdot, t) \text{ is non-decreasing, } \ \sup_{r\in\mathbb{R}^+} \lim_{t\to\infty} \Phi(r,t) \le C; \end{aligned} \tag{13}$$

any such function will be called "estimate of attractor type" ((E.A.T.), for short). Notice that, upon replacing $\Phi(r,t)$ by $\sup_{s\ge t} \Phi(r,s)$, we can always think of $\Phi(r,\cdot)$ as being non-increasing.

The argument of $\Psi(\cdot)$ and of $\Phi(\cdot,t)$ will have the meaning of $\|U^0\|_{L^\infty(\Omega)}$.

Finally, in case $C, \Psi$ or $\Phi$ depend on additional parameters introduced below (namely, on $p \in [1, +\infty)$ and/or on $\delta \in (0, +\infty)$), we will indicate these parameters as subscripts.

Notice that with this notation, we have e.g.

$$\begin{aligned} &C_\delta + \Psi_p(r) = \Psi_{\delta,p}(r), \quad C + e^{-\delta t}\Psi_p(r) + \Phi(r,t) = \Phi_{\delta,p}(r,t), \quad \sup_{t\in\mathbb{R}} \Phi(r,t) = \Psi(r), \\ &C + \Phi_p(r,t) = \Phi_p(r,t), \quad \Psi(r)1\!\!1_{[0,2\delta)}(t) + \Phi(r, t-2\delta)1\!\!1_{[2\delta,+\infty)}(t) = \Phi_\delta(r,t) \end{aligned}$$

and so on. This kind of relation is often used in the proofs of Section 3.

By $u^+$, respectively by $u^-$, $u \wedge 1$ we denote $\max\{u, 0\}$ (resp., $\max\{-u, 0\}$, $\min\{u, 1\}$).

2.2. **Abstract problem and assumptions.** Let $\Omega$ be a given bounded domain of $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$. We consider the abstract reaction-diffusion problem $(S)$. In order to simplify the presentation, we gather the assumptions on the operators $A_i$ in the following definition.

**Definition 2.1.** We say that $A$ is an operator of class $\mathcal{A}$ if the following holds:

- $-A$ is the infinitesimal generator of an analytic semigroup $e^{-tA}$ on $L^2(\Omega)$
- the semigroup $e^{-tA}$ is positive, in the sense that $e^{-tA}u \ge 0$ for $u \ge 0$;

• $e^{-tA}$ is non-expansive on all spaces $L^p(\Omega)$, i.e., for all $t>0$,

$$\forall p\in[1,+\infty] \quad \|e^{-tA}u\|_p \le \|u\|_p \ \text{ for } u\in L^\infty(\Omega); \tag{14}$$

• $e^{-tA}$ is exponentially stable on $L^2(\Omega)$, i.e. there exists $\omega>0$ such that for all $t>0$,

$$\|e^{-tA}\|_{\mathcal{L}(L^2(\Omega))} \le e^{-\omega t}; \tag{15}$$

• $e^{-tA}$ is hypercontractive, i.e., there exist $\sigma>0$ and $C>0$ such that

$$\|e^{-tA}u\|_{L^\infty(\Omega)} \le \frac{C}{t^\sigma}\|u\|_{L^1(\Omega)}. \tag{16}$$

**Remark 1.** It is well known that the operators $A_i$ featuring in (1),(3) are of class $\mathcal{A}$, provided $\lambda_i<1$, $i=1..3$ (see in particular [Fr, Pa, Rot]). If $\lambda_i=1$, then for all $c>0$, $(A_i+cI)$ is of class $\mathcal{A}$.

More generally, following [BL2] consider an abstract operator $A$ defined by a symmetric positive definite bilinear form $a(\cdot,\cdot)$ on a Hilbert space $V$ under the norm $\|u\|_V^2=a(u,u)$. Assume

$$V \text{ is densely embedded in } L^q(\Omega) \text{ for some } q>2. \tag{17}$$

Then we have, in particular, the triple $V\subset L^2(\Omega)\subset V'$ with the operator $A: V\longrightarrow V'$ defined by duality, i.e. $< Au,\xi >_{V',V}= a(u,\xi)$ for all $\xi\in V$. Restricting the operator's domain, we have the operator $A$ on $L^2(\Omega)$ defined by

$$D(A)=\left\{u\in V\subset L^2(\Omega)\,\middle|\, \exists\, w=:Au\in L^2(\Omega)\ \forall\,\xi\in V\ \textstyle\int_\Omega w\xi=a(u,\xi)\right\};$$

then it is well known that the operator $-A$ generates an analytic exponentially stable semigroup on $L^2(\Omega)$ (see e.g. [B, Chap.IV],[A, Chapter 7.1]). Moreover, under the Beurling-Deny assumptions

$$u\in V \implies \left[\, u^+\in V \ \ u\wedge 1\in V \text{ and } a(u^+,u^-)\ge 0,\ a(u\wedge 1,(u-1)^+)\ge 0 \,\right],$$

the semigroup $e^{-tA}$ is positive and non-expansive in $L^p(\Omega)$, $p\in[1,+\infty]$ (see [A, Section 7.1] and references therein). Finally, the embedding (17) ensures the hypercontractivity property (see [A, Section 7.3.2]). Therefore such operators are of class $\mathcal{A}$.

We assume that the operators $A_i$ in $(S)$ fulfill the assumptions

$$(H) \qquad \left\{\begin{array}{l} \text{the operators } A_i \text{ in } (S),\ i=1..3, \text{ are of class } \mathcal{A};\\ \text{in addition, there exists an operator } B \text{ of class } \mathcal{A}\\ \text{which satisfies (11),(12) with } A=A_i,\ i=1..3.\end{array}\right.$$

We impose the restrictions (7)–(10) on the reaction terms. Finally, we assume that $\bar\alpha_i\in L^\infty(\Omega)$, $\bar\alpha_i\ge 0$, $i=1..3$.

2.3. **Definitions and results.** Let us first make precise the two notions of solution we use.

**Definition 2.2.** For $T\in(0,+\infty)$, a mild solution of $(S)$ on $[0,T)$ is a triplet $U=(u_1,u_2,u_3)$ of functions on $[0,T)$ such that $U\in C([0,T),L^2(\Omega))$ and the Duhamel formula represents $U(t)$ for all $t\in[0,T)$:

$$u_i(t)=\bar\alpha_i+e^{-tA_i}(u_i^0-\bar\alpha_i)+\int_0^t e^{-(t-s)A_i}f_i(U(s))\,ds. \tag{18}$$

A strong solution is a triplet $U = (u_1, u_2, u_3)$ of functions on $[0, T)$ such that $u_i \in W^{1,2}_{loc}((0,T); L^2(\Omega)) \cap C([0,T), L^2(\Omega))$ with $u_i(0) = u_i^0$, $u_i(t) \in D(A_i)$ for a.e. $t$ and the equation in $(S)$ is verified in $L^2(\Omega)$ for a.e. $t \in (0,T)$.

Inserting a strong solution into the right-hand side of the Duhamel formula (18) and taking the time derivative in the sense of $W^{1,2}(0,T; L^2(\Omega))$ space, it is easy to check that a strong solution is also a mild one. In the framework we are given, the converse is also true (see Remark 2 below).

In order to show global in time existence of a (strong) solution, we start with the standard local existence and uniqueness result for a mild solution with non-negative $L^\infty$ data:

**Theorem 2.3.** *For all $U^0 \in \left((L^\infty(\Omega))^+\right)^3$ there exists $T_{max} = T_{max}(\|U^0\|_\infty) \in (0, +\infty]$ such that $(S)$ with the initial datum $U^0$ admits a unique mild solution on $[0, T_{max})$ with values in $\left((L^\infty(\Omega))^+\right)^3$; moreover, $T_{max} = +\infty$ unless $\|U(\cdot)\|_{L^\infty}$ gets unbounded as $t \to T_{max} - 0$.*

The arguments of the proof are classical (cf. [Pa, Th.6.1.4] applied in $L^2(\Omega)$), except that we need additional $L^\infty$ growth estimates and the positivity control. Thus we combine the local Lipschitz continuity of $f_i$, $i = 1..3$, the non-expansiveness property (14) of $e^{-tA_i}$ in $L^\infty$, the Gronwall inequality, and the Banach fixed-point theorem to get a unique local mild solution with values in $\left(L^\infty(\Omega)\right)^3$. The positivity comes from the sign properties (7), the inequalities (6) and the Duhamel formula. Finally, the continuation principle is used to get the solution on a maximal interval $[0, T_{max})$.

Now let us state the main result of this paper.

**Theorem 2.4.** *Assume that the operators $A_i$, $i = 1..3$, satisfy the properties $(H)$. Assume that $\bar\alpha_i \in (L^\infty(\Omega))^+$ satisfy (6) and $f_i$, $i = 1..3$, are locally Lipschitz continuous functions satisfying (7)–(10). Then*

*(i) Any mild solution of $(S)$ for $L^\infty$ initial datum $U^0$ can be continued globally in time.*

*(ii) An $L^\infty(\Omega)$ estimate of attractor type holds: namely, there exists $\Phi$ satisfying (13) such that the solution $U = (u_1, u_2, u_3)$ of $(S)$ with initial datum $U^0 = (u_1^0, u_2^0, u_3^0)$ satisfies*

$$\forall t \in \mathbb{R}^+ \quad \|U(t)\|_{L^\infty(\Omega)} \le \Phi(\|U^0\|_{L^\infty(\Omega)}, t).$$

We also have the above result (i) in the case the exponential stability hypothesis (15) is suppressed (notice that by the contractivity assumption (14), (15) still holds with $\omega = 0$), provided the assumptions (9),(10) made in the introduction are replaced by (19),(20) below. In the case of system (1),(3), this corresponds to the Neumann BC imposed on some of the components of the solution; and (19),(20) are fulfilled. To be precise, a careful modification and simplification of the proof of Theorem 2.4 yields the following abstract result.

