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# Survival Probabilities for $N$ -ary Subtrees on a Galton-Watson Family Tree

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## Abstract

The family tree of a Galton-Watson branching process may contain  $N$ -ary subtrees, i.e. subtrees whose vertices have at least  $N \geq 1$  children. For family trees without infinite  $N$ -ary subtrees, we study how fast  $N$ -ary subtrees of height  $t$  disappear as  $t \rightarrow \infty$ .

*Keywords:* Branching process; Family tree;  $N$ -ary tree; Binary tree; Survival probability

## 1 Introduction, Statement of the Results and Related Studies

The family tree associated with a Bienaimé-Galton-Watson process describes the evolution of a population in which each individual, independently of the others, creates  $k$  new individuals with probability  $p_k$  ( $k = 0, 1, \dots$ ). We assume that at generation zero there is single ancestor, called root of the tree and let

$$f(s) = \sum_{k=0}^{\infty} p_k s^k \quad (1)$$

to denote the probability generating function (pgf) of the offspring distribution (with the convention that  $f(1) = 1$ ). We recall the well known construction of a Galton-Watson family

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tree noting that the individuals participating in the process and the parent-child relations between them define the vertex-set and arc-set of the tree, respectively (for more details see e.g. Harris (1963, Ch. 6)). By  $\{Z_t, t = 0, 1, \dots\}$  we denote the generation size process, defined by the following recurrence:

$$Z_0 = 1, Z_{t+1} = \begin{cases} X_1 + \dots + X_{Z_t} & \text{if } Z_t > 0, \\ 0 & \text{if } Z_t = 0, \end{cases} \quad t = 1, 2, \dots \quad (2)$$

Here  $\{X_j, j = 1, 2, \dots\}$  are independent copies of a random variable whose pgf is given by (1). In terms of trees  $\{Z_t, t = 0, 1, \dots\}$  present the sizes of the strata on the Galton-Watson family tree. That is,  $Z_t$  equals the number of vertices which are at distance  $t$  from the root. (We recall that the distance between two vertices in a tree is determined by the number of arcs in the path between them.)

The probabilities  $P(Z_t > 0)$  are often called survival probabilities of the process. Their asymptotic behavior, as  $t \rightarrow \infty$ , has been studied long ago by several authors. Let

$$\gamma_1 = \lim_{t \rightarrow \infty} P(Z_t = 0). \quad (3)$$

It turns out that the parameters

$$a_1 = f'(\gamma_1), b_1 = f''(\gamma_1) \quad (4)$$

play important role in the study of survival probabilities of the process. (If  $\gamma_1 = 1$ , here and further on, by  $f'(1)$ ,  $f''(1)$  and  $f'''(1)$  we denote the left derivatives of the power series (1) at the point 1; also note that  $f'(1)$  is the mean value of the offspring distribution and  $f''(1)$  is its second factorial moment.) The following two results are well known and valid for offspring distributions satisfying the inequality  $p_0 + p_1 < 1$ .

*Result 1.* [See Harris (1963, Ch. 1, Thms. 6.1 and 8.4).] (i) If  $f'(1) > 1$  and  $\gamma_1 > 0$ , then  $0 < a_1 < 1$ . (ii) For  $0 < a_1 < 1$ ,

$$P(Z_t > 0) = 1 - \gamma_1 + d_1 a_1^t + O(a_1^{2t}), \quad (5)$$

as  $t \rightarrow \infty$ , where  $d_1 > 0$  is certain constant.

*Result 2.* [See Harris (1963, Ch. 1, Thm. 6.1 and Sect. 10.2).] (i) If  $f'(1) = 1$ , then  $\gamma_1 = 1$ . (ii) If  $f'(1) = 1$  and  $f'''(1) < \infty$ , then

$$P(Z_t > 0) \sim \frac{2}{b_1 t}, t \rightarrow \infty. \quad (6)$$

*Remark 1.* Result 2 was first obtained by Kolmogorov (1938). It is also valid if  $f''(1) < \infty$ ; see e.g. Sevast'yanov (1971, Ch. 2, Sect. 2).

