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# Stochastic Approximation of the Factors of a Generalized Canonical Correlation Analysis

by

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## Abstract

We suppose that data of a generalized canonical correlation analysis are i.i.d. observations of a random vector  $Z$  which are taken sequentially. We define a recursive method of sequential estimation of the factors. This can be applied also when there is a great amount of non random data vectors.

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**Key Words and Phrases:** Stochastic approximation, Canonical correlation analysis

**AMS 1991 Subject Classifications. Primary:** 62; **Secondary:** L20, H20.

## 1 Introduction

Consider the generalized canonical correlation analysis (gCCA) as defined by Carroll (1968). Suppose that data are i.i.d. observations of a random vector  $Z$  which are taken sequentially : at time  $n$ ,  $Z_n$  is observed. We define a recursive method of sequential estimation of the general and canonical factors of gCCA by using a stochastic approximation process derived from that of Krasulina (1970), whose convergence has been studied by Bouamaine and Monnez (1998). This method can be used also when there is a great amount of non random data vectors: at time  $n$ , a data vector is randomly chosen.

We present gCCA in the second section, stochastic approximation of the parameter of a linear regression function in the third and stochastic approximation of the factors in the fourth. In the following,  $\rho$  denotes a linear correlation coefficient,  $A'$  the transposed matrix of  $A$ ,  $\lambda_{\min}(A)$  the smallest eigenvalue of  $A$ ,  $\mathbb{R}^{p*}$  the dual space of  $\mathbb{R}^p$ . The abbreviation *a.s.* means almost surely.

## 2 Generalized canonical correlation analysis of a random vector

Consider two random vectors  $Z^1$  and  $Z^2$  defined on the same probability space and suppose that there is no affine relation between the components of  $Z = (Z^1, Z^2)$ . The canonical correlation analysis (CCA) consists in determining at step  $l \in \{1, \dots, r\}$  for  $k = 1, 2$  a linear combination of variance 1 of the centered components of  $Z^k$ ,

$V_l^k = (\eta_l^k)'(Z^k - E[Z^k])$ , uncorrelated with  $V_{l-1}^k, \dots, V_1^k$ , which maximizes  $\rho(V_l^1, V_l^2)$ .

This analysis permits one to put into relation two sets of variables or to explain one set of variables by the other.

A generalization of CCA to three or more sets of variables is gCCA. As it will be seen, gCCA can be interpreted as a principal component analysis (PCA) of a random vector  $Z = (Z^1, \dots, Z^q)$  in  $\mathbb{R}^p$ ,  $Z^1, \dots, Z^q$  being  $q$  random vectors which take in some sense a similar part in the determination of the factors.

More precisely, suppose that the set of components of a random vector  $Z$  in  $\mathbb{R}^p$  is divided into  $q$  sets of real random variables  $\{Z^{k1}, \dots, Z^{km_k}\}$ ,  $k = 1, \dots, q$ ,  $\sum_{k=1}^q m_k = p$ . Denote  $Z^k$  the random vector in  $\mathbb{R}^{m_k}$  whose components are  $Z^{k1}, \dots, Z^{km_k}$ . Suppose that there is no affine relation between the components of  $Z$ .

Consider the following problem : for  $l = 1, \dots, r$ , determine at step  $l$  a linear combination of all the centered variables  $Z^{kj}$ ,  $U_l = (\theta_l)'(Z - E[Z])$ ,  $\theta_l \in \mathbb{R}^{p^*}$ , named  $l^{th}$  general component, which is of variance 1 and uncorrelated with the preceding ones, and for  $k = 1, \dots, q$ , a linear combination of variance 1 of the centered variables of the  $k^{th}$  set,  $V_l^k = (\eta_l^k)'(Z^k - E[Z^k])$ ,  $\eta_l^k \in \mathbb{R}^{m_k^*}$ , named  $l^{th}$  canonical component of the  $k^{th}$  set, which maximize  $\sum_{k=1}^q \rho^2(U_l, V_l^k)$ .  $\theta_l$  is named the  $l^{th}$  general factor and  $\eta_l^k$  the  $l^{th}$  canonical factor of the  $k^{th}$  set.