**Theorem 2.5.** *Assume that $f_i$, $i = 1..3$, are locally Lipschitz continuous functions satisfying (7),(8), that the following weaker condition substitutes (9):*

$$\exists a \ge 0 \quad f_i(u_1, u_2, u_3) \le a(1 + u_1 + u_2 + u_3), \quad i = 1, 2; \tag{19}$$

*and assume that the following stronger condition substitutes* (10)*:*

(20) $$\exists b \geq 0, \beta \geq 0, \gamma \geq 0 \quad f_3(u_1, u_2, u_3) + cu_3 \leq b(1 + u_1^\beta + u_2^\gamma).$$

*with some* $c > 0$*. Assume that* $\bar{\alpha}_i \in (L^\infty(\Omega))^+$ *satisfy* (6)*.*
*Assume that hypotheses* $(H)$ *are replaced by the assumptions on* $A_i, c := A_i + cI$ *and the associated preconditioning operator* $B_c = B + cI$*:*

$(H_c)$ $$\begin{cases} \text{the operators } A_{i,c} \text{ in } (S),\ i = 1..3, \text{ are of class } \mathcal{A}; \\ \text{in addition, there exists an operator } B \text{ such that } B_c \text{ is of class } \mathcal{A} \\ \text{and } B_c \text{ satisfies } (11),(12) \text{ with } A = A_{i,c},\ i = 1..3, \end{cases}$$

*where* $B$ *is the infinitesimal generator of an analytic semigroup of positive linear operators* $e^{-tA_i}$ *on* $L^2(\Omega)$ *satisfying the* $L^p$*-contractivity property* (14) *and the hypercontractivity property* (16)*.*

*Then the mild solution of* $(S)$ *for any* $((L^\infty(\Omega)^+)^3$ *initial datum* $U^0$ *is defined globally in time.*

**Corollary 1.** *There exists a nonlinear semigroup on the positive cone* $\left((L^\infty(\Omega)^+\right)^3$ *given by Theorem* 2.4 *or by Theorem* 2.5*, under the respective assumptions of the theorems. We denote this semigroup by* $\{S(t)\}_{t\geq 0}$*.*

Clearly, the (E.A.T.) of Theorem 2.4(ii) implies the existence of a bounded absorbing set $\mathcal{E}$ in $((L^\infty(\Omega))^+)^3$ for the semigroup $\{S(t)\}_{t\geq 0}$; e.g., one can take

$$\mathcal{E} = \left\{U = (u_1, u_2, u_3)\,\middle|,\ \forall\, i = 1..3\ \|u_i\|_{L^\infty(\Omega)} \leq C + 1\right\}$$

where is given by (13) in the $(L^\infty(\Omega))^3$ (E.A.T.) of $U(t)$. By the general result (see Bénilan and Labani [BL1]; cf. Temam [Tem])), under the additional assumption of asymptotic compactness of the solution semigroup,

(21) $$\mathcal{M} = \bigcap_{t\geq 0} \overline{\bigcup_{\delta>0} S(t+\delta)\mathcal{E}}$$

is the maximal attractor for the semigroup. In the definition of $\mathcal{M}$, the closure is taken in $(L^2(\Omega))^3$; yet, as the semigroup is also compact in $(L^\infty(\Omega))^3$ (see Corollary 2(i) below), this closure could be taken in $(L^\infty(\Omega))^3$.

**Corollary 2.** *With the assumptions of Theorem* 2.4 *suppose in addition that for* $i = 1..3$*, the semigroups* $e^{-tA_i}$*,* $i = 1..3$*, are compact in* $L^\infty(\Omega)$*, for all* $t > 0$*. Then*
- (i) *the nonlinear semigroup* $\{S(t)\}_{t\geq 0}$ *associated with problem* $(S)$ *is compact in* $(L^\infty(\Omega))^3$*;*
- (ii) $\mathcal{M}$ *given by* (21) *is the maximal attractor for the semigroup* $\{S(t)\}_{t\geq 0}$ *in* $((L^\infty(\Omega))^+)^3$*.*

Notice that in general, a maximal attractor in the framework of Theorem 2.5 may not exist (cf. the asymptotic behaviour results of [Rot, E]).

The result of Corollary 2 is almost classical (see e.g. Temam [Tem]), except for the fact that we replace the assumption of continuity of the semigroup on $L^\infty$ by the continuity in a weaker topology (see [BL1, BL2]). For the sake of completeness, in Section 3.5 we give a proof adapted to our setting.

## 3. Proofs

The main arguments are those of Section 3.3; Section 3.2 gives the guidelines.

3.1. **Preliminary statements.** Let us first recall a few well-known properties of semigroups generated by operators of class $\mathcal{A}$; these properties will be used throughout the proofs.

**Lemma 3.1.** *Assume that $A$ is of class $\mathcal{A}$. Then*

(i) *$A^{-1}$ is bounded, and for all $u \geq 0$, one has $A^{-1}u \geq 0$;*

(ii) *there exists $C > 0$ such that for all $t > 0$,*

$$\|Ae^{-tA}\|_{\mathcal{L}(L^2(\Omega))} \leq \frac{C}{t}. \tag{22}$$

(iii) *the unique mild solution of the evolution problem*

$$\frac{d}{dt}u + Au = 0, \quad u(0) = u_0 \in L^2(\Omega)$$

*verifies the equation in the classical sense in $L^2(\Omega)$, for all $t > 0$. Namely, $u - u_0 \in C_0([0,\infty); L^2(\Omega)) \cap C^1((0,\infty); L^2(\Omega))$, both $\frac{d}{dt}u$ and $Au$ belong to $C((0,+\infty), L^2(\Omega))$, and the equality $\frac{d}{dt}u + Au = f$ holds in $L^2(\Omega)$ for all $t > 0$.*

(iv) *there exist $\sigma > 0$ and $C > 0$ such that for all $p > 2$ there exists $\lambda_p > 0$ such that*

$$\|e^{-tA}u\|_{L^p(\Omega)} \ \leq \ C\, e^{-\lambda_p t}\, t^{-\frac{\sigma}{p}} \|u\|_{L^{p/2}(\Omega)}; \tag{23}$$

*in addition, for all $q > 1$*

$$\|e^{-tA}u\|_{L^\infty(\Omega)} \ \leq \ C\, t^{-\frac{\sigma}{q}} \|u\|_{L^q(\Omega)}. \tag{24}$$

(v) *for all $\delta > 0$ there exists a constant $C_\delta$ such that*

$$\forall\, p \in [2,+\infty) \ \ \forall t \geq \delta \quad \|Ae^{-tA}\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))} \leq C_\delta. \tag{25}$$

(vi) *$A^{-1}1 \in L^p(\Omega)$ for all $p < +\infty$.*

Let us briefly indicate the arguments of the proof. The points (i),(ii) and (iii) are classical (see e.g. [Pa, Ch. 1, Remark 5.4], [Pa, Ch. 2, Theorem 5.2] and [Pa, Ch. 4, Theorem 3.5] respectively). The item (vi) is a straightforward combination of the estimates (14) (for $t \leq 1$) and (23) (for $t > 1$) with the inversion formula $A^{-1} = \int_0^{+\infty} e^{-tA}\, dt$.

The point (iv) follows by the Riesz-Thorin interpolation theorem (see e.g. [DS]) from (14),(15) and (16). Indeed, we first interpolate (14) and (15) (where we take either $p = \infty$, or $p = 1$). We get for all $r \in (1,\infty)$,

$$\|e^{-tA}\|_{\mathcal{L}(L^r(\Omega))} \leq e^{-\frac{2\omega}{\max\{r,r'\}}\, t}. \tag{26}$$

Then we interpolate (16) and (26) (where we choose $r = 1 + \frac{p}{q'}$). We find that for all $p \geq q > 1$ there exist $\lambda_{p,q} > 0$ such that

$$\|e^{-tA}u\|_{L^p(\Omega)} \ \leq \ C\, e^{-\lambda_{p,q} t}\, t^{-\sigma(\frac{1}{q} - \frac{1}{p})} \|u\|_{L^q(\Omega)}; \tag{27}$$

On the one hand, the choice $q = p/2$ yields (23). On the other hand, as $p \to \infty$ in (27) and $q$ remains fixed, estimate (24) follows.

For the proof of (v), we write $Ae^{-tA} = e^{-(t-\delta)A} e^{-\frac{\delta}{2}A}\big(Ae^{-\frac{\delta}{2}A}\big)$ and get

$$\begin{aligned}
&\|Ae^{-tA}\|_{\mathcal{L}(L^2(\Omega), L^\infty(\Omega))} \\
&\qquad \leq \|e^{-(t-\delta)A}\|_{\mathcal{L}(L^\infty(\Omega))} \|e^{-\frac{\delta}{2}A}\|_{\mathcal{L}(L^2(\Omega), L^\infty(\Omega))} \|Ae^{-\frac{\delta}{2}A}\|_{\mathcal{L}(L^2(\Omega))} \leq \frac{C}{\delta^{1+\sigma/2}}
\end{aligned}$$

thanks to (14), (24) and (22). Then we use the Hölder inequality for $p > 2$ and infer (25).

Finally, recall the following maximal regularity statement (see Lamberton [La]).

