



**HAL**  
open science

# Besov algebras on Lie groups of polynomial growth and related results

Isabelle Gallagher, Yannick Sire

► **To cite this version:**

Isabelle Gallagher, Yannick Sire. Besov algebras on Lie groups of polynomial growth and related results. 2010. hal-00522665v1

**HAL Id: hal-00522665**

**<https://hal.science/hal-00522665v1>**

Preprint submitted on 1 Oct 2010 (v1), last revised 9 Oct 2012 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# BESOV ALGEBRAS ON LIE GROUPS OF POLYNOMIAL GROWTH AND RELATED RESULTS

ISABELLE GALLAGHER AND YANNICK SIRE

ABSTRACT. In this paper we prove an algebra property under pointwise multiplication for Besov spaces defined on Lie groups of polynomial growth. When the setting is restricted to the case of H-type groups, this algebra property is generalized to paraproduct estimates. Some considerations on pseudodifferential calculus are also provided in that setting.

## 1. INTRODUCTION

**1.1. Lie groups of polynomial growth.** In this paper  $\mathbb{G}$  is an *unimodular* connected Lie group endowed with the Haar measure. By “unimodular” we mean that the Haar measure is left and right-invariant. If we denote by  $\mathcal{G}$  the Lie algebra of  $\mathbb{G}$ , we consider a family

$$\mathbb{X} = \{X_1, \dots, X_k\}$$

of left-invariant vector fields on  $\mathbb{G}$  satisfying the Hörmander condition, i.e.  $\mathcal{G}$  is the Lie algebra generated by the  $X_i$ 's. A standard metric on  $\mathbb{G}$ , called the Carnot-Carathéodory metric, is naturally associated with  $\mathbb{X}$  and is defined as follows: let  $\ell : [0, 1] \rightarrow \mathbb{G}$  be an absolutely continuous path. We say that  $\ell$  is admissible if there exist measurable functions  $a_1, \dots, a_k : [0, 1] \rightarrow \mathbb{C}$  such that, for almost every  $t \in [0, 1]$ , one has

$$\ell'(t) = \sum_{i=1}^k a_i(t) X_i(\ell(t)).$$

If  $\ell$  is admissible, its length is defined by

$$|\ell| = \int_0^1 \left( \sum_{i=1}^k |a_i(t)|^2 dt \right)^{\frac{1}{2}}.$$

For all  $x, y \in \mathbb{G}$ , define  $d(x, y)$  as the infimum of the lengths of all admissible paths joining  $x$  to  $y$  (such a curve exists by the Hörmander

---

The first author is partially supported by the ANR project ANR-08-BLAN-0301-01 “Mathocéan”, as well as by the Institut Universitaire de France. The second author is supported by the ANR project “PREFERRED”.

condition). This distance is left-invariant. For short, we denote by  $|x|$  the distance between  $e$ , the neutral element of the group, and  $x$  so that the distance from  $x$  to  $y$  is equal to  $|y^{-1}x|$ .

For all  $r > 0$ , denote by  $B(x, r)$  the open ball in  $\mathbb{G}$  with respect to the Carnot-Carathéodory distance and by  $V(r)$  the Haar measure of any ball. There exists  $d \in \mathbb{N}^*$  (called the local dimension of  $(G, \mathbb{X})$ ) and  $0 < c < C$  such that, for all  $r \in ]0, 1[$ ,

$$cr^d \leq V(r) \leq Cr^d,$$

see [27]. When  $r > 1$ , two situations may occur (see [20]):

- Either there exist  $c, C, D > 0$  such that, for all  $r > 1$ ,

$$cr^D \leq V(r) \leq Cr^D$$

where  $D$  is called the dimension at infinity of the group (note that, contrary to  $d$ ,  $D$  does not depend on  $\mathbb{X}$ ). The group is said to have polynomial volume growth.

- Or there exist  $c_1, c_2, C_1, C_2 > 0$  such that, for all  $r > 1$ ,

$$c_1e^{c_2r} \leq V(r) \leq C_1e^{C_2r}$$

and the group is said to have exponential volume growth.

When  $\mathbb{G}$  has *polynomial volume growth*, it is plain to see that there exists a constant  $C > 0$  such that for all  $r > 0$ ,  $V(2r) \leq CV(r)$ . In turn this implies that there exist  $C > 0$  and  $\kappa > 0$  such that for all  $r > 0$  and all  $\theta > 1$ ,  $V(\theta r) \leq C\theta^\kappa V(r)$ .

**1.2. Nilpotent Lie groups.** A Lie group is said to be *nilpotent* if its Lie algebra  $\mathcal{G}$  is nilpotent: more precisely writing  $\mathcal{G}^1 = \mathcal{G}$  and defining inductively  $\mathcal{G}^{k+1} = [\mathcal{G}^k, \mathcal{G}^k]$ , then there is  $n$  such that  $\mathcal{G}^n = \{0\}$ . It can be shown that such groups are always of polynomial growth (see for instance [16]).

**1.3. Stratified (Carnot) and H-type groups.** Carnot groups are a particular version of nilpotent groups, which admit a *stratified* structure and for which  $V(r) \sim r^Q$  for some positive  $Q$ , for all  $r > 0$ . One advantage of this additional structure is that such groups admit dilations. Important examples of such groups are H-type groups, a particular example being the Heisenberg group (in those cases there is actually an explicit formula for the Fourier transform).

More precisely, the Lie group  $\mathbb{G}$  is called a *Carnot group* if  $\mathbb{G}$  is simply connected and the Lie algebra of  $\mathbb{G}$  admits a stratification, i.e. there exist linear subspaces  $V_1, \dots, V_r$  of  $\mathcal{G}$  such that

$$\mathcal{G} = V_1 \oplus \dots \oplus V_r$$

which satisfy  $[V_1, V_i] = V_{i+1}$  for  $i = 1, \dots, r-1$  and  $[V_1, V_r] = 0$ . By  $[V_1, V_i]$  we mean the subspace of  $\mathcal{G}$  generated by the elements  $[X, Y]$  where  $X \in V_1$  and  $Y \in V_i$ . Carnot groups are nilpotent. Furthermore, via the exponential map,  $\mathbb{G}$  and  $\mathcal{G}$  can be identified as manifolds. The dilations  $\gamma_\delta$  ( $\delta > 0$ ) are then defined (on the Lie algebra level) by

$$\gamma_\delta(x_1 + \dots + x_r) = \delta x_1 + \delta^2 x_2 + \dots + \delta^r x_r, \quad x_i \in V_i.$$

We then define the homogeneous dimension to be

$$Q = \dim V_1 + 2 \dim V_2 + \dots + r \dim V_r.$$

Moreover, if  $\mathbb{G}$  is a Carnot group, we have for all  $r > 0$ ,

$$(1.1) \quad V(r) \sim r^Q$$

(see [17]). We will say that the  $Q$ -dimensional Carnot group is of step  $r$  with  $k$  generators. For instance the Heisenberg group  $\mathcal{H}^d$  is a Carnot group and  $Q = 2d + 2$ .

We denote  $\Delta_{\mathbb{G}} = \sum_{i=1}^k X_i^2$  the sub-laplacian on  $\mathbb{G}$ , and the associated gradient  $\nabla_{\mathbb{G}} = (X_1, \dots, X_k)$ .

The previous abstract definition of Carnot groups is not always very practical. It is however possible to prove (see [9]) that any  $N$ -dimensional Carnot group of step 2 with  $m$  generators is isomorphic to  $(\mathbb{R}^N, \circ)$  with the law given by ( $N = m + n$ ,  $x^{(1)} \in \mathbb{R}^m$ ,  $x^{(2)} \in \mathbb{R}^n$ )

$$(x^{(1)}, x^{(2)}) \circ (y^{(1)}, y^{(2)}) = \left( \begin{array}{l} x_j^{(1)} + y_j^{(1)}, \quad j = 1, \dots, m \\ x_j^{(2)} + y_j^{(2)} + \frac{1}{2} \langle x^{(1)}, U^{(j)} y^{(1)} \rangle, \quad j = 1, \dots, n \end{array} \right),$$

where  $U^{(j)}$  are  $m \times m$  linearly independent skew-symmetric matrices.

With this at hand, one can give the definition of a group of Heisenberg-type (H-type henceforth). These groups are two-step stratified nilpotent Lie groups whose Lie algebra carries a suitably compatible inner product, see [24]. One of these groups is the nilpotent Iwasawa subgroup of semi-simple Lie groups of split rank one (see [25]). More precisely, an H-type group is a Carnot group of step 2 with the following property: the Lie algebra  $\mathcal{G}$  of  $\mathbb{G}$  is endowed with an inner product  $\langle \cdot, \cdot \rangle$  such that if  $\mathcal{Z}$  is the center of  $\mathcal{G}$ , then

$$[\mathcal{Z}^\perp, \mathcal{Z}^\perp] = \mathcal{Z}$$

and moreover for every  $z \in \mathcal{Z}$ , the map  $J_z : \mathcal{Z}^\perp \rightarrow \mathcal{Z}^\perp$  defined by

$$\langle J_z(v), w \rangle = \langle z, [v, w] \rangle$$

for every  $w \in \mathcal{Z}^\perp$  is an orthogonal map whenever  $\langle z, z \rangle = 1$ .

If  $m = \dim(\mathcal{Z}^\perp)$  and  $n = \dim(\mathcal{Z})$ , then any H-type group is canonically isomorphic to  $\mathbb{R}^{m+n}$  with the above group law, where the matrices  $U^{(j)}$  satisfy the additional property

$$U^{(r)}U^{(s)} + U^{(s)}U^{(r)} = 0$$

for every  $r, s \in \{1, \dots, n\}$  with  $r \neq s$ . Whenever the center of the group is one-dimensional, the group is canonically isomorphic to the Heisenberg group on  $\mathbb{R}^{m+1}$ . We will always identify  $\mathcal{Z}^\perp$  with  $\mathbb{C}^\ell$  with  $2\ell = m$  and  $\mathcal{Z}$  to  $\mathbb{R}^n$  thanks to the discussion above. Note that the homogeneous dimension of a H-type group so defined is  $Q = 2\ell + n$ .