Remark that when  $m_k = 1$  for  $k = 1, \dots, p$  ( $q = p$ ), this analysis is equivalent to the normed principal component analysis of  $Z$  i.e. PCA of  $Z$  with the inner product

in  $\mathbb{R}^p$  defined by the weight matrix

$$\begin{pmatrix} \frac{1}{(\sigma^1)^2} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \frac{1}{(\sigma^p)^2} \end{pmatrix}, (\sigma^k)^2 = E \left[ (Z^k - E[Z^k])^2 \right].$$

In this PCA, the random variables  $Z^1, \dots, Z^p$  play in some sense a similar part in the definition of the squared distance in  $\mathbb{R}^p$  between two data vectors  $Z(\omega_1)$  and  $Z(\omega_2)$  :

$$d^2(Z(\omega_1), Z(\omega_2)) = \sum_{k=1}^p \left( \frac{Z^k(\omega_1) - Z^k(\omega_2)}{\sigma^k} \right)^2.$$

In the general case, denote  $C^k$  the covariance matrix of  $Z^k$ ,  $C$  that of  $Z$  and define the covariance matrices

$$C^{kl} = E \left[ (Z^k - E[Z^k])(Z^l - E[Z^l])' \right]$$

$$C^{k\cdot} = E \left[ (Z^k - E[Z^k])(Z - E[Z])' \right] = (C^{k1} \dots C^{kq}).$$

Let  $M$  be the block diagonal matrix

$$M = \begin{pmatrix} (C^1)^{-1} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & (C^q)^{-1} \end{pmatrix}.$$

Let  $\theta_l = (\theta_l^1, \dots, \theta_l^q)'$ ,  $\theta_l^k \in \mathbb{R}^{m_k}$  for  $k = 1, \dots, q$ . Consider the matrix  $B = MC$  which can be divided into  $q^2$  blocks, the block  $(k, l)$  being the  $(m_k, m_l)$  matrix  $(C^k)^{-1}C^{kl}$ .

**Theorem 1**  $\theta_l$  is a  $C$ -normed eigenvector of the matrix  $B = MC$  corresponding to the  $l^{\text{th}}$  greatest eigenvalue  $\lambda_l$  and for  $k = 1, \dots, q$ , there exists  $\alpha_l^k \in \mathbb{R}$  such that  $\eta_l^k = \alpha_l^k \theta_l^k$ .

**Proof.** Assume  $U_l$  to be determined. Then :  $\sum_{k=1}^q \rho^2(U_l, V_l^k) \max \iff \rho^2(U_l, V_l^k) \max$ ,  $k = 1, \dots, q$ . Thus  $V_l^k$  is obtained by making the linear regression of  $U_l$  with respect to  $Z^k$ . Therefore there exists  $\beta_l^k \in \mathbb{R}$  such that

$$\begin{aligned} V_l^k &= \beta_l^k \Pi^k U_l, & \Pi^k U_l &= (Z^k - E[Z^k])' b_l^k, \\ b_l^k &= (C^k)^{-1} E[(Z^k - E[Z^k]) U_l] = (C^k)^{-1} C^k \cdot \theta_l. \\ \sum_{k=1}^q \rho^2(U_l, V_l^k) &= \sum_{k=1}^q E[U_l \Pi^k U_l] = \sum_{k=1}^q (\theta_l)' (C^k)' (C^k)^{-1} C^k \cdot \theta_l = (\theta_l)' C M C \theta_l. \end{aligned}$$

$\theta_l$  maximizes  $(\theta_l)' C M C \theta_l$  under the constraints  $\theta_l' C \theta_j = 0$ ,  $j = 1, 2, \dots, l-1$  and  $\theta_l' C \theta_l = 1$ . Then  $\theta_l$  is eigenvector of  $MC$  corresponding to the eigenvalue  $\lambda_l$  and  $\eta_l^k = \beta_l^k b_l^k = \beta_l^k (C^k)^{-1} C^k \cdot \theta_l = \beta_l^k \lambda_l \theta_l^k$ . ■

Remark that  $\theta_l$  is the  $l^{\text{th}}$  factor of the principal component analysis of  $Z$ , the inner product in  $\mathbb{R}^p$  being defined by  $M$ . In this PCA, the random vectors  $Z^1, \dots, Z^q$  play in some sense a similar part in the definition of the squared distance in  $\mathbb{R}^p$  between

two data vectors  $Z(\omega_1)$  and  $Z(\omega_2)$  :

$$d^2(Z(\omega_1), Z(\omega_2)) = \sum_{k=1}^q (Z^k(\omega_1) - Z^k(\omega_2))' (C^k)^{-1} (Z^k(\omega_1) - Z^k(\omega_2)).$$