**Theorem 3.2.** *Assume that $-A$ is the infinitesimal generator of an analytic semigroup on $L^2(\Omega)$ verifying* (14)*. Let $p \in (1, +\infty)$.*

*Then the unique mild solution of the evolution problem*

$$\frac{d}{dt}u + Au = f \in L^p_{loc}([0, +\infty) \times \Omega), \quad u(0) = 0$$

*verifies the equation in the strong sense in $L^p(\Omega)$. Namely, $u \in W_0^{1,p}([0, \infty); L^p(\Omega))$ (so that the mapping $t \mapsto u(t) \in L^p(\Omega)$ is continuous and $u(0) = 0$), both $\frac{d}{dt}u$ and $Au$ (defined a.e. on $(0, +\infty)$) belong to $L^p(\Omega)$, and the equality $\frac{d}{dt}u + Au = f$ holds in $L^p(\Omega)$ for a.e. $t > 0$.*

*Moreover, there exists $C_p > 0$ such that for all $T > 0$, the maximal regularity estimate holds true:*

$$\|\frac{d}{dt}u\|_{L^p([0,T]\times\Omega)} + \|Au\|_{L^p([0,T]\times\Omega)} \ \leq \ C_p \, \|f\|_{L^p([0,T]\times\Omega)}.$$

By our assumptions, the operators $A_i$, $i = 1..3$, as well as the "preconditioning operator" $B$ verify the properties stated in Lemma 3.1 and in Theorem 3.2. In particular, we have the following remark concerning solutions with $L^\infty$ initial data.

**Remark 2.** The notions of mild and strong solutions are equivalent. Indeed, we only have to notice that a mild solution is also a strong one. Separating the right-hand side of the Duhamel formula (18), using Lemma 3.1(iii) and Theorem 3.2 we see that the unique mild solution $U(t) = (u_1(t), u_2(t), u_3(t))$ to problem $(S)$ verifies the equation in the strong sense, i.e. each term in $(S)$ makes sense in $L^2((0, T) \times \Omega)$ and the equality holds in $L^2(\Omega)$ for a.e. $t \in (0, T)$.

3.2. **Auxiliary statements and proof of Theorem 2.4.** The proof of Theorem 2.4 is based upon the three following lemmas.

**Lemma 3.3.** *Under the assumptions of Theorem* 2.4*, the following estimates of attractor type hold:*

$$\forall i,j = 1..3 \ \forall p \in [1, +\infty) \ \forall t < T_{max} \quad \|A_j^{-1} u_i(t)\|_{L^p(\Omega)} \leq \Phi_p(\|U^0\|_{L^\infty(\Omega)}, t); \tag{28}$$

$$\forall i = 1..3 \ \forall \delta > 0 \ \forall t \geq \delta \ \forall \tau < T_{max} \ \|e^{-tA_i} u_i(\tau)\|_{L^\infty(\Omega)} \leq \Phi_\delta(\|U^0\|_{L^\infty(\Omega)}, \tau). \tag{29}$$

**Lemma 3.4.** *Under the assumptions of Theorem* 2.4*, the following estimate of attractor type holds:*

$$\begin{gathered} \forall i = 1, 2 \ \forall p \in [1, +\infty) \ \forall \delta > 0 \ \forall \tau < T_{max} - 2\delta \\ \|u_i(\tau + \cdot)\|_{L^p((\delta, 2\delta)\times\Omega)} \leq \Phi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)}, \tau). \end{gathered} \tag{30}$$

*Moreover, if $T_{max} > 2\delta$, then*

$$\forall i = 1, 2 \ \forall p \in [1, +\infty) \ \forall \delta > 0 \ \forall t \leq 2\delta \quad \|u_i(\cdot)\|_{L^p((0,t)\times\Omega)} \leq \Psi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)}). \tag{31}$$

**Lemma 3.5.** *Under the assumptions of Theorem* 2.4*, the following estimate of attractor type holds:*

$$\forall i = 1..3 \ \forall \delta > 0 \ \forall \tau \in [2\delta, T_{max}) \quad \|u_i(\tau)\|_{L^\infty(\Omega)} \leq \Phi_\delta(\|U^0\|_{L^\infty(\Omega)}, \tau). \tag{32}$$

*Moreover, if $T_{max} > 2\delta$, then*

$$\forall i = 1..3 \ \forall \delta > 0 \ \forall \tau \in [0, 2\delta] \quad \|u_i(\tau)\|_{L^\infty(\Omega)} \leq \Psi_\delta(\|U^0\|_{L^\infty(\Omega)}). \tag{33}$$

The proofs are given in the next subsection; with Lemmas 3.3, 3.4, 3.5 in hand, we justify the claims of Theorem 2.4 as follows.

PROOF OF THEOREM 2.4:

(i) Take a solution of $(S)$ defined on the maximal interval $[0, T_{max})$; if $T_{max} < +\infty$, then $\|U(t)\|_{L^\infty}$ gets unbounded as $t$ approaches $T_{max}$. Pick $\delta < T_{max}$. Then the bound (32) in Lemma 3.5 contradicts the unboudnedness of $\|U(t)\|_{L^\infty}$. Thus $T_{max} = +\infty$.

(ii) Now fix e.g $\delta = 1/2$. Then by (i) and the two estimates of Lemma 3.5, for $i = 1..3$ we have the estimate

$$\forall t \in \mathbb{R}^+ \quad \|u_i(t)\|_{L^\infty(\Omega)} \le \Psi_{1/2}(\|U^0\|_{L^\infty(\Omega)})1\!\!1_{[0,1[}(t) + \Phi_{1/2}(\|U^0\|_{L^\infty(\Omega)}, t)1\!\!1_{[1,+\infty)}(t),$$

which is an (E.A.T.). $\diamond$

3.3. **Proofs of Lemmas 3.3, 3.4, 3.5.** Now we turn to the proofs. Notice that by the Hölder inequality, it is sufficient to prove the estimates of Lemmas 3.3,3.4 for $p$ satisfying

$$p \; > \; p_0 := \max\{2, \sigma\}. \tag{34}$$

PROOF OF LEMMA 3.3:

We work with the local solution $U(t) = (u_1(t), u_2(t), u_3(t))$; recall that it is defined, for $t \in [0, T_{max})$, $T_{max} \le +\infty$, as an element of $\big(L^\infty(\Omega)^+\big)^3$.

• Step 1. We obtain an $L^p(\Omega)$ E.A.T. on $B^{-1}u_i(t)$, for $i = 1, 3$.

For $0 \le t < T_{max}$, define $w(t) := B^{-1}\big((u_1 - \bar\alpha_1) + (u_3 - \bar\alpha_3)\big)$.

Apply the operator $(\frac{d}{dt} + B)$ to $w(\cdot)$. As pointed out hereabove, equations $(S)$ are satisfied in the strong sense. In particular, $u_i \in E := W^{1,2}_{loc}((0, T], L^2(\Omega)) \cap C([0, T], L^2(\Omega))$. The operator $B^{-1}$ being bounded and $\bar\alpha_i \in L^\infty(\Omega)$ being independent of $t$, we have $B^{-1}(u_i - \bar\alpha_i) \in E$. Thus we can apply the operator $B^{-1}$ term per term in equations $(S)$; it follows that $w$ verifies

$$\begin{cases} \dfrac{d}{dt}w + Bw = B^{-1}(f_1 + f_3) + \sum_{i=1,3}\big(I - B^{-1}A_i\big)(u_i - \bar\alpha_i) \\ w(0) = B^{-1}\big((u_1^0 - \bar\alpha_1) + (u_3^0 - \bar\alpha_3)\big) \end{cases} \tag{35}$$

in the strong sense.

The first term in the right-hand side of the above equation is non-positive thanks to assumption (8) and the positivity of $B^{-1}$. Now we want to benefit from hypotheses (11) and (12), by splitting $s_i := (u_i - \bar\alpha_i) \in D(A_i)$ into its positive and negative parts $s_i^\pm$.

We cannot do it directly, because $s_i^\pm$ may not belong to $D(A_i)$. Therefore we regularize $s^\pm$ by setting

$$s_{i,\rho}^\pm := (I - \rho^{-1}A_i)^{-1}s_i^\pm, \quad s_{i,\rho} := (I - \rho^{-1}A_i)^{-1}s_i \equiv s_{i,\rho}^+ - s_{i,\rho}^-.$$

It is well known (see e.g. [Pa, Ch. 1, Lemma 3.2]) that as $\rho \to \infty$,

$$\forall z \in L^2(\Omega) \equiv \overline{D(A_i)} \quad (I - \rho^{-1}A_i)^{-1}z \to z \text{ in } L^2(\Omega).$$

Because $B^{-1}$ is continuous on $L^2(\Omega)$ and $A_i$ commutes with $(I - \rho^{-1}A_i)^{-1}$, we infer that

$$\big(I - B^{-1}A_i\big)\, s_{i,\rho} \to \big(I - B^{-1}A_i\big)\, s_i \text{ as } \rho \to \infty.$$

In addition, $e^{-tA_i}$ being positive and non-expansive in $L^\infty$ for all $t>0$, from the representation

$$(I-\rho^{-1}A_i)^{-1}=\rho\int_0^\infty e^{-\rho t}e^{-tA_i}\,dt$$

we infer that $0\le s^+_{i,\rho}$ and $0\le s^-_{i,\rho}\le \|(u_i-\bar\alpha_i)^-\|_{L^\infty(\Omega)}$. Now, taking into account the positivity of $u_i$ and of $\bar\alpha_i$, we can write

$$0\le s^\pm_{i,\rho}\quad\text{and}\quad \|s^-_{i,\rho}\|_{L^\infty(\Omega)}\le \|\bar\alpha_i\|_{L^\infty(\Omega)}. \tag{36}$$