We will study special kinds of subtrees of a Galton-Watson family tree. We will consider only rooted subtrees and call two such subtrees disjoint if they do not have a common vertex different from the root. Next, for fixed integer  $N \geq 1$ , we define a complete infinite  $N$ -ary tree to be the family tree of a deterministic branching process with offspring pgf  $f(s) = s^N$ . For a branching process  $\{Z_t, t = 0, 1, \dots\}$  defined by (2), we introduce the random variable  $V_N$ , equal to the number of complete disjoint and infinite  $N$ -ary subtrees rooted at the ancestor.

If we restrict the process up to its  $t$ th generation ( $t = 0, 1, \dots$ ), or, equivalently, if we assume that the Galton-Watson family tree is cut off at height  $t$ , we can similarly define the random variable  $V_{N,t}$  to be the count of the complete disjoint  $N$ -ary subtrees of height at least  $t$ , which are rooted at the ancestor. It is clear that  $V_N = \lim_{t \rightarrow \infty} V_{N,t}$  with probability 1.

Further on we will assume that the offspring distribution  $\{p_k, k = 0, 1, \dots\}$  is such that  $p_k < 1$  for all  $k$  and  $p_k > 0$  for some  $k > N$ . For  $N \geq 1$ , we also let

$$\gamma_{N,0} = 0, \gamma_{N,t} = P(V_{N,t} = 0), t = 1, 2, \dots, \quad (7)$$

and

$$\gamma_N = P(V_N = 0) = \lim_{t \rightarrow \infty} \gamma_{N,t}. \quad (8)$$

The last limit exists since the probabilities  $\gamma_{N,t}$  monotonically increase in  $t$ .

In particular, for  $N = 1$ , the event  $\{V_1 > 0\}$  implies that the family tree contains an infinite unary subtree (infinite path), which means that the generations of the Galton-Watson process never die. In the same way, event  $\{V_2 > 0\}$  can be interpreted as the set of trajectories of the process whose family trees grow faster than binary splitting.

Another important observation follows from the well known extinction criterion; Harris (1963, Ch. 1, Thm. 6.1). We give it here in terms of the pgf (1) and probabilities (8) as follows: a necessary and sufficient condition for  $\gamma_1 = 1$  is  $f'(1) \leq 1$ ; if  $f'(1) > 1$ , then  $\gamma_1$  is the unique solution in  $[0, 1]$  of the equation

$$s = f(s). \quad (9)$$

This enables one to restate Results 1(ii) and 2(ii) in terms of counts of unary subtrees (recall (3) - (6) and Remark 1).

*Result 1'.* If  $a_1 < 1$ , then

$$P(V_{1,t} > 0 \mid V_1 = 0) = d_1 a_1^t + O(a_1^{2t})$$

as  $t \rightarrow \infty$ , where  $d_1 > 0$  is certain constant.

*Result 2'.* If  $a_1 = 1$  and  $f''(1) < \infty$ , then

$$P(V_{1,t} > 0 \mid V_1 = 0) \sim \frac{2}{b_1 t}, t \rightarrow \infty.$$

The main purpose of this present note is to study the survival of complete  $N$ -ary subtrees on a Galton-Watson family tree. We extend Results 1' and 2' to integer values of  $N$  greater than 1. Below we give the brief history of this problem.

The question how to compute the probability that the Galton-Watson process possesses "the binary splitting property" was first raised, settled and solved by Dekking (1991). The general ( $N \geq 2$ ) case was subsequently investigated by Pakes and Dekking (1991), who showed that the probability  $\gamma_N$ , defined by (8), is the smallest solution in  $[0, 1]$  of the equation

