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Sébastien Palcoux

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# Neveu-Schwarz and operators algebras I

## *Vertex operators superalgebras*

Sébastien Palcoux

### Abstract

This paper is the first of a series giving a self-contained way from the Neveu-Schwarz algebra to a new series of irreducible subfactors. Here we present an elementary, progressive and self-contained approach to vertex operator superalgebra. We then build such a structure from the loop algebra  $L\mathfrak{g}$  of any simple finite dimensional Lie algebra  $\mathfrak{g}$ . The Neveu-Schwarz algebra  $\mathfrak{Vir}_{1/2}$  emerges naturally on. As application, we obtain unitary actions of  $\mathfrak{Vir}_{1/2}$  on the unitary series of  $L\mathfrak{g}$ .

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## 1 Introduction

### 1.1 Background of the series

In the 90's, V. Jones and A. Wassermann started a program whose goal is to understand the unitary conformal field theory from the point of view of operator algebras (see [5], [16]). In [17], Wassermann defines and computes

the Connes fusion of the irreducible positive energy representations of the loop group  $LSU(n)$  at fixed level  $\ell$ , using primary fields, and with consequences in the theory of subfactors. In [14] V. Toledano Laredo proves the Connes fusion rules for  $LSpin(2n)$  using similar methods. Now, let  $\text{Diff}(\mathbb{S}^1)$  be the diffeomorphism group on the circle, its Lie algebra is the Witt algebra  $\mathfrak{W}$  generated by  $d_n$  ( $n \in \mathbb{Z}$ ), with  $[d_m, d_n] = (m - n)d_{m+n}$ . It admits a unique central extension called the Virasoro algebra  $\mathfrak{Vir}$ . Its unitary positive energy representation theory and the character formulas can be deduced by a so-called Goddard-Kent-Olive (GKO) coset construction from the theory of  $LSU(2)$  and the Kac-Weyl formulas (see [18], [2]). In [10], T. Loke uses the coset construction to compute the Connes fusion for  $\mathfrak{Vir}$ . Now, the Witt algebra admits two supersymmetric extensions  $\mathfrak{W}_0$  and  $\mathfrak{W}_{1/2}$  with central extensions called the Ramond and the Neveu-Schwarz algebras, noted  $\mathfrak{Vir}_0$  and  $\mathfrak{Vir}_{1/2}$ .

In this series (this paper, [11] and [12] ), we naturally introduce  $\mathfrak{Vir}_{1/2}$  in the vertex superalgebra context of  $L\mathfrak{sl}_2$ , we give a complete proof of the classification of its unitary positive energy representations, we obtain directly their character; then we give the Connes fusion rules, and an irreducible finite depth type  $\text{II}_1$  subfactors for each representation of the discrete series. Note that we could do the same for the Ramond algebra  $\mathfrak{Vir}_0$ , using twisted vertex module over the vertex operator algebra of the Neveu-Schwarz algebra  $\mathfrak{Vir}_{1/2}$ , as R. W. Verrill [15] and Wassermann [19] do for twisted loop groups.

## 1.2 Overview of the paper

First, we look unitary, projective, positive energy representations of  $\mathfrak{W}_{1/2}$ . The projectivity gives 2-cocycles, so that  $\mathfrak{W}_{1/2}$  admits a unique central extension  $\mathfrak{Vir}_{1/2}$ . Such representations are completely reducible, and the irreducibles are given by the unitary highest weight representations of  $\mathfrak{Vir}_{1/2}$ : Verma modules  $V(c, h)$  quotiented by null vectors, in no-ghost cases.

From the fermion algebra on  $H = \mathcal{F}_{NS}$ , we build the fermion field  $\psi(z)$ . Locality and Dong's lemma permit, via OPE, to generate a set of fields  $\mathcal{S}$ , so that there is a  $1 - 1$  map  $V : H \rightarrow \mathcal{S}$ , with  $Id = V(\Omega)$  and a Virasoro field  $L = V(\omega)$ . Then, we give vertex operator superalgebra's axioms, permitting to come so far, in a general framework  $(H, V, \Omega, \omega)$ , with  $H$  prehilbert.

Let  $\mathfrak{g}$  a simple finite-dimensional Lie algebra,  $\widehat{\mathfrak{g}}_+$  the  $\mathfrak{g}$ -boson algebra (central extension of the loop algebra  $L\mathfrak{g}$ ) and  $\widehat{\mathfrak{g}}_-$  the  $\mathfrak{g}$ -fermion algebra.

We build a module vertex operator superalgebra from  $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \ltimes \widehat{\mathfrak{g}}_-$  on  $H = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ , so that  $\mathfrak{Vir}_{1/2}$  acts on with  $(c, h) = (\frac{3}{2} \cdot \frac{\ell+1/3g}{\ell+g} \dim(\mathfrak{g}), \frac{c_{V_\lambda}}{2(\ell+g)})$ , with  $g$  the dual Coxeter number and  $c_{V_\lambda}$  the Casimir number.

### 1.3 The Neveu-Schwarz algebra

We start with  $\mathfrak{W}_{1/2}$ , the Witt superalgebra of sector (NS):

$$\begin{cases} [d_m, d_n] &= (m-n)d_{m+n} & m, n \in \mathbb{Z} \\ [\gamma_m, d_n] &= (m - \frac{n}{2})\gamma_{m+n} & m \in \mathbb{Z} + \frac{1}{2}, n \in \mathbb{Z} \\ [\gamma_m, \gamma_n]_+ &= 2d_{m+n} & m, n \in \mathbb{Z} + \frac{1}{2} \end{cases}$$

together with  $d_n^* = d_{-n}$  and  $\gamma_m^* = \gamma_{-m}$ ; we study representations which are:

(a) Unitary:  $\pi(A)^* = \pi(A^*)$

(b) Projective:  $A \mapsto \pi(A)$  is linear and  $[\pi(A), \pi(B)] - \pi([A, B]) \in \mathbb{C}$ .

(c) Positive energy :  $H$  admits an orthogonal decomposition  $H = \bigoplus_{n \in \frac{1}{2}\mathbb{N}} H_n$  such that  $\exists D$  acting on  $H_n$  as multiplication by  $n$ ,  $H_0 \neq \{0\}$ ,  $\dim(H_n) < +\infty$ . Here,  $\exists h \in \mathbb{C}$  such that  $D = \pi(d_0) - hI$ .

Now, the projectivity gives the 2-cocycles and we see that  $H_2(\mathfrak{W}_{1/2}, \mathbb{C})$  is 1-dimensional,  $\mathfrak{W}_{1/2}$  admits a unique central extension up to equivalence:

$$0 \rightarrow H_2(\mathfrak{W}_{1/2}, \mathbb{C}) \rightarrow \mathfrak{Vir}_{1/2} \rightarrow \mathfrak{W}_{1/2} \rightarrow 0$$

$\mathfrak{Vir}_{1/2}$  is the SuperVirasoro (of sector NS) or Neveu-Schwarz algebra:

$$\begin{cases} [L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n} \\ [G_m, L_n] &= (m - \frac{n}{2})G_{m+n} \\ [G_m, G_n]_+ &= 2L_{m+n} + \frac{C}{3}(m^2 - \frac{1}{4})\delta_{m+n} \end{cases}$$

with  $L_n^* = L_{-n}$ ,  $G_m^* = G_{-m}$  and  $C = cI$ ,  $c \in \mathbb{C}$  called the **central charge**.

The representations are completely reducible, the irreducibles are determined by the two numbers  $c, h$ , and are completely given by unitary highest weight representations of  $\mathfrak{Vir}_{1/2}$ , described as follows: The Verma modules  $H = V(c, h)$  are freely generated by:  $0 \neq \Omega \in H$  (cyclic vector),  $C\Omega = c\Omega$ ,  $L_0\Omega = h\Omega$  and  $\mathfrak{Vir}_{1/2}^+\Omega = \{0\}$ . Now,  $(\Omega, \Omega) = 1$ ,  $\pi(A)^* = \pi(A^*)$  and  $(u, v) = \overline{(v, u)}$  give the sesquilinear form  $(\cdot, \cdot)$ .  $V(c, h)$  can admit ghost:  $(u, u) < 0$ , and null vectors:  $(u, u) = 0$ . In no ghost case, the set of null vectors is  $K(c, h)$  the kernel of  $(\cdot, \cdot)$ , the maximal proper submodule.

Let  $L(c, h) = V(c, h)/K(c, h)$ , the unitary highest weight representations.

Theorem 1.2 of [11] will be proved classifying no ghost cases.

## 1.4 Vertex operators superalgebras

Our approach of vertex operators superalgebras is freely inspired by the following references: Borcherds [1], Goddard [3], Kac [9]. We start by working on the fermion algebra:  $[\psi_m, \psi_n]_+ = \delta_{m+n}I$  and  $\psi_n^* = \psi_{-n}$  ( $m, n \in \mathbb{Z} + \frac{1}{2}$ ). As for  $\mathfrak{W}_{1/2}$ , we build its Verma module  $H = \mathcal{F}_{NS}$  and the sesquilinear form  $(\cdot, \cdot)$ , which is a scalar product.  $H$  is a prehilbert space, the unique unitary highest weight representation of the fermion algebra. Let the formal power series  $\psi(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1}$  called fermion field. We inductively defined operators  $D$  giving positive energy structure ( $\Leftrightarrow [D, \psi] = z.\psi' + \frac{1}{2}\psi$ ) and  $T$  giving derivation ( $[T, \psi] = \psi'$ ). We compute  $(\psi(z)\psi(w)\Omega, \Omega) = \frac{1}{z-w}$  ( $|z| > |w|$ ), which permits to prove inductively an anticommutation relation shortly written as:  $\psi(z)\psi(w) = -\psi(w)\psi(z)$ . We define this relation in a general framework as locality: Let  $H$  prehilbert space, and let  $A \in (EndH)[[z, z^{-1}]]$  formal power series of the form  $A(z) = \sum_{n \in \mathbb{Z}} A(n)z^{-n-1}$  with  $A(n) \in End(H)$ . Such fields  $A$  and  $B$  are **local** if  $\exists \varepsilon \in \mathbb{Z}_2, \exists N \in \mathbb{N}$  such that  $\forall c, d \in H, \exists X(A, B, c, d) \in (z-w)^{-N}\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$  such that:

$$X(A, B, c, d)(z, w) = \begin{cases} (A(z)B(w)c, d) & \text{if } |z| > |w| \\ (-1)^\varepsilon (B(w)A(z)c, d) & \text{if } |w| > |z| \end{cases}$$

Now, using locality and a contour integration argument, we can explicitly construct a field  $A_n B$  from  $A$  and  $B$ , with  $(A_n B)(m) =$

$$\begin{cases} \sum_{p=0}^n (-1)^p C_n^p [A(n-p), B(m+p)]_\varepsilon & \text{if } n \geq 0 \\ \sum_{p \in \mathbb{N}} C_{p-n-1}^p (A(n-p)B(m+p) - (-1)^{\varepsilon+n} B(m+n-p)A(p)) & \text{if } n < 0 \end{cases}$$

We obtain the operator product expansion (OPE) shortly written as  $A(z)B(w) \sim \sum_{n=0}^{N-1} \frac{(A_n B)(w)}{(z-w)^{n+1}}$ ; and by an other contour integration argument:

$$[A(m), B(n)]_\varepsilon = \begin{cases} \sum_{p=0}^{N-1} C_m^p (A_p B)(m+n-p) & \text{if } m \geq 0 \\ \sum_{p=0}^{N-1} (-1)^p C_{p-m-1}^p (A_p B)(m+n-p) & \text{if } m < 0 \end{cases}$$

Thanks to Dong's lemma, the operation  $(A, B) \mapsto A_n B$  permits to generate many fields. To have a good behaviour, we define a system of generators as:  $\{A_1, \dots, A_r\} \subset (EndH)[[z, z^{-1}]]$  with  $D, T \in End(H), \Omega \in H$  such that:

(a)  $\forall i, j$   $A_i$  and  $A_j$  are local with  $N = N_{ij}$  and  $\varepsilon = \varepsilon_{ij} = \varepsilon_{ii}.\varepsilon_{jj}$

- (b)  $\forall i [T, A_i] = A'_i$
  - (c)  $H = \bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  for  $D$ ,  $\dim(H_n) < \infty$
  - (d)  $\forall i [D, A_i] = z.A'_i + \alpha_i A_i$  with  $\alpha_i \in \mathbb{N} + \frac{\varepsilon_i i}{2}$
  - (e)  $\Omega \in H_0$ ,  $\|\Omega\| = 1$ , and  $\forall i \forall m \in \mathbb{N}$ ,  $A_i(m)\Omega = D\Omega = T\Omega = 0$
  - (f)  $\mathcal{A} = \{A_i(m), \forall i \forall m \in \mathbb{Z}\}$  acts irreducibly on  $H$ , so that  $\langle \mathcal{A} \rangle . \Omega = H$
- Hence, we generate a space  $\mathcal{S}$ , with  $V : H \rightarrow \mathcal{S}$  a state-field correspondence linear map.  $V(a)(z)$  is noted  $V(a, z)$  and  $V(a, z)\Omega|_{z=0} = a$ .

Now,  $\{\psi\}$  is a system of generator, we generate  $\mathcal{S}$  and the map  $V$  with  $\psi(z) = V(\psi_{-\frac{1}{2}}\Omega, z)$ ; but,  $\psi(z)\psi(w) \sim \frac{Id}{z-w} + 2L(w)$ , with  $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = \frac{1}{2}\psi_{-2}\psi(z) = V(\omega, z)$  with  $\omega = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$ . Then, using OPE and Lie bracket, we find that  $D = L_0$ ,  $T = L_{-1}$ ,  $L(z)L(w) \sim \frac{(c/2)Id}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{(z-w)}$ , and  $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}$  with  $c = 2\|L_{-2}\Omega\|^2 = \frac{1}{2}$ , the central charge. As corollary,  $\mathfrak{Vir}$  acts on  $H = \mathcal{F}_{NS}$ , and admits its unitary highest weight representation  $L(c, h) = L(\frac{1}{2}, 0)$  as minimal submodule containing  $\Omega$ . We call  $\omega \in H$  the Virasoro vector, and  $L$  the Virasoro field.

We are now able to define vertex operators superalgebras in general.

A vertex operator superalgebra is an  $(H, V, \Omega, \omega)$  with:

- (a)  $H = H_0 \oplus H_1$  a prehilbert superspace.
- (b)  $V : H \rightarrow (EndH)[[z, z^{-1}]]$  a linear map.
- (c)  $\Omega, \omega \in H$  the vacuum and Virasoro vectors.

Let  $\mathcal{S}_\varepsilon = V(H_\varepsilon)$ ,  $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$  and  $A(z) = V(a, z) = \sum_{n \in \mathbb{Z}} A(n)z^{-n-1}$ , then  $(H, V, \Omega, \omega)$  satisfies the followings axioms:

- (1)  $\forall n \in \mathbb{N}$ ,  $\forall A \in \mathcal{S}$ ,  $A(n)\Omega = 0$ ,  $V(a, z)\Omega|_{z=0} = a$ , and  $V(\Omega, z) = Id$
- (2)  $\mathcal{A} = \{A(n) | A \in \mathcal{S}, n \in \mathbb{Z}\}$  acts irreducibly on  $H$ , so that  $\mathcal{A}.\Omega = H$ .
- (3)  $\forall A \in \mathcal{S}_{\varepsilon_1}$ ,  $\forall B \in \mathcal{S}_{\varepsilon_2}$ ,  $A$  and  $B$  are local with  $\varepsilon = \varepsilon_1.\varepsilon_2$ ,  $A_n B \in \mathcal{S}_{\varepsilon_1 + \varepsilon_2}$
- (4)  $V(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ ,  $[L_m, L_n] = (m-n)L_{m+n} + \frac{\|2\omega\|^2}{12}m(m^2-1)\delta_{m+n}$
- (5)  $H = \bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  for  $L_0$ ,  $\dim(H_n) < \infty$  and  $H_\varepsilon = \bigoplus_{n \in \mathbb{N} + \frac{\varepsilon}{2}} H_n$
- (6)  $[L_0, V(a, z)] = z.V'(a, z) + \alpha.V(a, z)$  for  $a \in H_\alpha$
- (7)  $[L_{-1}, V(a, z)] = V'(a, z) = V(L_{-1}.a, z) \in \mathcal{S}$

As corollaries, we have that a system of generators, generating a Virasoro field  $L \in \mathcal{S}$ , with  $D = L_0$  and  $T = L_{-1}$ , generates a vertex operator superalgebra; the fermion field  $\psi$  and the Virasoro field  $L$  generate one, each; and we prove the Borcherds associativity:  $V(a, z)V(b, w) = V(V(a, z-w)b, w)$ .