Now we come back to (35). Then for $i=1..3$,

$$\begin{aligned}\big(I-B^{-1}A_i\big)(u_i-\bar\alpha_i) &= \lim_{\rho\to+\infty}\big(I-B^{-1}A_i\big)\,s_{i,\rho}\\ &= \lim_{\rho\to+\infty}\Big[\big(I-B^{-1}A_i\big)\,s^+_{i,\rho}-\big(I-B^{-1}A_i\big)\,s^-_{i,\rho}\Big]\le \lim_{\rho\to+\infty}\Big[\big(B^{-1}A_i-I\big)s^-_{i,\rho}\Big],\end{aligned}$$

by assumption (11). Finally, from assumption (12) and the uniform in $\rho$ bound (36) we infer

$$\frac{d}{dt}w+Bw\le g(t),\quad \|g(t)\|_{L^q(\Omega)}\le C_q\ \text{ for all } q<+\infty.$$

Being a strong solution of problem (35), $w$ is also its mild solution. By the Duhamel formula and the positivity of $e^{-tB}$, we have

$$w(t)\le e^{-tB}w(0)+\int_0^t e^{-(t-s)B}g(s)\,ds,\quad \|g(t)\|_{L^q(\Omega)}\le C_q. \tag{37}$$

The first term is the operator $e^{-tB}$ applied to the function $w(0)=B^{-1}\big((u_1^0-\bar\alpha_1)+(u_3^0-\bar\alpha_3)\big)$ which belongs to $L^p(\Omega)$, thanks to Lemma 3.1(vi). By (14), for all $p<+\infty$ the $L^p$ norm of $e^{-tB}w(0)$ is globally bounded on $[0,T_{max})$ by a constant depending on $\|U^0\|_{L^\infty(\Omega)}$; by (23), $\|e^{-tB}w(0)\|_{L^p(\Omega)}$ decreases to zero if $T_{max}=+\infty$, $t\to+\infty$. Thus for all $p\in[1,+\infty)$, we have the (E.A.T.)

$$\|e^{-tB}w(0)\|_{L^p(\Omega)}\le \Phi_p(\|U^0\|_{L^\infty(\Omega)},t).$$

For all $p$ satisfying (34), $\frac{\sigma}{p}<1$, so that (37) and (23) provide the bound

$$\begin{aligned}\|w^+(t)\|_{L^p(\Omega)} &\le \Phi_p(\|U^0\|_{L^\infty(\Omega)},t)+C\int_0^t e^{\lambda_p(t-s)}(t-s)^{-\frac{\sigma}{p}}\|g(s)\|_{L^{p/2}(\Omega)}\,ds\\ &\le \Phi_p(\|U^0\|_{L^\infty(\Omega)},t)+C_p\int_0^t e^{-\lambda_p z}z^{-\frac{\sigma}{p}}\,dz\\ &\le \Phi_p(\|U^0\|_{L^\infty(\Omega)},t)+C_p.\end{aligned} \tag{38}$$

This yields the (E.A.T.) $\|w^+(t)\|_{L^p(\Omega)}\le \Phi_p(\|U^0\|_{L^\infty(\Omega)},t)$.

It remains to notice that by the positivity of $B^{-1}$ and of $u_i$, $w^-(t)$ is bounded by $B^{-1}(\bar\alpha_1+\bar\alpha_3)$; thanks to Lemma 3.1(vi), $\|w^-(t)\|_{L^p(\Omega)}\le C_p$. Finally, using again Lemma 3.1(v) and the positivity of $u_i$ we infer the (E.A.T.)

$$\begin{aligned}\|B^{-1}u_1(t)\|_{L^p(\Omega)}+\|B^{-1}u_3(t)\|_{L^p(\Omega)} &\le 2\|B^{-1}(\bar\alpha_1+\bar\alpha_3)\|_{L^p(\Omega)}+\|w^+(t)\|_{L^p(\Omega)}\\ &\le C_p+\Phi_p(\|U^0\|_{L^\infty(\Omega)},t)=\Phi_p(\|U^0\|_{L^\infty(\Omega)},t).\end{aligned}$$

• Step 2. We deduce the same (E.A.T.) for $A_j^{-1}u_i(t)$, for $i=1,3$ and $j=1..3$.

Indeed, from the positivity of $u_i$, the positivity of $A_j^{-1}$ and assumption (11), we have $0 \leq A_j^{-1}u_i$ and $0 \geq (I - B^{-1}A_j)(A_j^{-1}u_i) = A_j^{-1}u_i - B^{-1}u_i$, so that $0 \leq A_j^{-1}u_i \leq B^{-1}u_i$; this implies the desired (E.A.T.).

• Step 3. We deduce the $L^p(\Omega)$ (E.A.T.) on $A_j^{-1}u_2(t)$, for $j = 1..3$.

To this end, we now assign $w(t) := B^{-1}(u_2 - \bar{\alpha}_2)$. As in Step 1, we infer

$$\frac{d}{dt}w + Bw = B^{-1}f_2 + \big(I - B^{-1}A_2\big)(u_2 - \bar{\alpha}_2), \quad w(0) = B^{-1}(u_2^0 - \bar{\alpha}_2). \tag{39}$$

By the growth assumption (9) on $f_2$, we can dominate the first term in the right-hand side of (39) by the quantity $aB^{-1}(1 + u_3(t))$, which is already estimated in $L^q(\Omega)$ for all $q < +\infty$, thanks to Lemma 3.1(vi) and to Step 1. Splitting $(u_2 - \bar{\alpha}_2)$ and using (11) and (12) as in Step 1, we deduce

$$\frac{d}{dt}w + Bw \leq g(t), \quad \|g(t)\|_{L^q(\Omega)} \leq \Phi_q(\|U^0\|_{L^\infty(\Omega)}, t). \tag{40}$$

Similarly to the reasoning in Step 1, for all $p < +\infty$ from the Duhamel formula we deduce the $L^p(\Omega)$ (E.A.T.) of $w$, and then of $B^{-1}u_2(t)$. Indeed, the integral

$$I(t) = \int_0^t C_p\, e^{-\lambda_p(t-s)}(t-s)^{-\sigma/p}\Phi_{p/2}(\|U^0\|_{L^\infty(\Omega)}, s)\, ds$$

appearing in this calculation is estimated by splitting it into integrals over $[0, \frac{t}{2}]$ and $[\frac{t}{2}, t]$. Because $\Phi_{p/2}(r,t) \leq \Psi_{p/2}(r) := \sup_{t\in\mathbb{R}^+}\Phi_{p/2}(r,t)$, and because $\Phi_{p/2}(r,\cdot)$ can be assumed non-increasing, we infer for all $t \in [0, T_{max})$, $t \geq 2$

$$\begin{aligned} I(t) &\leq C_p \ \Big( \big(\tfrac{t}{2}\big)^{1-\sigma/p} e^{-\lambda_p t/2}\Psi_{p/2}(\|U^0\|_{L^\infty(\Omega)}) + \Phi_{p/2}(\|U^0\|_{L^\infty(\Omega)}, \tfrac{t}{2})\Big) \\ &= \Phi_p(\|U^0\|_{L^\infty(\Omega)}, t). \end{aligned}$$

For $t \leq \min\{T_{max}, 2\}$, we simply have

$$I_t \leq C_p\, \Phi_{p/2}(\|U^0\|_{L^\infty(\Omega)}, 0) \int_0^2 (t-s)^{-\sigma/p}\, ds = \Psi_p(\|U^0\|_{L^\infty(\Omega)}).$$

Finally, $I_t \leq \Psi_p(r)1\!\!1_{[0,2]}(t) + \Phi_p(r,t)1\!\!1_{[2,+\infty)}(t) = \Phi_p(r,t),\ r = \|U^0\|_{L^\infty(\Omega)}$, which is the desired (E.A.T.) estimate for $B^{-1}(u_2 - \bar{\alpha}_2)$. Since $B^{-1}\bar{\alpha}_2$ is bounded by Lemma 3.1(vi), we proceed as in Step 2 to deduce the same (E.A.T.) for $A_j^{-1}u_2(t)$.

• Step 4. We deduce the estimates on $e^{-tA_i}u_i(\tau)$, for $i = 1..3$ and $t \geq \delta > 0$.

Fix some $p < +\infty$. Because for $t > 0$, $Ran(e^{-tA_i}) \subset D(A_i)$ and $A_i^{-1}$ commutes with $e^{-tA_i}$, using the regularization property (25) we get

$$\begin{aligned} \|e^{-tA_i}u_i(\tau)\|_{L^\infty(\Omega)} &= \|A_i e^{-tA_i}\big(A_i^{-1}u_i(\tau)\big)\|_{L^\infty(\Omega)} \\ &\leq C_\delta\|A_i^{-1}u_i(\tau)\|_{L^p(\Omega)} = \Phi_\delta(\|U^0\|_{L^\infty(\Omega)}, \tau). \end{aligned}$$ ◇

**Remark 3.** In the above proof, the assumption (12) on $B^{-1}A_i$ can be dropped when $\bar{\alpha}_i = 0$ (see (36)).

Proof of Lemma 3.4: We exploit the bound (9), the E.A.T. estimates shown in Lemma 3.3, and the maximal regularity for the operator $\left(\frac{d}{dt}+A_1\right)$. Let us prove the estimates for $u_1$; the case of $u_2$ is entirely similar.