Remark too that this analysis in the case  $q = 2$  is the usual canonical correlation analysis.  $\theta_l = \begin{pmatrix} \theta_l^1 \\ \theta_l^2 \end{pmatrix}$  is eigenvector of  $\begin{pmatrix} I & (C^1)^{-1}C^{12} \\ (C^2)^{-1}C^{21} & I \end{pmatrix}$ ;  $\theta_l^k$  is colinear to the  $l^{\text{th}}$  canonical factor in  $Z^k$  for  $k = 1, 2$ .

### 3 Stochastic approximation of a linear regression function

#### 3.1 Proposition

Let  $R$  be a random vector in  $\mathbb{R}^r$  and suppose :

(H1) *There is no affine relation between the components of  $R$ .*

Let  $S$  be a random vector in  $\mathbb{R}^s$ . Denote  $CovarR$  the covariance matrix of  $R$  and  $Covar(R, S) = E[(R - E[R])(S - E[S])']$ . Let  $A$  be a  $(r, s)$  matrix and  $D$  a  $(s, 1)$  matrix.  $(CovarR)^{-1}Covar(R, S)$  is the solution in  $A$  of the following linear regression problem : find  $(A, D)$  such that  $E[\|S - A'R - D\|^2]$  be minimal,  $\|\cdot\|$  being the usual euclidean norm.

Denote  $A_1$  the  $(r+1, s)$  matrix such that  $A_1' = \begin{pmatrix} A' & D \end{pmatrix}$  and  $R_1$  the random vector in  $\mathbb{R}^{r+1}$  such that  $R_1' = \begin{pmatrix} R' & 1 \end{pmatrix}$ .  $E[R_1R_1']$  is positive definite by H1.  $(CovarR)^{-1}Covar(R, S)$  can be obtained as it follows.



**Proposition 2**  $A = (\text{Covar}(R))^{-1}\text{Covar}(R, S)$  and  $D = E[S] - A'E[R]$  form the solution of the linear system  $E[R_1 R_1'] A_1 = E[R_1 S']$ .

### 3.2 Stochastic approximation of $A_1$

Consider an i.i.d sample  $((R_n, S_n))$  of  $(R, S)$ . Denote for all  $n$   $R_{1n}$  the random vector in  $\mathbb{R}^{r+1}$  such that  $R_{1n}' = \begin{pmatrix} R_n' & 1 \end{pmatrix}$ . Consider the stochastic approximation process  $(A_{1n}, n \geq 1)$  of  $A_1 = (E[R_1 R_1'])^{-1} E[R_1 S']$  in the set of  $(r+1, s)$  matrices such that

$$A_{1,n+1} = A_{1n} - a_n(R_{1n}R_{1n}'A_{1n} - R_{1n}S_n'), \quad a_n \in \mathbb{R}^{+*}.$$

Denote  $A_{1n} = \begin{pmatrix} A_n \\ D_n \end{pmatrix}$ ,  $A_n$  and  $D_n$  being respectively a  $(r, s)$  and a  $(s, 1)$  random matrix.  $(A_n, n \geq 1)$  is a stochastic approximation process of

$$A = (\text{Covar}(R))^{-1}\text{Covar}(R, S).$$

We use in the proof of the convergence the Hilbert-Schmidt inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$  in the set of  $(r+1, s)$  matrices :

$$\langle A, B \rangle = \text{Trace}(A'B) = \sum_{i=1}^{r+1} \sum_{j=1}^s A(i, j)B(i, j).$$

Suppose :

(H2)  $(R, S)$  has 4<sup>th</sup> order moments.

(H3)  $a_n > 0$ ,  $\sum_1^\infty a_n = \infty$ ,  $\sum_1^\infty a_n^2 < \infty$ .

(H3')  $\left( a_n = \frac{a}{n^\alpha}, \quad a > 0, \quad \frac{2}{3} < \alpha < 1 \right)$  or  $\left( a_n = \frac{a}{n}, \quad a > \frac{1}{2\lambda_{\min}(E[R_1 R_1'])} \right)$ .