## 1.5 Vertex $\mathfrak{g}$ -superalgebras and modules

Let  $\mathfrak{g}$  be a simple Lie algebra of dimension  $N$ , a basis  $(X_a)$ , well normalized (see remark 4.2), such that  $[X_a, X_b] = i \sum_c \Gamma_{ab}^c X_c$  with  $\Gamma_{ab}^c \in \mathbb{R}$  totally antisymmetric. Let its dual coxeter number  $g = \frac{1}{4} \sum_{a,c} (\Gamma_{ac}^b)^2$ :

$\mathfrak{g}$	$A_n$	$B_n$	$C_n$	$D_n$	$E_6$	$E_7$	$E_8$	$F_4$	$G_2$
$\dim(\mathfrak{g})$	$n^2 + 2n$	$2n^2 + n$	$2n^2 + n$	$2n^2 - n$	78	133	248	52	14
$g$	$n + 1$	$2n - 1$	$n + 1$	$2n - 2$	12	18	30	9	4

For example,  $\mathfrak{g} = A_1 = \mathfrak{sl}_2$ ,  $\dim(\mathfrak{g}) = 3$  and  $g = 2$ .

Let  $\widehat{\mathfrak{g}}_+$  the  $\mathfrak{g}$ -boson algebra:  $[X_m^a, X_n^b] = [X_a, X_b]_{m+n} + m\delta_{ab}\delta_{m+n}\mathcal{L}$ , unique central extension (by  $\mathcal{L}$ ) of the loop algebra  $L\mathfrak{g} = C^\infty(\mathbb{S}^1, \mathfrak{g})$  (see [18] p 43). The unitary highest weight representations of  $\widehat{\mathfrak{g}}_+$  are  $H = L(V_\lambda, \ell)$ , with  $\ell \in \mathbb{N}$  such that  $\mathcal{L}\Omega = \ell\Omega$  (the level of  $H$ ),  $H_0 = V_\lambda$  irreducible representation of  $\mathfrak{g}$  such that  $(\lambda, \theta) \leq \ell$  with  $\lambda$  the highest weight and  $\theta$  the highest root (see [18] p 45). The category  $\mathcal{C}_\ell$  of representations for fixed  $\ell$  is finite. For example  $\mathfrak{g} = \mathfrak{sl}_2$ ,  $H = L(j, \ell)$ , with  $V_\lambda = V_j$  representations of  $\mathfrak{spin} j \leq \frac{\ell}{2}$ .

We define the  $\mathfrak{g}$ -fermion algebra  $\widehat{\mathfrak{g}}_-$  and the fermion fields, composed by  $N$  fermions; and as for  $N = 1$ , we generate a vertex operator superalgebra, but now, it contains  $\mathfrak{g}$ -boson fields ( $S^a$ ) whose related algebra is represented with  $L(V_0, g)$ ; and thanks to  $\widehat{\mathfrak{g}}_-$ -vertex background, the fields ( $S^a$ ) generate a vertex operator superalgebra; by this way, we are able to generate one, from  $\widehat{\mathfrak{g}}_+$  and  $H = L(V_0, \ell)$ ,  $\forall \ell \in \mathbb{N}$ . Remark that because of the vacuum axiom, the vertex structure need to take  $V_\lambda = V_0$  trivial representation; in general, we have vertex modules (see further).

Now, let  $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \times \widehat{\mathfrak{g}}_-$  the  $\mathfrak{g}$ -supersymmetric algebra; we prove it admits  $H = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  as unitary highest weight representations. We generate a vertex operator superalgebra, with a Virasoro field  $L$ , and also a SuperVirasoro field  $G$ , which gives the supersymmetry boson-fermion: Let  $B^a = X^a + S^a$  boson fields of level  $d = \ell + g$ , then:

$$B^a(z)G(w) \sim d^{\frac{1}{2}} \frac{\psi^a(w)}{(z-w)^2} \quad \text{and} \quad \psi^a(z)G(w) \sim d^{-\frac{1}{2}} \frac{B^a(w)}{(z-w)}.$$

Finally, from  $H^\lambda = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ , we define the vertex module  $(H^\lambda, V^\lambda)$  over  $(H^0, V, \Omega, \omega)$ , and we prove that  $\mathfrak{Vir}_{\frac{1}{2}}$  acts unitarily on  $H^\lambda$  and admits  $L(c, h)$  as minimal submodule containing the cyclic vector  $\Omega^\lambda$ , with  $c = \frac{3}{2} \cdot \frac{\ell + \frac{1}{2}g}{\ell + g} \dim(\mathfrak{g})$ ,  $h = \frac{c_{V_\lambda}}{2(\ell + g)}$  and  $c_{V_\lambda}$  the Casimir number of  $V_\lambda$ .

## 2 The Neveu-Schwarz algebra

### 2.1 Witt superalgebras and representations

**Definition 2.1.** A Lie superalgebra is a  $\mathbb{Z}_2$ -graded vector space  $\mathfrak{d} = \mathfrak{d}_{\bar{0}} \oplus \mathfrak{d}_{\bar{1}}$ , together with a graded Lie bracket  $[\cdot, \cdot] : \mathfrak{d} \times \mathfrak{d} \rightarrow \mathfrak{d}$ , such that  $[\cdot, \cdot]$  is a bilinear map with  $[\mathfrak{d}_i, \mathfrak{d}_j] \subseteq \mathfrak{d}_{i+j}$ , and for homogeneous elements  $X \in \mathfrak{d}_x, Y \in \mathfrak{d}_y, Z \in \mathfrak{d}_z$  :

- $[X, Y] = -(-1)^{xy}[Y, X]$
- $(-1)^{xz}[X, [Y, Z]] + (-1)^{xy}[Y, [Z, X]] + (-1)^{yz}[Z, [X, Y]] = 0$

**Definition 2.2.** The Witt algebra  $\mathfrak{W}$  is the Lie  $\star$ -algebra of vector fields on the circle, generated by  $d_n = ie^{i\theta n} \frac{d}{d\theta}$  ( $n \in \mathbb{Z}$ ).

**Remark 2.3.**  $\mathfrak{W}$  admits two supersymmetric extensions,  $\mathfrak{W}_0$  the Ramond sector (R) and  $\mathfrak{W}_{1/2}$  the Neveu-Schwarz sector (NS) ((see [8], [4] chap 9).

Here, we treat only the (NS) sector.

**Definition 2.4.** Let  $\mathfrak{d} = \mathfrak{W}_{1/2}$  the Witt superalgebra with:

$$\begin{cases} [d_m, d_n] = (m - n)d_{m+n} \\ [\gamma_m, d_n] = (m - \frac{n}{2})\gamma_{m+n} \\ [\gamma_m, \gamma_n]_+ = 2d_{m+n} \end{cases}$$

together with the  $\star$ -structure,  $d_n^* = d_{-n}$  and  $\gamma_m^* = \gamma_{-m}$ , and the super-structure:  $\mathfrak{d}_{\bar{0}} = \mathfrak{W} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}d_n$ ,  $\mathfrak{d}_{\bar{1}} = \bigoplus_{m \in \mathbb{Z} + 1/2} \mathbb{C}\gamma_m$

Now we investigate representations  $\pi$  of  $\mathfrak{W}_{1/2}$ , which are :

**Definition 2.5.** Let  $H$  be a prehilbert space.

- (a) *Unitary:*  $\pi(A)^* = \pi(A^*)$
- (b) *Projective:*  $A \mapsto \pi(A)$  is linear and  $[\pi(A), \pi(B)] - \pi([A, B]) \in \mathbb{C}$ .
- (c) *Positive energy :*  $H$  admits an orthogonal decomposition  $H = \bigoplus_{n \in \frac{1}{2}\mathbb{N}} H_n$  such that,  $\exists D$  acting on  $H_n$  as multiplication by  $n$ ,  $H_0 \neq \{0\}$  and  $\dim(H_n) < +\infty$ . Here,  $\exists h \in \mathbb{C}$  such that  $D = \pi(d_0) - hI$ .



## 2.2 Investigation

**Definition 2.6.** Let  $b : \mathfrak{W}_{1/2} \times \mathfrak{W}_{1/2} \rightarrow \mathbb{C}$  be the bilinear map defined by

$$[\pi(A), \pi(B)] - \pi([A, B]) = b(A, B)I \quad (b \text{ is a 2-cocycle})$$

**Definition 2.7.** Let  $f : \mathfrak{W}_{1/2} \rightarrow \mathbb{C}$  be a  $\star$ -linear form.

$\partial f = (A, B) \mapsto f([A, B])$  is a 2-coboundary.

**Remark 2.8.**  $A \mapsto \pi(A) + f(A)I$  define also a projective, unitary, positive energy representation, where  $b(A, B)$  becomes  $b(A, B) - f([A, B])$ .

**Proposition 2.9.** (SuperVirasoro extension)  $\mathfrak{W}_{1/2}$  has a unique central extension, up to equivalent, i.e.  $H_2(\mathfrak{W}_{1/2}, \mathbb{C})$  is 1-dimensional. This extension admits the basis  $(L_n)_{n \in \mathbb{Z}}, (G_m)_{m \in \mathbb{Z} + \frac{1}{2}}, C$  central, with  $L_n^* = L_{-n}, G_m^* = G_{-m}, C = cI, c \in \mathbb{C}$  called the **central charge**; and relations:

$$\begin{cases} [L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n} \\ [G_m, L_n] = (m - \frac{n}{2})G_{m+n} \\ [G_m, G_n]_+ = 2L_{m+n} + \frac{C}{3}(m^2 - \frac{1}{4})\delta_{m+n} \end{cases}$$

*Proof.*

Let  $L_n = \pi(d_n)$  and  $G_m = \pi(\gamma_m)$  then:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + b(d_m, d_n)I \\ [G_m, L_n] &= (m - \frac{n}{2})G_{m+n} + b(\gamma_m, d_n)I \\ [G_m, G_n]_+ &= 2L_{m+n} + b(\gamma_m, \gamma_n)I \end{aligned}$$

In particular:

$$\begin{aligned} [L_0, L_n] &= -nL_n + b(d_0, d_n)I \\ [L_0, G_n] &= -nG_n + b(d_0, \gamma_n)I \\ [L_1, L_{-1}] &= 2L_0 + b(d_1, d_{-1})I \end{aligned}$$

We choose :

$$\begin{aligned} f(d_n) &= -n^{-1}b(d_0, d_n) \\ f(\gamma_m) &= -m^{-1}b(d_0, \gamma_m) \\ f(d_0) &= \frac{1}{2}b(d_1, d_{-1}) \end{aligned}$$

Then, after adjustment by  $f$ :

$$\begin{aligned} [L_0, L_n] &= -nL_n \\ [L_0, G_n] &= -nG_n \\ [L_1, L_{-1}] &= 2L_0 \end{aligned}$$

Now  $D = L_0 - hI$  and if  $v \in H_k$ ,  $Dv = kv$ , then:  
 $DL_nv = L_nDv + [D, L_n]v = kL_nv + [L_0, L_n]v = (k - n)L_nv$   
 So,  $L_n : H_k \rightarrow H_{k-n}$  ( $= \{0\}$  if  $n > k$  ).  
 Similary,  $G_m : H_k \rightarrow H_{k-m}$ , then:

$$\begin{cases} [L_m, L_n] - (m - n)L_{m+n} : H_{m+n+k} \rightarrow H_k \\ [G_m, L_n] - (m - \frac{n}{2})G_{m+n} : H_{m+n+k} \rightarrow H_k \\ [G_m, G_n]_+ - 2L_{m+n} : H_{m+n+k} \rightarrow H_k \end{cases}$$

But  $b(d_m, d_n)I$ ,  $b(\gamma_m, d_n)I$ ,  $b(\gamma_m, \gamma_n)I : H_{m+n+k} \rightarrow H_{m+n+k}$ , so:

$$\begin{cases} b(d_m, d_n) = A(m)\delta_{m+n} \\ b(\gamma_m, d_n) = B(m)\delta_{m+n} = 0 \text{ because } 0 \notin \mathbb{Z} + 1/2 \ni m + n \\ b(\gamma_m, \gamma_n) = C(m)\delta_{m+n} \end{cases}$$

Now, on  $\mathfrak{W} = \mathfrak{d}_0$ ,  $b(A, B) = -b(B, A)$ , so,  $A(m) = -A(-m)$  and  $A(0) = 0$ , and Jacobi identity implies  $b([A, B], C) + b([B, C], A) + b([C, A], B) = 0$ , then, for  $d_k, d_n, d_m$  with  $k + n + m = 0$  :

$$(n - m)A(k) + (m - k)A(n) + (k - n)A(m) = 0$$

Now, with  $k = 1$  and  $m = -n - 1$ ,  $(n - 1)A(n + 1) = (n + 2)A(n) - (2n + 1)A(1)$ . Then  $A(n)$  is completely determined by the knowledge of  $A(1)$  and  $A(2)$ , and so, the solutions are a 2-dimensional space.

Now,  $n$  and  $n^3$  are solutions, so  $A(n) = a.n + b.n^3$  .

Finally, because  $[L_1, L_{-1}] = 2L_0$ ,  $A(1) = 0$  and  $a + b = 0$ , we obtain:

$$A(n) = b(n^3 - n) = \frac{c}{12}(n^3 - n), \quad c \in \mathbb{C} \text{ the central charge.}$$

### Process 2.9.

$[[A, B]_+, C] = [A, [B, C]]_+ + [B, [A, C]]_+$  then:

$$\begin{aligned} [[G_r, G_s]_+, L_n] &= [G_r, [G_s, L_n]]_+ + [G_s, [G_r, L_n]]_+ \\ &= [2L_{r+s}, L_n] = [G_r, (s - \frac{1}{2}n)G_{n+s}]_+ + [G_s, (r - \frac{1}{2}n)G_{n+r}]_+ \\ &= 2(r + s - n)L_{r+s+n} - \delta_{r+s+n} \frac{c}{6}(n^3 - n) \end{aligned}$$

$$= (s - \frac{1}{2}n)(2L_{r+s+n} + C(r)\delta_{r+s+n}) - (r - \frac{1}{2}n)(2L_{r+s+n} + C(s)\delta_{r+s+n})$$

Then taking  $r + s + n = 0$ ,  $\frac{c}{6}(n^3 - n) + (s - \frac{1}{2}n)C(r) + (r - \frac{1}{2}n)C(s) = 0$ .

Finally, with  $n = 2s$  and  $r = -3s$ ,  $C(s) = \frac{c}{3}(s^2 - \frac{1}{4})$ .  $\square$

**Definition 2.10.** *The central extension of  $\mathfrak{W}_{1/2}$  is called  $\mathfrak{Vir}_{1/2}$ , the Super-Virasoro algebra (on sector NS), also called Neveu-Schwarz algebra.*

**Theorem 2.11.** *(Complete reducibility)*

- (a) *If  $H$  is a unitary, projective, positive energy representation of  $\mathfrak{W}_{1/2}$ , then any non-zero vector  $v$  in the lowest energy subspace  $H_0$  generates an irreducible submodule.*
- (b)  *$H$  is an orthogonal direct sum of irreducibles such representations.*

*Proof.* (a) Let  $K$  be the minimal  $\mathfrak{W}_{1/2}$ -submodule containing  $v$ .

Clearly, since  $L_n v = G_m v = 0$  for  $m, n > 0$  and  $L_0 v = h v$ , we see that  $K$  is spanned by all products  $R.v$  with :

$$R = G_{-j_\beta} \dots G_{-j_1} L_{-i_\alpha} \dots L_{-i_1}, \quad 0 < i_1 \leq \dots \leq i_\alpha, \quad \frac{1}{2} \leq j_1 < \dots < j_\beta$$

But then,  $K_0 = \mathbb{C}v$ . Let  $K'$  be a submodule of  $K$ , and let  $p$  be the orthogonal projection onto  $K'$ . By unitarity,  $p$  commutes with the action of  $\mathfrak{W}_{1/2}$ , and hence with  $D$ . Thus  $p$  leaves  $K_0 = \mathbb{C}v$  invariant, so  $p v = 0$  or  $v$ .