Fix $\delta>0$ with $2\delta<T_{max}$. Fix $\tau>0$ such that $\tau+2\delta<T_{max}$ (the case $\tau=0$ will be considered separately). Set $w(t):=u_1(t+\tau)-\bar\alpha_1$. Because the system is autonomous, we deduce that $w$ is a strong and thus also a mild solution of

$$\frac{d}{dt}w+A_1w=f_1(U(\cdot+\tau)),\quad w(0)=u_1(\tau)-\bar\alpha_1. \tag{41}$$

By the bound (9) on $f_1$ and the positivity of $e^{-tA_1}$, we have

$$\text{for } t\in[0,2\delta],\quad 0\le u_1(\tau+t)\le\bar\alpha_1+e^{-tA_1}(u_1(\tau)-\bar\alpha_1)+\tilde w(t), \tag{42}$$

where $\tilde w$ is the mild solution, on $[0,T_{max}-\tau)$, of the auxiliary homogeneous problem

$$\frac{d}{dt}\tilde w+A_1\tilde w=a(1+u_3(\tau+\cdot)),\quad \tilde w(0)=0. \tag{43}$$

Now we set $\tilde W:=A_1^{-1}\tilde w$; $\tilde W$ is a strong solution of

$$\frac{d}{dt}\tilde W+A_1\tilde W=aA_1^{-1}(1+u_3(\tau+\cdot)),\quad \tilde W(0)=0. \tag{44}$$

By the maximal regularity result of Theorem 3.2, for all $p\in(1,+\infty)$ we have

$$\begin{aligned}\|A_1\tilde W\|_{L^p((0,2\delta)\times\Omega)}\ ,\ \|\frac{d}{dt}\tilde W\|_{L^p((0,2\delta)\times\Omega)}&\\ \le\ C_p\|A_1^{-1}u_3(\tau+\cdot)\|_{L^p((0,2\delta)\times\Omega)}&+a(2\delta)^{1/p}\|A_1^{-1}1\|_{L^p(\Omega)}.\end{aligned} \tag{45}$$

In particular, using the estimate (28) of Lemma 3.3 and the uniform on $[0,2\delta]$ bound $\Phi_p(\|U^0\|_{L^\infty(\Omega)},\tau+\cdot)\le\Phi_p(\|U^0\|_{L^\infty(\Omega)},\tau)$ (recall that without loss of generality, $\Phi_p(r,\cdot)$ can be assumed non-increasing), with the help of Lemma 3.1(vi) we get

$$\|\tilde w\|_{L^p((0,2\delta)\times\Omega)}=\|A_1\tilde W\|_{L^p((0,2\delta)\times\Omega)}\le\ \delta^{1/p}\left(\Phi_p(\|U^0\|_{L^\infty(\Omega)},\tau)+C_p\right). \tag{46}$$

Furthermore, using estimate (29) with $t\in[\delta,2\delta]$, using (14), we get

$$\begin{aligned}&\|e^{-\,\cdot\,A_1}(u_1(\tau)-\bar\alpha_1)\|_{L^p((\delta,2\delta)\times\Omega)}\\ &\qquad\le C_p\,\delta^{1/p}\big(\Phi_\delta(\|U^0\|_{L^\infty(\Omega)},\,\tau)+\|\bar\alpha_1\|_{L^\infty(\Omega)}\big)=\Phi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)},\,\tau).\end{aligned}$$

Gathering the obtained estimates, from inequality (42) and the boundedness of $\bar\alpha_i$ we deduce the required (E.A.T.) (30).

Now let us prove (31). Fix $\tau=0$, start with (42) and use the same technique except for the term $e^{-tA_1}(u_1^0-\bar\alpha_1)$. This term is estimated by $\|u_1^0\|_{L^\infty(\Omega)}+\|\bar\alpha_1\|_{L^\infty(\Omega)}$ uniformly in $t$, thanks to (14). Therefore (42) and the bound (46) yield

$$\begin{aligned}\|u_1(\cdot)\|_{L^p((0,2\delta)\times\Omega)}&\le\delta^{1/p}\,\Phi_p(\|U^0\|_{L^\infty(\Omega)},0)+C_p\,\delta^{1/p}(\|u_1^0\|_{L^\infty(\Omega)}+\|\bar\alpha_1\|_{L^\infty(\Omega)})\\ &=\Psi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)}).\end{aligned}$$

◇

Proof of Lemma 3.5: The proof is split into two steps.

• Step 1. We prove the required (E.A.T.) for $u_3(t)$.

We use the third equation of the system and exploit the Duhamel formula, the estimates of Lemma 3.4, and the polynomial growth restriction (10).

First, take any $\tau \in [2\delta, T_{max})$. We have

$$0 \le u_3(t+\tau-\delta) = \bar{\alpha}_3 + e^{-tA_3}(u_3(\tau-\delta) - \bar{\alpha}_3) + \tilde{w}(t), \tag{47}$$

where $\tilde{w}$ solves the problem

$$\frac{d}{dt}\tilde{w} + A_3\tilde{w} = g_\tau, \quad \tilde{w}(0) = 0, \tag{48}$$

with $g_\tau(\cdot) := f_3(U(\cdot+\tau-\delta))$; we have

$$g_\tau(\cdot) \ \le \ b\left(1 + |u_1|^\beta(\cdot+\tau-\delta) + |u_2|^\gamma(\cdot+\tau-\delta)\right) \tag{49}$$

thanks to the growth assumption (10). Hence estimate (30) implies the (E.A.T.)

$$\forall p \in [1,+\infty) \ \forall \tau \in [2\delta, T_{max}) \quad \|g_\tau^+\|_{L^p((0,\delta)\times\Omega)} \le \Phi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)}, \tau-\delta). \tag{50}$$

Using the positivity of $e^{-tA_3}$, the Duhamel formula and the $L^p - L^\infty$ regularizing effect (24), we infer for $p$ such that $\sigma\frac{p'}{p} < 1$ (in particular, for $p$ as in (34)) the inequality

$$\begin{aligned} \|\tilde{w}^+(\delta)\|_{L^\infty(\Omega)} &\le \int_0^\delta C_p(\delta-s)^{-\frac{\sigma}{p}}\|g_\tau^+(s)\|_{L^p(\Omega)}\,ds \\ &\le C_p\left(\int_0^\delta \left((\delta-s)^{-\frac{\sigma}{p}}\right)^{p'}\right)^{1/p'}\|g_\tau^+\|_{L^p((0,\delta)\times\Omega)} \\ &\le C_{\delta,p}\Phi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)}, \tau-\delta) = \Phi_{\delta,p}(\|U^0\|_{L^\infty(\Omega)}, \tau). \end{aligned} \tag{51}$$

In addition, using estimate (29) for $t = \tau - \delta$ and using (14), we get

$$\|e^{-\delta A_3}(u_3(\tau-\delta)-\bar{\alpha}_3)\|_{L^\infty(\Omega)} \le \Phi_\delta(\|U^0\|_{L^\infty(\Omega)}, \tau-\delta) + \|\bar{\alpha}_3\|_{L^\infty(\Omega)} = \Phi_\delta(\|U^0\|_{L^\infty(\Omega)}, \tau).$$

Gathering the obtained estimates, from (47) (with $t = \delta$) we infer (32) for $i = 3$.

With the analogous reasoning, taking $\tau = \delta$ in (47), replacing estimate (30) by the uniform in $t \in [0, 2\delta]$ bound (31), and using the bound

$$\|e^{-tA}(u_3^0 - \bar{\alpha}_3)\|_{L^\infty(\Omega)} \le \|u_3^0\|_{L^\infty(\Omega)} + \|\bar{\alpha}_3\|_{L^\infty(\Omega)} = \Psi(\|U^0\|_{L^\infty(\Omega)}),$$

we infer (33) for $i = 3$.

• Step 2. We deduce the required (E.A.T.) for $u_1(t)$ and $u_2(t)$.

Thanks to (9) and the bound obtained in Step 1, we have

$$\forall i = 1,2 \quad \forall t > 0 \ \ \|f_i^+(U(t))\|_{L^\infty(\Omega)} \le \Phi(\|U^0\|_{L^\infty(\Omega)}, t).$$

Therefore we can repeat the reasoning of Step 1, replacing $u_3$ with $u_1$ and $u_2$. ⋄

3.4. **The Neumann case.** In this subsection, we prove Theorem 2.5 used in the case some of the operators $A_i$ are not invertible (in practice, this corresponds to the Neumann boundary conditions in (3)).

Proof of Theorem 2.5: We indicate the modifications to the statements and the proofs analogous to those of Lemmas 3.3, 3.4, 3.5.

Because we only get time-dependent bounds, denote by $\Theta(r,t)$ a generic function from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$ which is non-decreasing in each of the two variables; additional subscripts $c, p$ of $\delta$ denote the dependence of $\Theta(\cdot,\cdot)$ on these parameters.

Similarly to Lemma 3.3, we first prove, in the place of (28) and (29), the bounds

$$\forall i,j = 1..3 \ \forall p \in [1,+\infty) \ \ \forall t < T_{max} \ \|A_{j,c}^{-1}u_i(t)\|_{L^p(\Omega)} \le \Theta_{c,p}(\|U^0\|_{L^\infty(\Omega)}, t); \tag{52}$$

(53) $\forall i\!=\!1..3\ \forall\delta\!>\!0\ \ \forall t\!\ge\!\delta\ \ \forall\tau\!<\!T_{max}\ \|e^{-tA_{i,c}}u_i(\tau)\|_{L^\infty(\Omega)}\!\le\!\Theta_{c,\delta}(\|U^0\|_{L^\infty(\Omega)},\,\tau).$

We follow step by step the proof of Lemma 3.3.

• Steps 1 and 2. We use $w_c := B_c^{-1}\big((u_1-\bar\alpha_1)+(u_3-\bar\alpha_3)\big)$ and remark that $w_c$ satisfies the inequations

$$\frac{d}{dt}w_c + B_c w_c \le c\,w_c + \sum\nolimits_{i=1,3}\big(I-B_c^{-1}A_{i,c}\big)(u_i-\bar\alpha_i),$$

which leads to $\frac{d}{dt}w_c + Bw_c \le g(t)$ with $\|g(t)\|_{L^q(\Omega)} \le C_{c,q}$ for all $q<+\infty$.