On an H-type group  $\mathbb{G}$ , the vector-fields in the algebra  $\mathcal{G}$  are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^{2\ell} z_l U_{l,j}^{(k)} \frac{\partial}{\partial t_k} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^{2\ell} z_l U_{l,j+\ell}^{(k)} \frac{\partial}{\partial t_k}$$

for  $j = 1, \dots, \ell$ ,  $z = (x, y) \in \mathbb{R}^{2\ell}$  and  $t \in \mathbb{R}^n$ . In the following we shall denote by  $\mathcal{X}$  any element of the family  $(X_1, \dots, X_\ell, Y_1, \dots, Y_\ell)$ .

The hypo-elliptic Kohn Laplacian on H-type groups writes

$$\Delta_{\mathbb{G}} = \sum_{j=1}^m \frac{\partial^2}{\partial x_j^2} + \frac{1}{4} |x|^2 \sum_{s=1}^n \frac{\partial^2}{\partial t_s^2} + \sum_{s=1}^n \sum_{i,j=1}^m x_i U_{ij}^{(s)} \frac{\partial^2}{\partial t_s \partial x_j}.$$

In the general case of Carnot groups, the Laplacian allows to define the Sobolev space  $H^s(\mathbb{G})$ , for  $s \in \mathbb{N}$ , as the space of  $L^2(\mathbb{G})$  functions such that  $(\text{Id} - \Delta_{\mathbb{G}})f$  belongs to  $L^2(\mathbb{G})$ . The case  $s \in \mathbb{R}$  follows by interpolation and duality. We also define the Schwartz space  $\mathcal{S}(\mathbb{G})$  as the set of smooth functions  $f$  on  $\mathbb{G}$  such that for any  $\kappa \in \mathbb{N}$ ,

$$\|f\|_{\kappa, \mathcal{S}} = \sup_{\substack{|\alpha| \leq \kappa, N \leq \kappa \\ w \in \mathbb{G}}} |\mathcal{X}^\alpha (|w|^{2N} f(w))| < \infty,$$

where we denote  $\mathcal{X}^\gamma$  any vector-field of the form

$$(1.2) \quad \mathcal{X}^\gamma = X_{j_1} \dots X_{j_{|\gamma|}},$$

where  $j_m$  belongs to  $\{1, \dots, 2\ell\}$  with the convention that  $X_{j+\ell} = Y_j$ .

**1.4. Main results and structure of the paper.** In [14], the authors investigate the algebra properties of the Bessel space  $L_\alpha^p(\mathbb{G})$  with

$$L_\alpha^p(\mathbb{G}) = \{f \in L^p(\mathbb{G}), (-\Delta_{\mathbb{G}})^{\frac{\alpha}{2}} f \in L^p(\mathbb{G})\},$$

where  $\mathbb{G}$  is any unimodular Lie group. We address the same type of question here for the scale of Besov spaces, when the group has polynomial growth. As pointed out below in the definition of Besov

spaces (see Section 2), we need to distinguish the different cases of homogeneous and non homogeneous spaces.

We prove the following theorems. The first one considers low regularity inhomogeneous Besov spaces.

**Theorem 1.** *Let  $\mathbb{G}$  be a Lie group with polynomial volume growth. For every  $0 < s \leq 1$  and  $1 \leq p, q \leq \infty$ , the space  $B_{p,q}^s(\mathbb{G}) \cap L^\infty(\mathbb{G})$  is an algebra under pointwise multiplication.*

The next theorem deals with the higher order case, and is restricted to the case of nilpotent groups.

**Theorem 2.** *Let  $\mathbb{G}$  be a nilpotent Lie group. For every  $s > 1$  and  $1 < p, q < \infty$ , the space  $B_{p,q}^s(\mathbb{G}) \cap L^\infty(\mathbb{G})$  is an algebra under pointwise multiplication.*

**Remark 1.1.** *Notice in Theorem 2 the limitation on the range of  $p$  and  $q$ . Such a restriction (on the index  $p$ ) is also present in [14]. Note that we shall also prove a generalization of Theorems 1 and 2 to the case when  $L^\infty(\mathbb{G})$  is replaced by  $L^r(\mathbb{G})$  (see Proposition 3.3).*

In the context of Carnot groups, one can define homogeneous Besov spaces (see Section 2). We have the following corollary of Theorems 1 and 2 (recall that Carnot groups are nilpotent).

**Corollary 1.2.** *Let  $\mathbb{G}$  be a Carnot group.*

- (1) *For every  $0 < s \leq 1$  (resp.  $0 < s < Q/p$ ) and  $1 \leq p, q \leq \infty$ , the space  $B_{p,q}^s(\mathbb{G}) \cap L^\infty(\mathbb{G})$  (resp.  $\dot{B}_{p,q}^s(\mathbb{G}) \cap L^\infty(\mathbb{G})$ ) is an algebra under pointwise multiplication.*
- (2) *For every  $s > 1$  and  $1 < p, q < \infty$ , the spaces  $B_{p,q}^s(\mathbb{G}) \cap L^\infty(\mathbb{G})$  and  $\dot{B}_{p,q}^s(\mathbb{G}) \cap L^\infty(\mathbb{G})$  (for  $s < Q/p$ ) are algebras under pointwise multiplication.*

Finally, in the context of H-type groups, thanks to paraproduct techniques, one can enlarge the range of admissible  $p$  and  $q$  and prove the following result.

**Theorem 3.** *Let  $\mathbb{G}$  be an H-type group. For every  $s > 0$  and for every  $1 \leq p, q \leq \infty$ , the space  $B_{p,q}^s(\mathbb{G}) \cap L^\infty(\mathbb{G})$  (resp.  $\dot{B}_{p,q}^s(\mathbb{G}) \cap L^\infty(\mathbb{G})$ ) if  $0 < s < Q/p$  is an algebra under pointwise multiplication.*

Besov spaces are defined in the coming section, and Theorems 1 and 2 as well as Corollary 1.2 are proved in Section 3. We present the proof of Theorem 3 in Section 4, by paraproduct techniques, and we also provide a few direct applications in the spirit of [5]. Actually Section 4 develops not only paraproduct techniques, but also gives definitions and

the main properties of a pseudodifferential calculus on H-type groups, following results of [4] for the special case of the Heisenberg group.

In the following we shall write  $A \lesssim B$  if there is a universal constant  $C$  such that  $A \leq CB$ . Similarly we shall write  $A \sim B$  if  $A \lesssim B$  and  $B \lesssim A$ .

**Remark 1.3.** *It has to be noticed that a careful look at the construction in [18] shows that one could provide the same arguments in the context of Riemannian manifolds with doubling measure and Ricci tensor bounded from below. This would allow to define Besov spaces in such a context and prove an analogous algebra property.*

## 2. LITTLEWOOD-PALEY DECOMPOSITION ON GROUPS OF POLYNOMIAL GROWTH, AND BESOV SPACES

This section is devoted to a presentation of the Littlewood-Paley decomposition on groups of polynomial growth, together with some standard applications. A general approach to the Littlewood-Paley decomposition on Lie groups with polynomial growth is investigated in [18]. We also refer to [7] or [5] for the case of the Heisenberg group. We recall here the construction of the homogeneous and inhomogeneous decompositions. For details and proofs of the results presented in this section we refer to [11], [18] and [22].

**2.1. Littlewood-Paley decomposition.** We first review the dyadic decomposition constructed in [18]. Let  $\chi \in C^\infty(\mathbb{R})$  be an even function such that  $0 \leq \chi \leq 1$  and  $\chi = 1$  on  $[0, 1/4]$ ,  $\chi = 0$  on  $[1, \infty[$ . Define  $\psi(x) = \chi(x/4) - \chi(x)$ , so that the support of  $\psi$  is included in  $[1/4, 4]$ . The following holds:

$$\forall \tau \in \mathbb{R}^*, \quad \sum_{j \in \mathbb{Z}} \psi(2^{-2j}\tau) = 1 \quad \text{and} \quad \chi(\tau) + \sum_{j \geq 0} \psi(2^{-2j}\tau) = 1, \quad \forall \tau \in \mathbb{R}.$$

Let us introduce the spectral decomposition of the hypo-elliptic Laplacian

$$-\Delta_{\mathbb{G}} = \int_0^\infty \lambda dE_\lambda.$$

Then we have

$$\chi(-\Delta_{\mathbb{G}}) = \int_0^\infty \chi(\lambda) dE_\lambda \quad \text{and} \quad \psi(-2^{-2j}\Delta_{\mathbb{G}}) = \int_0^\infty \psi(2^{-2j}\lambda) dE_\lambda.$$

We then define for  $j \in \mathbb{N}$  the operators

$$S_0 f = \chi(-\Delta_{\mathbb{G}}) f \quad \text{and} \quad \Delta_j f = \psi(-2^{-2j}\Delta_{\mathbb{G}}) f.$$

The homogeneous Littlewood-Paley decomposition of  $f$  in  $\mathcal{S}'(\mathbb{G})$  is then given by

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f,$$

while the inhomogeneous one is given by

$$f = S_0 f + \sum_{j=0}^{\infty} \Delta_j f.$$

Furioli *et al* (see [18]) proved the following theorem.