But  $p R v = R p v$ , hence  $K' = 0$  or  $K$  and  $K$  is irreducible.

(b) Take the irreducible module  $M_1$  generated by a vector of lowest energy. Now (changing  $h$  into  $h' = h + m$  if necessary), we repeat this process for  $M_1^\perp$ , to get  $M_2, M_3, \dots$ . The positive energy assumption shows that  $H = \oplus M_i$   $\square$

**Theorem 2.12.** *(Uniqueness) If  $H$  and  $H'$  are irreducibles with  $c = c'$  and  $h = h'$ , then they are unitarily equivalents as  $\mathfrak{W}_{1/2}$ -modules.*

*Proof.*  $H_0 = \mathbb{C}u$  and  $H'_0 = \mathbb{C}u'$  with  $u, u'$  unitary.

Let  $U : H \rightarrow H', Au \mapsto Au'$ , we want to prove that  $U^*U = UU^* = Id$ .

Let  $Au \in H_n, Bu \in H_m$ :

If  $n \neq m$ , for example,  $n < m$ , then  $B^*Au \in H_{n-m} = 0$  and

$$(Au, Bu) = (B^*Au, u) = 0 = (Au', Bu').$$

If  $n = m$ , then  $D = B^*A$  is a constant energy operator, so in  $\mathbb{C}L_0 \oplus \mathbb{C}C$ .

Now,  $(L_0 u, u) = h = (L_0 u', u')$  iff  $h = h'$  and  $(Cu, u) = c = (Cu', u')$  iff  $c = c'$ .

Finally,  $(v, w) = (Uv, Uw) \forall v, w \in H$  and  $(v', w') = (U^*v', U^*w') \forall v', w' \in H'$  iff  $h = h'$  and  $c = c'$ .

So,  $U^*U = UU^* = Id$ , ie,  $H$  and  $H'$  are unitarily equivalents.  $\square$

**Definition 2.13.**  $\mathfrak{Vir}_{1/2} = \mathfrak{Vir}_{1/2}^- \oplus \mathfrak{Vir}_{1/2}^0 \oplus \mathfrak{Vir}_{1/2}^+$  with  $\mathfrak{Vir}_{1/2}^0 = \mathbb{C}L_0 \oplus \mathbb{C}C$

$$\mathfrak{Vir}_{1/2}^+ = \bigoplus_{m,n>0} \mathbb{C}L_m \oplus \mathbb{C}G_n \quad \mathfrak{Vir}_{1/2}^- = \bigoplus_{m,n<0} \mathbb{C}L_m \oplus \mathbb{C}G_n$$

**Remark 2.14.** *This decomposition pass to the universal envelopping :*

$$\mathcal{U}(\mathfrak{Vir}_{1/2}) = \mathcal{U}(\mathfrak{Vir}_{1/2}^-) \cdot \mathcal{U}(\mathfrak{Vir}_{1/2}^0) \cdot \mathcal{U}(\mathfrak{Vir}_{1/2}^+)$$

**Remark 2.15.** *We see that an irreducible, unitary, projective, positive energy representation of  $\mathfrak{W}_{1/2}$  is exactly given by a unitary highest weight representation of  $\mathfrak{Vir}_{1/2}$  (see the following section).*

### 2.3 Unitary highest weight representations

**Definition 2.16.** *Let the Verma module  $H = V(c, h)$  be the  $\mathfrak{Vir}_{1/2}$ -module freely generated by followings conditions:*

- (a)  $\Omega \in H$ , called the cyclic vector ( $\Omega \neq 0$ ).
- (b)  $L_0\Omega = h\Omega$ ,  $C\Omega = c\Omega$  ( $h, c \in \mathbb{R}$ )
- (c)  $\mathfrak{Vir}_{1/2}^+\Omega = \{0\}$

**Lemma 2.17.**  $\mathcal{U}(\mathfrak{Vir}_{1/2}^-)\Omega = H$  and a set of generators is given by:  
 $G_{-j_\beta} \dots G_{-j_1} L_{-i_\alpha} \dots L_{-i_1} \Omega$ ,  $0 < i_1 \leq \dots \leq i_\alpha$ ,  $\frac{1}{2} \leq j_1 < \dots < j_\beta$

*Proof.* It's clear. □

**Lemma 2.18.**  $V(c, h)$  admits a canonical sesquilinear form  $(., .)$ , completely defined by:

- (a)  $(\Omega, \Omega) = 1$
- (b)  $\pi(A)^* = \pi(A^*)$
- (c)  $(u, v) = \overline{(v, u)} \quad \forall u, v \in H$  (in particular  $(u, u) = \overline{(u, u)} \in \mathbb{R}$ ).

*Proof.* It's clear. □

**Definition 2.19.**  $u \in V(c, h)$  is a ghost if  $(u, u) < 0$ .

**Lemma 2.20.** *If  $V(c, h)$  admits no ghost then  $c, h \geq 0$*

*Proof.* Since  $L_n L_{-n} \Omega = L_{-n} L_n \Omega + 2nh\Omega + c \frac{n(n^2-1)}{12} \Omega$ ,

we have  $(L_{-n} \Omega, L_{-n} \Omega) = 2nh + \frac{n(n^2-1)}{12} c \geq 0$ .

Now, taking  $n$  first equal to 1 and then very large, we obtain the lemma.  $\square$

**Definition 2.21.** *Let  $K(c, h) = \ker(\cdot, \cdot) = \{x \in V(c, h); (x, y) = 0 \forall y\}$  the maximal proper submodule of  $V(c, h)$ , and  $L(c, h) = V(c, h)/K(c, h)$ , irreducible highest weight representation of  $\mathfrak{Vir}_{1/2}$ , with  $(\cdot, \cdot)$  well-defined on.*

**Definition 2.22.**  *$u \in V(c, h)$  is a null vector if  $(u, u) = 0$ .*

**Lemma 2.23.** *On no ghost case, the set of null vectors is  $K(c, h)$ .*

*Proof.* Let  $x$  be a null vector, and  $y \in V(c, h)$ .

By assumption  $\forall \alpha, \beta \in \mathbb{C}, (\alpha x + \beta y, \alpha x + \beta y) \geq 0$ . We develop it, with  $\alpha = (y, y)$  and  $\beta = -(x, y)$ , we obtain :  $|(x, y)|^2 (y, y) \leq (x, x) (y, y)^2 = 0$ .

So if  $y$  is not a null vector then  $(x, y) = 0$ . Else,  $(x, x) = (y, y) = 0$ , so taking  $\alpha = 1$  and  $\beta = -(x, y)$ , we obtain  $2|(x, y)|^2 \leq 0$  and so  $(x, y) = 0$   $\square$

**Corollary 2.24.**  *$L(c, h)$  is a unitary highest weight representation.*

*Proof.* Without ghost,  $(\cdot, \cdot)$  is a scalar product on  $L(c, h)$ .  $\square$

**Remark 2.25.** *Theorem 1.2 of [11] will be proved classifying no ghost cases.*

### 3 Vertex operators superalgebras

We give a progressive introduction to vertex operators superalgebras structure. We start with the fermion algebra as example. We work on to obtain, at the end of the section, vertex axioms naturally.

#### 3.1 Investigation on fermion algebra

**Definition 3.1.** *Let the fermion algebra (of sector NS), generated by  $(\psi_n)_{n \in \mathbb{Z} + \frac{1}{2}}$ , and  $I$  central, with the relations:*

$$[\psi_m, \psi_n]_+ = \delta_{m+n} I \quad \text{and} \quad \psi_n^* = \psi_{-n}$$

**Definition 3.2.** *(Verma module) Let  $H = \mathcal{F}_{NS}$  freely generated by:*

- (a)  $\Omega \in H$  is called the vacuum vector,  $\Omega \neq 0$ .
- (b)  $\psi_m \Omega = 0 \quad \forall m > 0$
- (c)  $I \Omega = \Omega$

**Lemma 3.3.** *A set of generators of  $H$  is given by:*  
 $\psi_{-m_1} \dots \psi_{-m_r} \Omega \quad m_1 < \dots < m_r \quad r \in \mathbb{N}, \quad m_i \in \mathbb{N} + \frac{1}{2}$

*Proof.* It's clear. □

**Lemma 3.4.**  *$H$  admits the sesquilinear form  $(\cdot, \cdot)$  completely defined by :*

- (a)  $(\Omega, \Omega) = 1$
  - (b)  $(u, v) = \overline{(v, u)} \quad \forall u, v \in H$
  - (c)  $(\psi_n u, v) = (u, \psi_{-n} v) \quad \forall u, v \in H \quad \text{ie } \pi(\psi_n)^* = \pi(\psi_n^*)$
- $(\cdot, \cdot)$  is a scalar product and  $H$  is a prehilbert space.*

*Proof.* It's clear. □

**Remark 3.5.**  *$H$  is an irreducible representation of the fermion algebra. It is its unique unitary highest weight representation.*

**Remark 3.6.**  $\psi_n^2 = \frac{1}{2}[\psi_n, \psi_n]_+ = 0$  if  $n \neq 0$

**Definition 3.7.** (Operator  $D$ ) Let  $D \in \text{End}(H)$  inductively defined by :

(a)  $D\Omega = 0$

(b)  $D\psi_{-m}a = \psi_{-m}Da + m\psi_{-m}a \quad \forall m \in \mathbb{N} + \frac{1}{2} \text{ and } \forall a \in H$

**Lemma 3.8.**  $D$  decomposes  $H$  into  $\bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  with  $D\xi = n\xi$   
 $\forall \xi \in H_n$ ,  $\dim(H_n) < \infty$  and  $H_n \perp H_m$  if  $n \neq m$

*Proof.* Let  $a = \psi_{-m_1} \dots \psi_{-m_r} \Omega$  be a generic element of the base of  $H$ , then  $D.a = (\sum m_i)a$ . □

**Remark 3.9.**  $[D, \psi_m] = -m\psi_m$  and  $\Omega \in H_0$ , so  $\psi_m : H_{m+n} \rightarrow H_n$ .

**Definition 3.10.** (Operator  $T$ ) Let  $T \in \text{End}(H)$  inductively defined by :

(a)  $T\Omega = 0$

(b)  $T\psi_{-m}a = \psi_{-m}Ta + (m - \frac{1}{2})\psi_{-m-1}a \quad \forall m \in \mathbb{N} + \frac{1}{2} \text{ and } \forall a \in H$

**Remark 3.11.**  $[T, \psi_m] = -(m - \frac{1}{2})\psi_{m-1}$ .

**Definition 3.12.** Let  $\psi(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}}.z^{-n-1}$  the fermion operator.

**Remark 3.13.**  $\psi \in (\text{End}H)[[z, z^{-1}]]$  is a formal power series.

**Lemma 3.14.** (Relations with  $\psi_n$ ,  $D$  and  $T$ )

(a)  $[\psi_{m+\frac{1}{2}}, \psi]_+ = z^m$

(b)  $[D, \psi] = z.\psi' + \frac{1}{2}\psi$

(c)  $[T, \psi] = \psi'$

*Proof.*  $[\psi_{m+\frac{1}{2}}, \psi(z)]_+ = \sum [\psi_{m+\frac{1}{2}}, \psi_{n+\frac{1}{2}}]_+.z^{-n-1} = z^m$

$[D, \psi(z)] = \sum (-n - \frac{1}{2})\psi_{n+\frac{1}{2}}.z^{-n-1} = z.\psi'(z) + \frac{1}{2}\psi(z)$

$[T, \psi(z)] = \sum (-n)\psi_{n-\frac{1}{2}}.z^{-n-1} = \sum (-n-1)\psi_{n+\frac{1}{2}}.z^{-n-2} = \psi'(z)$  □

**Remark 3.15.**  $(.,.)$  induces  $(\psi(z_1)\dots\psi(z_n)c, d) \in \mathbb{C}[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$ ,  $\forall c, d \in H$ .

**Lemma 3.16.**  $(\psi(z)\Omega, \Omega) = 0$  and  $(\psi(z)\psi(w)\Omega, \Omega) = \frac{1}{z-w}$  if  $|z| > |w|$ .

*Proof.*  $(\psi(z)\Omega, \Omega) = \sum_{n \in \mathbb{Z}} (\psi_{n+\frac{1}{2}}\Omega, \Omega) \cdot z^{-n-1} = 0$   
 $(\psi(z)\psi(w)\Omega, \Omega) = \sum_{m, n \in \mathbb{Z}} (\psi_{m+\frac{1}{2}}\Omega, \psi_{-n-\frac{1}{2}}\Omega) \cdot z^{-n-1} w^{-m-1}$   
 $= \sum_{m, n \in \mathbb{Z}} (\psi_{m-\frac{1}{2}}\Omega, \psi_{-n-\frac{1}{2}}\Omega) \cdot z^{-n-1} w^{-m} = \sum_{n \in \mathbb{N}} (\psi_{-n-\frac{1}{2}}\Omega, \psi_{-n-\frac{1}{2}}\Omega) \cdot z^{-n-1} w^n$   
 $= z^{-1} \sum_{n \in \mathbb{N}} \left(\frac{w}{z}\right)^n = \frac{1}{z-w}$  if  $|z| > |w|$   $\square$

**Lemma 3.17.**  $\forall c, d \in H, (\psi(z)c, d) \in \mathbb{C}[z, z^{-1}]$ .

*Proof.*  $(\psi(z)\psi_{-n-\frac{1}{2}}c, d) = (c, d) \cdot z^{-n-1} - (\psi(z)c, \psi_{n+\frac{1}{2}}d)$   
 $(\psi(z)c, \psi_{-n-\frac{1}{2}}d) = (c, d) \cdot z^n - (\psi(z)\psi_{n+\frac{1}{2}}c, d)$   
Then, the result follows by lemma 3.16 and induction.  $\square$

**Proposition 3.18.**  $\forall c, d \in H, \exists X(c, d) \in (z-w)^{-1}\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$  such that:

$$X(c, d)(z, w) = \begin{cases} (\psi(z)\psi(w)c, d) & \text{if } |z| > |w| \\ -(\psi(w)\psi(z)c, d) & \text{if } |w| > |z| \end{cases}$$

*Proof.*  $(\psi(z)\psi(w)\psi_{-n-\frac{1}{2}}c, d) = (\psi(z)c, d)w^{-n-1} - (\psi(w)c, d)z^{-n-1} + (\psi(z)\psi(w)c, \psi_{n+\frac{1}{2}}d)$   
 $(\psi(z)\psi(w)c, \psi_{-n-\frac{1}{2}}d) = (\psi(w)c, d)z^n - (\psi(z)c, d)w^n + (\psi(z)\psi(w)\psi_{n+\frac{1}{2}}c, d)$   
Then, the result follows by lemma 3.16, 3.17, symmetry and induction.  $\square$

## 3.2 General framework

**Definition 3.19.** Let  $H$  prehilbert and  $A \in (\text{End}H)[[z, z^{-1}]]$  a formal power series defined as  $A(z) = \sum_{n \in \mathbb{Z}} A(n)z^{-n-1}$  with  $A(n) \in \text{End}(H)$ .