Notice that $B$ satisfies the properties (i)–(v) of Lemma 3.1 with the value $\lambda_p=0$ in (23). Then we can still write (38) with $\lambda_p=0$; the remaining arguments do not change, and we conclude to (52) with $i=1,3$.

• Step 3. We put $w(t) := B_c^{-1}(u_2-\bar\alpha_2)$ and get (40) with $\Phi_q(\|U^0\|_{L^\infty(\Omega)},t)$ replaced by $\Theta_{c,q}(\|U^0\|_{L^\infty(\Omega)},t)$. Because of the weaker hypothesis (19), we have to use the Gronwall lemma to control the growth of $B_c^{-1}u_2(t)$ in $L^p(\Omega)$.

• Step 4. There is no change to this argument; we get (53).

Now we follow the proof of Lemma 3.4. In the place of (41), we write

$$\frac{d}{dt}w + A_{1,c}w = f_1(U(\cdot+\tau)) + cu_1(\cdot+\tau), \quad w(0)=u_1(\tau)-\bar\alpha_1.$$

for $w(t)=u_1(t+\tau)-\bar\alpha_1$; then we use growth assumption (19) on $f_1$, the estimates (52) for $j=1$, $i=1,2,3$; we base the calculation upon the semigroup $e^{-tA_{1,c}}$ and exploit the maximal regularity of the operator $A_{1,c}$. We do the same with $A_{1,c}$ replaced by $A_{2,c}$. In the place of (30), we obtain the bound

$$\begin{aligned}&\forall i\!=\!1,2\ \ \forall p\!\in\![1,+\infty)\ \ \forall\delta\!>\!0\ \ \forall\tau\!<\!T_{max}\!-\!2\delta\\ &\|u_i(\tau+\cdot)\|_{L^p((\delta,2\delta)\times\Omega)} \le \Theta_{c,\delta,p}(\|U^0\|_{L^\infty(\Omega)},\,\tau).\end{aligned} \tag{54}$$

Finally, we follow the proof of Lemma 3.5. In the place of (32), we get the bound

$$\forall i\!=\!1..3\ \ \forall\delta\!>\!0\ \ \forall\tau\!\in\![2\delta,T_{max})\quad \|u_i(\tau)\|_{L^\infty(\Omega)} \le \Theta_{c,\delta}(\|U^0\|_{L^\infty(\Omega)},\,\tau). \tag{55}$$

In Step 1 of the proof, while considering $\tilde w$ introduced in (48), we need the assumption (20) that allows to write

$$\frac{d}{dt}\tilde w + A_{3,c}\tilde w \le g_\tau, \quad \tilde w(0)=0, \tag{56}$$

with the bound (49). We conclude to (55) for $i=3$. Finally, in Step 2 of the proof, we use again the Gronwall lemma to limit the growth of $(\|u_1\|_{L^\infty(\Omega)}\!+\!\|u_2\|_{L^\infty(\Omega)})(t)$.

We eventually arrive at a finite bound $\|U(t)\|_{L^\infty(\Omega)} \le \Theta_{c,\delta}(\|U^0\|_{L^\infty(\Omega)},\,t)$ for $t<T_{max}$; this amouts to global existence of a mild solution to $(S)$. ⋄

3.5. **Existence of a maximal attractor.** The result is classical, except that $\{S(t)\}_{t\ge 0}$ is not continuous in the topology $(L^\infty(\Omega))^3$; thus the $L^2$-continuity and the $L^2-L^\infty$ regularizing effect are used instead (cf. the general statement of Bénilan and Labani [BL2]).

Proof of Corollary 2:

(i) For $T>0$, $r>0$, let $(U^{0,k})_{k\in\mathbb{N}}$ be a sequence in $((L^\infty(\Omega))^+)^3$ such that $\|U^{0,k}\|_{L^\infty}\le K$. We have to show that $(U^k(T))_{k\in\mathbb{N}} := (S(T)U^{0,k})_{k\in\mathbb{N}}$ is relatively compact in $(L^\infty(\Omega))^3$. For $t\le T$, by the Duhamel formula (18), $u_i^k(t)\!-\!\bar\alpha_i$ is the sum of two terms: the term $e^{-tA_i}(u_i^{0,k}\!-\!\bar\alpha_i)$ which is compact by the assumption, and the term $T_i^k(t) := \int_0^t e^{-sA_i} f_i^k(t-s)\,ds$ with $f_i^k(t) := f_i(U^k(t))$, for $i = 1..3$. Because Theorem 2.4 implies a uniform $L^\infty$ bound on the source terms $f_i^k$, the families $(T_i^k(\cdot))_k$ are compact in $C([0,T]; L^2(\Omega))$ by the result of Baras, Hassan and Véron [BHV]. Consequently, from (18) we get $u_i^k(t) \to u_i(t)$ and $f_i^k(\cdot) \to f_i = f_i(U(\cdot))$ in $C([0,T]; L^2(\Omega))$ for a (not relabelled) subsequence $k \to \infty$; here we denote the limit by $U := (u_1, u_2, u_3)$.

Moreover, by the continuity of $e^{-sA_i}$ from $L^2(\Omega)$ to $L^\infty(\Omega)$ (we use (16)), for all $s > 0$ we have

$$e^{-sA_i} f_i^k(t-s) \to e^{-sA_i} f_i(t-s) \text{ in } L^\infty(\Omega) \text{ as } k \to \infty.$$

Applying the Lebesgue dominated convergence theorem (note (14) with $p = \infty$), we get the convergence of $T_i^k(t)$ to $\int_0^t e^{-sA_i} f_i(U(t-s))\,ds$ also in the $L^\infty(\Omega)$. Now from (18), we deduce that, for a subsequence, $S(t)U^{0,k} \to U(t)$ in $(L^\infty(\Omega))^3$, which had to be proved.

(ii) Recall that a maximal attractor in $((L^\infty(\Omega))^+)^3$ is a compact set that is invariant for the semigroup $\{S(t)\}_{t\geq 0}$ on $((L^\infty(\Omega))^+)^3$ and satisfies

$$\forall\, r > 0 \ \lim_{t\to\infty} \sup\nolimits_{U^0 \in ((L^\infty(\Omega))^+)^3,\, \|U^0\|_\infty \leq r} \operatorname{dist}\big(S(t)U^0, \mathcal{M}\big) = 0. \tag{57}$$

We first show that for all $\tau > 0$, $S(\tau)\mathcal{M} = \mathcal{M}$. First, by the definition of $\mathcal{M}$, $U \in \mathcal{M}$ if and only if there exists a sequence $(t_k)_k$ going to infinity and a sequence $(U^k)_k \in \mathcal{E}$ such that $S(t_k)U^k \longrightarrow U$ in $(L^\infty(\Omega))^3$. By the $L^2$ continuity of $\{S(t)\}_{t\geq 0}$, we have

$$S(\tau)U = L^2\text{–}\lim_{k\to\infty} S(t_k + \tau)U^k \in \mathcal{M}.$$

Further, by (i) there exists a (not relabelled) subsequence $(t_k)_k$ such that $t_k \geq \tau$ and $S(t_k - \tau)U^k \longrightarrow V$ in $(L^\infty(\Omega))^3$. By the definition of $\mathcal{M}$, we then have $V \in \mathcal{M}$; and

$$U = L^2\text{–}\lim_{k\to\infty} S(t_k)U^k = S(\tau)V \in S(\tau)\mathcal{M}.$$

It remains to show (57). Reasoning by contradiction, assume that $(U^k)_k$ is a bounded sequence in $((L^\infty(\Omega))^+)^3$ such that $\operatorname{dist}\big(S(t_k)U^k, \mathcal{M}\big) \geq const > 0$ for some sequence $(t_k)_k$ going to infinity. By (i), up to extraction of a subsequence we have $S(t_k)U^k \longrightarrow U$ in $(L^\infty(\Omega))^3$. Yet for $\tau$ large enough, the (E.A.T.) estimate ensures that $S(\tau)U^k \in \mathcal{E}$, therefore we obtain a contradiction from the fact that

$$U = \lim_{k\to\infty} S(t_k)U^k = \lim_{k\to\infty} S(t_k - \tau)\left(S(\tau)U^k\right) \in \lim_{k\to\infty} S(t_k - \tau)\,\mathcal{E} \subset \mathcal{M}.$$

It remains to notice that $\mathcal{M}$ is bounded because it is included in $\mathcal{E}$, and $\mathcal{M}$ is closed by construction; thus from (i) and the identity $S(\tau)\mathcal{M} = \mathcal{M}$, we see that $\mathcal{M}$ is compact in $(L^\infty(\Omega))^3$. ◇

## 4. Example of a preconditioning operator

In this section we prove existence of a preconditioning operator for the Laplace operator $-\Delta$ with homogeneous Robin or Dirichlet boundary conditions. Notice that the case of the Dirichlet boundary condition has to be treated apart. Let us also point out that in the case where $A$ is the Laplace operator with the Neumann boundary condition, preconditioning operator $B$ satisfying (11) cannot exist, because $B^{-1}Au = 0 < u$ for $u \equiv 1$.