**Theorem 4** ([18]). *Let  $\mathbb{G}$  be a Lie group with polynomial growth. Then  $u$  belongs to  $L^p(\mathbb{G})$  if and only if  $S_0 u$  and  $\sqrt{\sum_{j=0}^{\infty} |\Delta_j u|^2}$  belong to  $L^p(\mathbb{G})$ . Moreover, we have*

$$\|u\|_{L^p(\mathbb{G})} \sim \|S_0 u\|_{L^p(\mathbb{G})} + \left\| \left( \sum_{j=0}^{\infty} |\Delta_j u|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{G})}.$$

In the following we shall denote by  $\Psi_j$  the kernel of the operator  $\psi(2^{-2j} \Delta_{\mathbb{G}})$ . One can show that  $\Psi_j$  is mean free (see Corollary 5.1 of [10] for Carnot groups, and Theorem 7.1.2 of [11] for an extension to groups of polynomial growth). In the context of Carnot groups,  $\Psi_j$  satisfies the dilation property

$$\Psi_j(x) = 2^{Qj} \Psi_0(2^j x).$$

In the more general context of groups of polynomial growth, this does not hold but one has nevertheless the following important estimates: let  $\alpha \in \mathbb{N}$  be given, as well as  $I \in \bigcup_{\beta \in \mathbb{N}} \{1, \dots, k\}^{\beta}$  and  $p \in [1, \infty]$ . The following result is due to [18]:

$$(2.1) \quad \forall j \geq 0, \quad \|(1 + |\cdot|)^{\alpha} X^I \Psi_j\|_{L^p(\mathbb{G})} \lesssim 2^{j(\frac{\alpha}{p} + |I|)},$$

where  $1/p + 1/p' = 1$ . We have denoted  $X^I = X_{i_1} \dots X_{i_{\beta}}$  and  $|I| = \beta$ . Moreover as proved in [11], Theorem 7.1.2,

$$(2.2) \quad \forall j \in \mathbb{Z}, \quad \|X_i \Psi_j\|_{L^1(\mathbb{G})} \lesssim 2^j.$$

Finally putting together classical estimates on the heat kernel (see [14] or [29] for instance) and the methods of [18] allows to write that for any  $\alpha \geq 0$ ,

$$(2.3) \quad \forall j \in \mathbb{Z}, \quad \|\cdot\|^{\alpha} \Psi_j\|_{L^1(\mathbb{G})} \lesssim 2^{j\alpha}.$$



**2.2. Besov spaces.** As a standard application of the Littlewood-Paley decomposition, one can define Besov spaces on Lie groups with polynomial volume growth in the following way : let  $s \in \mathbb{R}$  and  $1 \leq p, q \leq +\infty$ , then

$$B_{p,q}^s(\mathbb{G}) = \left\{ f \in \mathcal{S}'(\mathbb{G}), \|S_0 f\|_{L^p(\mathbb{G})} + \left( \sum_{j=0}^{\infty} (2^{js} \|\Delta_j f\|_{L^p(\mathbb{G})})^q \right)^{1/q} < \infty \right\}$$

with the obvious adaptation if  $q = \infty$ . When  $p = q = 2$  one recovers the usual Sobolev spaces (see for instance [5] for a proof in the case of the Heisenberg group). We define

$$\|f\|_{B_{p,q}^s(\mathbb{G})} = \|S_0 f\|_{L^p(\mathbb{G})} + \left( \sum_{j=0}^{\infty} (2^{js} \|\Delta_j f\|_{L^p(\mathbb{G})})^q \right)^{1/q}.$$

Note that when  $s > 0$  one sees easily that  $\|S_0 f\|_{L^p(\mathbb{G})}$  may be replaced by  $\|f\|_{L^p(\mathbb{G})}$ . One can also define the homogeneous counterpart

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{G})} = \left( \sum_{j \in \mathbb{Z}} (2^{js} \|\Delta_j f\|_{L^p(\mathbb{G})})^q \right)^{1/q}$$

but proving that this does provide a norm is not an easy matter, and is actually not true in general if  $s$  is too large. In the context of Carnot group, the homogeneous space  $\dot{B}_{p,q}^s(\mathbb{G})$  can be defined as the set of functions in  $\mathcal{S}'(\mathbb{G})$  such that the above norm is finite, and this provides a Banach space if  $s < Q/p$  (we refer for instance to [2] for a proof in the euclidean case, or to [5] for the Heisenberg group).

Actually Besov spaces are often rather defined using the heat flow (the advantage being that it does not require the Littlewood-Paley machinery). In [18], the authors prove that if  $s \in \mathbb{R}$ , then  $u \in B_{p,q}^s(\mathbb{G})$  is equivalent to: for all  $t > 0$ , the function  $e^{t\Delta_{\mathbb{G}}} u$  belongs to  $L^p(\mathbb{G})$  and

$$\left( \int_0^1 t^{-sq/2} \|(t(-\Delta_{\mathbb{G}}))^{m/2} e^{t\Delta_{\mathbb{G}}} f\|_{L^p(\mathbb{G})}^q \frac{dt}{t} \right)^{1/q} < \infty$$

for  $m \geq 0$  greater than  $s$ . Moreover one can similarly prove (see Theorem 4 of [6] in the case of the Heisenberg group) that

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{G})} \sim \left( \int_0^{\infty} t^{-sq/2} \|(t(-\Delta_{\mathbb{G}}))^{m/2} e^{t\Delta_{\mathbb{G}}} f\|_{L^p(\mathbb{G})}^q \frac{dt}{t} \right)^{1/q}.$$

### 3. PROOFS OF THEOREMS 1, 2 AND COROLLARY 1.2

**3.1. Proof of Theorem 1 and Corollary 1.2(1).** To prove the results, we use an idea already present in [14] and which consists in representing the norm on Besov spaces by suitable functionals. More

precisely, we introduce the following functionals (note that they differ from those used in [14]):

$$\mathcal{S}_s f(w) = \frac{\|\tau_{-w}f - f\|_{L^p(\mathbb{G})}}{|w|^s} \quad \text{and} \quad \mathcal{D}f(w) = \frac{\|\tau_{-w}f + \tau_w f - 2f\|_{L^p(\mathbb{G})}}{|w|}$$

where  $\tau_{-w}f(w') = f(w'w)$ . We prove the following theorem.

**Theorem 5.** *Let  $\mathbb{G}$  be a Lie group with polynomial growth. Then for any  $s \in ]0, 1[$  and  $p, q \in [1, +\infty]$ , we have*

$$\|f\|_{B_{p,q}^s(\mathbb{G})} \sim \|f\|_{L^p} + \|\mathcal{S}_s f\|_{L^q(\mathbb{G}, \frac{\mathbf{1}_{|y| \leq 1} dy}{V(|y|)})}$$

and if  $s = 1$

$$\|f\|_{B_{p,q}^1(\mathbb{G})} \sim \|f\|_{L^p} + \|\mathcal{D}f\|_{L^q(\mathbb{G}, \frac{\mathbf{1}_{|y| \leq 1} dy}{V(|y|)})}.$$

**Remark 3.1.** *By the same arguments one can prove that*

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{G})} \sim \|\mathcal{S}_s f\|_{L^q(\mathbb{G}, \frac{dy}{V(|y|)})}$$

and

$$\|f\|_{\dot{B}_{p,q}^1(\mathbb{G})} \sim \|\mathcal{D}f\|_{L^q(\mathbb{G}, \frac{dy}{V(|y|)})}.$$

Once Theorem 5 is proved, the algebra property follows immediately. Indeed, let  $f, g$  belong to the space  $(B_{p,q}^s \cap L^\infty)(\mathbb{G})$  for  $s \neq 1$ . It is easy to see that

$$(3.1) \quad \mathcal{S}_s(fg) \leq \|f\|_{L^\infty} \mathcal{S}_s g + \|g\|_{L^\infty} \mathcal{S}_s f,$$

hence the result using the equivalence in Theorem 5. The case  $s = 1$  is similar. Theorem 5 and Remark 3.1 therefore provide Theorem 1 and the first part of Corollary 1.2.  $\square$

**Remark 3.2.** *Actually (3.1) can easily be extended to*

$$\|\mathcal{S}_s(fg)\|_{L^p} \leq \|f\|_{L^{a_1}} \|\mathcal{S}_s g\|_{L^{b_1}} + \|g\|_{L^{a_2}} \|\mathcal{S}_s f\|_{L^{b_2}}$$

with  $1/a_i + 1/b_i = 1/p$ .

We now prove Theorem 5. We need to prove that for  $s \in ]0, 1[$

$$\sum_{j \in \mathbb{N}} (2^{js} \|\Delta_j f\|_{L^p(\mathbb{G})})^q \sim \int_{\mathbb{G}} \mathbf{1}_{|w| \leq 1} \frac{\|\tau_{-w}f - f\|_{L^p(\mathbb{G})}^q}{V(|w|)|w|^{sq}} dw$$

and for  $s = 1$

$$\sum_{j \in \mathbb{N}} (2^j \|\Delta_j f\|_{L^p(\mathbb{G})})^q \sim \int_{\mathbb{G}} \mathbf{1}_{|w| \leq 1} \frac{\|\tau_{-w}f + \tau_w f - 2f\|_{L^p(\mathbb{G})}^q}{V(|w|)|w|^q} dw,$$

with the obvious modification if  $q = \infty$ . We shall only detail the proof of the  $s \neq 1$  case, and leave the other one to the reader. The proof is very much inspired by the euclidean case (see for instance [2]), although one is missing the usual dilation property, which will be replaced by estimate (2.1). The classical proof also uses a Taylor expansion at order one, which we must adapt to our context in order to use only horizontal vector fields (which alone appear in (2.1)).

Let us start by bounding the quantity  $\|\tau_{-w}\Delta_j f - \Delta_j f\|_{L^p(\mathbb{G})}$ . Recalling that  $\Delta_j = \sum_{|j'-j|\leq 1} \Delta_j \Delta_{j'}$ , we have

$$\tau_{-w}\Delta_j f - \Delta_j f = \sum_{|j'-j|\leq 1} \Delta_{j'} f \star (\tau_{-w}\Psi_j - \Psi_j),$$

where  $\Psi_j$  is the kernel associated with  $\psi(2^{-2j}\Delta_{\mathbb{G}})$ . It follows by Young's inequality that

$$\|\tau_{-w}\Delta_j f - \Delta_j f\|_{L^p(\mathbb{G})} \leq \sum_{|j'-j|\leq 1} \|\Delta_{j'} f\|_{L^p} \|\tau_{-w}\Psi_j - \Psi_j\|_{L^1}.$$

Now let us estimate  $\|\tau_{-w}\Psi_j - \Psi_j\|_{L^1}$ . We have

$$\begin{aligned} \tau_{-w}\Psi_j(x) - \Psi_j(x) &= \int_0^1 \frac{d}{ds} \Psi_j(x\varphi(s)) ds \\ &= \sum_{\ell=1}^k \int_0^1 c_\ell(s) (X_\ell(x\varphi(s))\Psi_j)(x\varphi(s)) ds, \end{aligned}$$

where  $\varphi$  is an admissible path linking  $e$  to  $w$ . It follows that

$$\begin{aligned} \|\tau_{-w}\Psi_j - \Psi_j\|_{L^1} &\leq \int_{\mathbb{G}} \sum_{\ell=1}^k \int_0^1 |c_\ell(s)| |(X_\ell(x\varphi(s))\Psi_j)(x\varphi(s))| ds dx \\ &\leq \sum_{\ell=1}^k \int_0^1 |c_\ell(s)| ds \|X_\ell \Psi_j\|_{L^1} \end{aligned}$$

by the Fubini theorem and a change of variables. Using (2.2) we get

$$\forall j \in \mathbb{N}, \quad \|\tau_{-w}\Psi_j - \Psi_j\|_{L^1} \lesssim 2^j \sum_{\ell=1}^k \int_0^1 |c_\ell(s)| ds$$

so by definition of  $|w|$  and by the Cauchy-Schwarz inequality we find

$$\forall j \in \mathbb{N}, \quad \|\tau_{-w}\Psi_j - \Psi_j\|_{L^1} \lesssim 2^j |w|.$$