**Definition 3.20.** Let  $A, B \in (\text{End}H)[[z, z^{-1}]]$

$A$  and  $B$  are **local** if  $\exists \varepsilon \in \mathbb{Z}_2, \exists N \in \mathbb{N}$  such that  $\forall c, d \in H$ :  
 $\exists X(A, B, c, d) \in (z-w)^{-N}\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$  such that:

$$X(A, B, c, d)(z, w) = \begin{cases} (A(z)B(w)c, d) & \text{if } |z| > |w| \\ (-1)^\varepsilon (B(w)A(z)c, d) & \text{if } |w| > |z| \end{cases}$$

**Example 3.21.**  $\psi$  is local with itself, with  $N = 1$  and  $\varepsilon = \bar{1}$

**Notation 3.22.**  $[X, Y]_\varepsilon = \begin{cases} XY - YX & \text{if } \varepsilon = \bar{0} \\ XY + YX & \text{if } \varepsilon = \bar{1} \end{cases}$

**Remark 3.23.** Let  $n \in \mathbb{N}$ , then,  $(z-w)^n = \sum_{p=0}^n C_n^p (-1)^p w^p z^{n-p}$  and,

$$(z-w)^{-n} = \begin{cases} \sum_{p \in \mathbb{N}} C_{p+n-1}^p w^p z^{-p-n} & \text{if } |z| > |w| \\ (-1)^n \sum_{p \in \mathbb{N}} C_{p+n-1}^p z^p w^{-p-n} & \text{if } |w| > |z| \end{cases}$$



**Proposition 3.24.** *Let  $A, B$  local and  $c, d \in H$  then:*

$$X(A, B, c, d)(z, w) = \sum_{n \in \mathbb{Z}} X_n(A, B, c, d)(w)(z - w)^{-n-1},$$

$$X_n(A, B, c, d)(w) = (A_n B(w)c, d),$$

$$A_n B(w) = \sum_{m \in \mathbb{Z}} (A_n B)(m)w^{-m-1} \text{ and } (A_n B)(m) =$$

$$\begin{cases} \sum_{p=0}^n (-1)^p C_n^p [A(n-p), B(m+p)]_\varepsilon & \text{if } n \geq 0 \\ \sum_{p \in \mathbb{N}} C_{p-n-1}^p (A(n-p)B(m+p) - (-1)^{\varepsilon+n} B(m+n-p)A(p)) & \text{if } n < 0 \end{cases}$$

*Proof.*  $X(A, B, c, d) \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}, (z-w)^{-1}]$ , we develop it around  $z = w$ :

$$X(A, B, c, d)(z, w) = \sum_{n \in \mathbb{Z}} X_n(A, B, c, d)(w)(z - w)^{-n-1}$$

$$\text{with } X_n(A, B, c, d)(w) = \frac{1}{2\pi i} \oint_w (z-w)^n X(A, B, c, d)(z, w) dz.$$

By contour integration argument ( $\oint_w = \int_{|z|=R>|w|} - \int_{|z|=r<|w|}$ ), we obtain:

$$\begin{aligned} X_n(A, B, c, d)(w) &= \frac{1}{2\pi i} \left( \int_{|z|=R>|w|} - \int_{|z|=r<|w|} \right) (z-w)^n X(A, B, c, d)(z, w) dz \\ &= \frac{1}{2\pi i} \int_{|z|=R>|w|} (z-w)^n (A(z)B(w)c, d) dz - \frac{(-1)^\varepsilon}{2\pi i} \int_{|z|=r<|w|} (z-w)^n (B(w)A(z)c, d) dz \\ &= \frac{1}{2\pi i} \sum_{q \in \mathbb{Z}, p=0}^n \left( \int_{|z|=R>|w|} C_n^p (-1)^p z^{n-p} w^p (A(q)B(w)c, d) z^{-q-1} dz \right. \\ &\quad \left. - (-1)^\varepsilon \int_{|z|=r<|w|} C_n^p (-1)^p z^{n-p} w^p (B(w)A(q)c, d) z^{-q-1} dz \right) \\ &= \left( \sum_{p=0}^n (-1)^p w^p C_n^p [A(n-p), B(w)]_\varepsilon c, d \right), \text{ with } n \in \mathbb{N}. \end{aligned}$$

$$\begin{aligned} X_{-n}(A, B, c, d)(w) &= \frac{1}{2\pi i} \sum_{q \in \mathbb{Z}, p \in \mathbb{N}} \left( \int_{|z|=R>|w|} C_{p+n-1}^p z^{-n-p} w^p (A(q)B(w)c, d) z^{-q-1} dz \right. \\ &\quad \left. - (-1)^\varepsilon \int_{|z|=r<|w|} C_{p+n-1}^p (-1)^n w^{-n-p} z^p (B(w)A(q)c, d) z^{-q-1} dz \right) \\ &= \left( \sum_{p \in \mathbb{N}} C_{p+n-1}^p (w^p A(-n-p)B(w) - (-1)^{\varepsilon+n} w^{-n-p} B(w)A(p)) c, d \right) \quad \square \end{aligned}$$

**Definition 3.25.** *Let the operation  $(A, B) \rightarrow A_n B$  as for proposition 3.24.*

**Formula 3.26.** *The formula of  $(A_n B)(m)$  on proposition 3.24.*

**Corollary 3.27.** *(Operator product expansion) Let  $A, B$  local, and  $c, d \in H$ :*

$$(A(z)B(w)c, d) \sim \left( \sum_{n=0}^{N-1} \frac{(A_n B)(w)}{(z-w)^{n+1}} c, d \right) \text{ near } z = w$$

*Proof.*  $X(A, B, c, d)(z, w) = \sum_{n \in \mathbb{Z}} (A_n B)(w)c, d)(z - w)^{-n-1}$

$\in (z-w)^{-N} \mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ , so  $A_n B = 0$  for  $-n-1 < -N$  ie  $n \geq N$ .  $\square$

**Remark 3.28.** We write OPE as:  $A(z)B(w) \sim \sum_{n=0}^{N-1} \frac{(A_n B)(w)}{(z-w)^{n+1}}$ .

**Remark 3.29.**  $z^m = \begin{cases} \sum_{k=0}^m C_m^k (z-w)^k w^{m-k} & \text{if } m \geq 0 \\ \sum_{k \in \mathbb{N}} (-1)^k C_{k-m-1}^k (z-w)^k w^{m-k} & \text{if } m < 0 \end{cases}$

**Proposition 3.30.** (Lie bracket) Let  $A, B$  local, with  $\varepsilon \in \mathbb{Z}_2$  then:

$$[A(m), B(n)]_\varepsilon = \begin{cases} \sum_{p=0}^{N-1} C_m^p (A_p B)(m+n-p) & \text{if } m \geq 0 \\ \sum_{p=0}^{N-1} (-1)^p C_{p-m-1}^p (A_p B)(m+n-p) & \text{if } m < 0 \end{cases}$$

*Proof.*  $\forall c, d \in H, ([A(m), B(n)]_\varepsilon c, d) = \frac{1}{(2\pi i)^2} (\int \int_{|z|=R>|w|} - \int \int_{|z|=r<|w|}) z^m w^n X(A, B, c, d)(z, w) dz dw$

By contour integration argument ( $\int \int_{|z|=R>|w|} - \int \int_{|z|=r<|w|} = \oint_0 \oint_w$ ):

$$([A(m), B(n)]_\varepsilon c, d) = \frac{1}{2\pi i} \oint_0 w^n \frac{1}{2\pi i} \oint_w z^m (\sum_{p=0}^{N-1} \frac{(A_p B)(w)}{(z-w)^{p+1}} c, d) dz dw$$

$$\begin{aligned} & \text{We suppose } m \geq 0, \text{ then by previous remark, } ([A(m), B(n)]_\varepsilon c, d) = \\ &= \frac{1}{2\pi i} \oint_0 w^n \frac{1}{2\pi i} \oint_w \sum_{k=0}^m C_m^k w^{m-k} (\sum_{p=0}^{N-1} \frac{(A_p B)(w)}{(z-w)^{p+1-k}} c, d) dz dw \\ &= \frac{1}{2\pi i} \oint_0 (\sum_{p=0}^{N-1} w^{n+m-p} C_m^p (A_p B)(w) c, d) dw \\ &= \frac{1}{2\pi i} \oint_0 (\sum_{r \in \mathbb{Z}, p=0}^{N-1} w^{n+m-p-r-1} C_m^p (A_p B)(r) c, d) dw \\ &= (\sum_{p=0}^{N-1} C_m^p (A_p B)(m+n-p) c, d) \quad (\text{we take } C_m^p = 0 \text{ if } p > m). \end{aligned}$$

Similarly for  $m < 0 \dots$ , and the result follows.  $\square$

**Formula 3.31.** The formula of  $[A(m), B(n)]_\varepsilon$  on proposition 3.30.

**Definition 3.32.** (Operator  $D$ ) Let  $D \in \text{End}(H)$  decomposing  $H$  into  $\bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  with  $D\xi = n\xi \forall \xi \in H_n, \dim(H_n) < \infty$  and  $H_n \perp H_m$  if  $n \neq m$ .

**Notation 3.33.** Let  $A'(z) = \frac{d}{dz} A(z) = \sum_{n \in \mathbb{Z}} (-n) A(n-1) z^{-n-1}$ .

**Definition 3.34.**  $A \in (\text{End} H)[[z, z^{-1}]]$  is graded if:  
 $\exists \alpha \in \frac{1}{2}\mathbb{N}$  such that  $[D, A(z)] = zA'(z) + \alpha A(z)$

**Lemma 3.35.**  $A$  is graded with  $\alpha \iff$   
 $A(n) : H_m \rightarrow H_{m-n+\alpha-1} \quad \forall n \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{N}$

*Proof.*  $[D, A(z)] = zA'(z) + \alpha A(z) = \sum_{n \in \mathbb{Z}} (\alpha - 1 - n)A(n)z^{-n-1}$   
 $\iff [D, A(n)] = (\alpha - 1 - n)A(n) \quad \forall n \in \mathbb{Z}$   
 $\iff \forall n \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{N}, \forall \xi \in H_m$   
 $DA(n)\xi = A(n)D\xi + [D, A(n)]\xi = (m - n + \alpha - 1)A(n)\xi$   
 $\iff A(n) : H_m \rightarrow H_{m-n+\alpha-1} \quad \forall n \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{N}. \quad \square$

**Lemma 3.36.** *Let  $A, B$  local and graded with  $\alpha$  and  $\beta$  then:*  
 $[D, A_n B(z)] = z(A_n B)'(z) + (\alpha + \beta - n - 1)A_n B(z).$

*Proof.*  $A(n) : H_m \rightarrow H_{m-n+\alpha-1}$  and  $B(n) : H_m \rightarrow H_{m-n+\beta-1}$   
Now, by formula 3.26,  $A_p B(n) : H_m \rightarrow H_{m-n+(\alpha+\beta-p-1)-1}$   
The result follows by the previous lemma.  $\square$

**Lemma 3.37.** *Let  $A, B \in (\text{End}H)[[z, z^{-1}]]$ , graded with  $\alpha$  and  $\beta$ , then:*  
 *$A$  and  $B$  are local  $\iff \exists \varepsilon \in \mathbb{Z}_2, \exists N \in \mathbb{N}$  such that  $\forall c, d \in H$ :*  
 $(z-w)^N (A(z)B(w)c, d) = (-1)^\varepsilon (z-w)^N (B(w)A(z)c, d)$  *as formal series.*

*Proof.*  $(\Rightarrow)$  True by definition.  
 $(\Leftarrow)$  Let  $c \in H_p, d \in H_q$   
 $A(n)c \in H_{p-n+\alpha-1} = 0$  for  $n > p + \alpha - 1$ ,  
 $B(m)c \in H_{p-m+\beta-1} = 0$  for  $m > p + \beta - 1$ ,  
 $A(n)B(m)c, B(m)A(n)c \in H_{p-(m+n)+\alpha+\beta-2}, d \in H_q$  and  $H_r \perp H_q$  if  $q \neq r$ .

Let  $S = \{(m, n) \in \mathbb{Z}^2; m + n = p - q + \alpha + \beta - 2, m \leq p + \beta - 1\}$   
and  $S' = \{(m, n) \in \mathbb{Z}^2; m + n = p - q + \alpha + \beta - 2, n \leq p + \alpha - 1\}$

$$(z-w)^N (A(z)B(w)c, d) = \sum_{S, k=0}^N C_N^k (A(n)B(m)c, d) z^{-n-1-k} w^{-m-1+N-k}$$

$$(z-w)^N (B(w)A(z)c, d) = (-1)^\varepsilon \sum_{S', k=0}^N C_N^k (B(m)A(n)c, d) z^{-n-1-k} w^{-m-1+N-k}$$

But,  $S \cap S'$  is a finite subset of  $\mathbb{Z}^2$ , so the formal series is a polynomial:  
 $P(A, B, c, d) \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ ; now, using remark 3.23, and the fact that  
 $A(n)c = 0$  for  $n > p + \alpha - 1$  and  $B(m)c = 0$  for  $m > p + \beta - 1$ , then:

$$(z, w)^{-N} P(A, B, c, d)(z, w) = \begin{cases} (A(z)B(w)c, d) & \text{if } |z| > |w| \\ (-1)^\varepsilon (B(w)A(z)c, d) & \text{if } |w| > |z| \end{cases} \quad \square$$

**Remark 3.38.** (*associativity*)  $(A_n B)_m C = A_n (B_m C) = A_n B_m C$

**Lemma 3.39.** *Let  $A_1, \dots, A_R$  graded,  $A_i$  and  $A_j$  local with  $N = N_{ij} \in \mathbb{N}$ . Then,  $\forall c, d \in H$ :*

$$\prod_{i < j} (z_i - z_j)^{N_{ij}} (A_1(z_1) \dots A_R(z_R) c, d) \in \mathbb{C}[z_1^{\pm 1}, \dots, z_R^{\pm 1}]$$

*Proof.* It is exactly as the previous lemma:

We can put each  $A_i(z_i)$  on the first place by commutations.

We obtain equalities between  $R$  series with support  $S_i \cup T$ , with  $T$  the support due to  $\prod_{i < j} (z_i - z_j)^{N_{ij}}$  (finite), and as the previous lemma:

$$S_i = \{(m_1, \dots, m_R) \in \mathbb{Z}^R; m_1 + \dots + m_R = K, m_i \leq k_i\}$$

So,  $\bigcap S_i$  is a finite subset of  $\mathbb{Z}^R$ , and the result follows.  $\square$

**Lemma 3.40.** (*Dong's lemma*) *Let  $A, B, C$  graded and pairwise local, then  $A_n B$  and  $C$  are local.*

*Proof.* Let  $Q(z_1, z_2, z_3) = \prod_{i < j} (z_i - z_j)^{N_{ij}}$ , by lemma 3.39,  $\forall d, e \in H$ :

$$Q.(A(z_1)B(z_2)C(z_3)d, e) = Q.(-1)^{\varepsilon_1 + \varepsilon_2} (C(z_3)A(z_1)B(z_2)d, e) \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$$

Now, we divide this polynomial by  $Q$ , we fix  $z_2$  and we develop around  $z_1 = z_2$ .

Then  $\exists N \in \mathbb{N}$  such that  $\forall n \in \mathbb{Z}$  if  $P_n$  is the coefficient of  $(z_1 - z_2)^{-n-1}$  then  $S_n = (z_2 - z_3)^N P_n \in \mathbb{C}[z_2^{\pm 1}, z_3^{\pm 1}]$ .

Now, on one hand  $S_n = (z_2 - z_3)^N (A_n B(z_2) C(z_3) d, e)$  and on the other hand  $S_n = (-1)^\varepsilon (z_2 - z_3)^N (C(z_3) A_n B(z_2) d, e)$ , with  $\varepsilon = \varepsilon_1 + \varepsilon_2$ .

