



**HAL**  
open science

# Neveu-Schwarz and operators algebras I: Vertex operators superalgebras.

Sébastien Palcoux

► **To cite this version:**

Sébastien Palcoux. Neveu-Schwarz and operators algebras I: Vertex operators superalgebras.. 2010.  
hal-00522526v1

**HAL Id: hal-00522526**

**<https://hal.science/hal-00522526v1>**

Preprint submitted on 30 Sep 2010 (v1), last revised 6 Oct 2010 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Neveu-Schwarz and operators algebras I

## *Vertex operators superalgebras*

Sébastien Palcoux

### Abstract

This paper is the first of a series giving a self-contained way from the Neveu-Schwarz algebra to a new series of irreducible subfactors. Here we present an elementary, progressive and self-contained approach to vertex operator superalgebra. We then build such a structure from the loop algebra  $L\mathfrak{g}$  of any simple finite dimensional Lie algebra  $\mathfrak{g}$ . The Neveu-Schwarz algebra  $\mathfrak{Vir}_{1/2}$  emerges naturally on. As application, we obtain a unitary action of  $\mathfrak{Vir}_{1/2}$  on the unitary discrete series of  $L\mathfrak{g}$ .

## Contents

1	Introduction	1
2	The Neveu-Schwarz algebra	7
3	Vertex operators superalgebras	13
4	Vertex $\mathfrak{g}$ -superalgebras and modules	28

## 1 Introduction

### 1.1 Background of the series

In the 90's, V. Jones and A. Wassermann started a program whose goal is to understand the unitary conformal field theory from the point of view of

operator algebras (see [5], [16]). In [17], Wassermann defines and computes the Connes fusion of the irreducible positive energy representations of the loop group  $LSU(n)$  at fixed level  $\ell$ , using primary fields, and with consequences in the theory of subfactors. In [14] V. Toledano Laredo proves the Connes fusion rules for  $LSpin(2n)$  using similar methods. Now, let  $\text{Diff}(\mathbb{S}^1)$  be the diffeomorphism group on the circle, its Lie algebra is the Witt algebra  $\mathfrak{W}$  generated by  $d_n$  ( $n \in \mathbb{Z}$ ), with  $[d_m, d_n] = (m - n)d_{m+n}$ . It admits a unique central extension called the Virasoro algebra  $\mathfrak{Vir}$ . Its unitary positive energy representation theory and the character formulas can be deduced by a so-called Goddard-Kent-Olive (GKO) coset construction from the theory of  $LSU(2)$  and the Kac-Weyl formulas (see [18], [2]). In [10], T. Loke uses the coset construction to compute the Connes fusion for  $\mathfrak{Vir}$ . Now, the Witt algebra admits two supersymmetric extensions  $\mathfrak{W}_0$  and  $\mathfrak{W}_{1/2}$  with central extensions called the Ramond and the Neveu-Schwarz algebras, noted  $\mathfrak{Vir}_0$  and  $\mathfrak{Vir}_{1/2}$ .

In this series (this paper, [11] and [12] ), we naturally introduce  $\mathfrak{Vir}_{1/2}$  in the vertex superalgebra context of  $L\mathfrak{sl}_2$ , we give a complete proof of the classification of its unitary positive energy representations, we obtain directly their character; then we give the Connes fusion rules, and an irreducible finite depth type  $\text{II}_1$  subfactors for each representation of the discrete series. Note that we could do the same for the Ramond algebra  $\mathfrak{Vir}_0$ , using twisted vertex module over the vertex operator algebra of the Neveu-Schwarz algebra  $\mathfrak{Vir}_{1/2}$ , as R. W. Verrill [15] and Wassermann [19] do for twisted loop groups.

## 1.2 Overview of the paper

First, we look unitary, projective, positive energy representations of  $\mathfrak{W}_{1/2}$ . The projectivity gives 2-cocycles, so that  $\mathfrak{W}_{1/2}$  admits a unique central extension  $\mathfrak{Vir}_{1/2}$ . Such representations are completely reducible, and the irreducibles are given by the unitary highest weight representations of  $\mathfrak{Vir}_{1/2}$ : Verma modules  $V(c, h)$  quotiented by null vectors, in no-ghost cases.

From the fermion algebra on  $H = \mathcal{F}_{NS}$ , we build the fermion field  $\psi(z)$ . Locality and Dong's lemma permit, via OPE, to generate a set of fields  $\mathcal{S}$ , so that there is a 1 - 1 map  $V : H \rightarrow \mathcal{S}$ , with  $Id = V(\Omega)$  and a Virasoro field  $L = V(\omega)$ . Then, we give vertex operator superalgebra's axioms, permitting to come so far, in a general framework  $(H, V, \Omega, \omega)$ , with  $H$  prehilbert.

Let  $\mathfrak{g}$  a simple finite-dimensional Lie algebra,  $\widehat{\mathfrak{g}}_+$  the  $\mathfrak{g}$ -boson algebra

(central extension of the loop algebra  $L\mathfrak{g}$ ) and  $\widehat{\mathfrak{g}}_-$  the  $\mathfrak{g}$ -fermion algebra. We build a module vertex operator superalgebra from  $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \times \widehat{\mathfrak{g}}_-$  on  $H = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ , so that  $\mathfrak{Vir}_{1/2}$  acts on with  $(c, h) = (\frac{3}{2} \cdot \frac{\ell+1/3g}{\ell+g} \dim(\mathfrak{g}), \frac{c_{V_\lambda}}{2(\ell+g)})$ , with  $g$  the dual Coxeter number and  $c_{V_\lambda}$  the Casimir number.