**Theorem 3** Assume H1, H2, H3 ; then  $\|A_{1n} - A_1\|$  and  $\|A_n - A\|$  converge to 0 a.s. and in  $L^2$  ; if H3 is replaced by H3', then moreover  $\sum_1^\infty a_n \|A_{1n} - A_1\| < \infty$  and  $\sum_1^\infty a_n \|A_n - A\| < \infty$  a.s.

**Proof.** Denote  $B_n = R_{1n}R'_{1n}$ ,  $F_n = R_{1n}S'_n$ ,  $B = E[R_1R'_1]$ ,  $F = E[R_1S']$ . As  $A_1 = B^{-1}F$ , we have :

$$\begin{aligned} A_{1,n+1} - A_1 &= A_{1n} - A_1 - a_n B(A_{1n} - A_1) - a_n U_n, \\ U_n &= (B_n - B)A_{1n} + F - F_n. \\ \|A_{1,n+1} - A_1\|^2 &= \|A_{1n} - A_1\|^2 + a_n^2 \|B(A_{1n} - A_1)\|^2 + a_n^2 \|U_n\|^2 \\ &\quad - 2a_n \langle A_{1n} - A_1, B(A_{1n} - A_1) \rangle - 2a_n \langle A_{1n} - A_1, U_n \rangle \\ &\quad + 2a_n^2 \langle B(A_{1n} - A_1), U_n \rangle. \end{aligned}$$

Denote  $\mathcal{T}_n$  the  $\sigma$ -field generated by  $(R_j, S_j)$ ,  $j < n$  and  $A_{11}$ . As  $E(U_n | \mathcal{T}_n) = 0$ , we have a.s. :

$$\begin{aligned} E(\|A_{1,n+1} - A_1\|^2 | \mathcal{T}_n) &\leq \|A_{1n} - A_1\|^2 (1 + a_n^2 \|B\|^2) + a_n^2 E(\|U_n\|^2 | \mathcal{T}_n) \\ &\quad - 2a_n \langle A_{1n} - A_1, B(A_{1n} - A_1) \rangle. \end{aligned}$$

By H2, there exist  $c$  and  $d > 0$  such that :

$$E(\|U_n\|^2 | \mathcal{T}_n) = E(\|(B_n - B)A_{1n} + (F - F_n)\|^2 | \mathcal{T}_n) \leq c \|A_{1n} - A_1\|^2 + d.$$

Denote  $M^{(j)}$  the  $j^{\text{th}}$  column of a matrix  $M$  and  $\lambda$  the smallest eigenvalue of  $B$ .

Then :

$$\begin{aligned} \langle A_{1n} - A_1, B(A_{1n} - A_1) \rangle &= \sum_{j=1}^s \langle A_{1n}^{(j)} - A_1^{(j)}, B(A_{1n}^{(j)} - A_1^{(j)}) \rangle \\ &\geq \lambda \sum_{j=1}^s \|A_{1n}^{(j)} - A_1^{(j)}\|^2 = \lambda \|A_{1n} - A_1\|^2. \end{aligned}$$

Therefore we have *a.s.* :

$$E(\|A_{1,n+1} - A_1\|^2 | \mathcal{T}_n) \leq (1 + \|B\|^2 a_n^2 + ca_n^2) \|A_{1n} - A_1\|^2 + da_n^2 - 2\lambda a_n \|A_{1n} - A_1\|^2.$$

By H3, applying the Robbins-Siegmund lemma (1971) gives the *a.s.* convergence of  $\|A_{1n} - A_1\|$  and of  $\sum_1^\infty a_n \|A_{1n} - A_1\|^2$ . As  $\sum_1^\infty a_n = \infty$ , we have  $\|A_{1n} - A_1\| \rightarrow 0$  *a.s.* Taking the expectation of  $E(\|A_{1,n+1} - A_1\|^2 | \mathcal{T}_n)$  gives in the same way

$$E[\|A_{1n} - A_1\|^2] \rightarrow 0.$$

Thus there exist  $N \in \mathbb{N}$  and  $f > 0$  such that for  $n > N$  :

$$E[\|A_{1,n+1} - A_1\|^2] \leq (1 - 2\lambda a_n) E[\|A_{1n} - A_1\|^2] + fa_n^2.$$