Then, the result follows by lemmas 3.36 and 3.37.  $\square$

**Proof's corollary 3.41.** *If in addition,  $A$  and  $C$  are local with  $\varepsilon_1 \in \mathbb{Z}_2$ , and,  $B$  and  $C$ , local with  $\varepsilon_2$ , then,  $A_n B$  and  $C$  are local with  $\varepsilon = \varepsilon_1 + \varepsilon_2$ .*

**Lemma 3.42.** *If  $A$  and  $B$  are local with  $\varepsilon \in \mathbb{Z}_2$ , so is  $A'$  and  $B$*

$$\text{Proof. } (z - w)^N (A(z)B(w)c, d) = (-1)^\varepsilon (z - w)^N (B(w)A(z)c, d)$$

Then, applying  $\frac{d}{dz}$  and the lemma 3.37, the result follows.  $\square$

**Definition 3.43.** (*Operator  $T$* ) *Let  $T \in \text{End}(H)$ .*

**Lemma 3.44.** *Let  $A, B$  local such that  $[T, A] = A'$  and  $[T, B] = B'$ .*

*Then,  $[T, A_n B] = (A_n B)' = A'_n B + A_n B'$  and  $[T, A'] = A''$*

*Proof.*  $(z-w)^N([T, A(z)B(w)]c, d) = (z-w)^N((A'(z)B(w) + A(z)B'(w))c, d)$

$$\begin{aligned} &= (z-w)^N \sum_{n \in \mathbb{Z}} ((A'_n B + A_n B')(w)c, d)(z-w)^{-n-1} \quad \text{on one hand} \\ &= (z-w)^N \left( \frac{d}{dz} + \frac{d}{dw} \right) \left( \sum_{n \in \mathbb{Z}} A_n B(w)(z-w)^{-n-1}c, d \right) \quad \text{on the other hand} \end{aligned}$$

$$\begin{aligned} &= (z-w)^N \left[ \left( \sum_{n \in \mathbb{Z}} (-n-1)A_n B(w)(z-w)^{-n-2}c, d \right) + \right. \\ &\quad \left. \left( \sum_{n \in \mathbb{Z}} (A_n B)'(w)(z-w)^{-n-1}c, d \right) + \left( \sum_{n \in \mathbb{Z}} (n+1)A_n B(w)(z-w)^{-n-2}c, d \right) \right] \\ &= (z-w)^N \sum_{n \in \mathbb{Z}} ((A_n B)'(w)c, d)(z-w)^{-n-1} \end{aligned}$$

By identification:  $[T, A_n B] = (A_n B)' = A'_n B + A_n B'$

Now,  $[T, A] = A' \Rightarrow [T, A(n)] = -nA(n-1)$ , so  $[T, A'] = A''$  □

**Lemma 3.45.** *Let  $\Omega \in H$ ;  $A, B$  local with  $A(m)\Omega = B(m)\Omega = 0 \forall m \in \mathbb{N}$ , then  $A'(m)\Omega = A_n B(m)\Omega = 0 \forall m \in \mathbb{N}, \forall n \in \mathbb{Z}$ .*

*Proof.*  $A'(m) = -mA(m-1)$ , so  $A'(m)\Omega = 0 \forall m \in \mathbb{N}$

On the formula 3.26,  $A(n-p)\Omega = B(m+p)\Omega = A(p)\Omega = 0$  because  $n-p, m+p, p \in \mathbb{N}$ , then,  $A_n B(m)\Omega = 0 \forall m \in \mathbb{N}, \forall n \in \mathbb{Z}$ . □

### 3.3 System of generators

**Definition 3.46.** *Let  $H$  prehilbert space;  $\{A_1, \dots, A_r\} \subset (\text{End}H)[[z, z^{-1}]]$  is a system of generators if  $\exists D, T \in \text{End}(H), \Omega \in H$  such that:*

- (a)  $\forall i, j$   $A_i$  and  $A_j$  are local with  $N = N_{ij}$  and  $\varepsilon = \varepsilon_{ij} = \varepsilon_{ii} \cdot \varepsilon_{jj}$
- (b)  $\forall i$   $[T, A_i] = A'_i$
- (c)  $D$  decomposes  $H = \bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  with  $D\xi = n\xi \forall \xi \in H_n, \dim(H_n) < \infty$ ,  $H_n \perp H_m$  if  $n \neq m$  and  $\forall i$   $A_i$  is graded with  $\alpha_i \in \mathbb{N} + \frac{\varepsilon_{ii}}{2}$
- (d)  $\Omega \in H_0, \|\Omega\| = 1$ , and  $\forall i \forall m \in \mathbb{N}, A_i(m)\Omega = D\Omega = T\Omega = 0$
- (e)  $\mathcal{A} = \{A_i(m), \forall i \forall m \in \mathbb{Z}\}$  acts irreducibly on  $H$ , so that  $H$  is the minimal space containing  $\Omega$  and stable by the action of  $\mathcal{A}$

**Definition 3.47.** *Let  $S \subset (\text{End}H)[[z, z^{-1}]]$ , the minimal subset containing  $Id, A_1, \dots, A_r$ , stable by the operations:*

$$(A, B) \mapsto (A_n B) \quad (\forall n \in \mathbb{Z}) \quad , \quad A \mapsto A'$$

Let  $\mathcal{S}_\varepsilon = \{A \in \mathcal{S} \mid A \text{ is local with itself with } \varepsilon \in \mathbb{Z}_2\}$ , so that  $\mathcal{S} = \mathcal{S}_0 \amalg \mathcal{S}_1$ .

Let  $\mathcal{S}_\varepsilon = \text{lin} \langle \mathcal{S}_\varepsilon \rangle$  and  $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$ .

**Remark 3.48.** All is well defined by previous lemmas.

**Lemma 3.49.**  $\forall A, B \in \mathcal{S}$ , they are local,  $A_n B \in \mathcal{S}$  and  $[T, A] = A' \in \mathcal{S}$

*Proof.* By previous lemmas and linearizing Dong's lemma.  $\square$

**Lemma 3.50.** Let  $E \in \mathcal{S}_{\varepsilon_1}$  and  $F \in \mathcal{S}_{\varepsilon_2}$  then:

(a)  $E_n F \in \mathcal{S}_{\varepsilon_1 + \varepsilon_2}$

(b)  $E$  and  $F$  are local with  $\varepsilon = \varepsilon_1 \cdot \varepsilon_2$

*Proof.* (a)  $E$  and  $F$  are local with an  $\varepsilon \in \mathbb{Z}_2$ .

We use the corollary 3.41 with  $A = E, B = F, C = E$ , with  $A = E, B = F, C = F$  and finally with  $A = E, B = F, C = E_n F$ . Then we see that  $E_n F$  is local with itself with  $\varepsilon' = \varepsilon_1 + \varepsilon + \varepsilon_2 + \varepsilon = \varepsilon_1 + \varepsilon_2$ , so,  $E_n F \in \mathcal{S}_{\varepsilon_1 + \varepsilon_2}$

(b) By induction:

Base case:  $\forall i, j A_i \in \mathcal{S}_{\varepsilon_{ii}}, A_j \in \mathcal{S}_{\varepsilon_{jj}}$  and are local with  $\varepsilon = \varepsilon_{ij} = \varepsilon_{ii} \cdot \varepsilon_{jj}$  by definition 3.46.

Inductive step: We suppose the property for  $E \in \mathcal{S}_{\varepsilon_1}, F \in \mathcal{S}_{\varepsilon_2}$  and  $G \in \mathcal{S}_{\varepsilon_3}$ .

We prove it for  $E_n F$  and  $G$ :

$E$  and  $G$  are local with  $\varepsilon = \varepsilon_1 \cdot \varepsilon_3$

$F$  and  $G$  are local with  $\varepsilon = \varepsilon_2 \cdot \varepsilon_3$

Now,  $E_n F \in \mathcal{S}_{\varepsilon_1 + \varepsilon_2}, G \in \mathcal{S}_{\varepsilon_3}$  and by corollary 3.41 with  $A = E, B = F, C = G, E_n F$  and  $G$  are local with  $\varepsilon = \varepsilon_1 \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 = (\varepsilon_1 + \varepsilon_2) \cdot \varepsilon_3$

The following lemma completes the proof.  $\square$

**Lemma 3.51.**  $A \in \mathcal{S}_\varepsilon \Rightarrow A' \in \mathcal{S}_\varepsilon$

*Proof.* By lemma 3.42, if  $A$  and  $B$  are local with  $\varepsilon \in \mathbb{Z}_2$ , so is  $A'$  and  $B$ .

The result follows by taking  $B = A$  and then  $B = A'$ .  $\square$

**Definition 3.52.** (well defined by lemma 3.45)

$$\begin{array}{ccc} R: \mathcal{S} & \longrightarrow & H \\ A & \longmapsto & a := A(z)\Omega|_{z=0} \end{array} \quad \text{linear.}$$

**Examples 3.53.**

- (a)  $R(Id) = \Omega, \quad R(A) = A(-1)\Omega$
- (b)  $R(A') = A(-2)\Omega = T.R(A)$
- (c)  $R(A_n B) = A(n)R(B)$  (by formula 3.26)
- (d)  $R(A_n Id) = A(n)\Omega$

**Lemma 3.54.**  $A$  is graded with  $\alpha \iff R(A) \in H_\alpha$

*Proof.* By lemma 3.35 and 3.36, inductions and linear combinations. □

**State-Field correspondence:**

**Lemma 3.55.** (Existence)  $\forall a \in H, \exists A \in \mathcal{S}$  such that  $R(A) = a$ .

*Proof.*  $R((A_{i_1})_{m_1}(A_{i_2})_{m_2}\dots(A_{i_k})_{m_k} Id) = A_{i_1}(m_1)R((A_{i_2})_{m_2}\dots(A_{i_k})_{m_k} Id)$   
 $= \dots = A_{i_1}(m_1)\dots A_{i_k}(m_k)\Omega$

Now, the action of the  $A_i(m)$  on  $\Omega$  generates  $H$  by definition 3.46. □

**Lemma 3.56.** Let  $A \in \mathcal{S}$ , then  $A(z)\Omega = e^{zT}R(A)$ .

*Proof.* Let  $F_A(z) = A(z)\Omega = \sum_{n \in \mathbb{N}} A(-n-1)\Omega z^n$ ,

Then,  $\forall b \in H, (F_A(z), b) \in \mathbb{C}[z]$

Now,  $\frac{d}{dz}(F_A(z), b) = (\frac{d}{dz}F_A(z), b) = (A'(z)\Omega, b)$

$= ([T, A(z)]\Omega, b) = (T.A(z)\Omega, b) = (T.F_A(z)\Omega, b)$

But,  $F_A(0) = R(A)$ , so we see that:  $(F_A(z), b) = (e^{zT}R(A), b) \forall b \in H$

Finally,  $F_A(z) = e^{zT}R(A)$  □

**Lemma 3.57.** (Unicity)  $R(A) = R(B) \Rightarrow A = B$ .

*Proof.* Let  $C = A - B$ , then  $R(C) = R(A) - R(B) = 0$

and  $F_C(z) = e^{zT}R(C) = 0$

Now,  $\forall e \in H, \exists E \in \mathcal{S}$  such that  $R(E) = e$ .

Then  $\forall f \in H, \exists N \in \mathbb{N} \exists \varepsilon \in \mathbb{Z}_2$  such that :

$(z-w)^N(C(z)E(w)\Omega, f) = (-1)^\varepsilon(z-w)^N(E(w)C(z)\Omega, f)$

Now,  $(E(w)C(z)\Omega, f) = (E(w)F_C(z), f) = 0 = (C(z)E(w)\Omega, f)$

So,  $(C(z)E(w)\Omega, f)|_{w=0} = (C(z)e, f) = 0 \forall e, f \in H$

Finally,  $C = 0$  and  $A = B$  □

Now, we can well defined:

**Definition 3.58.** (*State-Field correspondence map*)

$$V : \begin{array}{ccc} H & \longrightarrow & \mathcal{S} \\ a & \longmapsto & V(a) \end{array} \quad \text{linear.}$$

$$\text{such that : } \begin{cases} \forall a \in H & R(V(a)) = a \\ \forall A \in \mathcal{S} & V(R(A)) = A \end{cases}$$

**Notation 3.59.**  $V(a)(z)$  is noted  $V(a, z)$  and  $A(z) = V(R(A), z)$

**Examples 3.60.**

(a)  $V(0, z) = 0, \quad V(\Omega, z) = Id$

(b)  $V'(a, z) = V(T.a, z)$

(c)  $(A_n B)(z) = V(A(n)R(B), z)$

**Definition 3.61.** Let  $H_\varepsilon = \bigoplus_{n \in \mathbb{N} + \frac{\varepsilon}{2}} H_n$  so that  $H = H_0 \oplus H_1$ .

**Lemma 3.62.**  $R(\mathcal{S}_\varepsilon) = H_\varepsilon \quad (\varepsilon \in \mathbb{Z}_2)$

*Proof.* Base step: by definition 3.46 and lemma 3.54,

$\forall i \ A_i \in \mathcal{S}_{\varepsilon_{ii}}$  and  $R(A_i) \in H_{\alpha_i}$  with  $\alpha_i \in \mathbb{N} + \frac{\varepsilon_{ii}}{2}$

Inductive step: by lemma 3.50 □

**Corollary 3.63.** (*Relation with T and D*) Let  $a \in H_\alpha$ , we have that:

(a)  $[T, V(a, z)] = V'(a, z) = V(T.a, z) \in \mathcal{S}$

(b)  $[D, V(a, z)] = z.V'(a, z) + \alpha.V(a, z) \quad (\notin \mathcal{S} \text{ in general})$

### 3.4 Application to fermion algebra

$$H = \mathcal{F}_{NS}, \quad \psi(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1} \text{ with } [\psi_m, \psi_n]_+ = \delta_{m+n} Id.$$

**Proposition 3.64.**  $\{\psi\}$  is a system of generator.



*Proof.*  $\psi$  is local with itself with  $N = 1$  and  $\varepsilon = \bar{1} = \bar{1}.\bar{1}$  (see definition 3.46)  
 We have construct  $D$  and  $T$  (p 14),  $\Omega \in H_0$ ,  $\|\Omega\| = 1$ ,  $D\Omega = T\Omega = 0$ .  
 $[T, \psi(z)] = \psi'(z)$ ,  $[D, \psi(z)] = z.\psi'(z) + \frac{1}{2}\psi(z)$  and  $\frac{1}{2} \in \mathbb{N} + \frac{1}{2}$   
 Finally,  $\{\psi_n, n \in \frac{1}{2}\mathbb{N}\}$  acts irreducibly on  $H$   $\square$

**Corollary 3.65.**  $\{\psi\}$  generates an  $\mathcal{S}$  with a state-field correspondence with:

$$R(\psi) = \psi_{-\frac{1}{2}}\Omega \text{ and } \psi(z) = V(\psi_{-\frac{1}{2}}\Omega, z)$$

**Lemma 3.66.** (OPE)  $\psi(z)\psi(w) \sim \frac{Id}{z-w}$

*Proof.*  $\psi_n\psi(w) = V(\psi_{n+\frac{1}{2}}\psi_{-\frac{1}{2}}\Omega, w) = 0$  if  $n \geq 1$  ( here  $N = 1$  )

Now, for  $0 \leq n \leq N - 1$  i.e  $n = 0$  :

$$\psi_{\frac{1}{2}}\psi_{-\frac{1}{2}}\Omega = ([\psi_{\frac{1}{2}}, \psi_{-\frac{1}{2}}]_+ - \psi_{-\frac{1}{2}}\psi_{\frac{1}{2}})\Omega = \Omega, \text{ so } \psi_0\psi(w) = Id \quad \square$$

**Remark 3.67.** (Next operator)  $\psi_{-\frac{1}{2}}\psi_{-\frac{1}{2}}\Omega = 0$ , so  $\psi_{-1}\psi = 0$ ; and the next

operator of the expansion is  $2L(w) := \psi_{-2}\psi(w) = 2 \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$

Now,  $R(L) = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$ , then  $L(w) = V(\frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega, w)$ .

**Remark 3.68.**  $L(n) = L_{n-1}$  so,  $L_0\Omega = L_{-1}\Omega = 0$  by lemma 3.45.

**Lemma 3.69.** (OPE)  $\psi(z)L(w) \sim \frac{1/2\psi(w)}{(z-w)^2} - \frac{1/2\psi'(w)}{(z-w)}$

*Proof.*  $\psi_n L(w) = \frac{1}{2}V(\psi_{n+\frac{1}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega, w) = 0$  if  $n \geq 2$  ( here  $N = 2$  )

Now,  $\psi_{\frac{1}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega = -\psi_{-\frac{3}{2}}\Omega = R(\psi')$  ,  $\psi_{\frac{3}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega = \psi_{-\frac{1}{2}}\Omega = R(\psi')$   $\square$

**Lemma 3.70.** (Lie bracket)  $[L_m, \psi_n] = -(n + \frac{1}{2}m)\psi_{m+n}$

*Proof.* By lemma 3.50,  $\psi$  and  $L$  are local with  $\varepsilon = \bar{0}$ , and by formula 3.31:

$$\begin{aligned} [\psi(m), L(n+1)] &= -\frac{1}{2}C_m^0\psi'(m+n+1) + \frac{1}{2}C_m^1\psi(m+n+1-1) \\ &= \frac{1}{2}(m+n+1)\psi(m+n) + \frac{1}{2}m\psi(m+n) = (m + \frac{1}{2} + \frac{1}{2}n)\psi(m+n) \end{aligned}$$

We have computed for  $m \geq 0$ , we find the same result for  $m < 0$ .