**Proposition 1.** *Let $A$ be the operator associated with $-d\Delta$ on $\Omega$ with the boundary condition $\lambda\partial_n u+(1-\lambda)u=0$ on $\partial\Omega$ with parameter $\lambda\in[0,1)$. Take $e\in(0,d]$ and $\mu\in[\lambda,1)$. Consider the operator $B$ associated with $-e\Delta$ on $\Omega$ with the homogeneous boundary condition on $\partial\Omega$ with parameter $\mu$.*

(i) *The operator $B$ is of class $\mathcal{A}$ and satisfies property* (11).
(ii) *Assume that either $\lambda=\mu=0$, or $\lambda>0$. Then property* (12) *holds.*

Remark that (ii) holds also for $\mu=0$ and $\lambda>0$ (see [BL2]; cf. the $L^1$ estimate of $\partial_n z$ in [DL, Vol.2, II.6.4, Proposition 9]); yet the constraint $\mu\geq\lambda$ is required for (i).

Proof :

(i) It is well known that $-B$ generates a positive, analytic, exponentially stable semigroup on $L^2(\Omega)$ that is $L^p$-non-expansive and hypercontractive (see in particular Friedman [Fr], Pazy [Pa], Rothe [Rot]; in particular, the generator of the semigroup is the linear operator induced by a bilinear form satisfying the properties listed in Remark 1 and [BL2]).

For the proof of the inequality (11), consider $u\in D(A)$, $u\geq 0$. Then there exists $h\in L^2(\Omega)$ such that $u$ and $v:=B^{-1}Au$ verify

$$\left\{\begin{array}{l} -d\Delta u=h \ \text{ in } \Omega \\ \lambda\partial_n u+(1-\lambda)u=0 \ \text{ on } \partial\Omega, \end{array}\right. \qquad \left\{\begin{array}{l} -e\Delta v=h \ \text{ in } \Omega \\ \mu\partial_n v+(1-\mu)v=0 \ \text{ on } \partial\Omega. \end{array}\right.$$

Set $k:=d/e$ (notice that $k\geq 1$) and $z:=ku-v$. We have two cases.

• If $\lambda>0$, then an easy calculation shows that $z$ verifies

$$\left\{\begin{array}{l} -\Delta z=0 \ \text{ in } \Omega \\ \mu\partial_n z+(1-\mu)z=k\,\frac{\lambda-\mu}{\lambda}\,u \ \text{ on } \partial\Omega. \end{array}\right.$$

Because $u\geq 0$ and $\lambda-\mu\leq 0$, $z$ is a subsolution of the problem $Bw=0$ with the Robin boundary condition with parameter $\mu$. By the maximum principle, $z\leq 0$. Then $u-v\leq ku-v\leq 0$, because $k\geq 1$ and $u\geq 0$.

• If $\lambda=0$, then $z$ verifies

$$\left\{\begin{array}{l} -\Delta z=0 \ \text{ in } \Omega \\ \mu\partial_n z+(1-\mu)z=k\,\mu\,\partial_n u \ \text{ on } \partial\Omega. \end{array}\right.$$

The maximum principle yields $\partial_n u\leq 0$ (in the weak sense) on $\partial\Omega$; we conclude in the same way as above.

In both cases, we have found that $u\leq v$, that also reads as $(I-B^{-1}A)u\leq 0$.

(ii) The proof is the duality reasoning of Martin and Pierre [MP2] and Bénilan and Labani [BL2]; we give it here in a simplified setting.

First we fix $p\in(1,+\infty)$ and $u\in D(A)\cap L^\infty(\Omega)$, $u\geq 0$. Let $v$ and $h$ have the same meaning as in (i). Without loss of generality, we may assume that $d=e=1$. Notice that thanks to (i), we already have $0\leq u\leq v$. Now we consider the auxiliary problem

$$\left\{\begin{array}{l} -\Delta z=v^{p-1} \ \text{ in } \Omega \\ \mu\partial_n z+(1-\mu)z=0. \end{array}\right. \tag{58}$$

Once more, we consider separately the two cases.

• If $\lambda > 0$, then using the Green-Gauss formula (or the symmetry of $B$ in $L^p$), we get

$$\|v\|^p_{L^p(\Omega)} = \int_\Omega v\, v^{p-1} = -\int_\Omega v\,\Delta z = -\int_\Omega \Delta v\, z = \int_\Omega h\, z$$
$$= -\int_\Omega \Delta u\, z = -\int_\Omega u\Delta z + \int_{\partial\Omega} \big(u\partial_n z - z\partial_n u\big) = \int_\Omega u v^{p-1} + \int_{\partial\Omega} \big(u\partial_n z - z\partial_n u\big).$$

Because $\lambda > 0$ and $\mu < 1$, we can write $z\partial_n u = \frac{\mu(\lambda-1)}{\lambda(\mu-1)} u\partial_n z$. As in [MP2], we notice that the combination of the Calderón-Zygmund $W^{2,q}$ regularity estimate (see e.g. [GT]) and of the trace embedding ensure that the solution of (58) verifies the $L^{p'}$ estimate

$$\|\partial_n z\|_{L^{p'}(\partial\Omega)} \le C_p\,\|v^{p-1}\|_{L^{p'}(\Omega)} = C_p\,\|v\|^{p-1}_{L^p(\Omega)} \tag{59}$$

with a constant $C_p$ that depends on $p$ and $\Omega$ but that is independent of $v$.

Therefore, with the help of the Hölder inequality we deduce that

$$\|v\|^p_{L^p(\Omega)} \le C_p\,\|u\|_{L^\infty(\Omega)}\|v\|^{p-1}_{L^p(\Omega)}.$$

Hence $\|v\|_{L^p(\Omega)} \le C_p\,\|u\|_{L^\infty(\Omega)}$, which means that $\|B^{-1}Au\|_{L^p(\Omega)} \le C\|u\|_{L^\infty(\Omega)}$.

• If $\lambda = \mu = 0$, then $A = \frac{d}{e}B$, and (12) (even with $p = \infty$) is evident. $\diamond$

**Remark 4** (cf. Martin and Pierre [MP2])**.** In the setting of Proposition 1, estimate (12) may fail in the case $\lambda = 0$ and $\mu > 0$. To give an example, consider the one-dimensional case with $\Omega = (0,1)$ and (with the notation of the above proof) consider the family $(u_{n,m})_{m\ge n}$ solving the Dirichlet Laplacian problem

$$-(u)'' = h_{n,m}, \quad u(0) = 0 = u(1) \qquad \text{with } h_{n,m}(x) = m\rho(m(x - \tfrac{1}{n})),$$

where $\rho \in C_0^\infty([-1,1];\mathbb{R}^+)$ is the standard function used for construction of sequences of mollifiers. As $m \to \infty$, $h_{n,m}$ goes to the Dirac measure concentrated at $x_n = \frac{1}{n}$, and $u_{n,m}$ goes to the function $u_n(\cdot) := G(\cdot, x_n)$, where $G(\cdot,\cdot)$ is the Green function of the Dirichlet Laplacian on $(0,1)$. Explicit calculation shows that $\lim_{n\to\infty}\lim_{m\to\infty}\|u_{n,m}\|_{L^\infty((0,1))} = 0$. Moreover, $\lim_{n\to\infty}\lim_{m\to\infty} u'_{n,m}(0) = +\infty$, which forces the solutions $v_{n,m}$ of the Robin Laplacian problem

$$-(v)'' = h_{n,m}, \quad (-\mu v' + (1-\mu)v)(0) = 0 = (\mu v' + (1-\mu)v)(1)$$

to go to infinity (e.g. in $L^1(0,1)$) as $m \to \infty$ and then $n \to \infty$; this is easily seen from the fact that the difference $v_{n,m} - u_{n,m}$ is an affine function. Thus the ratio $\|v_{m,n}\|_{L^1((0,1))}/\|u_{n,m}\|_{L^\infty((0,1))}$ is unbounded, which contradicts the statement (12).

**Remark 5.** The above Remark 4 (together with Remark 3) corresponds to the third case of the assumption (5); this case excludes the possibility to have Robin BC on some component(s) of $U = (u_1, u_2, u_3)$ and *non-homogeneous* Dirichlet BC on some other component(s). We guess that this restriction is a technical one. Indeed, the above restriction is needed to get the upper bound on $(I - B^{-1}A_i)(u_i(t) - \bar\alpha_i)$ in the proof of Lemma 3.3; we stress that the difficulty comes from $(u_i(t) - \bar\alpha_i)^-$, i.e. from the small values of $u$ and not from the large ones.

**Proposition 2.** *Let $A$ be the operator associated with $-d\Delta$ on $\Omega$ with the boundary condition $\lambda\partial_n u + (1-\lambda)u = 0$ on $\partial\Omega$ with parameter $\lambda \in [0,1]$. Take $e \in (0,d]$. Consider the operator $B$ associated with $-e\Delta$ on $\Omega$ with the homogeneous Neumann boundary condition on $\partial\Omega$. Take $c > 0$.*

(i) *The operator $(B+cI)$ is of class $\mathcal{A}$ and satisfies property* (11) *with $A, B$ replaced by $(A+cI), (B+cI)$, respectively.*
(ii) *Assume that $\lambda > 0$. Then property* (12) *holds with $A, B$ replaced by $(A+cI), (B+cI)$, respectively.*

The proof is left to the reader; the arguments are the same as for Proposition 1.

## 5. Examples and extensions

### 5.1. A $3 \times 3$ system with Dirichlet or Robin boundary conditions.

**Theorem 5.1.** *Consider system* (1) *with the Dirichlet or Robin boundary conditions* (3) *with $\lambda_i$ that satisfy one of the assumptions* (5). *Assume in addition that the boundary source terms $\alpha_i$ belong to the class $H^{1/2}(\partial\Omega) \cap (L^\infty(\partial\Omega))^+$ (if $\lambda_i = 0$) or to the class $H^{-1/2}(\partial\Omega) \cap (L^\infty(\partial\Omega))^+$ (if $\lambda_i > 0$).*

*Assume that the locally Lipschitz reaction terms $f_i$, $i = 1..3$, satisfy* (7)–(10).