Applying for instance lemmas 2 and 3 in Monnez (1992) gives for  $a_n = \frac{a}{n^\alpha}$  with  $0 < \alpha < 1$ , or  $\alpha = 1$  and  $a > \frac{1}{2\lambda}$ ,  $\overline{\lim} \frac{1}{a_n} E[\|A_{1n} - A_1\|^2] < \infty$ . Hence there exists  $g > 0$  such that

$$\sum_1^\infty a_n E[\|A_{1n} - A_1\|] \leq g \sum_1^\infty a_n^{\frac{3}{2}} < \infty$$

for  $\alpha > \frac{2}{3}$ ; it follows that  $\sum_1^\infty a_n \|A_{1n} - A_1\| < \infty$  *a.s.* ■

## 4 Stochastic approximation of general factors

### 4.1 Definition of the process

Let  $((Z_n^1, \dots, Z_n^q), n \geq 1)$  be an i.i.d. sample of the random vector  $Z = (Z^1, \dots, Z^q)$  in  $\mathbb{R}^p$ . For  $k = 1, \dots, q$  and  $n \geq 1$ , let

$$\overline{Z}_n^k = \frac{1}{n} \sum_{i=1}^n Z_i^k, \quad M_n^k = \frac{1}{n} \sum_{i=1}^n Z_i^k Z_i^{k'} - \overline{Z}_n^k \left( \overline{Z}_n^k \right)';$$

$M_n^k$  is a consistent estimator of the covariance matrix  $C^k$  of  $Z^k$ . For  $n \geq 1$ , define the block-diagonal  $(p, p)$  matrix

$$M_n = \begin{pmatrix} M_n^1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & M_n^q \end{pmatrix};$$

$M_n$  is a consistent estimator of  $M^{-1}$ . Denote  $\langle \cdot, \cdot \rangle_n$  the inner product defined by  $M_n$  in the dual space  $\mathbb{R}^{p*}$  of  $\mathbb{R}^p$  and  $\|\cdot\|_n$  the associated norm.

In order to estimate  $B^{ij} = (C^i)^{-1} C^{ij}$  for  $i = 1, \dots, q$  and  $j \neq i$ , define the stochastic approximation processes  $(B_{1n}^{ij})$  in the set of  $(m_i + 1, m_j)$  matrices in the same way as in paragraph 3.2,  $(F_n^{ij})$  in the set of  $(m_j, 1)$  matrices and  $(B_n^{ij})$  in the set of  $(m_i, m_j)$

matrices :

$$B_{1,n+1}^{ij} = B_{1n}^{ij} - a_n (Z_{1n}^i (Z_{1n}^i)' B_{1n}^{ij} - Z_{1n}^i (Z_n^j)'),$$

$$Z_{1n}^i = \begin{pmatrix} Z_n^i \\ 1 \end{pmatrix}, B_{1n}^{ij} = \begin{pmatrix} B_n^{ij} \\ (F_n^{ij})' \end{pmatrix}.$$

Define in the set of  $(p, p)$  matrices the stochastic approximation process  $(B_n)$  of the matrix  $B = MC$  such that the block  $(i, j)$  of  $B_n$  is  $B_n^{ij}$ :

$$B_n = \begin{pmatrix} I & B_n^{12} & \cdot & \cdot & B_n^{1q} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ B_n^{q1} & B_n^{q2} & \cdot & \cdot & I \end{pmatrix}.$$

In order to estimate  $r$  general factors of gCCA, define for  $l = 1, 2, \dots, r$ , the Krasulina-type stochastic approximation process  $(X_n^l)$  of the general factor  $\theta_l$  in  $\mathbb{R}^{p*}$  (Krasulina, 1970, Bouamaine and Monnez, 1998) :

$$X_{n+1}^l = \text{orth}_{M_n}(Y_{n+1}^l)$$

$$Y_{n+1}^l = X_n^l + a_n (B_n - F_n(X_n^l)I) X_n^l$$

$$F_n(X_n^l) = \frac{\langle B_n X_n^l, X_n^l \rangle_{n-1}}{\|X_n^l\|_{n-1}^2};$$

$X_{n+1}^l = \text{orth}_{M_n}(Y_{n+1}^l)$  means that  $(X_{n+1}^1, \dots, X_{n+1}^l)$  is obtained by a Gram-Schmidt orthogonalization with respect to  $M_n$  of  $(Y_{n+1}^1, \dots, Y_{n+1}^l)$ .