Now,  $\psi(m) = \psi_{m+\frac{1}{2}}$  and  $L(n+1) = L_n$ , so the result follows.  $\square$

**Lemma 3.71.**  $D = L_0$  and  $T = L_{-1}$

*Proof.*  $[L_0, \psi_n] = -n\psi_n = [D, \psi_n]$  ,  $[L_{-1}, \psi_n] = -(n - \frac{1}{2})\psi_{n-1} = [T, \psi_n]$

So, by irreducibility and Schur's lemma,  $L_0 - D$  and  $L_{-1} - T \in \mathbb{C}Id$

Now,  $L_0\Omega = D\Omega = L_{-1}\Omega = T\Omega = 0$ , then,  $D = L_0$  and  $T = L_{-1}$   $\square$

**Corollary 3.72.**  $\forall a \in H_s$ :

(a)  $[L_{-1}, V(a, z)] = V'(a, z) = V(L_{-1}.a, z) \in \mathcal{S}$

(b)  $[L_0, V(a, z)] = z.V'(a, z) + s.V(a, z)$

**Remark 3.73.**  $\forall A \in \mathcal{S}$ ,  $A' = (L_0A)$ , so, by Dong's lemma, we finally don't need here to  $A \mapsto A'$  for the construction of  $\mathcal{S}$ .

**Lemma 3.74.** (OPE)  $L(z)L(w) \sim \frac{(c/2)Id}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{(z-w)}$

*Proof.*  $L_n L(w) = V(L(n)L(-1)\Omega, w) = V(L_{n-1}L_{-2}\Omega, w) = 0$  if  $n \geq 4$ .

Then, here,  $N = 4$ , so, for  $0 \leq n \leq N - 1$ :

(a)  $V(L_{-1}L_{-2}\Omega, w) = L'(w)$

(b)  $L_0L_{-2}\Omega = 2L_{-2}\Omega = 2R(L)$  because  $L_{-2}\Omega \in H_2$

(c)  $L_1L_{-2}\Omega = \frac{1}{2}L_1\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega = \frac{1}{2}[L_1, \psi_{-\frac{3}{2}}]\psi_{-\frac{1}{2}}\Omega = \frac{1}{2}\psi_{-\frac{1}{2}}^2\Omega = 0$

(d)  $L_2L_{-2}\Omega \in H_0 = \mathbb{C}\Omega$ , so,  $L_2L_{-2}\Omega = K\Omega$  with  $K = \|L_{-2}\Omega\|^2$

□

**Notation 3.75.**  $c := 2\|L_{-2}\Omega\|^2$ , the central charge.

(here  $c = \frac{1}{2}(\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega, \psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega) = \frac{1}{2}$ )

**Notation 3.76.** Let  $\delta_k = \begin{cases} 0 & \text{if } k \neq 0 \\ Id & \text{if } k = 0 \end{cases}$

**Lemma 3.77.** (Lie bracket)  $[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n}$ .

*Proof.* By lemma 3.50,  $L \in \mathcal{S}_0$ , and by formula 3.31:

If  $m + 1 \geq 0$ , then:  $[L(m + 1), L(n + 1)] =$

$$C_{m+1}^0 L'(m + n + 2) + 2C_{m+1}^1 L(m + n + 2 - 1) + \frac{c}{2}C_{m+1}^3 Id(m + n + 2 - 3)$$

$$= -(m + n + 2)L(m + n + 2) + 2(m + 1)L(m + n + 1) + \frac{c}{2}\frac{m(m^2-1)}{6}\delta_{m+n}$$

$$= (m - n)L(m + n + 1) + \frac{c}{12}m(m^2 - 1)\delta_{m+n}$$

We find the same result for  $m + 1 < 0$

□

**Remark 3.78.**  $L_m^* = L_{-m}$

*Proof.*  $[\psi_{-n}, L_m^*] = [L_m, \psi_n]^* = -(n + \frac{1}{2}m)\psi_{-m-n} = [\psi_{-n}, L_{-m}]$ , then the result follows by irreducibility, Schur's lemma and grading. □

**Remark 3.79.** The  $(L_n)$  generate a Virasoro algebra  $\mathfrak{Vir}$ .

**Corollary 3.80.**  $\mathfrak{Vir}$  acts on  $H = \mathcal{F}_{NS}$ , and admits  $L(c, h) = L(\frac{1}{2}, 0)$  as minimal submodule containing  $\Omega$ .

**Definition 3.81.** Let call  $L$  the Virasoro operator, and  $\omega = R(L) = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$ , the Virasoro vector.

### 3.5 Vertex operator superalgebra

**Definition 3.82.** A vertex operator superalgebra is an  $(H, V, \Omega, \omega)$  with:

- (a)  $H = H_{\bar{0}} \oplus H_{\bar{1}}$  a prehilbert superspace.
- (b)  $V : H \rightarrow (\text{End}H)[[z, z^{-1}]]$  a linear map.
- (c)  $\Omega, \omega \in H$  the vacuum and Virasoro vectors.

Let  $\mathcal{S}_\varepsilon = V(H_\varepsilon)$ ,  $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$  and  $A(z) = V(a, z) = \sum_{n \in \mathbb{Z}} A(n)z^{-n-1}$ , then  $(H, V, \Omega, \omega)$  satisfies the followings axioms:

1. (vacuum axioms):  $\forall A \in \mathcal{S}$  and  $\forall n \in \mathbb{N}$ ,  $A(n)\Omega = 0$ ,  
 $V(a, z)\Omega|_{z=0} = a$  and  $V(\Omega, z) = Id$
2. (irreducibility axiom): Let  $\mathcal{A} = \{A(n) | A \in \mathcal{S}, n \in \mathbb{Z}\}$  then,  
 $\mathcal{A}.$  acts irreducibly on  $H$ , so that  $\mathcal{A}.\Omega = H$
3. (locality axiom):  $\forall A \in \mathcal{S}_{\varepsilon_1}$ ,  $\forall B \in \mathcal{S}_{\varepsilon_2}$ ,  $A$  and  $B$  are local  
 (see definition 3.20 and lemma 3.37), with  $\varepsilon = \varepsilon_1.\varepsilon_2$  and  $A_n B \in \mathcal{S}_{\varepsilon_1 + \varepsilon_2}$
4. (Virasoro axiom):  $V(\omega, z) = L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$  Virasoro operator  
 $(L_0\Omega = L_{-1}\Omega = 0$  and  $\omega = L_{-2}\Omega)$ . Let  $c = 2\|\omega\|^2$  the central charge:  
 $[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n}$
5. ( $L_0$  axioms)  $L_0$  decomposes  $H$  into  $\bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  with  $\dim(H_n) < \infty$ ,  
 $H_n \perp H_m$  if  $n \neq m$ ,  $H_\varepsilon = \bigoplus_{n \in \mathbb{N} + \frac{\varepsilon}{2}} H_n$ ,  $\Omega \in H_0$ ,  $\omega \in H_2$ , and  
 $\forall a \in H_\alpha$ ,  $[L_0, V(a, z)] = z.V'(a, z) + \alpha.V(a, z)$
6. ( $L_{-1}$  axioms):  $[L_{-1}, V(a, z)] = V'(a, z) = V(L_{-1}.a, z) \in \mathcal{S}$

**Corollary 3.83.** *A system of generators, generating a Virasoro operator  $L \in \mathcal{S}$ , with  $D = L_0$  and  $T = L_{-1}$ , generates a vertex operator superalgebra.*

**Corollary 3.84.** *The fermion operator  $\psi$  generates a vertex operator superalgebra, with Virasoro vector  $\omega = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$ .*

**Remark 3.85.** *The Virasoro operator  $L$  alone, generates the minimal vertex operator (super)algebra.*

**Remark 3.86.** *Let  $A(z) = V(a, z)$  and  $B(w) = V(b, w)$ ; the formula 3.26 is general, so similarly, by vacuum axioms,  $A_n B(w) = V(A(n)b, w)$ .*

**Proposition 3.87.** *(Borcherds associativity)  $\exists N \in \mathbb{N}$  such that  $\forall c, d \in H$ :  $(z - w)^N (V(a, z)V(b, w)c, d) = (z - w)^N (V(V(a, z - w)b, w)c, d)$*

*Proof.* To simplify the proof, we don't write:  
 $\exists N \in \mathbb{N}$  such that  $\forall c, d \in H (z - w)^N (\cdot, c, d)$ , but it is implicit.

$$\begin{aligned} V(a, z)V(b, w) &= A(z)B(w) = \sum A_n B(w)(z - w)^{-n-1} \\ &= \sum V(A(n)b, w)(z - w)^{-n-1} = V(\sum A(n)b(z - w)^{-n-1}, w) \\ &= V(\sum A(n)(z - w)^{-n-1}b, w) = V(V(a, z - w)b, w). \end{aligned}$$

□

## 4 Vertex $\mathfrak{g}$ -superalgebras and modules

### 4.1 Preliminaries

#### 4.1.1 Simple Lie algebra $\mathfrak{g}$

Let  $\mathfrak{g}$  be a simple Lie algebra of dimension  $N$ , a basis  $(X_a)$  with  $[X_a, X_b] = i \sum_c \Gamma_{ab}^c X_c$  with  $\Gamma_{ab}^c \in \mathbb{R}$  totally antisymmetric.

**Lemma 4.1.** *Let  $\mathcal{C} = \sum_b X_b^2$ , then  $[\mathfrak{g}, \mathcal{C}] = 0$*

*Proof.* It suffices to prove  $[X_a, \mathcal{C}] = 0$  for each  $X_a$ .

$$[X_a, \mathcal{C}] = \sum_b [X_a, X_b^2] = \sum_b ([X_a, X_b]X_b + X_b[X_a, X_b]) = i \sum_{b,c} \Gamma_{ab}^c X_c X_b + i \sum_{b,c} \Gamma_{ab}^c X_b X_c = i \sum_{b,c} (\Gamma_{ab}^c + \Gamma_{ac}^b) X_c X_b = 0 \quad \text{by antisymmetry.} \quad \square$$

**Remark 4.2.**  $\mathcal{C}$  is a multiple of the **Casimir** of  $\mathfrak{g}$ . We suppose to have well normalized the basis such that  $\mathcal{C}$  is exactly the Casimir.

**Corollary 4.3.** *By Schur's lemma,  $\mathcal{C}$  acts as multiplicative constant  $c_V$  on each irreducible representation  $V$ .*

**Example 4.4.**  $\mathfrak{g}$  is simple, it acts irreducibly on  $V = \mathfrak{g}$  with  $ad$ .

**Lemma 4.5.**  $\sum_{a,c} \Gamma_{ac}^b \cdot \Gamma_{ac}^d = \delta_{bd} c_{\mathfrak{g}}$

$$\begin{aligned} \text{Proof. } (\sum_a ad_{X_a}^2)(X_b) &= c_{\mathfrak{g}} X_b = \sum_a [X_a, [X_a, X_b]] \\ &= i^2 \sum_{a,c,d} \Gamma_{ab}^c \cdot \Gamma_{ac}^d X_d = \sum_{a,c,d} \Gamma_{ac}^b \cdot \Gamma_{ac}^d X_d. \end{aligned}$$

$$\text{Then, } \sum_{a,c} \Gamma_{ac}^b \cdot \Gamma_{ac}^d = \delta_{bd} c_{\mathfrak{g}} \quad \square$$

**Definition 4.6.**  $g = \frac{c_{\mathfrak{g}}}{2}$  is called the dual Coxeter number.

**Example 4.7.**  $\mathfrak{g} = A_1 = \mathfrak{sl}_2$ ,  $dim(\mathfrak{g}) = 3$

$$[E, F] = H, [H, E] = 2E, [H, F] = -2F, \text{ with Casimir } EF + FE + \frac{1}{2}H^2$$

We choose the basis:  $X_1 = \frac{i\sqrt{2}}{2}(E - F)$ ,  $X_2 = \frac{\sqrt{2}}{2}(E + F)$ ,  $X_3 = \frac{\sqrt{2}}{2}H$ ,  
with relations:  $[X_1, X_2] = i\sqrt{2}X_3$ ,  $[X_3, X_1] = i\sqrt{2}X_2$ ,  $[X_2, X_3] = i\sqrt{2}X_1$   
 $\mathcal{C} = \sum_a X_a^2 = EF + FE + \frac{1}{2}H^2$  and  $g = \frac{1}{2} \sum_{a,b} (\Gamma_{ab}^c)^2 = 2$

Table (see [7] p 111)

$\mathfrak{g}$	$A_n$	$B_n$	$C_n$	$D_n$	$E_6$	$E_7$	$E_8$	$F_4$	$G_2$
$dim(\mathfrak{g})$	$n^2 + 2n$	$2n^2 + n$	$2n^2 + n$	$2n^2 - n$	78	133	248	52	14
$g$	$n + 1$	$2n - 1$	$n + 1$	$2n - 2$	12	18	30	9	4

### 4.1.2 Loop algebra $L\mathfrak{g}$

**Definition 4.8.** Let  $L\mathfrak{g} = C^\infty(\mathbb{S}^1, \mathfrak{g})$  the loop algebra of  $\mathfrak{g}$ . It's an infinite dimensional Lie  $\star$ -algebra, admitting the  $X_n^a = X_a e^{in\theta}$  as basis, with  $n \in \mathbb{Z}$  and  $(X_a)$  the base of  $\mathfrak{g}$ ; so:

$$[X_m^a, X_n^b] = [X_a, X_b]_{m+n} \quad \text{and} \quad (X_n^a)^\star = X_{-n}^a$$

**Proposition 4.9.** (Boson cocycle)  $L\mathfrak{g}$  has a unique central extension, up to equivalent, i.e.  $H_2(L\mathfrak{g}, \mathbb{C})$  is 1-dimensional.  $H_2(L\mathfrak{g}, \mathbb{C})$  is 1-dimensional. Let  $\mathcal{L}$  the central element and  $\widehat{\mathfrak{g}}_+ = L\mathfrak{g} \oplus \mathbb{C}\mathcal{L}$  called  $\mathfrak{g}$ -boson algebra, then:

$$[X_m^a, X_n^b] = [X_a, X_b]_{m+n} + m\delta_{ab}\delta_{m+n}\mathcal{L}$$

*Proof.* See [13] or [18] p 46. □

**Theorem 4.10.** The unitary highest weight representations of  $\widehat{\mathfrak{g}}_+$  are  $H = L(V_\lambda, \ell)$  with:

- (a)  $\ell \in \mathbb{N}$  such that  $\mathcal{L}\Omega = \ell\Omega$  (the level of  $H$ ).
- (b)  $H_0 = V_\lambda$  irreducible representation of  $\mathfrak{g}$  such that:  
 $(\lambda, \theta) \leq \ell$  with  $\lambda$  the highest weight and  $\theta$  the highest root.

*Proof.* See [13] or [18] p 48. □

**Remark 4.11.** Let  $\mathcal{C}_\ell$  the category of such representations for  $\ell$  fixed.  $\mathcal{C}_\ell$  is a finite set and  $\mathcal{C}_\ell \subset \mathcal{C}_{\ell+1}$

**Remark 4.12.** The irreducible unitary projective positive energy representations of  $L\mathfrak{g}$  are given by the unitary highest weight representation of  $\widehat{\mathfrak{g}}_+$ .