$$s = g_N(s), \quad (10)$$

where

$$g_N(s) = \sum_{j=0}^{N-1} (1-s)^j f^{(j)}(s) / j! \quad (11)$$

and  $f(s)$  is the offspring pgf (1). We also point out that a particular case arising from a study of Mandelbrot's percolation process was previously considered by Chayes et al. (1988). Their problem is equivalent to finding a condition on  $p$  for  $\gamma_8 < 1$  when  $f(s) = (1 - p + ps)^9$ . Furthermore, note that  $g_1(s) = f(s)$  and thus, for  $N = 1$ , eq. (10) reduces to (9). For particular offspring distributions, Pakes and Dekking (1991) encountered the following phenomenon: if  $N \geq 2$ , then there is a critical value  $m_N^c$  for the offspring mean  $f'(1)$  such that  $\gamma_N = 1$  if  $f'(1) < m_N^c$  and  $\gamma_N < 1$  if  $f'(1) \geq m_N^c$ . Let  $\gamma_N^c$  be the critical probability obtained when  $f'(1) = m_N^c$ . It turns out, for instance, that if  $N = 2$  and the offspring distribution is geometric, then  $m_2^c = 4$  and  $\gamma_2^c = .75$ ; for a Poisson offspring distribution the same parameters are:  $m_2^c = 3.3509$ ,  $\gamma_2^c = .4648$ . (Further numerical results in this direction can be found in Pakes and Dekking (1991) and Yanev and Mutafchiev (2006).) This phenomenon is qualitatively different from what happens for  $N = 1$  where the extinction probability  $\gamma_1 = 1$  if  $f'(1) \leq m_1^c = 1$ , except for the trivial case where  $f(s) = s$  and  $\gamma_1 < 1$  if  $f'(1) > 1$ . The case  $N \geq 2$  seems to be studied surprisingly later than the classical one when  $N = 1$ . In fact, first assertions of the fundamental theorem on the existence of infinite unary subtrees on a Galton-Watson family tree appeared about 120 - 150 years earlier and the problem was definitely settled around 1930. For more historical details, see e.g. Harris (1963) and Sevastyanov (1971). Recently, Yanev and Mutafchiev (2006) derived the probability distributions of the random variables  $V_{N,t}$  and  $V_N$ . The result for  $V_{N,t}$  is given in a form of recurrence. Furthermore, the expression for the probability distribution of  $V_N$  turns out to be very simple: its probability mass function equals the difference between two particular neighbor partial sums of the Taylor's expansion of  $f(1)$  around the point  $\gamma_N$ .

To state our main results in an appropriate form, we extend notations (4) to integer values of  $N \geq 2$ . We set

$$a_N = g'_N(\gamma_N), b_N = g''_N(\gamma_N) \quad (12)$$

and also recall definitions (7), (8) and (11).

**Theorem 1** *If  $\gamma_N \in (0, 1)$  is the smallest solution of eq. (10), then,*

- (i) *for  $N \geq 2$ , we have  $a_N \leq 1$ .*
- (ii) *If  $a_N < 1$ , then*

$$P(V_{N,t} > 0 \mid V_N = 0) = d_N a_N^t + O(a_N^{2t})$$

*as  $t \rightarrow \infty$ , where  $d_N > 0$  is certain constant.*

- (iii) *If  $a_N = 1$ , then*
- (iiia)  *$b_N > 0$ , and,*
- (iiib) *for  $N \geq 2$  and finite  $b_N$ ,*

$$P(V_{N,t} > 0 \mid V_N = 0) \sim \frac{2}{\gamma_N b_N t}, t \rightarrow \infty.$$

Our paper is organized as follows. The proofs of the results are presented in next Section 2. We recall there some old and classical methods used in the theory of branching processes. Section 3 contains few numerical results for particular offspring distributions.

We conclude our introduction with a remark on studies which are closely related to our model.