### 1.3 The Neveu-Schwarz algebra

We start with  $\mathfrak{W}_{1/2}$ , the Witt superalgebra of sector (NS):

$$\begin{cases} [d_m, d_n] = (m-n)d_{m+n} & m, n \in \mathbb{Z} \\ [\gamma_m, d_n] = (m - \frac{n}{2})\gamma_{m+n} & m \in \mathbb{Z} + \frac{1}{2}, n \in \mathbb{Z} \\ [\gamma_m, \gamma_n]_+ = 2d_{m+n} & m, n \in \mathbb{Z} + \frac{1}{2} \end{cases}$$

together with  $d_n^* = d_{-n}$  and  $\gamma_m^* = \gamma_{-m}$ ; we study representations which are:

(a) Unitary:  $\pi(A)^* = \pi(A^*)$

(b) Projective:  $A \mapsto \pi(A)$  is linear and  $[\pi(A), \pi(B)] - \pi([A, B]) \in \mathbb{C}$ .

(c) Positive energy :  $H$  admits an orthogonal decomposition  $H = \bigoplus_{n \in \frac{1}{2}\mathbb{N}} H_n$  such that  $\exists D$  acting on  $H_n$  as multiplication by  $n$ ,  $H_0 \neq \{0\}$ ,  $\dim(H_n) < +\infty$

Here,  $\exists h \in \mathbb{C}$  such that  $D = \pi(d_0) - hI$ .

Now, the projectivity gives the 2-cocycles and we see that  $H_2(\mathfrak{W}_{1/2}, \mathbb{C})$  is 1-dimensional,  $\mathfrak{W}_{1/2}$  admits a unique central extension up to equivalence:

$$0 \rightarrow H_2(\mathfrak{W}_{1/2}, \mathbb{C}) \rightarrow \mathfrak{Vir}_{1/2} \rightarrow \mathfrak{W}_{1/2} \rightarrow 0$$

$\mathfrak{Vir}_{1/2}$  is the SuperVirasoro (of sector NS) or Neveu-Schwarz algebra:

$$\begin{cases} [L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n} \\ [G_m, L_n] = (m - \frac{n}{2})G_{m+n} \\ [G_m, G_n]_+ = 2L_{m+n} + \frac{C}{3}(m^2 - \frac{1}{4})\delta_{m+n} \end{cases}$$

with  $L_n^* = L_{-n}$ ,  $G_m^* = G_{-m}$  and  $C = cI$ ,  $c \in \mathbb{C}$  called the **central charge**.

The representations are completely reducible, the irreducibles are determined by the two numbers  $c, h$ , and are completely given by unitary highest weight representations of  $\mathfrak{Vir}_{1/2}$ , described as follows: The Verma modules  $H = V(c, h)$  are freely generated by:  $0 \neq \Omega \in H$  (cyclic vector),  $C\Omega = c\Omega$ ,  $L_0\Omega = h\Omega$  and  $\mathfrak{Vir}_{1/2}^+\Omega = \{0\}$ . Now,  $(\Omega, \Omega) = 1$ ,  $\pi(A)^* = \pi(A^*)$  and  $(u, v) = \overline{(v, u)}$  give the sesquilinear form  $(\cdot, \cdot)$ .  $V(c, h)$  can admit ghost:  $(u, u) < 0$ , and null vectors:  $(u, u) = 0$ . In no ghost case, the set of null vectors is  $K(c, h)$  the kernel of  $(\cdot, \cdot)$ , the maximal proper submodule.

Let  $L(c, h) = V(c, h)/K(c, h)$ , the unitary highest weight representations.

The theorem ?? will be proved classifying no ghost cases.

## 1.4 Vertex operators superalgebras

Our approach of vertex operators superalgebras is freely inspired by the following references: Borcherds [1], Goddard [3], Kac [9]. We start by working on the fermion algebra:  $[\psi_m, \psi_n]_+ = \delta_{m+n}I$  and  $\psi_n^* = \psi_{-n}$  ( $m, n \in \mathbb{Z} + \frac{1}{2}$ ). As for  $\mathfrak{W}_{1/2}$ , we build its Verma module  $H = \mathcal{F}_{NS}$  and the sesquilinear form  $(\cdot, \cdot)$ , which is a scalar product.  $H$  is a prehilbert space, the unique unitary highest weight representation of the fermion algebra. Let the formal power series  $\psi(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1}$  called fermion field. We inductively defined operators  $D$  giving positive energy structure ( $\Leftrightarrow [D, \psi] = z.\psi' + \frac{1}{2}\psi$ ) and  $T$  giving derivation ( $[T, \psi] = \psi'$ ). We compute  $(\psi(z)\psi(w)\Omega, \Omega) = \frac{1}{z-w}$  ( $|z| > |w|$ ), which permits to prove inductively an anticommutation relation shortly written as:  $\psi(z)\psi(w) = -\psi(w)\psi(z)$ . We define this relation in a general framework as locality: Let  $H$  prehilbert space, and let  $A \in (EndH)[[z, z^{-1}]]$  formal power series of the form  $A(z) = \sum_{n \in \mathbb{Z}} A(n)z^{-n-1}$  with  $A(n) \in End(H)$ . Such fields  $A$  and  $B$  are **local** if  $\exists \varepsilon \in \mathbb{Z}_2, \exists N \in \mathbb{N}$  such that  $\forall c, d \in H, \exists X(A, B, c, d) \in (z-w)^{-N}\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$  such that:

$$X(A, B, c, d)(z, w) = \begin{cases} (A(z)B(w)c, d) & \text{if } |z| > |w| \\ (-1)^\varepsilon (B(w)A(z)c, d) & \text{if } |w| > |z| \end{cases}$$

Now, using locality and a contour integration argument, we can explicitly construct a field  $A_n B$  from  $A$  and  $B$ , with  $(A_n B)(m) =$

$$\begin{cases} \sum_{p=0}^n (-1)^p C_n^p [A(n-p), B(m+p)]_\varepsilon & \text{if } n \geq 0 \\ \sum_{p \in \mathbb{N}} C_{p-n-1}^p (A(n-p)B(m+p) - (-1)^{\varepsilon+n} B(m+n-p)A(p)) & \text{if } n < 0 \end{cases}$$

We obtain the operator product expansion (OPE) shortly written as  $A(z)B(w) \sim \sum_{n=0}^{N-1} \frac{(A_n B)(w)}{(z-w)^{n+1}}$ ; and by an other contour integration argument:

$$[A(m), B(n)]_\varepsilon = \begin{cases} \sum_{p=0}^{N-1} C_m^p (A_p B)(m+n-p) & \text{if } m \geq 0 \\ \sum_{p=0}^{N-1} (-1)^p C_{p-m-1}^p (A_p B)(m+n-p) & \text{if } m < 0 \end{cases}$$

Thanks to Dong's lemma, the operation  $(A, B) \mapsto A_n B$  permits to generate many fields. To have a good behaviour, we define a system of generators as:  $\{A_1, \dots, A_r\} \subset (EndH)[[z, z^{-1}]]$  with  $D, T \in End(H), \Omega \in H$  such that:

(a)  $\forall i, j$   $A_i$  and  $A_j$  are local with  $N = N_{ij}$  and  $\varepsilon = \varepsilon_{ij} = \varepsilon_{ii}.\varepsilon_{jj}$

- (b)  $\forall i [T, A_i] = A'_i$
  - (c)  $H = \bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  for  $D$ ,  $\dim(H_n) < \infty$
  - (d)  $\forall i [D, A_i] = z.A'_i + \alpha_i A_i$  with  $\alpha_i \in \mathbb{N} + \frac{\varepsilon_i i}{2}$
  - (e)  $\Omega \in H_0$ ,  $\|\Omega\| = 1$ , and  $\forall i \forall m \in \mathbb{N}$ ,  $A_i(m)\Omega = D\Omega = T\Omega = 0$
  - (f)  $\mathcal{A} = \{A_i(m), \forall i \forall m \in \mathbb{Z}\}$  acts irreducibly on  $H$ , so that  $\langle \mathcal{A} \rangle . \Omega = H$
- Hence, we generate a space  $\mathcal{S}$ , with  $V : H \rightarrow \mathcal{S}$  a state-field correspondence linear map.  $V(a)(z)$  is noted  $V(a, z)$  and  $V(a, z)\Omega|_{z=0} = a$ .

Now,  $\{\psi\}$  is a system of generator, we generate  $\mathcal{S}$  and the map  $V$  with  $\psi(z) = V(\psi_{-\frac{1}{2}}\Omega, z)$ ; but,  $\psi(z)\psi(w) \sim \frac{Id}{z-w} + 2L(w)$ , with  $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = \frac{1}{2}\psi_{-2}\psi(z) = V(\omega, z)$  with  $\omega = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$ . Then, using OPE and Lie bracket, we find that  $D = L_0$ ,  $T = L_{-1}$ ,  $L(z)L(w) \sim \frac{(c/2)Id}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{(z-w)}$ , and  $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}$  with  $c = 2\|L_{-2}\Omega\|^2 = \frac{1}{2}$ , the central charge. As corollary,  $\mathfrak{Vir}$  acts on  $H = \mathcal{F}_{NS}$ , and admits its unitary highest weight representation  $L(c, h) = L(\frac{1}{2}, 0)$  as minimal submodule containing  $\Omega$ . We call  $\omega \in H$  the Virasoro vector, and  $L$  the Virasoro field.

We are now able to define vertex operators superalgebras in general.

A vertex operator superalgebra is an  $(H, V, \Omega, \omega)$  with:

- (a)  $H = H_0 \oplus H_1$  a prehilbert superspace.
- (b)  $V : H \rightarrow (EndH)[[z, z^{-1}]]$  a linear map.
- (c)  $\Omega, \omega \in H$  the vacuum and Virasoro vectors.