**Example 4.13.** We take  $\mathfrak{g} = \mathfrak{sl}_2$ , then  $H = L(j, \ell)$  with:

- $\mathcal{L}\Omega = \ell\Omega, \quad \ell \in \mathbb{N}$
- $H_0 = V_j$  with  $j \in \frac{1}{2}\mathbb{N}$  the spin and  $j \leq \frac{\ell}{2}$ , such that  
 $\mathcal{C}\Omega = c_{V_j}\Omega$  with  $\mathcal{C} = \sum_a (X_0^a)^2$  the Casimir and  $c_{V_j} = 2j^2 + 2j$

## 4.2 $\mathfrak{g}$ -vertex operator superalgebras

### 4.2.1 $\mathfrak{g}$ -fermion

**Definition 4.14.** Let  $\widehat{\mathfrak{g}}_-$  be the  $\mathfrak{g}$ -fermion algebra, generated by  $(\psi_m^a)$  with  $a \in \{1, \dots, N\}$ ,  $N = \dim(\mathfrak{g})$ ,  $m \in \mathbb{Z} + \frac{1}{2}$  and relations:

$$[\psi_m^a, \psi_n^b]_+ = \delta_{ab} \delta_{m+n} \quad \text{and} \quad (\psi_m^a)^* = \psi_{-m}^a$$

**Remark 4.15.** As for the fermion algebra of section 3.1, we generate the Verma module  $H = \mathcal{F}_{NS}^{\mathfrak{g}}$ , and the sesquilinear form  $(\cdot, \cdot)$  which is a scalar product;  $\pi(\psi_n^a)^* = \pi((\psi_n^a)^*)$ ,  $\mathcal{F}_{NS}^{\mathfrak{g}}$  is a prehilbert space, an irreducible representation of  $\widehat{\mathfrak{g}}_-$  and its unique unitary highest weight representation.

**Definition 4.16.** Let  $\psi^a(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}}^a \cdot z^{-n-1}$  the fermion operators.

**Remark 4.17.**  $\psi^a(z)\psi^b(w) \sim \frac{\delta_{ab}}{(z-w)}$

**Remark 4.18.** As for the single fermion operator  $\psi$ , of section 3.4,  $\{\psi^a, a \in \{1, \dots, N\}\}$  generates a vertex operator superalgebra with:

$$\omega = \frac{1}{2} \sum_a \psi_{-\frac{3}{2}}^a \psi_{-\frac{1}{2}}^a \Omega \quad \text{and} \quad c = 2\|\omega\|^2 = \frac{\dim(\mathfrak{g})}{2}$$

**Definition 4.19.** Let  $S^c(z) = V(s^c, z) = \sum_{n \in \mathbb{Z}} S_n^c z^{-n-1}$  with:

$$s^c = -\frac{i}{2} \sum_{a,b} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \Omega \in H_1 \subset H_{\bar{0}}$$

**Lemma 4.20.** (OPE and Lie bracket)

$$\psi^a(z)S^b(w) \sim \frac{i \sum_c \Gamma_{ab}^c \psi^c(w)}{(z-w)} \quad \text{and} \quad [\psi_m^a, S_n^b] = i \sum_c \Gamma_{ab}^c \psi_{m+n}^c = [S_m^a, \psi_n^b]$$

*Proof.*  $\psi_{n+\frac{1}{2}}^d \cdot s^c = 0$  if  $n \geq 1$  and  $\psi_{\frac{1}{2}}^d \cdot s^c = i \sum_a \Gamma_{dc}^a \psi_{-\frac{1}{2}}^a \Omega$ . □

**Remark 4.21.**  $[S_m^a, \psi_n^a] = 0$

**Lemma 4.22.**  $(S_m^b)^* = S_{-m}^b$

*Proof.*  $[(S_n^b)^*, \psi_{-m}^a] = [\psi_m^a, S_n^b]^* = -i \sum_c \Gamma_{ab}^c \psi_{-m-n}^c = [S_{-n}^b, \psi_{-m}^a]$   
The result follows by irreducibility, Schur's lemma and grading. □

**Remark 4.23.** (Jacobi)  $[X_a, [X_b, X_c]] = [[X_a, X_b], X_c] + [X_b, [X_a, X_c]]$   
 $\Leftrightarrow \sum_d \Gamma_{bc}^d \Gamma_{ad}^e = \sum_d (\Gamma_{ab}^d \Gamma_{dc}^e + \Gamma_{ac}^d \Gamma_{bd}^e) \Leftrightarrow \sum_e (\Gamma_{ab}^e \Gamma_{cd}^e + \Gamma_{da}^e \Gamma_{ce}^e + \Gamma_{db}^e \Gamma_{ac}^e) = 0$

**Notation 4.24.**  $[S^a, S^b] := i \sum_c \Gamma_{ab}^c S^c$

**Lemma 4.25.** (OPE and Lie bracket)

$$S^a(z)S^b(w) \sim \frac{[S^a, S^b](w)}{(z-w)} + \frac{g \cdot \delta_{ab}}{(z-w)^2}$$

$$\text{and } [S_m^a, S_n^b] = [S^a, S^b](m+n) + \ell \cdot m \delta_{ab} \delta_{m+n} \quad (\text{with } \ell = g \in \mathbb{N})$$

*Proof.*  $S_n^d S^c = 0$  if  $n \geq 2$  and:

$$\begin{aligned} \text{(a)} \quad S_0^d S^c &= -\frac{i}{2} \sum_{a,b} \Gamma_{ab}^c S_0^d \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \Omega \\ &= -\frac{i}{2} (i \sum_{a,b,e} \Gamma_{ab}^c \Gamma_{da}^e \psi_{-\frac{1}{2}}^e \psi_{-\frac{1}{2}}^b \Omega + i \sum_{a,b,e} \Gamma_{ab}^c \Gamma_{db}^e \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^e \Omega) \\ &= -\frac{i}{2} (i \sum_{a,b,e} (\Gamma_{eb}^c \Gamma_{de}^a + \Gamma_{ae}^c \Gamma_{de}^b) \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \Omega) \\ &= i \sum_e \Gamma_{dc}^e \frac{-i}{2} \sum_{a,b} \Gamma_{ab}^e \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \Omega = i \sum_e \Gamma_{dc}^e S^e = [S^d, S^c](-1) \end{aligned}$$

$$\text{(b)} \quad S_1^d S^c = -\frac{i}{2} i \sum_{a,b,e} \Gamma_{ab}^c \Gamma_{da}^e \psi_{\frac{1}{2}}^e \psi_{-\frac{1}{2}}^b \Omega = \frac{1}{2} \sum_{a,b} \Gamma_{ab}^c \Gamma_{ab}^d = g \cdot \delta_{cd}$$

□

**Corollary 4.26.**  $(S_m^a)$  is the basis of a  $\mathfrak{g}$ -boson algebra.

It admits  $L(V_0, g)$  as minimal submodule of  $\mathcal{F}_{NS}^{\mathfrak{g}}$  containing  $\Omega$  (with  $V_0 = \mathbb{C}$  the trivial representation of  $\mathfrak{g}$ ).

**Lemma 4.27.**  $\sum_a (S_{-1}^a)^2 \Omega = 4g\omega$

$$\begin{aligned} \text{Proof.} \quad \sum_e (S_{-1}^e)^2 \Omega &= -\frac{i}{2} \sum_{a,b,e} \Gamma_{ab}^c S_{-1}^e \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \Omega \\ &= -\frac{1}{4} \sum_{a,b,c,d,e} \Gamma_{ab}^e \Gamma_{cd}^e \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \psi_{-\frac{1}{2}}^c \psi_{-\frac{1}{2}}^d \Omega - \frac{i}{2} \sum_{a,b,c} \Gamma_{ab}^e [S_{-1}^e, \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b] \Omega \\ &= -\frac{1}{12} \sum_{a,b,c,d} (\sum_e (\Gamma_{ab}^e \Gamma_{cd}^e + \Gamma_{da}^e \Gamma_{cb}^e + \Gamma_{db}^e \Gamma_{ac}^e) \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \psi_{-\frac{1}{2}}^c \psi_{-\frac{1}{2}}^d \Omega) \\ &+ \sum_{a,b,c,e} \Gamma_{ea}^b \Gamma_{ea}^c \psi_{-\frac{3}{2}}^c \psi_{-\frac{1}{2}}^b \Omega = 4g\omega \end{aligned}$$

□

**Lemma 4.28.** (OPE and Lie bracket)

$$S^a(z)L(w) \sim \frac{S^a(w)}{(z-w)^2} \quad \text{and} \quad [L_m, S_n^a] = -n S_{m+n}^a$$



*Proof.*  $S_n^a \cdot \omega = 0$  for  $n \geq 3$  and:

$$(a) \quad S_0^a \cdot \omega = \frac{1}{4g} \sum_b S_0^a (S_{-1}^b)^2 \Omega = \frac{1}{4g} \sum_b ([S_0^a, S_{-1}^b] S_{-1}^b \Omega + S_{-1}^b [S_0^a, S_{-1}^b] \Omega) \\ = \frac{i}{4g} \sum_{b,c} (\Gamma_{ab}^c + \Gamma_{ac}^b) S_{-1}^c S_{-1}^b \Omega = 0$$

$$(b) \quad S_2^a \cdot \omega = \frac{1}{4g} \sum_b ([S_2^a, S_{-1}^b] S_{-1}^b \Omega + S_{-1}^b [S_2^a, S_{-1}^b] \Omega) \\ = \frac{i}{4g} \sum_{b,c} \Gamma_{ab}^c S_1^c S_{-1}^b \Omega = \frac{i}{4g} \sum_{b,c} \Gamma_{ab}^c \delta_{bc} \ell = 0$$

$$(c) \quad S_1^a \cdot \omega = \frac{1}{4g} \sum_b ([S_1^a, S_{-1}^b] S_{-1}^b \Omega + S_{-1}^b [S_1^a, S_{-1}^b] \Omega) \\ = \frac{i}{4g} (2\ell + i \sum_{b,c} \Gamma_{ab}^c S_0^c S_{-1}^b \Omega) = \frac{2(\ell+g)}{4g} S_{-1}^a \Omega = S_{-1}^a \Omega \quad (\star)$$

□

**Corollary 4.29.**  $(S^a)$  generate a vertex operator (super)algebra with  $\omega = \frac{1}{4g} \sum_a (S_{-1}^a)^2 \Omega$  as Virasoro vector.

#### 4.2.2 $\mathfrak{g}$ -boson

**Definition 4.30.** Let  $X^a(z) = \sum_{n \in \mathbb{Z}} X_n^a z^{-n-1}$  the boson operators with  $[X_m^a, X_n^b] = [X^a, X^b]_{m+n} + m \delta_{ab} \delta_{m+n} \cdot \mathcal{L}$

**Corollary 4.31.** The  $\mathfrak{g}$ -boson algebra  $\widehat{\mathfrak{g}}_+$  generates a vertex operator (super)algebra on  $H = L(V_0, g)$ , and also on  $H = L(V_0, \ell)$  for any  $\ell \in \mathbb{N}$ , with  $\omega = \frac{1}{2(\ell+g)} \sum_a (X_{-1}^a)^2 \Omega$  as Virasoro vector; and:

$$X^a(z) X^b(w) \sim \frac{[X^a, X^b](w)}{(z-w)} + \frac{g \cdot \delta_{ab}}{(z-w)^2}$$

$$X^a(z) L(w) \sim \frac{X^a(w)}{(z-w)^2} \quad \text{and} \quad [L_m, X_n^a] = -n X_{m+n}^a$$

*Proof.* By the previous work on  $(S^a)$  and  $(\star)$ . □

**Lemma 4.32.**  $c = 2\|\omega\|^2 = \frac{\ell \dim(\mathfrak{g})}{\ell+g}$

$$\begin{aligned} \text{Proof.} \quad & 4(\ell+g)^2 \|\omega\|^2 = \sum_{a,b} ((X_{-1}^a)^2 \Omega, (X_{-1}^b)^2 \Omega) = \sum_{a,b} (\Omega, (X_1^a)^2 (X_{-1}^b)^2 \Omega) \\ & = \sum_{a,b} (\Omega, X_1^a X_{-1}^b [X_1^a, X_{-1}^b] \Omega + X_1^a [X_1^a, X_{-1}^b] X_{-1}^b \Omega) \\ & = (\sum_{a,b,c} i \Gamma_{ab}^c (\Omega, X_1^a X_0^c X_{-1}^b \Omega)) + 2\ell \sum_a (\Omega, X_1^a X_{-1}^a \Omega) \\ & = (\sum_{a,b,c,d} (-1) \Gamma_{ab}^c \Gamma_{cb}^d (\Omega, X_1^a X_{-1}^d \Omega) + 2\ell^2 \dim(\mathfrak{g})) \\ & = (2g \ell \dim(\mathfrak{g}) + 2\ell^2 \dim(\mathfrak{g})) = 2\ell \dim(\mathfrak{g}) (\ell+g) \end{aligned} \quad \square$$

**Remark 4.33.** *By vacuum axiom of vertex operator superalgebra,  $X_0^a \Omega = 0$ , then, the representation  $H_0 = V_\lambda$  of  $\mathfrak{g}$  is necessary the trivial one  $V_0$ . At section 4.3, we see that general  $L(V_\lambda, \ell)$  admits the structure of vertex module over  $L(V_0, \ell)$ .*

### 4.2.3 $\mathfrak{g}$ -supersymmetry

By lemma 4.20, the  $\mathfrak{g}$ -boson algebra  $\widehat{\mathfrak{g}}_+$  acts on the  $\mathfrak{g}$ -fermion algebra  $\widehat{\mathfrak{g}}_-$ , then, we can build their semi-direct product:

**Definition 4.34.** *Let  $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \ltimes \widehat{\mathfrak{g}}_-$  the  $\mathfrak{g}$ -supersymmetric algebra.*

**Proposition 4.35.** *The unitary highest weight representations (irreducible) of  $\widehat{\mathfrak{g}}$  are  $H = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  (see [6]).*

*Proof.* Let  $H$  be such a representation of  $\widehat{\mathfrak{g}}$ , then,  $\widehat{\mathfrak{g}}_-$  acts on, but it admits a unique irreducible representation:  $\mathcal{F}_{NS}^{\mathfrak{g}}$ , so  $H = M \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ , with  $M$  a multiplicity space. Now,  $\widehat{\mathfrak{g}}_+$  acts on  $H$  and on  $\mathcal{F}_{NS}^{\mathfrak{g}}$  (corollary 4.26), and the difference commutes with  $\widehat{\mathfrak{g}}_-$ ; but  $\widehat{\mathfrak{g}}_-$  acts irreducibly on  $\mathcal{F}_{NS}^{\mathfrak{g}}$ , so, the commutant of  $\widehat{\mathfrak{g}}_-$  is  $End(M) \otimes \mathbb{C}$  by Schur's lemma. So,  $\widehat{\mathfrak{g}}_+$  acts on  $M$ , and this action is necessarily irreducible. Finally, by unitary highest weight context,  $\exists \lambda$  such that  $M = L(V_\lambda, \ell)$ .  $\square$

**Remark 4.36.** *Using the previous notations,  $\widehat{\mathfrak{g}}_+$  acts on  $L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  as  $B_n^a = X_n^a + S_n^a$ , bosons of level  $d = \ell + g$ .*

**Corollary 4.37.** *From  $(\psi^a(z))$  and  $(B^a(z))$ , we generate  $S^a(z)$  and  $X^a = B^a - S^a$  a vertex operator superalgebra on  $H = L(V_0, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  with the Virasoro vector:*

$$\omega = \frac{1}{2} \sum_a \psi_{-\frac{3}{2}}^a \psi_{-\frac{1}{2}}^a \Omega + \frac{1}{2(\ell + g)} \sum_a (X_{-1}^a)^2 \Omega \quad \text{and :}$$

$$c = 2\|\omega\|^2 = \frac{dim(\mathfrak{g})}{2} + \frac{\ell dim(\mathfrak{g})}{\ell + g} = \frac{3}{2} \cdot \frac{\ell + \frac{1}{3}g}{\ell + g} dim(\mathfrak{g})$$

**Definition 4.38.** *(SuperVirasoro operator)*

*Let  $\tau_1 = \sum_a \psi_{-\frac{1}{2}}^a X_{-1}^a \Omega$ ,  $\tau_2 = \frac{1}{3} \sum_a \psi_{-\frac{1}{2}}^a S_{-1}^a \Omega$  and  $\tau = (\ell + g)^{-\frac{1}{2}} (\tau_1 + \tau_2)$ . Let  $G(z) = V(\tau, z) = \sum_{n \in \mathbb{Z}} G_{n-\frac{1}{2}} z^{-n-1} = \sum_{n \in \mathbb{Z} + \frac{1}{2}} G_n z^{-n-\frac{3}{2}}$*