## 4.2 Principle of the study of convergence

The details of the proof of convergence of this type of process are given by Bouamaine and Monnez (1997, 1998).

Denote  $\wedge$  the exterior product of elements of  $\mathbb{R}^{p^*}$  and for  $j = 1, \dots, p$ ,  ${}^j\wedge\mathbb{R}^{p^*}$  the exterior power of order  $j$  of  $\mathbb{R}^{p^*}$  :  $(e_1, \dots, e_p)$  being a basis of  $\mathbb{R}^{p^*}$ , the set of  $C_p^j$  exterior products  $e_{i_1} \wedge \dots \wedge e_{i_j}$  with  $1 \leq i_1 < \dots < i_j \leq p$  is a basis of  ${}^j\wedge\mathbb{R}^{p^*}$ .

Define the inner product in  ${}^j\wedge\mathbb{R}^{p^*}$  derived from that in  $\mathbb{R}^{p^*}$  defined by  $M^{-1}$  ; in this definition  $G_j$  is the set of permutations of  $\{k_1, \dots, k_j\}$ ,  $s(\sigma)$  is the number of inversions in the permutation  $\sigma$  and  $\varepsilon(\sigma) = (-1)^{s(\sigma)}$  :

$$\langle e_{i_1} \wedge \dots \wedge e_{i_j}, e_{k_1} \wedge \dots \wedge e_{k_j} \rangle = \sum_{\sigma \in G_j} \varepsilon(\sigma) \langle e_{i_1}, e_{\sigma(k_1)} \rangle_{M^{-1}} \dots \langle e_{i_j}, e_{\sigma(k_j)} \rangle_{M^{-1}}.$$

Suppose that the  $r$  greatest eigenvalues of the endomorphism  $B = MC$  in  $\mathbb{R}^{p^*}$  are different :  $\lambda_1 > \dots > \lambda_r$ . Define for  $j = 1, \dots, r$ , the endomorphism  ${}^jB$  in  ${}^j\wedge\mathbb{R}^{p^*}$  by :

$${}^jB(x^1 \wedge \dots \wedge x^j) = \sum_{h=1}^j x^1 \wedge \dots \wedge Bx^h \wedge \dots \wedge x^j, x^l \in \mathbb{R}^{p^*}, l = 1, \dots, j.$$

If  $V^1, \dots, V^j$  are eigenvectors of  $B$  corresponding respectively to  $\lambda_1, \dots, \lambda_j$ ,  $V^1 \wedge \dots \wedge V^j$  is eigenvector of  ${}^jB$  corresponding to the greatest eigenvalue  $\lambda_{1j} = \sum_{l=1}^j \lambda_l$ . Denote  ${}^jS_1$  the eigensubspace corresponding to  $\lambda_{1j}$  and  $({}^jS_1)^\perp$  its orthogonal.

The proof of convergence is made in two steps. Denote  ${}^jX_n = X_n^1 \wedge \dots \wedge X_n^j$ . It is proved first that for  $j = 1, \dots, r$ ,  $\frac{{}^jX_n}{\|{}^jX_n\|}$  converges *a.s.* in a set  ${}^jE$  to  $V^1 \wedge \dots \wedge V^j \in {}^jS_1$ .

It is proved then that, for  $l = 1, \dots, r$ ,  $\frac{X_n^l}{\|X_n^l\|}$  converges *a.s.* in  $\cap_{j=1}^l {}^j E$  to  $V^l$ . The definition of the convergence set  ${}^j E$  is as follows.

In the deterministic case  $B_n = B$ ,  $M_n = M^{-1}$ , define

$$h_j({}^j x) = \frac{\langle {}^{j1} B {}^j x, {}^j x \rangle}{\langle {}^j x, {}^j x \rangle}, {}^j x \in {}^j \wedge \mathbb{R}^{p^*},$$

and the process  $({}^j U_n)$  in  ${}^j \wedge \mathbb{R}^{p^*}$  by

$${}^j U_{n+1} = (I + a_n ({}^{j1} B - h_j({}^j U_n) I)) {}^j U_n.$$

In this case  ${}^j E$  is the set  $\{ {}^j X_1 \notin ({}^j S_1)^\perp \}$ ;  $X_1^1, \dots, X_1^j$  may not be orthogonal to the subspace generated by the eigenvectors of  $B$  corresponding to its  $j$  greatest eigenvalues.