Let  $\mathcal{S}_\varepsilon = V(H_\varepsilon)$ ,  $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$  and  $A(z) = V(a, z) = \sum_{n \in \mathbb{Z}} A(n)z^{-n-1}$ , then  $(H, V, \Omega, \omega)$  satisfies the followings axioms:

- (1)  $\forall n \in \mathbb{N}$ ,  $\forall A \in \mathcal{S}$ ,  $A(n)\Omega = 0$ ,  $V(a, z)\Omega|_{z=0} = a$ , and  $V(\Omega, z) = Id$
- (2)  $\mathcal{A} = \{A(n) | A \in \mathcal{S}, n \in \mathbb{Z}\}$  acts irreducibly on  $H$ , so that  $\mathcal{A}.\Omega = H$ .
- (3)  $\forall A \in \mathcal{S}_{\varepsilon_1}$ ,  $\forall B \in \mathcal{S}_{\varepsilon_2}$ ,  $A$  and  $B$  are local with  $\varepsilon = \varepsilon_1.\varepsilon_2$ ,  $A_n B \in \mathcal{S}_{\varepsilon_1 + \varepsilon_2}$
- (4)  $V(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ ,  $[L_m, L_n] = (m-n)L_{m+n} + \frac{\|2\omega\|^2}{12}m(m^2-1)\delta_{m+n}$
- (5)  $H = \bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  for  $L_0$ ,  $\dim(H_n) < \infty$  and  $H_\varepsilon = \bigoplus_{n \in \mathbb{N} + \frac{\varepsilon}{2}} H_n$
- (6)  $[L_0, V(a, z)] = z.V'(a, z) + \alpha.V(a, z)$  for  $a \in H_\alpha$
- (7)  $[L_{-1}, V(a, z)] = V'(a, z) = V(L_{-1}.a, z) \in \mathcal{S}$

As corollaries, we have that a system of generators, generating a Virasoro field  $L \in \mathcal{S}$ , with  $D = L_0$  and  $T = L_{-1}$ , generates a vertex operator superalgebra; the fermion field  $\psi$  and the Virasoro field  $L$  generate one, each; and we prove the Borcherds associativity:  $V(a, z)V(b, w) = V(V(a, z-w)b, w)$ .

## 1.5 Vertex $\mathfrak{g}$ -superalgebras and modules

Let  $\mathfrak{g}$  be a simple Lie algebra of dimension  $N$ , a basis  $(X_a)$ , well normalized (see remark 4.2), such that  $[X_a, X_b] = i \sum_c \Gamma_{ab}^c X_c$  with  $\Gamma_{ab}^c \in \mathbb{R}$  totally antisymmetric. Let its dual coxeter number  $g = \frac{1}{4} \sum_{a,c} (\Gamma_{ac}^b)^2$ :

$\mathfrak{g}$	$A_n$	$B_n$	$C_n$	$D_n$	$E_6$	$E_7$	$E_8$	$F_4$	$G_2$
$\dim(\mathfrak{g})$	$n^2 + 2n$	$2n^2 + n$	$2n^2 + n$	$2n^2 - n$	78	133	248	52	14
$g$	$n + 1$	$2n - 1$	$n + 1$	$2n - 2$	12	18	30	9	4

For example,  $\mathfrak{g} = A_1 = \mathfrak{sl}_2$ ,  $\dim(\mathfrak{g}) = 3$  and  $g = 2$ .

Let  $\widehat{\mathfrak{g}}_+$  the  $\mathfrak{g}$ -boson algebra:  $[X_m^a, X_n^b] = [X_a, X_b]_{m+n} + m\delta_{ab}\delta_{m+n}\mathcal{L}$ , unique central extension (by  $\mathcal{L}$ ) of the loop algebra  $L\mathfrak{g} = C^\infty(\mathbb{S}^1, \mathfrak{g})$  (see [18] p 43). The unitary highest weight representations of  $\widehat{\mathfrak{g}}_+$  are  $H = L(V_\lambda, \ell)$ , with  $\ell \in \mathbb{N}$  such that  $\mathcal{L}\Omega = \ell\Omega$  (the level of  $H$ ),  $H_0 = V_\lambda$  irreducible representation of  $\mathfrak{g}$  such that  $(\lambda, \theta) \leq \ell$  with  $\lambda$  the highest weight and  $\theta$  the highest root (see [18] p 45). The category  $\mathcal{C}_\ell$  of representations for fixed  $\ell$  is finite. For example  $\mathfrak{g} = \mathfrak{sl}_2$ ,  $H = L(j, \ell)$ , with  $V_\lambda = V_j$  representations of  $\mathfrak{spin} j \leq \frac{\ell}{2}$ .

We define the  $\mathfrak{g}$ -fermion algebra  $\widehat{\mathfrak{g}}_-$  and the fermion fields, composed by  $N$  fermions; and as for  $N = 1$ , we generate a vertex operator superalgebra, but now, it contains  $\mathfrak{g}$ -boson fields ( $S^a$ ) whose related algebra is represented with  $L(V_0, g)$ ; and thanks to  $\widehat{\mathfrak{g}}_-$ -vertex background, the fields ( $S^a$ ) generate a vertex operator superalgebra; by this way, we are able to generate one, from  $\widehat{\mathfrak{g}}_+$  and  $H = L(V_0, \ell)$ ,  $\forall \ell \in \mathbb{N}$ . Remark that because of the vacuum axiom, the vertex structure need to take  $V_\lambda = V_0$  trivial representation; in general, we have vertex modules (see further).