**Proposition 4.39.** (*Supersymmetry boson-fermion*)

$$B^a(z)G(w) \sim d^{\frac{1}{2}} \frac{\psi^a(w)}{(z-w)^2} \quad \text{and} \quad \psi^a(z)G(w) \sim d^{-\frac{1}{2}} \frac{B^a(w)}{(z-w)}$$

$$[G_m, B_n^a] = -nd^{\frac{1}{2}}\psi_{m+n}^a \quad \text{and} \quad [G_m, \psi_n^a]_+ = d^{-\frac{1}{2}}B_{m+n}^a$$

*Proof.*  $\psi_{n+\frac{1}{2}}^a \tau_i = 0$  for  $n \geq 2$  and:

(a)  $\psi_{\frac{1}{2}}^a \tau_1 = X_{-1}^a \Omega$

(b)  $\psi_{\frac{1}{2}}^a \tau_2 = \frac{1}{3}(S_{-1}^a \Omega - \sum_b \psi_{-\frac{1}{2}}^b \psi_{\frac{1}{2}}^a S_{-1}^b \Omega) = \frac{1}{3}(S_{-1}^a \Omega - i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^b \psi_{-\frac{1}{2}}^c \Omega)$   
 $= S_{-1}^a \Omega$

(c)  $\psi_{\frac{3}{2}}^a \tau_1 = \psi_{\frac{3}{2}}^a \tau_2 = 0.$

$S_n^a \tau_i, X_n^a \tau_i = 0$  for  $n \geq 2$  and:

(a)  $S_0^a \tau_1 = \sum_b S_0^a \psi_{-\frac{1}{2}}^b X_{-1}^b \Omega = i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^c X_{-1}^b \Omega$

(b)  $S_0^a \tau_2 = \frac{1}{3} \sum_b S_0^a \psi_{-\frac{1}{2}}^b S_{-1}^b \Omega = \frac{1}{3}(i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^c S_{-1}^b \Omega + \sum_b \psi_{-\frac{1}{2}}^b S_0^a S_{-1}^b \Omega)$   
 $= \frac{1}{3}(i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^c S_{-1}^b \Omega + i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^b S_{-1}^c \Omega)$   
 $= \frac{i}{3} \sum_{b,c} (\Gamma_{ab}^c + \Gamma_{ac}^b) \psi_{-\frac{1}{2}}^c S_{-1}^b \Omega = 0$

(c)  $X_0^a \tau_1 = \sum_b \psi_{-\frac{1}{2}}^b X_0^a X_{-1}^b \Omega = i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^b X_{-1}^c \Omega = -S_0^a \tau_1$

(d)  $X_0^a \tau_2 = X_1^a \tau_2 = S_1^a \tau_1 = 0$

(e)  $X_1^a \tau_1 = \ell \psi_{-\frac{1}{2}}^a \Omega$

(f)  $S_1^a \tau_2 = \frac{1}{3} \sum_b S_1^a \psi_{-\frac{1}{2}}^b S_{-1}^b \Omega = \frac{1}{3}(i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^c S_{-1}^b \Omega + \sum_b \psi_{-\frac{1}{2}}^b S_1^a S_{-1}^b \Omega)$   
 $= \frac{1}{3}(\sum_{b,c,d} \Gamma_{ab}^c \Gamma_{bc}^d \psi_{-\frac{1}{2}}^d \Omega + g \psi_{-\frac{1}{2}}^a \Omega) = g \psi_{-\frac{1}{2}}^a \Omega$

□

**Remark 4.40.**  $G_m^* = G_{-m}$  (as lemma 4.22)

**Lemma 4.41.** (*OPE and Lie bracket*)

$$L(z)G(w) \sim \frac{G'(w)}{(z-w)} + \frac{\frac{3}{2}G(w)}{(z-w)^2} \quad \text{and} \quad [G_m, L_n] = (m - \frac{1}{2}n)G_{m+n}$$

*Proof.*  $L(n)\tau = L_{n-1}\tau = 0$  for  $n \geq 3$  and:

(a)  $L_{-1}\tau = R(G')$  (see  $L_{-1}$  axioms and definition 3.52)

(b)  $L_0\tau = \frac{3}{2}R(G)$  (see  $L_0$  axioms)

(c)  $L_1(\tau_1 + \tau_2) = \sum_a L_1\psi_{-\frac{1}{2}}^a(X_{-1}^a + \frac{1}{3}S_{-1}^a)\Omega = \sum_a \psi_{-\frac{1}{2}}^a L_1(X_{-1}^a + \frac{1}{3}S_{-1}^a)\Omega$   
 $= \sum_a \psi_{-\frac{1}{2}}^a(X_0^a + \frac{1}{3}S_0^a)\Omega = 0$

□

**Remark 4.42.**  $[[A, B]_+, C] = [A, [B, C]_+] + [B, [A, C]_+]$   
 $= [A, [B, C]]_+ + [B, [A, C]]_+$

**Lemma 4.43.** (*OPE and Lie bracket*)

$$G(z)G(w) \sim \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2L(w)}{(z-w)} \quad \text{and} \quad [G_m, G_n]_+ = 2L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n}$$

*Proof.* By supersymmetry:

(a)  $[[G_m, G_n]_+, B_r^a] = -2rB_{m+n+r}^a = [2L_{m+n}, B_r^a]$

(b)  $[[G_m, G_n]_+, \psi_r^a] = -2(r + \frac{1}{2}(m+n))\psi_{m+n+r}^a = [2L_{m+n}, \psi_r^a]$

Then,  $[[G_m, G_n]_+ - 2L_{m+n}, B_r^a] = [[G_m, G_n]_+ - 2L_{m+n}, \psi_r^a] = 0$ .

Now,  $(B_r^a), (\psi_r^a)$  act irreducibly on  $H$ , so by Schur's lemma:

$$[G_m, G_n]_+ - 2L_{m+n} = k_{m,n}I$$

Now, among the  $G_n\tau$ ,  $G_{\frac{3}{2}}\tau$  is the only to give a constant term and:

$$G_{\frac{3}{2}}\tau = (\ell + g)^{-1} \sum_a G_{\frac{3}{2}}\psi_{-\frac{1}{2}}^a(X_{-1}^a + \frac{1}{3}S_{-1}^a)\Omega$$

$$= (\ell + g)^{-1} \sum_a (X_1^a + S_1^a)(X_{-1}^a + \frac{1}{3}S_{-1}^a)\Omega$$

$$= (\ell + g)^{-1} \dim(\mathfrak{g})(\ell + \frac{1}{3}g)\Omega = \frac{2}{3}c\Omega.$$

Finally, by formulas 3.26 and 3.31,  $k_{m,n} = \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n}$ .

□

**Summary 4.44.**

$$\begin{cases} L(z)L(w) \sim \frac{(c/2)}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{(z-w)} \\ L(z)G(w) \sim \frac{G'(w)}{(z-w)} + \frac{\frac{3}{2}G(w)}{(z-w)^2} \\ G(z)G(w) \sim \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2L(w)}{(z-w)} \end{cases}$$

and:

$$\begin{cases} [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} \\ [G_m, L_n] = (m - \frac{n}{2})G_{m+n} \\ [G_m, G_n]_+ = 2L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n} \end{cases}$$

$$L_n^* = L_{-n}, \quad G_m^* = G_{-m}, \quad \text{and } c = \frac{3}{2} \cdot \frac{\ell + \frac{1}{3}g}{\ell + g} \dim(\mathfrak{g})$$

the SuperVirasoro algebra of sector (NS), or Neveu-Schwarz algebra  $\mathfrak{Vir}_{1/2}$ .

**Corollary 4.45.**  $\mathfrak{Vir}_{\frac{1}{2}}$  acts unitarily on  $H = L(V_0, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  and admits  $L(c, 0)$  as minimal submodule containing  $\Omega$  (see definition 2.21).

### 4.3 Vertex modules

**Remark 4.46.** If  $\ell = 0$ , then  $\lambda = 0$  and  $L(V_0, 0) = \mathbb{C}$  trivial, and what we will show is ever proved by the previous section. So, we suppose  $\ell \in \mathbb{N}^*$  fixed.

#### 4.3.1 Summary

Let  $H = L(V_0, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ , the vacuum representation of the  $\mathfrak{g}$ -supersymmetric algebra  $\widehat{\mathfrak{g}}$ , with  $\pi : \widehat{\mathfrak{g}} \longrightarrow \text{End}(H)$ .

We have construct the vertex operator superalgebra  $(H, \Omega, \omega, V)$  with  $V : H \longrightarrow (\text{End}H)[[z, z^{-1}]]$  the state-field correspondance map.

$\mathcal{S} = V(H)$  is generated by  $(V(\psi_{-\frac{1}{2}}^a \Omega))_a$ ,  $(V(X_{-1}^b \Omega))_b$ , and  $V(\mathcal{L}\Omega)$ , pairwise local, with the operations,  $(A, B) \mapsto A_n B$  and linear combinations.

We write  $V(\psi_{-\frac{1}{2}}^a \Omega, z) = \sum_{n \in \mathbb{Z}} \pi(\psi_{n+\frac{1}{2}}^a) z^{-n-1}$ ,

$V(X_{-1}^b \Omega, z) = \sum_{n \in \mathbb{Z}} \pi(X_n^b) z^{-n-1}$  and  $V(\mathcal{L}\Omega, z) = \pi(\mathcal{L}) (= \ell Id_H)$ .

### 4.3.2 Modules

Let  $H^\lambda = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  a unitary highest weight representation of  $\widehat{\mathfrak{g}}$  and  $\pi^\lambda : \widehat{\mathfrak{g}} \rightarrow \text{End}(H^\lambda)$

**Remark 4.47.**  $H^\lambda$  is itself the minimal subspace containing  $\Omega^\lambda$  and stable by the action of  $\widehat{\mathfrak{g}}$ :  $\Omega^\lambda$  is the cyclic vector of  $H^\lambda$ .

On the vacuum representation,  $\Omega$  is called the vacuum vector.

**Lemma 4.48.**  $(\sum_{n \in \mathbb{Z}} \pi^\lambda(\psi_{n+\frac{1}{2}}^a) z^{-n-1})_a$ ,  $(\sum_{n \in \mathbb{Z}} \pi^\lambda(X_n^b) z^{-n-1})_b$  and  $\pi^\lambda(\mathcal{L})$  are pairwise local (definition 3.20).

*Proof.* Let  $A, B \in \widehat{\mathfrak{g}}[[z, z^{-1}]]$ ;  $\pi$  and  $\pi^\lambda$  are faithful representations of  $\widehat{\mathfrak{g}}$ .

Then, as formal power series, with  $N \in \mathbb{N}$  and  $\varepsilon \in \mathbb{Z}_2$ :

$$\begin{aligned} & (z-w)^N \pi(A(z)) \pi(B(w)) c, d \\ &= (-1)^\varepsilon (z-w)^N (\pi(B(w)) \pi(A(z)) c, d) \quad \forall c, d \in H \quad \text{if and only if} \\ & (z-w)^N (\pi^\lambda(A(z)) \pi^\lambda(B(w)) e, f) \\ &= (-1)^\varepsilon (z-w)^N (\pi^\lambda(B(w)) \pi^\lambda(A(z)) e, f) \quad \forall e, f \in H^\lambda \end{aligned}$$

We generate inductively an operator  $D$  decomposing  $H^\lambda$  into  $\bigoplus H_n^\lambda$  by:  
 $D\Omega^\lambda = 0$ ,  $D\psi_{-m}^a \xi = \psi_{-m}^a D\xi + m\psi_{-m}^a \xi$ ,  $DX_{-n}^b \xi = X_{-n}^b D\xi + nX_{-n}^b \xi$ ,  $\xi \in H^\lambda$ ,  
clearly well defined; but,  $\psi_m^a : H_p^\lambda \rightarrow H_{p-m}^\lambda$  and  $X_n^b : H_p^\lambda \rightarrow H_{p-n}^\lambda$ , so, by  
lemmas 3.35, 3.36, 3.37, the result follows.  $\square$

**Lemma 4.49.**  $D = L_0 - \frac{c_{V_\lambda}}{2(\ell+g)}$ ,  
with  $c_{V_\lambda}$  the Casimir number of  $V_\lambda$  (see corollary 4.3)

*Proof.*  $[L_0, \psi_n^a] = [D, \psi_n^a]$  and  $[L_0, X_n^a] = [D, X_n^a]$ , so, by irreducibility and Schur's lemma,  $L_0 - D \in \mathbb{C}Id_{H^\lambda}$ . Now,  $D\Omega^\lambda = 0$  and  $L_0\Omega^\lambda = h\Omega^\lambda \neq 0$  in general. Now, writing explicitly  $L_0$  with formula 3.26, we obtain:

$$2(\ell+g)L_0\Omega^\lambda = \sum_a (X_0^a)^2 \Omega^\lambda = \mathcal{C}.\Omega^\lambda = c_{V_\lambda}\Omega^\lambda \quad \square$$

**Theorem 4.50.**  $\mathfrak{Vir}_{\frac{1}{2}}$  acts unitarily on  $H^\lambda = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  and admits  $L(c, h)$  as minimal submodule containing  $\Omega^\lambda$ , with  $c = \frac{3}{2} \cdot \frac{\ell + \frac{1}{3}g}{\ell + g} \dim(\mathfrak{g})$  and  $h = \frac{cV_\lambda}{2(\ell + g)}$ .

*Proof.* We generate  $\mathcal{S}^\lambda$  from generators of previous lemma, with the operations  $(A, B) \mapsto A_n B$  (now available) and linear combinations. The formula 3.26 is independant of the choice between the faithful representations  $\pi$  and  $\pi^\lambda$ . So, we identify  $\mathcal{S}$  and  $\mathcal{S}^\lambda$ , which gives the isomorphism  $i : \mathcal{S} \rightarrow \mathcal{S}^\lambda$ ; we compose it with the state-field correspondence map  $V : H \rightarrow \mathcal{S}$  to give:

$$\begin{aligned} V^\lambda : H &\longrightarrow (End H^\lambda)[[z, z^{-1}]] \\ a &\longmapsto i(V(a)) \end{aligned} \quad (1)$$

Then,  $\sum_{n \in \mathbb{Z}} \pi^\lambda(\psi_{n+\frac{1}{2}}^a) z^{-n-1} = V^\lambda(\psi_{-\frac{1}{2}}^a \Omega, z)$ ,  
 $\sum_{n \in \mathbb{Z}} \pi^\lambda(X_n^b) z^{-n-1} = V^\lambda(X_{-1}^b \Omega, z)$  and  $\pi^\lambda(\mathcal{L}) = V^\lambda(\mathcal{L} \Omega, z)$   
Now,  $V(a)_n V(b) = V(V(a, n)b) \forall a, b \in H$ , so, by construction:

$$V^\lambda(a)_n V^\lambda(b) = V^\lambda(V(a, n)b) \quad (2)$$

Then,  $V^\lambda(\omega, z) = \sum L_n z^{-n-2}$ ,  $V^\lambda(\tau, z) = \sum G_{m-\frac{1}{2}} z^{-m-1}$ ,  $L_n^* = L_{-n}$  and  $G_m^* = G_{-m}$ , with  $(L_n), (G_m)$  verifying superVirasoro relations. (3)  $\square$

**Remark 4.51.**  $[L_m, \psi_n^a] = -(n + \frac{1}{2}m)\psi_{m+n}^a$  and  $[L_m, X_n^a] = -nX_{m+n}^a$ , so:

$$\begin{cases} [L_{-1}, V^\lambda(a, z)] = (V^\lambda)'(a, z) \\ [L_0, V^\lambda(a, z)] = z \cdot (V^\lambda)'(a, z) + rV^\lambda(a, z) \end{cases} \quad (a \in H_r) \quad (4)$$

**Remark 4.52.**  $V^\lambda(\Omega, z) = Id_{H^\lambda}$  because  $\pi$  and  $\pi^\lambda$  are at same level  $\ell$ . (5)

**Definition 4.53.** By (1)...(5),  $(H^\lambda, V^\lambda)$  is called a **vertex module** of  $(H, V, \Omega, \omega)$ .

We now apply the theorem 4.50 to GKO construction with  $\mathfrak{g} = \mathfrak{sl}_2$ .

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