This implies that there is a sequence  $(c_j)$  in the unit ball of  $\ell^q$  such that

$$(3.2) \quad \forall j \in \mathbb{N}, \quad \|\tau_{-w}\Delta_j f - \Delta_j f\|_{L^p(\mathbb{G})} \lesssim c_j |w| 2^{j(1-s)} \|f\|_{B_{p,q}^s(\mathbb{G})}.$$

On the other hand one has of course

$$(3.3) \quad \|\tau_{-w}\Delta_j f - \Delta_j f\|_{L^p(\mathbb{G})} \lesssim c_j 2^{-js} \|f\|_{B_{p,q}^s(\mathbb{G})}.$$

Now let us define  $j_w$  such that  $\frac{1}{|w|} \leq 2^{j_w} \leq \frac{2}{|w|}$  (which is possible since  $|w| \leq 1$ ). Then using (3.2) for low frequencies and (3.3) for high frequencies allows to write

$$\|\tau_{-w}f - f\|_{L^p(\mathbb{G})} \lesssim \|f\|_{B_{p,q}^s(\mathbb{G})} \left( \sum_{j \leq j_w} c_j 2^{j(1-s)} |w| + \sum_{j > j_w} c_j 2^{-js} \right).$$

Let us first consider the case  $q = \infty$ . Then one finds directly that

$$\|\tau_{-w}f - f\|_{L^p(\mathbb{G})} \lesssim |w|^s \|f\|_{B_{p,q}^s(\mathbb{G})}$$

which proves one side of the equivalence. The case  $q < \infty$  is slightly more technical but is very close to the euclidean case. We include it here for sake of completeness (see [2]). We have that

$$\left\| \frac{\|\tau_{-w}f - f\|_{L^p}}{|w|^s} \right\|_{L^q(\mathbb{G}, \frac{\mathbf{1}_{|w| \leq 1}}{V(|w|)})} \lesssim 2^q \|f\|_{B_{p,q}^s}^q (I_1 + I_2)$$

where

$$I_1 = \int_{\mathbb{G}} \mathbf{1}_{|w| \leq 1} \left( \sum_{j \leq j_w} c_j 2^{j(1-s)} \right)^q \frac{|w|^{q(1-s)} dw}{V(|w|)}$$

and

$$I_2 = \int_{\mathbb{G}} \mathbf{1}_{|w| \leq 1} \left( \sum_{j > j_w} c_j 2^{-js} \right)^q \frac{|w|^{-qs} dw}{V(|w|)}.$$

Hölder's inequality with the weight  $2^{j(1-s)}$  and the definition of  $j_w$  imply that

$$\left( \sum_{j \leq j_w} c_j 2^{j(1-s)} \right)^q \lesssim |w|^{-(1-s)(q-1)} \sum_{j \leq j_w} c_j^r 2^{j(1-s)}.$$

By Fubini's theorem, we deduce that

$$I_1 \lesssim \sum_{j \in \mathbb{N}} \int_{B(0, 2^{-j+1})} |w|^{1-s} \frac{dw}{V(|w|)} 2^{j(1-s)} c_j^q \lesssim 1,$$

since  $\|(c_j)\|_{\ell^q} = 1$ . The estimate on  $I_2$  is very similar. Note that it is crucial here that  $s \in ]0, 1[$ .

The converse inequality is easy to prove and only depends on the fact that the mean value of  $\Psi_j$  is zero. We write indeed

$$\begin{aligned}\Delta_j f(w) &= \int \tau_v f(w) \Psi_j(v) dv \\ &= \int (\tau_v f(w) - f(w)) \Psi_j(v) dv\end{aligned}$$

so that

$$2^{js} \|\Delta_j f\|_{L^p} \leq \sup_{v \in \mathbb{G}} \frac{\|\tau_v f - f\|_{L^p}}{|v|^s} \int 2^{js} |v|^s |\Psi_j(v)| dv.$$

So the case  $q = \infty$  simply follows from (2.3), while the case  $q < \infty$  is similar though a little more technical, as above. We leave the details to the reader. This proves Theorem 5.  $\square$

Using Remark 3.2, the same proof provides the following result, which will be useful in the next section.

**Proposition 3.3.** *Let  $\mathbb{G}$  be a Lie group with polynomial volume growth. For every  $0 < s \leq 1$  and  $1 \leq p, q \leq \infty$  one has*

$$\|fg\|_{B_{p,q}^s} \leq \|f\|_{L^{a_1}} \|g\|_{B_{b_1,q}^s} + \|g\|_{L^{a_2}} \|f\|_{B_{b_2,q}^s},$$

where  $1/p = 1/a_i + 1/b_i$ .

**3.2. Proof of Theorem 2 and Corollary 1.2(2).** We now deal with the case  $s \geq 1$ . To do so, we need the following proposition, which is new to our knowledge, even in the context of the Heisenberg group (see Section 4.2 below).

**Proposition 3.4.** *Let  $\mathbb{G}$  be a nilpotent Lie group and let  $s > 0$  and  $p \in ]1, \infty[$  be given. Then  $f \in B_{p,q}^{s+1}(\mathbb{G})$  if and only if  $f \in B_{p,q}^s(\mathbb{G})$  and for all  $i = 1, \dots, k$ , we have  $X_i f \in B_{p,q}^s(\mathbb{G})$ .*

**Remark 3.5.** *As in the case of Remark 3.1, one has a similar result for homogeneous quantities (when defined):*

$$\|f\|_{\dot{B}_{p,q}^{s+1}(\mathbb{G})} < \infty \iff \|X_i f\|_{\dot{B}_{p,q}^s(\mathbb{G})} < \infty, \forall i = 1, \dots, k.$$

*Proof.* On the one hand we need to prove that for all  $i = 1, \dots, k$  and all  $j \in \mathbb{N}$ ,

$$\|\Delta_j X_k f\|_{L^p} \lesssim 2^j \|\Delta_j f\|_{L^p}.$$

Since  $[\Delta_j, \Delta_{\mathbb{G}}] = 0$ , Bernstein's lemma (see Proposition 4.3 of [18]) implies that

$$(3.4) \quad \|\Delta_j (-\Delta_{\mathbb{G}})^{\frac{1}{2}} f\|_{L^p} \lesssim 2^j \|\Delta_j f\|_{L^p}.$$

By density of polynomials in the space of continuous functions it is then actually enough to prove that for all integers  $m$ ,

$$(3.5) \quad \|(-\Delta_{\mathbb{G}})^{\frac{m}{2}}(-\Delta_{\mathbb{G}})^{-\frac{1}{2}}X_k f\|_{L^p} \lesssim \|(-\Delta_{\mathbb{G}})^{\frac{m}{2}}f\|_{L^p}.$$

Indeed if (3.5) holds, then one also has

$$\|(-2^{2j}\Delta_{\mathbb{G}})^{\frac{m}{2}}(-\Delta_{\mathbb{G}})^{-\frac{1}{2}}X_k f\|_{L^p} \lesssim \|(-2^{2j}\Delta_{\mathbb{G}})^{\frac{m}{2}}f\|_{L^p}$$

so for smooth compactly supported function  $\varphi$ ,

$$(3.6) \quad \|\varphi(-2^{2j}\Delta_{\mathbb{G}})(-\Delta_{\mathbb{G}})^{-\frac{1}{2}}X_k f\|_{L^p} \lesssim \|\varphi(2^{2j}\Delta_{\mathbb{G}})f\|_{L^p}$$

hence

$$\begin{aligned} \|\Delta_j X_k f\|_{L^p} &= \|(-\Delta_{\mathbb{G}})^{\frac{1}{2}}\varphi(-2^{2j}\Delta_{\mathbb{G}})(-\Delta_{\mathbb{G}})^{-\frac{1}{2}}X_k f\|_{L^p} \\ &\lesssim 2^j \|\Delta_j f\|_{L^p} \end{aligned}$$

due to (3.4) and (3.6). So let us prove (3.5). Actually according to [26] the operator  $\mathcal{L}_m = (-\Delta_{\mathbb{G}})^{\frac{m-1}{2}}X_k(-\Delta_{\mathbb{G}})^{-\frac{m}{2}}$  is bounded over  $L^p(\mathbb{G})$  for  $1 < p < \infty$  (it is here that the assumption that  $\mathbb{G}$  is nilpotent is used, as if not that property is false, see for instance [1]). So writing

$$\begin{aligned} (-\Delta_{\mathbb{G}})^{\frac{m}{2}}(-\Delta_{\mathbb{G}})^{-\frac{1}{2}}X_k f &= (-\Delta_{\mathbb{G}})^{\frac{m}{2}}(-\Delta_{\mathbb{G}})^{-\frac{1}{2}}X_k(-\Delta_{\mathbb{G}})^{-\frac{m}{2}}(-\Delta_{\mathbb{G}})^{\frac{m}{2}}f \\ &= \mathcal{L}_m(-\Delta_{\mathbb{G}})^{\frac{m}{2}}f, \end{aligned}$$

the result follows.