Now, let  $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \times \widehat{\mathfrak{g}}_-$  the  $\mathfrak{g}$ -supersymmetric algebra; we prove it admits  $H = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  as unitary highest weight representations. We generate a vertex operator superalgebra, with a Virasoro field  $L$ , and also a SuperVirasoro field  $G$ , which gives the supersymmetry boson-fermion: Let  $B^a = X^a + S^a$  boson fields of level  $d = \ell + g$ , then:

$$B^a(z)G(w) \sim d^{\frac{1}{2}} \frac{\psi^a(w)}{(z-w)^2} \quad \text{and} \quad \psi^a(z)G(w) \sim d^{-\frac{1}{2}} \frac{B^a(w)}{(z-w)}.$$

Finally, from  $H^\lambda = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ , we define the vertex module  $(H^\lambda, V^\lambda)$  over  $(H^0, V, \Omega, \omega)$ , and we prove that  $\mathfrak{Vir}_{\frac{1}{2}}$  acts unitarily on  $H^\lambda$  and admits  $L(c, h)$  as minimal submodule containing the cyclic vector  $\Omega^\lambda$ , with  $c = \frac{3}{2} \cdot \frac{\ell + \frac{1}{2}g}{\ell + g} \dim(\mathfrak{g})$ ,  $h = \frac{c_{V_\lambda}}{2(\ell + g)}$  and  $c_{V_\lambda}$  the Casimir number of  $V_\lambda$ .

## 2 The Neveu-Schwarz algebra

### 2.1 Witt superalgebras and representations

**Definition 2.1.** A Lie superalgebra is a  $\mathbb{Z}_2$ -graded vector space  $\mathfrak{d} = \mathfrak{d}_{\bar{0}} \oplus \mathfrak{d}_{\bar{1}}$ , together with a graded Lie bracket  $[\cdot, \cdot] : \mathfrak{d} \times \mathfrak{d} \rightarrow \mathfrak{d}$ , such that  $[\cdot, \cdot]$  is a bilinear map with  $[\mathfrak{d}_i, \mathfrak{d}_j] \subseteq \mathfrak{d}_{i+j}$ , and for homogeneous elements  $X \in \mathfrak{d}_x, Y \in \mathfrak{d}_y, Z \in \mathfrak{d}_z$  :

- $[X, Y] = -(-1)^{xy}[Y, X]$
- $(-1)^{xz}[X, [Y, Z]] + (-1)^{xy}[Y, [Z, X]] + (-1)^{yz}[Z, [X, Y]] = 0$

**Definition 2.2.** The Witt algebra  $\mathfrak{W}$  is the Lie  $\star$ -algebra of vector fields on the circle, generated by  $d_n = ie^{i\theta n} \frac{d}{d\theta}$  ( $n \in \mathbb{Z}$ ).

**Remark 2.3.**  $\mathfrak{W}$  admits two supersymmetric extensions,  $\mathfrak{W}_0$  the Ramond sector (R) and  $\mathfrak{W}_{1/2}$  the Neveu-Schwarz sector (NS) ((see [8], [4] chap 9).

Here, we treat only the (NS) sector.

**Definition 2.4.** Let  $\mathfrak{d} = \mathfrak{W}_{1/2}$  the Witt superalgebra with:

$$\begin{cases} [d_m, d_n] = (m - n)d_{m+n} \\ [\gamma_m, d_n] = (m - \frac{n}{2})\gamma_{m+n} \\ [\gamma_m, \gamma_n]_+ = 2d_{m+n} \end{cases}$$

together with the  $\star$ -structure,  $d_n^* = d_{-n}$  and  $\gamma_m^* = \gamma_{-m}$ , and the super-structure:  $\mathfrak{d}_{\bar{0}} = \mathfrak{W} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}d_n$ ,  $\mathfrak{d}_{\bar{1}} = \bigoplus_{m \in \mathbb{Z} + 1/2} \mathbb{C}\gamma_m$

Now we investigate representations  $\pi$  of  $\mathfrak{W}_{1/2}$ , which are :

**Definition 2.5.** Let  $H$  be a prehilbert space.

- (a) *Unitary:*  $\pi(A)^* = \pi(A^*)$
- (b) *Projective:*  $A \mapsto \pi(A)$  is linear and  $[\pi(A), \pi(B)] - \pi([A, B]) \in \mathbb{C}$ .
- (c) *Positive energy :*  $H$  admits an orthogonal decomposition  $H = \bigoplus_{n \in \frac{1}{2}\mathbb{N}} H_n$  such that,  $\exists D$  acting on  $H_n$  as multiplication by  $n$ ,  $H_0 \neq \{0\}$  and  $\dim(H_n) < +\infty$ . Here,  $\exists h \in \mathbb{C}$  such that  $D = \pi(d_0) - hI$ .