The process  $({}^j X_n) = (X_n^1 \wedge \dots \wedge X_n^j)$  can be considered as resulting from a stochastic perturbation of  $({}^j U_n)$ ; denote

$$\begin{aligned} \Delta_{nj} &= 1 + a_n (\lambda_{1j} - h_j({}^j X_n)) \\ Q^j &= {}^j X_1 + \sum_{n=1}^{\infty} \frac{{}^j X_{n+1} - (I + a_n ({}^{j1} B - h_j({}^j X_n) I)) {}^j X_n}{\prod_{i=1}^n \Delta_{ij}}. \end{aligned}$$

Then  ${}^j E = \{ Q^j \notin ({}^j S_1)^\perp \}$ .

Remark that  $Q^j = {}^j X_1$  in the case of the deterministic process  $({}^j U_n)$ . In the case of the stochastic process  $({}^j X_n)$ , remark that we have  $P({}^j E) = 1$  under general assumptions on the law of  $Z$  if  ${}^j X_1$  is absolutely continuous.

### 4.3 Convergence of the process

Suppose :

(H1') *There is no affine relation between the components of  $Z$ .*

(H2')  *$Z$  has 4<sup>th</sup> order moments.*

(H3'')  $(a_n = \frac{a}{n^\alpha}, a > 0, \frac{2}{3} < \alpha < 1)$  or  $(a_n = \frac{a}{n}, a > \frac{1}{2 \min_i \lambda_{\min} E[Z_1^i Z_1^{i'}]}, Z_1^i = \begin{pmatrix} Z^i \\ 1 \end{pmatrix})$ .

**Theorem 4** *Assume H1', H2', H3'' ; then, for  $l = 1, \dots, r$ ,  $Q^l$  converges a.s. Suppose moreover that the  $r$  greatest eigenvalues of  $B$  are different; then, for  $l = 1, \dots, r$ ,  $X_n^l$  converges a.s. in  $\cap_{j=1}^l E^j$  to a random eigenvector of  $B$  corresponding to its  $l^{\text{th}}$  greatest eigenvalue.*

**Proof.** The convergence of this type of process is proved in theorem 4 of Bouamaine and Monnez (1998, p. 24) under the following assumptions :

- (1)  $\sum_1^\infty a_n \|B_n - B\| < \infty$  a.s.
- (2)  $M_n \longrightarrow M^{-1}, \sum_1^\infty a_n \|M_{n-1} - M^{-1}\| < \infty$  a.s.
- (3)  $a_n > 0, \sum_1^\infty a_n = \infty, \sum_1^\infty a_n^2 < \infty$ .

It suffices to verify (1) and (2).

By H1', H2', H3'' and theorem 3,  $\sum_{n=1}^\infty a_n \|B_n^{ij} - B^{ij}\| < \infty$  a.s. for every pair



$(i, j)$  ; then (1) is verified. For  $k = 1, \dots, q$  :

$$\|M_{n-1}^k - C^k\| \leq \left\| \frac{1}{n-1} \sum_{i=1}^{n-1} Z_i^k Z_i^{k'} - E[Z^k Z^{k'}] \right\| + \|\bar{Z}_n^k - E[Z^k]\| \left( \|\bar{Z}_n^k\| + \|E[Z^k]\| \right)$$

$$\bar{Z}_n^k \xrightarrow{a.s.} E[Z^k]; \sum_{n=1}^{\infty} a_n E \left[ \|\bar{Z}_n^k - E[Z^k]\| \right] = \sum_{n=1}^{\infty} a_n O \left( \frac{1}{\sqrt{n}} \right) < \infty.$$

Then :  $\sum_{n=1}^{\infty} a_n \|M_{n-1}^k - C^k\| < \infty$  for  $k = 1, \dots, q$ ,  $\sum_{n=1}^{\infty} a_n \|M_{n-1} - M^{-1}\| < \infty$  *a.s.* (2) is verified. ■

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