*Remark 2.* Pakes and Dekking (1991) noticed that there are links between complete infinite  $N$ -ary subtrees on a Galton-Watson family tree, Mandelbrot's percolation process studied by Chayes et al. (1988) and results obtained by Pemantle (1988) and related to a model of a reinforced random walk. In particular, Pemantle (1988) established the following criterion for  $\gamma_N < 1$  (see his Lemma 5 or Pakes and Dekking (1991, pp. 356-357)): if for some  $s_0 \in (0, 1)$  we have  $g_N(s_0) \leq s_0$ , then  $\gamma_N \leq s_0$ . Here we also indicate a relationship between the  $N$ -ary subtrees phenomenon and the existence of a  $k$ -core in a random graph. The  $k$ -core of a graph is the largest subgraph with minimum degree at least  $k$ . This concept was introduced by Bollobás (1984) in the context of finding large  $k$ -connected subgraphs of random graphs. He considered the Erdős-Rényi random graph  $G(n, p)$  with  $n$  vertices in which the possible arcs are present independently, each with probability  $p$ . If we set  $p = \lambda/n$ , where  $\lambda > 0$  is a constant, it is natural to ask: for  $k \geq 3$ , what is the critical value  $\lambda_c(k)$  of  $\lambda$  above which a (non-empty)  $k$ -core first appears in  $G(n, \lambda/n)$  with probability tending to 1 as  $n \rightarrow \infty$ . To answer this question Pittel et al. (1996) considered a Galton-Watson family tree rooted at a vertex  $x_0$  (ancestor) of the graph  $G(n, \lambda/n)$  and assume that the offspring distribution of the branching process is Poisson with mean  $\lambda$ . Let  $B_k$  denote the event that  $x_0$  has at least  $k$  children each of which has at least  $k-1$  children each of which has at least  $k-1$  children, and so on. It is clear that this assumption slightly modifies the concept of a complete infinite  $(k-1)$ -ary subtree (the only difference occurs in the assumption for the offspring number of the ancestor  $x_0$ ). Pittel et al. (1996) found the threshold  $\lambda_c(k)$  for the emergence of a non-trivial  $k$ -core in  $G(n, \lambda/n)$  and showed that, except at the critical value, the number of vertices in the  $k$ -core approaches  $P(B_k)n$  as  $n \rightarrow \infty$ . Their results also showed that a giant  $k$ -core appears suddenly when the number of arcs in the random graph reaches  $c_k n/2$ , where the constants  $c_k$  are explicitly computed. There is a remarkable coincidence between constants  $c_k$  and the critical means  $m_{k-1}^c$  ( $k = 3, 4, 5$ ) of the Poisson offspring distributions which yield existence of  $(k-1)$ -ary subtree on a Galton-Watson family tree given by Yanev and Mutafchiev (2006, p. 232). The idea of embedding a Poisson branching process in the random graph model was recently developed by Riordan (2007) who gave a new proof of the results of Pittel et al. (1996) and extended them to a general model of inhomogeneous random graphs with independence between their arcs.

## 2 Proofs of the Results

First, we recall Pakes and Dekking (1991) result: the probability  $\gamma_N$ , defined by (8), is the smallest solution in  $[0, 1]$  of eq. (10). To prove part (i) of the theorem, note that  $\gamma_N > 0$  implies that  $g_N(0) > 0$ . Therefore, for  $s \in [0, \gamma_N]$ , the graph of the function  $y = g_N(s)$  lies above the diagonal of the unit square in the coordinate system  $sOy$ . At  $s = \gamma_N$  the curve  $y = g_N(s)$  crosses or touches the diagonal  $y = s$ . If it touches it, then  $a_N = g'_N(\gamma_N) = 1$ . If  $y = g_N(s)$  crosses the diagonal, then, for some sufficiently small  $\epsilon \in (0, \gamma_N)$ , we have  $g_N(\gamma_N - \epsilon) \geq \gamma_N - \epsilon$  and  $g_N(\gamma_N + \epsilon) \leq \gamma_N + \epsilon$ . Hence  $g_N(\gamma_N + \epsilon) - g_N(\gamma_N - \epsilon) \leq 2\epsilon$ . Therefore the derivative

$$g'_N(s) = (1-s)^{N-1} f^{(N)}(s) / (N-1)!$$

should not exceed 1 for certain  $s = s_\epsilon \in (\gamma_N - \epsilon, \gamma_N + \epsilon)$ , by the mean value theorem. Letting  $\epsilon \rightarrow 0$  and using the continuity of  $g'_N(s)$ , we get assertion (i).