## 2.2 Investigation

**Definition 2.6.** Let  $b : \mathfrak{W}_{1/2} \times \mathfrak{W}_{1/2} \rightarrow \mathbb{C}$  be the bilinear map defined by

$$[\pi(A), \pi(B)] - \pi([A, B]) = b(A, B)I \quad (b \text{ is a 2-cocycle})$$

**Definition 2.7.** Let  $f : \mathfrak{W}_{1/2} \rightarrow \mathbb{C}$  be a  $\star$ -linear form.

$\partial f = (A, B) \mapsto f([A, B])$  is a 2-coboundary.

**Remark 2.8.**  $A \mapsto \pi(A) + f(A)I$  define also a projective, unitary, positive energy representation, where  $b(A, B)$  becomes  $b(A, B) - f([A, B])$ .

**Proposition 2.9.** (SuperVirasoro extension)  $\mathfrak{W}_{1/2}$  has a unique central extension, up to equivalent, i.e.  $H_2(\mathfrak{W}_{1/2}, \mathbb{C})$  is 1-dimensional. This extension admits the basis  $(L_n)_{n \in \mathbb{Z}}, (G_m)_{m \in \mathbb{Z} + \frac{1}{2}}, C$  central, with  $L_n^* = L_{-n}, G_m^* = G_{-m}, C = cI, c \in \mathbb{C}$  called the **central charge**; and relations:

$$\begin{cases} [L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n} \\ [G_m, L_n] = (m - \frac{n}{2})G_{m+n} \\ [G_m, G_n]_+ = 2L_{m+n} + \frac{C}{3}(m^2 - \frac{1}{4})\delta_{m+n} \end{cases}$$

*Proof.*

Let  $L_n = \pi(d_n)$  and  $G_m = \pi(\gamma_m)$  then:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + b(d_m, d_n)I \\ [G_m, L_n] &= (m - \frac{n}{2})G_{m+n} + b(\gamma_m, d_n)I \\ [G_m, G_n]_+ &= 2L_{m+n} + b(\gamma_m, \gamma_n)I \end{aligned}$$

In particular:

$$\begin{aligned} [L_0, L_n] &= -nL_n + b(d_0, d_n)I \\ [L_0, G_n] &= -nG_n + b(d_0, \gamma_n)I \\ [L_1, L_{-1}] &= 2L_0 + b(d_1, d_{-1})I \end{aligned}$$

We choose :

$$\begin{aligned} f(d_n) &= -n^{-1}b(d_0, d_n) \\ f(\gamma_m) &= -m^{-1}b(d_0, \gamma_m) \\ f(d_0) &= \frac{1}{2}b(d_1, d_{-1}) \end{aligned}$$

Then, after adjustment by  $f$ :

$$\begin{aligned} [L_0, L_n] &= -nL_n \\ [L_0, G_n] &= -nG_n \\ [L_1, L_{-1}] &= 2L_0 \end{aligned}$$

Now  $D = L_0 - hI$  and if  $v \in H_k$ ,  $Dv = kv$ , then:  
 $DL_nv = L_nDv + [D, L_n]v = kL_nv + [L_0, L_n]v = (k - n)L_nv$   
 So,  $L_n : H_k \rightarrow H_{k-n}$  ( $= \{0\}$  if  $n > k$  ).  
 Similary,  $G_m : H_k \rightarrow H_{k-m}$ , then:

$$\begin{cases} [L_m, L_n] - (m - n)L_{m+n} : H_{m+n+k} \rightarrow H_k \\ [G_m, L_n] - (m - \frac{n}{2})G_{m+n} : H_{m+n+k} \rightarrow H_k \\ [G_m, G_n]_+ - 2L_{m+n} : H_{m+n+k} \rightarrow H_k \end{cases}$$

But  $b(d_m, d_n)I$ ,  $b(\gamma_m, d_n)I$ ,  $b(\gamma_m, \gamma_n)I : H_{m+n+k} \rightarrow H_{m+n+k}$ , so:

$$\begin{cases} b(d_m, d_n) = A(m)\delta_{m+n} \\ b(\gamma_m, d_n) = B(m)\delta_{m+n} = 0 \text{ because } 0 \notin \mathbb{Z} + 1/2 \ni m + n \\ b(\gamma_m, \gamma_n) = C(m)\delta_{m+n} \end{cases}$$

Now, on  $\mathfrak{W} = \mathfrak{d}_0$ ,  $b(A, B) = -b(B, A)$ , so,  $A(m) = -A(-m)$  and  $A(0) = 0$ , and Jacobi identity implies  $b([A, B], C) + b([B, C], A) + b([C, A], B) = 0$ , then, for  $d_k, d_n, d_m$  with  $k + n + m = 0$  :

$$(n - m)A(k) + (m - k)A(n) + (k - n)A(m) = 0$$

Now, with  $k = 1$  and  $m = -n - 1$ ,  $(n - 1)A(n + 1) = (n + 2)A(n) - (2n + 1)A(1)$ . Then  $A(n)$  is completely determined by the knowledge of  $A(1)$  and  $A(2)$ , and so, the solutions are a 2-dimensional space.

Now,  $n$  and  $n^3$  are solutions, so  $A(n) = a.n + b.n^3$  .