*Then for all initial data $\left(u_1^0, u_2^0, u_3^0\right) \in \left((L^\infty(\Omega))^+\right)^3$ there exists a unique global in time mild (and also strong) solution to* (1),(3) *with values in $\left((L^\infty(\Omega))^+\right)^3$; moreover, there exists a maximal attractor in $\left((L^\infty(\Omega))^+\right)^3$ for system* (1),(3).

Proof : First, let us point out that the data $\alpha_i$ of the type we consider admit a $W^{1,2}(\Omega) \cap (L^\infty(\Omega))^+$ lift $\bar\alpha_i$ inside $\Omega$ defined by

$$\begin{cases} -d_i \Delta \bar\alpha_i = 0 & \text{in } \Omega \\ \lambda_i \partial_n \bar\alpha_i + (1-\lambda_i)\bar\alpha_i = \alpha_i & \text{on } \partial\Omega. \end{cases}$$

The so defined extension satisfy $e^{r\Delta}\bar\alpha_i = \bar\alpha_i$ for all $r > 0$, in particular (6) holds. Since we have the equality $-d_i\Delta u_i = -d_i\Delta(u_i - \bar\alpha_i)$ in $\Omega$ with $\lambda_i\partial_n(u_i - \bar\alpha_i) + (1-\lambda_i)(u_i - \bar\alpha_i) = 0$ on $\partial\Omega$, in the $W^{1,2}(\Omega)$ sense, we can recast the problem into the abstract form $(S)$ with the operators $A_i$ given by $-d_i\Delta$ with homogeneous Robin or Dirichlet boundary conditions. It is well known that the linear semigroups $e^{-tA_i}$ are compact in $L^2(\Omega)$ for $t > 0$ (see e.g. [Pa]), hence the compactness of $e^{-tA_i}$ in $L^\infty(\Omega)$ follows from the hypercontractivity (16).

Assume that the first or the second case of assumptions (5) occurs. Then, according to Proposition 1, condition $(H)$ holds true; indeed, we can choose for the preconditioning operator $B$ the operator $-e\Delta$ with $e = \min_{i=1..3} d_i$ with the homogeneous boundary condition (3) corresponding to $\lambda := \max_{i=1..3}\lambda_i$. In the last case of assumptions (5), we notice that $\alpha_i = 0$ implies $\bar\alpha_i = 0$, thus Remark 3 can be used in the place of Proposition 1(ii).

Therefore the conclusions follow by Theorem 2.4 and Corollary 2. ◇

### 5.2. The case of a Neumann boundary condition.

In a similar way, we get global existence for (1) when Neumann boundary conditions are imposed on some of the components.

**Theorem 5.2.** *Assume that $\lambda_i \in [0,1]$ satisfy*

$$\begin{array}{c} \text{either } \lambda_i \in (0,1],\ i=1..3,\ \text{ or } \lambda_1=\lambda_2=\lambda_3=0, \\ \text{or } \lambda_i \in [0,1] \text{ with } \alpha_i = 0 \text{ for } i \text{ such that } \lambda_i = 0. \end{array} \tag{60}$$

*Assume that $\alpha_i$ are of the same kind as in Theorem* 5.1*, and the locally Lipschitz reaction terms $f_i$, $i = 1..3$, satisfy* (7),(8) *and* (19),(20)*.*

*Then for all initial data $\left(u_1^0, u_2^0, u_3^0\right) \in \left((L^\infty(\Omega))^+\right)^3$ there exists a unique global in time mild (and also strong) solution to* (1),(3) *with values in $\left((L^\infty(\Omega))^+\right)^3$.*

Proof : The proof follows the lines of the previous one, using Theorem 2.5 and 2 in the place of Theorem 2.4(i) and Proposition 1, respectively. $\diamond$

5.3. **A** $5 \times 5$ **system.** It is easy to use the same approach on system (2),(3). We get the following result.

**Theorem 5.3.** *Consider system* (2) *with the Dirichlet or Robin boundary conditions* (3) *corresponding to $\lambda_i$ and $\alpha_i$ of the same kind as in Theorem* 5.1 *(but now for $i = 1..5$).*

*Then for all initial data $\left(u_i^0\right)_{i=1..5} \in \left((L^\infty(\Omega))^+\right)^5$ there exists a unique mild (and also strong) solution to* (2),(3) *with values in $\left((L^\infty(\Omega))^+\right)^5$; moreover, there exists a maximal attractor in $\left((L^\infty(\Omega))^+\right)^3$ for system* (2),(3)*.*

Proof : One follows the whole scheme of the proof of Theorem 2.4 (via Lemmas 3.3,3.4,3.5); then the claims follow exactly in the same way as in Theorem 5.1.

The main modification (which is a simplification) is in the proof of the analogue of Lemma 3.3. In the place of the function $w(\cdot)$ used in the proof of Lemma 3.3, here we use

$$w(t) := B^{-1}\big((u_1-\bar{\alpha}_1) + 2(u_2-\bar{\alpha}_2) + (u_3-\bar{\alpha}_3) + 2(u_4-\bar{\alpha}_4) + (u_5-\bar{\alpha}_5)\big).$$

Combining the five equations in (2) with the respective weights $1, 2, 1, 2, 1$, proceeding as in Lemma 3.3 we get directly the $L^\infty$ (E.A.T.) estimate on $\|B^{-1}u_i(t)\|_{L^p(\Omega)}$ for all $i$. Hence the estimates (28),(29) with $i, j = 1..5$ follow readily from the property (11) of the preconditioner $B$. Then, as in Lemma 3.4, we deduce the (E.A.T.) estimates (30),(31) for $i = 1$, $i = 3$ and $i = 5$. Finally, as in Lemma 3.5, with the $L^p$ technique of Martin and Pierre [MP1] we get the estimates (32),(33) for $i = 2$ and $i = 4$, whence the same estimates for $i = 1, 3, 5$ follow. $\diamond$

**Acknowledgement** We thank Wolfgang Arendt, Dieter Bothe, Mustapha Mokhtar-Kharroubi and Michel Pierre for stimulating discussions and for providing references.

## References

[AL] S. Amraoui and H. Labani. Global Existence and Maximal Attractor of Facilitated Diffusion Model. *Forum Math.* 15, pp.923–933, 2003.

[A] W. Arendt. *Semigroups and Evolution Equations: Functional Calculus, Regularity and Kernel Estimates.* Evolutionary equations. Vol. I, 1–85, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[B] V. Barbu. *Nonlinear semigroups and differential equations in Banach spaces.* Noordhoff International Publishing, Leiden, 1976.

[BHV] P. Baras, J.-C. Hassan and L. Véron. Compacité de l'opérateur définissant la solution d'une équation d'évolution non homogène. *C.R. Acad. Sci. Paris Sér. A* 284, no.14, pp.799-802.

[BL1] Ph. Bénilan and H. Labani. Systèmes de récation-diffusion abstraites (French). *Publications Mathématiques de Besançon. Analyse Non Linéaire* vol.14, 1994.

[BL2] Ph. Bénilan and H. Labani. Existence of attractors in $L^\infty(\Omega)$ for a class of reaction-diffusion systems. In *Nonlinear evolution equations and related topics*, 771–784, Birkhäuser, Basel, 2004.

[DL] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology.* Vol. 2, Springer-Verlag, Berlin, 1988.

[DS] N. Dunford and J. T. Schwartz. *Linear Operators. Parts I and II.* Wiley-Interscience, 1958.

[E] W. Ebel. Existence and asymptotic behaviour of solutions in a system of reaction diffusion equations. *Math. Z.* 193, no.1, 41–66, 1986.

[Fe1] W. Feng. Global existence and boundedness of the solution for a blood oxigenation model. *J. Math. Anal. Appl.* 181, 462–472, 1994.

[Fe2] W. Feng. Stability and asymptotic behaviour in a reaction-diffusion system. *M3AS Math. Models Meth. Appl. Sci.* 17, 155–169, 1994.

[Fr] A. Friedman. *Partial Differential Equations.* New York, 1969, 2008.

[GT] D. Gilbard and N. Trudinger. *Elliptic partial differential equations of second order.* Grundl. Math. Wiss. 224. Springer-Verlag, Berlin, 1983, 1998, 2001.

[L] H. Labani. *Comportement asymptotique de certaines équations de réaction-diffusion et d'une classe d'équations des ondes.* (French) Th́ese d'État, Marrakesh, 2002.

[La] D. Lamberton. Équations d'évolution lináires associés à des semi-groupes de contractions dans les espaces $L^p$. (French) *J. Funct. Anal.* 72, no. 2, 252–262, 1987.

[MP1] R.H. Martin and M. Pierre. Nonlinear reaction-diffusion systems. In *Nonlinear equations in the applied sciences*, 363–398, *Math. Sci. Engrg.*, 185, Academic Press, Boston, 1992.

[Mo] J. Morgan. Boundedness and decay results for reaction-diffusion systems. *SIAM J. Math. Anal.* 21, no. 5, 1172–1189, 1990.

[MP2] R. Martin and M. Pierre. Influence of mixed boundary conditions in some reaction-diffusion systems. *Proc. Roy. Soc. Edinburgh Sect. A* 127, no. 5, 1053–1066, 1997

[Pa] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Appl. Math. Sci. 44, Springer-Verlag, New York, 1983.

[Rot] F. Rothe. *Global solutions of reaction-diffusion systems.* Lect. Notes in Math. 1072, Springer-Verlag, Berlin, 1984.

[Tem] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics.* Appl. Math. Sci. 68, Springer-Verlag, New York, 1988, 1997.

*E-mail address*: `boris.andreianov@univ-fcomte.fr`