Assertion (ii) can be obtained using a result on iterations of functions due to Koenigs (1884) (see also Harris (1963, Ch. 1, Sect. 8.3)). Below we state a suitable modification of it as a separate lemma. The proof follows the same line of reasoning as in Harris (1963, Ch. 1, Thm. 8.4).

**Lemma 1** *Let*

$$h(s) = \sum_{j=0}^{\infty} h_j s^j$$

*( $h_j$  real) be a function, which is analytic in  $|s| < 1$ , strictly increasing in  $[0, 1]$  and such that  $h(1) = 1$ . Let*

$$h_0(s), h_1(s) = h(s), h_{t+1} = h(h_t(s)), t = 1, 2, \dots \quad (13)$$

*be the sequence of iterations of  $h(s)$ . Suppose that the equation*

$$s = h(s) \quad (14)$$

*has a solution in  $[0, 1]$  and let  $q$  be the least one in  $[0, 1]$ . If  $q$  satisfies  $h'(q) < 1$ , then*

$$h_t(0) = q - d[h'(q)]^t + O([h'(q)]^{2t})$$

*as  $t \rightarrow \infty$ , where  $d > 0$  denotes an absolute constant.*

We will apply Lemma 1 setting  $h(s) = g_N(s)$ . Define the iterations  $g_{N,t}(s)$  of the function  $g_N(s)$  as in (13). Also, recall that  $g_N(1) = 1$  and  $\gamma_{N,0} = 0$  (see definitions (11) and (7), respectively). We set  $q = \gamma_N$  in eq. (14). Then, we use the recurrence  $\gamma_{N,t} = g_N(\gamma_{N,t-1})$ ; see Yanev and Mutafchiev (2006, p. 227). Iterating  $t$  times as in (13), we get  $\gamma_{N,t} = P(V_{N,t} = 0) = g_{N,t}(0)$ . Hence, by Lemma 1 and notation (12<sub>1</sub>),

$$\gamma_{N,t} = \gamma_N - d'_N a_N^t + O(a_N^{2t})$$

as  $t \rightarrow \infty$ , where  $d'_N > 0$  denotes an absolute constant. Dividing both sides of this equality by  $\gamma_N$  and writing conditional probabilities for  $V_{N,t}$ , we obtain assertion (ii) with  $d_N = d'_N/\gamma_N$ .

To prove (iiia), let us assume that  $b_N \leq 0$  (see notation (12<sub>2</sub>)). This shows that  $g'_N(s)$  decreases in a neighborhood of  $s = \gamma_N$ . Hence, there exists a sufficiently small number  $\delta > 0$  such that, for any  $s \in (\gamma_N - \delta, \gamma_N]$ , we have  $g'_N(s) \geq g'_N(\gamma_N) = a_N = 1$ . Therefore,  $[g_N(s) - s]' \geq 0$ , and so, the function  $g_N(s) - s$  increases in  $(\gamma_N - \delta, \gamma_N]$ . Thus, for any  $s \in (\gamma_N - \delta, \gamma_N)$ , we have  $g_N(s) - s \leq g_N(\gamma_N) - \gamma_N = 0$ . Combining the inequalities  $g_N(s) \leq s$ ,  $g_N(0) > 0$  and using the continuity of  $g_N(s)$ , we conclude that there is some  $s_0 < \gamma_N$  that solves eq. (10). This contradicts the assumption that  $\gamma_N$  is the smallest solution in  $(0, 1]$  of eq. (10). So, (iiia) is proved.

The asymptotic given in assertion (iiib) will also follow from classical results on iterations of analytic functions, increasing on a segment of the real axis. One possible proof may use



a general result of Harris (1963, Ch. 1, Lemma 10.1) establishing uniform asymptotics for  $1/[1 - h_t(s)]$ , where  $h_t(s)$  denote the iterations defined by (13) and the complex variable  $s$  varies in some particular subsets of the unit disc. In our case it suffices, however, to consider the behavior of  $h_t(s)$  only at  $s = 0$ . The problem turns out to be similar to that for the critical branching process which was studied first by Kolmogorov (1938). The proof of the next lemma repeats the arguments given by Sevast'yanov (1971, Ch. 2, Sect. 2).