Finally, because  $[L_1, L_{-1}] = 2L_0$ ,  $A(1) = 0$  and  $a + b = 0$ , we obtain:

$$A(n) = b(n^3 - n) = \frac{c}{12}(n^3 - n), \quad c \in \mathbb{C} \text{ the central charge.}$$

### Process 2.9.

$[[A, B]_+, C] = [A, [B, C]]_+ + [B, [A, C]]_+$  then:

$$\begin{aligned} [[G_r, G_s]_+, L_n] &= [G_r, [G_s, L_n]]_+ + [G_s, [G_r, L_n]]_+ \\ &= [2L_{r+s}, L_n] = [G_r, (s - \frac{1}{2}n)G_{n+s}]_+ + [G_s, (r - \frac{1}{2}n)G_{n+r}]_+ \\ &= 2(r + s - n)L_{r+s+n} - \delta_{r+s+n} \frac{c}{6}(n^3 - n) \end{aligned}$$

$$= (s - \frac{1}{2}n)(2L_{r+s+n} + C(r)\delta_{r+s+n}) - (r - \frac{1}{2}n)(2L_{r+s+n} + C(s)\delta_{r+s+n})$$

Then taking  $r + s + n = 0$ ,  $\frac{c}{6}(n^3 - n) + (s - \frac{1}{2}n)C(r) + (r - \frac{1}{2}n)C(s) = 0$ .

Finally, with  $n = 2s$  and  $r = -3s$ ,  $C(s) = \frac{c}{3}(s^2 - \frac{1}{4})$ .  $\square$

**Definition 2.10.** *The central extension of  $\mathfrak{W}_{1/2}$  is called  $\mathfrak{Vir}_{1/2}$ , the Super-Virasoro algebra (on sector NS), also called Neveu-Schwarz algebra.*

**Theorem 2.11.** *(Complete reducibility)*

(a) *If  $H$  is a unitary, projective, positive energy representation of  $\mathfrak{W}_{1/2}$ , then any non-zero vector  $v$  in the lowest energy subspace  $H_0$  generates an irreducible submodule.*

(b)  *$H$  is an orthogonal direct sum of irreducibles such representations.*

*Proof.* (a) Let  $K$  be the minimal  $\mathfrak{W}_{1/2}$ -submodule containing  $v$ .

Clearly, since  $L_n v = G_m v = 0$  for  $m, n > 0$  and  $L_0 v = h v$ , we see that  $K$  is spanned by all products  $R.v$  with :

$$R = G_{-j_\beta} \dots G_{-j_1} L_{-i_\alpha} \dots L_{-i_1}, \quad 0 < i_1 \leq \dots \leq i_\alpha, \quad \frac{1}{2} \leq j_1 < \dots < j_\beta$$

But then,  $K_0 = \mathbb{C}v$ . Let  $K'$  be a submodule of  $K$ , and let  $p$  be the orthogonal projection onto  $K'$ . By unitarity,  $p$  commutes with the action of  $\mathfrak{W}_{1/2}$ , and hence with  $D$ . Thus  $p$  leaves  $K_0 = \mathbb{C}v$  invariant, so  $p v = 0$  or  $v$ .

But  $p R v = R p v$ , hence  $K' = 0$  or  $K$  and  $K$  is irreducible.

(b) Take the irreducible module  $M_1$  generated by a vector of lowest energy. Now (changing  $h$  into  $h' = h + m$  if necessary), we repeat this process for  $M_1^\perp$ , to get  $M_2, M_3, \dots$ . The positive energy assumption shows that  $H = \oplus M_i$   $\square$

**Theorem 2.12.** *(Uniqueness) If  $H$  and  $H'$  are irreducibles with  $c = c'$  and  $h = h'$ , then they are unitarily equivalents as  $\mathfrak{W}_{1/2}$ -modules.*

*Proof.*  $H_0 = \mathbb{C}u$  and  $H'_0 = \mathbb{C}u'$  with  $u, u'$  unitary.

Let  $U : H \rightarrow H', Au \mapsto Au'$ , we want to prove that  $U^*U = UU^* = Id$ .

Let  $Au \in H_n, Bu \in H_m$ :

If  $n \neq m$ , for example,  $n < m$ , then  $B^*Au \in H_{n-m} = 0$  and

$$(Au, Bu) = (B^*Au, u) = 0 = (Au', Bu').$$

If  $n = m$ , then  $D = B^*A$  is a constant energy operator, so in  $\mathbb{C}L_0 \oplus \mathbb{C}C$ .

Now,  $(L_0 u, u) = h = (L_0 u', u')$  iff  $h = h'$  and  $(Cu, u) = c = (Cu', u')$  iff  $c = c'$ .

Finally,  $(v, w) = (Uv, Uw) \forall v, w \in H$  and  $(v', w') = (U^*v', U^*w') \forall v', w' \in H'$  iff  $h = h'$  and  $c = c'$ .