On the other hand, using again the fact that polynomials are dense in the space of continuous functions, we also need to check that

$$\|(-\Delta_{\mathbb{G}})^{\frac{m+1}{2}}f\|_{L^p} \lesssim \|(-\Delta_{\mathbb{G}})^{\frac{m}{2}}X_k f\|_{L^p}.$$

To prove that we notice that we have just seen that

$$\|(-\Delta_{\mathbb{G}})^{\frac{m}{2}}X_k f\|_{L^p} \lesssim \|(-\Delta_{\mathbb{G}})^{\frac{m+1}{2}}f\|_{L^p},$$

so we need to prove the converse inequality: this is actually obtained in a classical way by duality (see for instance [8], pages 36-37), so we omit the proof. Proposition 3.4 is proved.  $\square$

Proposition 3.4 allows to obtain rather easily Theorem 2 (as well as Corollary 1.2(2)), using also Proposition 3.3. We follow the lines of [14], by writing, thanks to Proposition 3.4,

$$\|fg\|_{B_{p,q}^{s+1}} \sim \|fg\|_{B_{p,q}^s} + \sum_{i=1}^k \|X_i(fg)\|_{B_{p,q}^s}$$

and by arguing by induction.

On the one hand by Proposition 3.3 and the induction assumption we have

$$\|fg\|_{B_{p,q}^s} \lesssim \|f\|_{L^{a_1}} \|g\|_{B_{b_1,q}^s} + \|g\|_{L^{a_2}} \|f\|_{B_{b_2,q}^s}$$

so

$$\|fg\|_{B_{p,q}^s} \lesssim \|f\|_{L^{a_1}} \|g\|_{B_{b_1,q}^{s+1}} + \|g\|_{L^{a_2}} \|f\|_{B_{b_2,q}^{s+1}},$$

where

$$\frac{1}{p} = \frac{1}{a_i} + \frac{1}{b_i} \quad \text{for } i \in \{1, 2\}.$$

Then we write, by the Leibniz rule,

$$\|X_i(fg)\|_{B_{p,q}^s} \leq \|fX_i g\|_{B_{p,q}^s} + \|gX_i f\|_{B_{p,q}^s}$$

and we have, by the induction hypothesis again,

$$\|fX_i g\|_{B_{p,q}^s} \lesssim \|f\|_{B_{b_3,q}^s} \|X_i g\|_{L^{a_3}} + \|f\|_{L^{a_1}} \|X_i g\|_{B_{b_1,q}^s}.$$

The parameters have been chosen as follows:

$$\frac{1}{b_3} = \frac{s}{(s+1)b_2} + \frac{1}{(s+1)a_1} \quad \text{and} \quad \frac{1}{a_3} = \frac{s}{(s+1)a_2} + \frac{1}{(s+1)b_1}.$$

Notice that the requirement  $1/p = 1/b_3 + 1/a_3$  holds. Then we use the classical interpolation estimate

$$\|f\|_{B_{b_3,q}^s} \leq \|f\|_{B_{b_2,q}^{\frac{s}{s+1}}} \|f\|_{L^{a_1}^{\frac{1}{s+1}}}$$

as well as (using again the continuity of Riesz transforms)

$$\|X_i g\|_{L^{a_3}} \leq \|g\|_{B_{b_1,q}^{\frac{1}{s+1}}} \|g\|_{L^{a_2}^{\frac{s}{s+1}}},$$

which allows to conclude the proof of Theorem 2 and its corollary.  $\square$

#### 4. PARADIFFERENTIAL AND PSEUDODIFFERENTIAL CALCULUS ON H-TYPE GROUPS, AND APPLICATIONS

In this section, we describe several topics related to harmonic analysis on H-type groups, which we recall are particular cases of Carnot groups where it turns out that an explicit Fourier transform is available (see Section 1.3 in the introduction for definitions).

**4.1. Fourier transforms.** In order to construct para-differential and pseudo-differential calculus on H-type groups, one needs to introduce a suitable Fourier transform. This is classically done through the infinite-dimensional unitary irreducible representations on a suitable Hilbert space since H-type groups are non commutative. Two representations are available: the Bargmann representation (see [23] for instance) and the Schrödinger representation (see [13] for instance).

4.1.1. *General definitions.* Let us define generally what a Fourier transform is on non commutative groups. Consider a Hilbert space  $\mathcal{H}_\lambda(\mathbb{K}^\ell)$  of functions defined on  $\mathbb{G}$ . The irreducible unitary representations

$$\pi_\lambda : \mathbb{G} \rightarrow \mathcal{H}_\lambda(\mathbb{K}^\ell)$$

are parametrized by  $\lambda \in \mathbb{R}^n \setminus \{0\}$ . We have then the following definition.

**Definition 4.1.** *We define the Fourier transform on  $\mathbb{G}$  by the following formula: let  $f \in L^1(\mathbb{G})$ . Then the Fourier transform of  $f$  is the operator on  $\mathcal{H}_\lambda(\mathbb{K}^\ell)$  parametrized by  $\lambda \in \mathbb{R}^n \setminus \{0\}$  defined by*

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{G}} f(z, t) \pi_\lambda(z, t) dz dt.$$

Note that one has  $\mathcal{F}(f \star g)(\lambda) = \mathcal{F}(f)(\lambda) \circ \mathcal{F}(g)(\lambda)$ .

Let  $F_{\alpha, \lambda}$ ,  $\alpha \in \mathbb{N}^\ell$  be a Hilbert basis of  $\mathcal{H}_\lambda(\mathbb{K}^\ell)$ . We recall that an operator  $A(\lambda)$  of  $\mathcal{H}_\lambda$  such that

$$\sum_{\alpha \in \mathbb{N}^\ell} |(A(\lambda)F_{\alpha, \lambda}, F_{\alpha, \lambda})_{\mathcal{H}_\lambda}| < +\infty$$

is said to be of *trace-class*. One then sets

$$\text{tr}(A(\lambda)) = \sum_{\alpha \in \mathbb{N}^\ell} (A(\lambda)F_{\alpha, \lambda}, F_{\alpha, \lambda})_{\mathcal{H}_\lambda},$$

and the following inversion theorem holds.

**Theorem 6.** *If a function  $f$  satisfies*

$$\sum_{\alpha \in \mathbb{N}^d} \int_{\mathbb{R}^n} \|\mathcal{F}(f)(\lambda)F_{\alpha, \lambda}\|_{\mathcal{H}_\lambda} |\lambda|^\ell d\lambda < \infty$$

*then we have for almost every  $w$ ,*

$$f(w) = \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_{\mathbb{R}^n} \text{tr}(\pi_\lambda(w^{-1})\mathcal{F}(f)(\lambda)) |\lambda|^\ell d\lambda.$$

Following [30], the representation  $\pi_\lambda$  on  $\mathbb{G}$  determines a representation  $\pi_\lambda^*$  on its Lie algebra  $\mathcal{G}$  on the space of  $C^\infty$  vectors. The representation  $\pi_\lambda^*$  is defined by

$$\pi_\lambda^*(X)f = \left( \frac{d}{dt} \pi_\lambda(\exp(tX))f \right) \Big|_{t=0}$$

for every  $X$  in the Lie algebra  $\mathcal{G}$ . We can extend  $\pi_\lambda^*$  to the universal enveloping algebra of left-invariant differential operators on  $\mathbb{G}$ . Let  $\mathcal{K}$  be a left-invariant operator on  $\mathbb{G}$ , then we have

$$\mathcal{K}(\pi_\lambda f, g) = (\pi_\lambda \pi_\lambda^*(\mathcal{K})f, g)$$



where  $(\cdot, \cdot)$  stands for the  $\mathcal{H}_\lambda$  inner product.

4.1.2. *Bargmann representations on H-type groups.* Given  $\lambda \in \mathbb{R}^n \setminus \{0\}$ , consider the Hilbert space (called the Fock space)  $\mathcal{H}_\lambda(\mathbb{C}^\ell)$  of all entire holomorphic functions  $F$  on  $\mathbb{C}^\ell$  such that

$$\|F\|_{\mathcal{H}_\lambda}^2 = \left(\frac{2|\lambda|}{\pi}\right)^\ell \int_{\mathbb{C}^\ell} |F(\xi)|^2 e^{-|\lambda||\xi|^2} d\xi < \infty.$$

The corresponding irreducible unitary representation  $\pi_\lambda$  of the group  $\mathbb{G}$  is realized on  $\mathcal{H}_\lambda(\mathbb{C}^\ell)$  by (recall that  $t \in \mathbb{R}^n$  and  $z, \xi \in \mathbb{C}^\ell$ ) (see [15])

$$(\pi_\lambda(z, t)F)(\xi) = F(\xi - z)e^{i\langle \lambda, t \rangle - |\lambda|(|z|^2 + \langle z, \xi \rangle)}.$$

It is a well-known fact that the Fock space admits an orthonormal basis given by the monomials

$$F_{\alpha, \lambda}(\xi) = \frac{(\sqrt{2|\lambda|}\xi)^\alpha}{\sqrt{\alpha!}}, \quad \alpha \in \mathbb{N}^\ell.$$

A very important property for us is the following diagonalization result.

**Proposition 4.2.** *Let  $\mathcal{F}_B$  be the group Fourier transform associated to the Bargmann representation  $\pi$ . The following diagonalization property holds: for every  $f \in \mathcal{S}(\mathbb{G})$ ,*

$$\mathcal{F}_B(\Delta_{\mathbb{G}}f)(\lambda)F_{\alpha, \lambda} = -4|\lambda|(2|\alpha| + \ell)\mathcal{F}_B(f)(\lambda)F_{\alpha, \lambda}.$$

This allows to define the following formula, for every  $\rho \in \mathbb{R}$ :

$$\mathcal{F}_B((-\Delta_{\mathbb{G}})^\rho f)(\lambda)F_{\alpha, \lambda} = (4|\lambda|(2|\alpha| + \ell)^\rho)\mathcal{F}_B(f)(\lambda)F_{\alpha, \lambda}.$$

4.1.3. *The  $L^2$  representation on H-type groups.* Another useful representation is the so-called Schrödinger, or  $L^2$  representation. In this case, the unitary irreducible representations are given on  $L^2(\mathbb{R}^\ell)$  by, for  $\lambda \in \mathbb{R}^n$  (and writing  $z = (x, y)$ )

$$(\tilde{\pi}_\lambda(z, t)F)(\xi) = e^{i\langle \lambda, t \rangle + |\lambda|(\sum_{j=1}^\ell x_j \xi_j + \frac{1}{2}x_j y_j)} F(\xi + y).$$