**Lemma 2** *Suppose that  $h(s), h_t(s), t = 0, 1, \dots$  and  $q$  are the same as in Lemma 1. Furthermore, suppose that  $h'(q) = 1$  and  $h''(q) \in (0, \infty)$ . Then, we have*

$$\frac{1}{q - h_t(0)} = \frac{th''(q)}{2} + o(t) \quad (15)$$

as  $t \rightarrow \infty$ .

To show how assertion (iiib) follows from this lemma we set in both sides of (15):  $q = \gamma_N, h(s) = g_N(s), h_t(0) = g_{N,t}(0) = \gamma_{N,t}$  and  $h''(q) = g''_N(\gamma_N) = b_N$ . Thus, we obtain

$$\frac{1}{\gamma_N - \gamma_{N,t}} = \frac{tb_N}{2} + o(t), t \rightarrow \infty. \quad (16)$$

To complete the proof of (iiib) it remains to take the reciprocal of (16), divide both sides by  $\gamma_N$  and convert the ratio  $\gamma_{N,t}/\gamma_N$  into conditional probability for  $V_{N,t}$ .

### 3 Numerical Results

*Geometric distribution.* We look at the case, where

$$f(s) = \frac{1 - ps}{1 - p}, g_N(s) = 1 - \left[ \frac{p(1 - s)}{1 - ps} \right]^N, 0 < p < 1.$$

Pakes and Dekking (1991) established in this case that the critical mean for  $N = 2$  is  $m_2^c = 4$  which implies that the critical value for the parameter  $p$  is  $p_2^c = 4/5$ . It is easy to see that the least solution in  $[0, 1]$  of eq. (10) is  $\gamma_2^c = 3/4$ . Calculating the first two derivatives of  $g_2(s)$  at  $s = 3/4$ , we get  $a_2^c = 1, b_2^c = 2$ . Therefore, by assertion (iiib) of Theorem 1,

$$P(V_{2,t} > 0 \mid V_2 = 0) \sim \frac{4}{3t}, t \rightarrow \infty.$$

*Poisson distribution.* The Poisson offspring distribution has the pgf  $f(s) = e^{m(s-1)}, m > 0$ . Whence

$$g_N(s) = e^{m(s-1)} \sum_{j=0}^{N-1} [(1-s)m]^j / j!.$$

In this case  $m_2^c = 3.3509$  and the least solution in  $[0, 1]$  of eq. (10) is  $\gamma_2^c = .4648$  (see Yanev and Mutafchiev (2006)). This implies that  $a_2^c = .1127$  and by Theorem 1(ii),

$$P(V_{2,t} > 0 \mid V_2 = 0) = d_{2po}^c (.1127)^t + O((.1127)^{2t}), t \rightarrow \infty,$$

where  $d_{2po} > 0$  is certain constant.

*One-or-many distribution.* This is a two-parameter family of discrete distributions defined for some  $p \in (0, 1)$  and integer  $r > N > 1$  by the equalities:  $p_r = p, p_1 = 1 - p$ . Clearly,  $f(s) = (1 - p)s + ps^r$ , and hence

$$g_N(s) = 1 - p \sum_{j=N}^r \binom{r}{j} (1 - s)^j s^{r-j}.$$

Pakes and Dekking (1991) showed that if  $r = N + 1$ , then  $\gamma_N^c = 1/N^2$  and the threshold value of the parameter  $p$  is  $p_N^c = (1 - 1/N)(1 - 1/N^2)^{-N}$ . For  $r = 3$  and  $N = 2$ , we have  $g_2'(s) = 6ps(1 - s)$ ,  $p_2^c = 8/9$ ,  $\gamma_2^c = 1/4$ . Thus, we get  $a_2^c = 3/4$ , and hence by Theorem 1(ii),

$$P(V_{2,t} > 0 \mid V_2 = 0) = d_{2oo3}^c (3/4)^t + O((3/4)^{2t}), t \rightarrow \infty,$$

where  $d_{2oo3} > 0$  is again certain constant.

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