The intertwining operator between the Bargmann and the  $L^2$  representations is the Hermite-Weber transform  $K_\lambda : \mathcal{H}_\lambda \rightarrow L^2(\mathbb{R}^\ell)$  given by

$$(4.1) \quad (K_\lambda \phi)(\xi) = C_\ell |\lambda|^{\ell/4} e^{|\lambda| \frac{|\xi|^2}{2}} \phi \left( -\frac{1}{2|\lambda|} \frac{\partial}{\partial \xi} \right) e^{-|\lambda||\xi|^2},$$

which is unitary and satisfies  $K_\lambda \pi_\lambda(z, t) = \tilde{\pi}_\lambda(z, t) K_\lambda$ . Following [30] and the previous description, we have

$$\tilde{\pi}_\lambda^*(X_j) = i|\lambda|\xi_j \quad \text{and} \quad \tilde{\pi}_\lambda^*(Y_j) = \frac{\partial}{\partial \xi_j}$$

for  $j = 1, \dots, \ell$ , and similarly for  $k = 1, \dots, n$

$$(4.2) \quad \tilde{\pi}_\lambda^*(\partial_{t_k}) = i\lambda_k.$$

Therefore, we have

$$(4.3) \quad \tilde{\pi}_\lambda^*(-\Delta_{\mathbb{G}}) = -\sum_{j=1}^n \frac{\partial^2}{\partial \xi_j^2} + |\lambda|^2 |\xi|^2.$$

Notice that this is a Hermite operator and the eigenfunctions of  $\pi_\lambda^*(-\Delta_{\mathbb{G}})$  are

$$\Phi_\alpha^\lambda(\xi) = |\lambda|^{n/4} \Phi_\alpha(\sqrt{|\lambda|}\xi), \quad \alpha = (\alpha_1, \dots, \alpha_\ell)$$

where  $\Phi_\alpha(\xi)$  is the product  $\psi_{\alpha_1}(\xi_1) \dots \psi_{\alpha_\ell}(\xi_\ell)$  and  $\psi_{\alpha_j}(\xi_j)$  is the eigenfunction of  $-\frac{\partial^2}{\partial \xi_j^2} + \xi_j^2$  with eigenvalue  $2\alpha_j + 1$ . This leads to the following formula, where  $|\alpha| = \alpha_1 + \dots + \alpha_\ell$ :

$$\tilde{\pi}_\lambda^*(-\Delta_{\mathbb{G}})\Phi_\alpha^\lambda = (2|\alpha| + \ell)|\lambda|\Phi_\alpha^\lambda.$$

As a consequence, one has the following lemma.

**Lemma 4.3.** *Let  $\mathcal{F}_S$  be the group Fourier transform associated to the Schrödinger representation  $\tilde{\pi}$ . The following diagonalization property holds: for every  $f \in \mathcal{S}(\mathbb{G})$*

$$\mathcal{F}_S((-\Delta_{\mathbb{G}})f)\Phi_\alpha^\lambda = (2|\alpha| + \ell)|\lambda|\mathcal{F}_S(f)\Phi_\alpha^\lambda.$$

*Proof.* We have by definition

$$\begin{aligned} \mathcal{F}_S((-\Delta_{\mathbb{G}})f)\Phi_\alpha^\lambda &= \int_{\mathbb{G}} (-\Delta_{\mathbb{G}})f(z, t) \tilde{\pi}_\lambda(z, t) \Phi_\alpha^\lambda \\ &= \int_{\mathbb{G}} f(z, t) (-\Delta_{\mathbb{G}}) \tilde{\pi}_\lambda(z, t) \Phi_\alpha^\lambda. \end{aligned}$$

Using the definition of the dual representation, we have

$$\mathcal{F}_S((-\Delta_{\mathbb{G}})f)\Phi_\alpha^\lambda = \int_{\mathbb{G}} f(z, t) \pi_\lambda(z, t) \pi_\lambda^*(-\Delta_{\mathbb{G}})\Phi_\alpha^\lambda$$

and using the properties of the Hermite operator, this yields to the desired result.  $\square$

**4.2. A localization lemma.** As in [5], one can prove a localization lemma (also called Bernstein Lemma), which we choose to state here in the context of the Bargmann representation. The proof is omitted as it is identical to the Heisenberg situation treated in [5]. Note that with the result given in Proposition 3.4, the equivalence provided in the last statement of the lemma could be extended to iterated vector fields  $X^I$ . We do not write the estimate here as it is not needed in the sequel.

**Lemma 4.4.** *Let  $p$  and  $q$  be two elements of  $[1, \infty]$ , with  $p \leq q$ , and let  $u \in \mathcal{S}(\mathbb{G})$  satisfy for all  $\alpha \in \mathbb{N}^\ell$ ,*

$$(4.4) \quad \mathcal{F}_B(u)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{\lambda \in (2|\alpha|+\ell)^{-1}2^{2j}\mathcal{B}_0} \mathcal{F}_B(u)(\lambda)F_{\alpha,\lambda}.$$

*Then we have, with the notation (1.2)*

$$\forall k \in \mathbb{N}, \quad \sup_{\beta=k} \|\mathcal{X}^\beta u\|_{L^q(\mathbb{G})} \leq C_k 2^{Nj(\frac{1}{p}-\frac{1}{q})+kj} \|u\|_{L^p(\mathbb{G})}.$$

*On the other hand, if*

$$(4.5) \quad \mathcal{F}_B(u)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{\lambda \in (2|\alpha|+\ell)^{-1}2^{2j}\mathcal{C}_0},$$

*then for all  $\rho \in \mathbb{R}$ ,*

$$C_\rho^{-1} 2^{-j\rho} \|(-\Delta_{\mathbb{G}})^{\frac{\rho}{2}} u\|_{L^p(\mathbb{G})} \leq \|u\|_{L^p(\mathbb{G})} \leq C_\rho 2^{-j\rho} \|(-\Delta_{\mathbb{G}})^{\frac{\rho}{2}} u\|_{L^p(\mathbb{G})}.$$

**Remark 4.5.** *The same result holds under less restrictive assumptions than  $u \in \mathcal{S}(\mathbb{G})$ , but in that case (4.4) should be replaced by the assumption that  $u * f = 0$  for every radial function  $f \in \mathcal{S}(\mathbb{G})$  satisfying for all  $\alpha \in \mathbb{N}^\ell$ ,*

$$\mathcal{F}_B(f)(\lambda)F_{\alpha,\lambda} = 0 \quad \text{for } \lambda \in (2|\alpha| + \ell)^{-1}2^{2j}\mathcal{B}_0.$$

*Similarly (4.5) should be replaced by the assumption that  $u * g = 0$  for every radial function  $g \in \mathcal{S}(\mathbb{G})$  satisfying for every  $\alpha \in \mathbb{N}^\ell$ ,*

$$\mathcal{F}_B(g)(\lambda)F_{\alpha,\lambda} = 0 \quad \text{for } \lambda \in (2|\alpha| + \ell)^{-1}2^{2j}\mathcal{C}_0.$$

*We refer to [5] for more details.*

**4.3. Paraproduct on H-type groups.** In order to develop a paraproduct on H-type groups, one has to prove that the product of two functions is localized in frequencies whenever the functions are localized. This is the object of the next lemma, which is an extension of Proposition 4.2 of [5].

**Lemma 4.6.** *There is a constant  $M_1 \in \mathbb{N}$  such that the following holds. Consider  $f$  and  $g$  two functions of  $\mathcal{S}(\mathbb{G})$  such that*

$$\mathcal{F}_B(f)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{\lambda \in (2|\alpha|+\ell)2^{2m}\mathcal{C}} \mathcal{F}_B(f)(\lambda)F_{\alpha,\lambda} \quad \text{and}$$

$$\mathcal{F}_B(g)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{\lambda \in (2|\alpha|+\ell)2^{2m'}\mathcal{C}} \mathcal{F}_B(g)(\lambda)F_{\alpha,\lambda}$$

*for some integers  $m$  and  $m'$ . If  $m' - m > M_1$ , then there exists a ring  $\tilde{\mathcal{C}}$  such that*

$$\mathcal{F}_B(fg)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{\lambda \in (2|\alpha|+\ell)2^{2m'}\tilde{\mathcal{C}}} \mathcal{F}_B(fg)(\lambda)F_{\alpha,\lambda}.$$

*On the other hand, if  $|m' - m| \leq M_1$ , then there exists a ball  $\tilde{\mathcal{B}}$  such that*

$$\mathcal{F}_B(fg)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{\lambda \in (2|\alpha|+\ell)2^{2m'}\tilde{\mathcal{B}}} \mathcal{F}_B(fg)(\lambda)F_{\alpha,\lambda}.$$

**Remark 4.7.** *As pointed out in Remark 4.5, it is actually not necessary to assume as much smoothness on  $f$  and  $g$ , but the result is slightly more complicated to state in that case.*

We are now in position to introduce the following definition.

**Definition 4.8.** *We shall call paraproduct of  $v$  by  $u$  and shall denote by  $T_u v$  the following bilinear operator:*

$$T_u v = \sum_j S_{j-1} u \Delta_j v.$$

*We shall call remainder of  $u$  and  $v$  and shall denote by  $R(u, v)$  the following bilinear operator:*

$$R(u, v) = \sum_{|j-j'| \leq 1} \Delta_j u \Delta_{j'} v.$$

**Remark 4.9.** *Just by looking at the definition, it is clear that formally*

$$uv = T_u v + T_v u + R(u, v).$$

According to Lemma 4.6,  $S_{j-1} u \Delta_j v$  is frequency localized, in the sense of (4.5), in a ring of size  $2^j$ . But, for terms of the kind  $\Delta_j u \Delta_{j'} v$  with  $|j - j'| \leq 1$ , we have an accumulation of frequencies at the origin. Such terms are frequency localized, in the sense of (4.4), in a ball of size  $2^j$ .

One of the immediate consequences of Lemma 4.6 is the following result on the product of functions in Besov spaces.

**Corollary 4.10.** *Let  $\rho > 0$  and  $(p, r) \in [1, +\infty]^2$  be three real numbers.*

*If  $f$  and  $g$  belong to  $L^\infty \cap B_{p,r}^\rho(\mathbb{G})$ , then*

$$\|fg\|_{B_{p,r}^\rho(\mathbb{G})} \leq C(\|f\|_{L^\infty} \|g\|_{B_{p,r}^\rho(\mathbb{G})} + \|g\|_{L^\infty} \|f\|_{B_{p,r}^\rho(\mathbb{G})}).$$

*If  $\rho_1 + \rho_2 > 0$  and if  $p_1$  is such that  $\rho_1 < Q/p_1$ , then for all  $(p_2, r_2) \in [1, +\infty]^2$  and for all  $f$  and  $g$  in  $B_{p_1, \infty}^{\rho_1} \cap B_{p_2, r_2}^{\rho_2}(\mathbb{G})$ ,*

$$\|fg\|_{B_{p_2, r_2}^\rho(\mathbb{G})} \leq C(\|f\|_{B_{p_1, \infty}^{\rho_1}} \|g\|_{B_{p_2, r_2}^{\rho_2}} + \|g\|_{B_{p_1, \infty}^{\rho_1}} \|f\|_{B_{p_2, r_2}^{\rho_2}}),$$

*where  $\rho = \rho_1 + \rho_2 - Q/p_1$ .*

*Moreover, if  $\rho_1 + \rho_2 \geq 0$ ,  $\rho_1 < Q/p_1$  and  $\frac{1}{r_1} + \frac{1}{r_2} = 1$ , then for  $f$  and  $g$  in  $B_{p_1, r_1}^{\rho_1} \cap B_{p_2, r_2}^{\rho_2}(\mathbb{G})$ , one has*

$$\|fg\|_{B_{p, \infty}^\rho(\mathbb{G})} \leq C(\|f\|_{B_{p_1, r_1}^{\rho_1}} \|g\|_{B_{p_2, r_2}^{\rho_2}} + \|g\|_{B_{p_1, r_1}^{\rho_1}} \|f\|_{B_{p_2, r_2}^{\rho_2}}).$$

*Finally if  $\rho_1 + \rho_2 > 0$ ,  $\rho_j < Q/p_j$  and  $p \geq \max(p_1, p_2)$ , then for all  $(r_1, r_2)$ ,*

$$\|fg\|_{B_{p,r}^{\rho_1+\rho_2}(\mathbb{G})} \leq C\|f\|_{B_{p_1, r_1}^{\rho_1}} \|g\|_{B_{p_2, r_2}^{\rho_2}},$$

with  $\rho_{12} = \rho_1 + \rho_2 - Q(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p})$  and  $r = \max(r_1, r_2)$ , and if  $\rho_1 + \rho_2 \geq 0$ , with  $\rho_j < Q/p_j$  and with  $\frac{1}{r_1} + \frac{1}{r_2} = 1$ , then for all  $p \geq \max(p_1, p_2)$ ,

$$\|fg\|_{B_{p,\infty}^{\rho_{12}}(\mathbb{G})} \leq C \|f\|_{B_{p_1,r_1}^{\rho_1}} \|g\|_{B_{p_2,r_2}^{\rho_2}}.$$

Note that the same results hold in the case of homogeneous Besov spaces, up to obvious modifications.

**4.4. Applications: Refined Hardy and Sobolev estimates.** One interest of frequency localization is that it allows to prove rather simply Sobolev and Hardy inequalities, and actually even refined versions of those inequalities. This approach is presented here.

**Theorem 7.** *Let  $s$  be any real number in  $]0, Q/2[$ . Then the following inequality holds, with  $\frac{1}{r} = \frac{1}{2} - \frac{s}{Q}$*

$$\|f\|_{L^r(\mathbb{G})} \leq C \|f\|_{\dot{B}_{\infty,\infty}^{s-\frac{Q}{2}}(\mathbb{G})}^{\frac{2s}{Q}} \|f\|_{\dot{H}^s(\mathbb{G})}^{1-\frac{2s}{Q}}.$$

Moreover, let  $p$  and  $q$  be two real numbers in  $[1, \infty]$  such that  $2 \leq q < \frac{2Q}{Q-2s} < p \leq \infty$ . Then the following inequality holds, with  $\alpha = \frac{pq}{p-q} \left( \frac{1}{q} - \frac{1}{2} + \frac{s}{Q} \right)$

$$\left( \int_{\mathbb{G}} \frac{|u(w)|^2}{|w|^{2s}} dw \right)^{\frac{1}{2}} \leq C \|u\|_{\dot{B}_{p,2}^{s-Q(\frac{1}{2}-\frac{1}{p})}(\mathbb{G})}^{\alpha} \|u\|_{\dot{B}_{q,2}^{s-Q(\frac{1}{2}-\frac{1}{q})}(\mathbb{G})}^{1-\alpha}.$$

The first result (which should be understood as a refined Sobolev inequality) was first proved in the euclidian case by P. Gérard, Y. Meyer and F. Oru in [19] following the proof of Sobolev embeddings of J.-Y. Chemin and C.-J. Xu in [12], and in [5] in the Heisenberg case. The second result is a refined Hardy inequality, proved in [3]. We refer to [10] for recent extensions. Let us explain why they should be considered "refined": this is due to the following proposition, whose proof is identical to [3], up to obvious modifications.

**Proposition 4.11.** *Let  $\theta$  be a function in  $\mathcal{S}(\mathbb{G})$ ,  $p$  in  $[1, \infty]$ ,  $\sigma$  in  $] -Q(1 - 1/p), +\infty[$  and  $\varepsilon_0$  a positive real number. A constant  $C$  exists such that the oscillatory function  $f_\varepsilon(y) = \theta(y)e^{iy_1/\varepsilon}$  satisfies*

$$(4.6) \quad \forall \varepsilon \leq \varepsilon_0, \quad \|f_\varepsilon\|_{\dot{B}_{p,1}^\sigma(\mathbb{G})} \leq C\varepsilon^{-\sigma}.$$

Then one can construct a family  $(f_\varepsilon)_{\varepsilon>0}$  of smooth functions on  $\mathbb{G}$  such that, for any  $s$  in  $]0, Q/2[$  and any  $\beta < 2s/Q$ ,

$$\lim_{\varepsilon \rightarrow 0} \frac{\|f_\varepsilon\|_{L^{2Q-2s}}}{\|f_\varepsilon\|_{\dot{B}_{\infty, \infty}^{s-\frac{Q}{2}}}^\beta \|f_\varepsilon\|_{\dot{H}^s}^{1-\beta}} = +\infty \quad \text{and}$$

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\|f_\varepsilon\|_{\dot{B}_{\infty, \infty}^{s-\frac{Q}{2}}}^\beta \|f_\varepsilon\|_{\dot{H}^s}^{1-\beta}} \int \frac{f_\varepsilon^2}{|w|^{2s}} dw = +\infty.$$

Now let us turn to the proof of Theorem 7. The first estimate actually only relies on the localization lemma (Lemma 4.4), and the proof follows exactly the lines of [5]. As to the proof of the Hardy inequality, it relies on the paraproduct algorithm recalled in the previous section. The idea indeed consists simply in noticing that

$$\int_{\mathbb{G}} \frac{|u(w)|^2}{|w|^{2s}} dw = \langle \rho^{-2s}, u^2 \rangle$$

so it is enough to prove that  $\rho^{-2s}$  and  $u^2$  belongs to a pair of spaces in duality. One proves easily that  $|\cdot|^{-2s}$  belongs to the space  $\dot{B}_{1, \infty}^{Q-2s}$  and using product laws of Corollary 4.10 we conclude directly that

$$\left( \int_{\mathbb{G}} \frac{|u(w)|^2}{|w|^{2s}} dw \right)^{\frac{1}{2}} \leq C \|u\|_{\dot{H}^s(\mathbb{G})}.$$

The precised Hardy inequality is then obtained by studying more precisely each term of the paraproduct algorithm, and is derived exactly as in the case of the Heisenberg group or the euclidian case (see [3]).

#### 4.5. Pseudodifferential calculus.

4.5.1. *Introduction.* Once paradifferential operators have been constructed, with similar properties to the euclidian case, it is natural to address the question of the construction of pseudodifferential operators. It is out of the scope of this article to review the history of pseudodifferential operators in the euclidian setting (we refer the interested reader for instance to [21] or [28]), but let us simply point out that the study of such operators allowed spectacular progress in the study of linear and nonlinear PDEs in the last forty years. In this section we shall see how such a theory may be developed in the context of H-type groups. Actually this project was undertaken in [4] in the case of the Heisenberg group, and it is quite easy to adapt that work to our more general setting. We shall therefore not go through all the details of the construction and the proofs of the main properties, which turned out

in [4] to be quite technical, but rather explain the general principles and refer to [4] for details.

4.5.2. *Definitions.* In order to motivate the next definition, let us recall that according to (4.3), the sublaplacian is naturally associated with the Harmonic oscillator. It is therefore natural to construct symbols on a H-type group  $\mathbb{G}$  by using the Weyl-Hörmander calculus associated with the Harmonic oscillator. However this is not sufficient, since as noted in (4.2), functions of  $\lambda$  alone should also be considered as symbols. This motivates the following definition of symbols on a H-type group, and the next one giving pseudodifferential operators. As pointed out in [4] it is possible to consider low regularity symbols (in the  $\mathbb{G}$  variable): that regularity is measured by a positive index  $\rho$  which is arbitrary and is fixed from now on.

**Definition 1.** *A smooth function  $a$  defined on  $\mathbb{G} \times \mathbb{R}^n \times \mathbb{R}^{2\ell}$  is a symbol if there is a real number  $\mu$  such that for all  $N \in \mathbb{N}$ , the following semi norm is finite:*

$$\|a\|_{N;S_{\mathbb{G}}(\mu)} = \sup_{\substack{\lambda \neq 0 \\ \Theta \in \mathbb{R}^{2\ell}}} \sup_{|\beta|+|k| \leq N} |\lambda|^{-\frac{|\beta|}{2}} (1 + |\lambda|(1 + \Theta^2))^{\frac{|\beta|-\mu}{2}} \\ \times \|(\lambda \partial_{\lambda})^k \partial_{\Theta}^{\beta} a(\cdot, \lambda, \Theta)\|_{C^{\rho}(\mathbb{G})}.$$

Besides, one additionally requires that the function

$$(w, \lambda, \xi, \eta) \mapsto \sigma(a)(w, \lambda, \xi, \eta) = a\left(w, \lambda, \operatorname{sgn}(\lambda) \frac{\xi}{\sqrt{|\lambda|}}, \frac{\eta}{\sqrt{|\lambda|}}\right)$$

is smooth close to  $\lambda = 0$  for any  $(w, \xi, \eta) \in \mathbb{G} \times \mathbb{R}^{2\ell}$ . In that case we shall write  $a \in S_{\mathbb{G}}(\mu)$ .

Before giving the definition of a pseudodifferential operator associated with such a symbol, let us introduce the scaling operator

$$\forall f \in L^2(\mathbb{R}^d), \quad T_{\lambda} f(\xi) = |\lambda|^{-d/4} f(|\lambda|^{-1/2} \xi),$$

as well as the operator  $J_{\lambda} = T_{\lambda} K_{\lambda}$ , where  $K_{\lambda}$  is the intertwining operator defined in (4.1).

**Definition 2.** *To a symbol  $a$  of order  $\mu$  in the sense of Definition 1, we associate the pseudodifferential operator on  $\mathbb{G}$  defined in the following way: for any  $f \in \mathcal{S}(\mathbb{G})$  and any  $w \in \mathbb{G}$ ,*

$$\operatorname{Op}(a)f(w) = \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_{\mathbb{R}^n} \operatorname{tr}(\pi_{\lambda}(w^{-1}) \mathcal{F}_B(f)(\lambda) A_{\lambda}(w)) |\lambda|^{\ell} d\lambda,$$

where

$$A_\lambda(w) = J_\lambda^* \text{op}^w(a(w, \lambda, \xi, \eta)) J_\lambda.$$

4.5.3. *Examples.* It is easy to see, using the formulas of the Section 4.1, that the following result hold for  $1 \leq j \leq \ell$ :

$$\begin{aligned} X_j &= \text{Op}(2i \text{sgn}(\lambda) \sqrt{|\lambda|} \eta_j), \quad Y_j = -\text{Op}(2i \sqrt{|\lambda|} \xi_j), \\ \partial_{i_k} &= \text{Op}(i \lambda_k), \quad \text{and } -\Delta_{\mathbb{G}} = 4 \text{Op}(|\lambda|(\eta^2 + \xi^2)). \end{aligned}$$

4.5.4. *Some results.* The proofs of the following results are easy adaptations of the proofs of [4]. Due to their rather high technicality we choose here not to go through the proofs in this more general context.

**Theorem 8.** *If  $a$  is a symbol in  $S_{\mathbb{G}}(\mu)$  with  $\rho = +\infty$ , then  $\text{Op}(a)$  maps continuously  $\mathcal{S}(\mathbb{G})$  into itself.*

*Moreover, consider  $\text{Op}(a)$  and  $\text{Op}(b)$  two pseudodifferential operators of order  $\mu$  and  $\nu$  respectively.*

- *If  $\rho > 2(2\ell + n) + |\mu|$ , then the operator  $\text{Op}(a)^*$  is a pseudodifferential operator of order  $\mu$ .*
- *If  $\rho > 2(2\ell + n) + |\mu| + |\nu|$ , then the operator  $\text{Op}(a) \circ \text{Op}(b)$  is a pseudodifferential operator of order less or equal to  $\mu + \nu$ .*

Note that precise formulas for the symbols of  $\text{Op}(a)^*$  and  $\text{Op}(a) \circ \text{Op}(b)$  may be derived, as in [4].

Now let us consider the action of pseudodifferential operators on Sobolev spaces.

**Theorem 9.** *Let  $\mu$  be a real number, and  $\rho > 2(2\ell + n)$  be a noninteger real number. Consider a symbol  $a$  in  $S_{\mathbb{G}}(\mu)$  in the sense of Definition 1. Then the operator  $\text{Op}(a)$  is bounded from  $H^s(\mathbb{G})$  into  $H^{s-\mu}(\mathbb{G})$ , for any real number  $s$  such that  $|s - \mu| < \rho$ . More precisely there exists  $n \in \mathbb{N}$  such that*

$$\|\text{Op}(a)\|_{\mathcal{L}(H^s(\mathbb{G}), H^{s-\mu}(\mathbb{G}))} \leq C_n \|a\|_{N; S_{\mathbb{G}}(\mu)}.$$

*If  $\rho > 0$ , then the result holds for  $0 < s - \mu < \rho$ .*

## REFERENCES

- [1] G. Alexopoulos. An application of homogenization theory to harmonic analysis: Harnack inequalities and Riesz transforms on Lie groups of polynomial growth. *Canad. J. Math.*, 44(4):691–727, 1992.
- [2] Hajer Bahouri, Jean-Yves Chemin, and Raphael Danchin. *Fourier Analysis and Nonlinear PDEs*. Springer, to appear.
- [3] Hajer Bahouri, Jean-Yves Chemin, and Isabelle Gallagher. Refined Hardy inequalities. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 5(3):375–391, 2006.
- [4] Hajer Bahouri, Clotilde Fermanian-Kammerer, and Isabelle Gallagher. Phase space analysis and pseudodifferential calculus on the Heisenberg group. *accepted for publication, Astérisque*.



- [5] Hajer Bahouri and Isabelle Gallagher. Paraproduit sur le groupe de Heisenberg et applications. *Rev. Mat. Iberoamericana*, 17(1):69–105, 2001.
- [6] Hajer Bahouri and Isabelle Gallagher. Heat kernel on the Heisenberg group. In *Advances in Phase Space Analysis of Partial Differential Equations*, volume 78 of *Progress in nonlinear differential equations and their applications*. 2009.
- [7] Hajer Bahouri, Patrick Gerard, and Chao-Jiang Xu. Estimations de Strichartz généralisées sur le groupe de Heisenberg. In *Séminaire sur les Équations aux Dérivées Partielles, 1997–1998*, pages Exp. No. X, 13. École Polytech., Palaiseau, 1998.
- [8] Dominique Bakry. The Riesz transforms associated with second order differential operators. In *Seminar on Stochastic Processes, 1988 (Gainesville, FL, 1988)*, volume 17 of *Progr. Probab.*, pages 1–43. Birkhäuser Boston, Boston, MA, 1989.
- [9] A. Bonfiglioli and F. Uguzzoni. Nonlinear Liouville theorems for some critical problems on H-type groups. *J. Funct. Anal.*, 207(1):161–215, 2004.
- [10] Diego Chamorro. Improved Sobolev Inequalities and Muckenhoupt weights on stratified Lie groups. *Submitted*.
- [11] Diego Chamorro. *Inégalités de Gagliardo-Nirenberg précisées sur le groupe de Heisenberg*. PhD Thesis.
- [12] Jean-Yves Chemin and Chao-Jiang Xu. Inclusions de Sobolev en calcul de Weyl-Hörmander et champs de vecteurs sous-elliptiques. *Ann. Sci. École Norm. Sup...*, 30:719–751, 1997.
- [13] Lawrence J. Corwin and Frederick P. Greenleaf. *Representations of nilpotent Lie groups and their applications. Part I*, volume 18 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. Basic theory and examples.
- [14] T. Coulhon, E. Russ, and V. Tardivel-Nachef. Sobolev algebras on Lie groups and Riemannian manifolds. *Amer. J. Math.*, 123:283–342, 2001.
- [15] A. H. Dooley. Heisenberg-type groups and intertwining operators. *J. Funct. Anal.*, 212(2):261–286, 2004.
- [16] Nick Dungey, A. F. M. ter Elst, and Derek W. Robinson. *Analysis on Lie groups with polynomial growth*, volume 214 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2003.
- [17] G. Folland and E. M. Stein. *Hardy spaces on homogeneous groups*. Princeton Univ. Press, 1982.
- [18] G. Furioli, C. Melzi, and A. Veneruso. Littlewood-Paley decompositions and Besov spaces on Lie groups of polynomial growth. *Math. Nachr.*, 279(9-10):1028–1040, 2006.
- [19] Patrick Gérard, Yves Meyer, and Frédéric Oru. Inégalités de Sobolev précisées.
- [20] Y. Guivarc’h. Croissance polynomiale et période des fonctions harmoniques. *Bull. Soc. Math. France*, 101:333–379, 1973.
- [21] Lars Hörmander. *The analysis of linear partial differential equations*, volume 3 of *Springer Verlag*. 1985.
- [22] A Hulanicki. A functional calculus for Rockland operators on nilpotent Lie groups estimates. *Studia Mathematica*, 78:253–266, 1984.
- [23] A. Kaplan and F. Ricci. Harmonic analysis on groups of Heisenberg type. In *Harmonic analysis (Cortona, 1982)*, volume 992 of *Lecture Notes in Math.*, pages 416–435. Springer, Berlin, 1983.

- [24] Aroldo Kaplan. Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc.*, 258(1):147–153, 1980.
- [25] Adam Korányi. Geometric properties of Heisenberg-type groups. *Adv. in Math.*, 56(1):28–38, 1985.
- [26] Noël Lohoué and Nicolas Th. Varopoulos. Remarques sur les transformées de Riesz sur les groupes de Lie nilpotents. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(11):559–560, 1985.
- [27] A. Nagel, E. M. Stein, and S. Wainger. Balls and metrics defined by vector fields i: Basic properties. *Acta Math.*, 155:103–147, 1985.
- [28] Michael E. Taylor. *Pseudodifferential operators and nonlinear PDE*, volume 100 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1991.
- [29] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992.
- [30] Qiaohua Yang and Fuliu Zhu. The heat kernel on H-type groups. *Proc. Amer. Math. Soc.*, 136(4):1457–1464, 2008.

INSTITUT DE MATHÉMATIQUES UMR 7586, UNIVERSITÉ PARIS VII, 175, RUE  
DU CHEVALERET, 75013 PARIS, FRANCE  
*E-mail address:* `gallagher@math.jussieu.fr`

UNIVERSITÉ PAUL CÉZANNE, LATP, FACULTÉ DES SCIENCES ET TECHNIQUES,  
CASE COUR A, AVENUE ESCADRILLE NORMANDIE-NIEMEN, F-13397 MARSEILLE  
CEDEX 20, FRANCE, AND CNRS, LATP, CMI, 39 RUE F. JOLIOT-CURIE, F-  
13453 MARSEILLE CEDEX 13, FRANCE  
*E-mail address:* `sire@cmi.univ-mrs.fr`