So,  $U^*U = UU^* = Id$ , ie,  $H$  and  $H'$  are unitarily equivalents.  $\square$

**Definition 2.13.**  $\mathfrak{Vir}_{1/2} = \mathfrak{Vir}_{1/2}^- \oplus \mathfrak{Vir}_{1/2}^0 \oplus \mathfrak{Vir}_{1/2}^+$  with  $\mathfrak{Vir}_{1/2}^0 = \mathbb{C}L_0 \oplus \mathbb{C}C$

$$\mathfrak{Vir}_{1/2}^+ = \bigoplus_{m,n>0} \mathbb{C}L_m \oplus \mathbb{C}G_n \quad \mathfrak{Vir}_{1/2}^- = \bigoplus_{m,n<0} \mathbb{C}L_m \oplus \mathbb{C}G_n$$

**Remark 2.14.** This decomposition pass to the universal envelopping :

$$\mathcal{U}(\mathfrak{Vir}_{1/2}) = \mathcal{U}(\mathfrak{Vir}_{1/2}^-) \cdot \mathcal{U}(\mathfrak{Vir}_{1/2}^0) \cdot \mathcal{U}(\mathfrak{Vir}_{1/2}^+)$$

**Remark 2.15.** We see that an irreducible, unitary, projective, positive energy representation of  $\mathfrak{W}_{1/2}$  is exactly given by a unitary highest weight representation of  $\mathfrak{Vir}_{1/2}$  (see the following section).

### 2.3 Unitary highest weight representations

**Definition 2.16.** Let the Verma module  $H = V(c, h)$  be the  $\mathfrak{Vir}_{1/2}$ -module freely generated by followings conditions:

- (a)  $\Omega \in H$ , called the cyclic vector ( $\Omega \neq 0$ ).
- (b)  $L_0\Omega = h\Omega$ ,  $C\Omega = c\Omega$  ( $h, c \in \mathbb{R}$ )
- (c)  $\mathfrak{Vir}_{1/2}^+\Omega = \{0\}$

**Lemma 2.17.**  $\mathcal{U}(\mathfrak{Vir}_{1/2}^-)\Omega = H$  and a set of generators is given by:  
 $G_{-j_\beta} \dots G_{-j_1} L_{-i_\alpha} \dots L_{-i_1} \Omega$ ,  $0 < i_1 \leq \dots \leq i_\alpha$ ,  $\frac{1}{2} \leq j_1 < \dots < j_\beta$

*Proof.* It's clear. □

**Lemma 2.18.**  $V(c, h)$  admits a canonical sesquilinear form  $(., .)$ , completely defined by:

- (a)  $(\Omega, \Omega) = 1$
- (b)  $\pi(A)^* = \pi(A^*)$
- (c)  $(u, v) = \overline{(v, u)} \quad \forall u, v \in H$  (in particular  $(u, u) = \overline{(u, u)} \in \mathbb{R}$ ).

*Proof.* It's clear. □

**Definition 2.19.**  $u \in V(c, h)$  is a ghost if  $(u, u) < 0$ .

**Lemma 2.20.** *If  $V(c, h)$  admits no ghost then  $c, h \geq 0$*

*Proof.* Since  $L_n L_{-n} \Omega = L_{-n} L_n \Omega + 2nh\Omega + c \frac{n(n^2-1)}{12} \Omega$ ,

we have  $(L_{-n} \Omega, L_{-n} \Omega) = 2nh + \frac{n(n^2-1)}{12} c \geq 0$ .

Now, taking  $n$  first equal to 1 and then very large, we obtain the lemma.  $\square$

**Definition 2.21.** *Let  $K(c, h) = \ker(\cdot, \cdot) = \{x \in V(c, h); (x, y) = 0 \forall y\}$  the maximal proper submodule of  $V(c, h)$ , and  $L(c, h) = V(c, h)/K(c, h)$ , irreducible highest weight representation of  $\mathfrak{Vir}_{1/2}$ , with  $(\cdot, \cdot)$  well-defined on.*

**Definition 2.22.**  *$u \in V(c, h)$  is a null vector if  $(u, u) = 0$ .*

**Lemma 2.23.** *On no ghost case, the set of null vectors is  $K(c, h)$ .*

*Proof.* Let  $x$  be a null vector, and  $y \in V(c, h)$ .

By assumption  $\forall \alpha, \beta \in \mathbb{C}, (\alpha x + \beta y, \alpha x + \beta y) \geq 0$ . We develop it, with  $\alpha = (y, y)$  and  $\beta = -(x, y)$ , we obtain :  $|(x, y)|^2 (y, y) \leq (x, x) (y, y)^2 = 0$ .

So if  $y$  is not a null vector then  $(x, y) = 0$ . Else,  $(x, x) = (y, y) = 0$ , so taking  $\alpha = 1$  and  $\beta = -(x, y)$ , we obtain  $2|(x, y)|^2 \leq 0$  and so  $(x, y) = 0$   $\square$

**Corollary 2.24.**  *$L(c, h)$  is a unitary highest weight representation.*

*Proof.* Without ghost,  $(\cdot, \cdot)$  is a scalar product on  $L(c, h)$ .  $\square$

**Remark 2.25.** *Theorem 1.2 of [11] will be proved classifying no ghost cases.*

### 3 Vertex operators superalgebras

We give a progressive introduction to vertex operators superalgebras structure. We start with the fermion algebra as example. We work on to obtain, at the end of the section, vertex axioms naturally.

#### 3.1 Investigation on fermion algebra

**Definition 3.1.** *Let the fermion algebra (of sector NS), generated by  $(\psi_n)_{n \in \mathbb{Z} + \frac{1}{2}}$ , and  $I$  central, with the relations:*

$$[\psi_m, \psi_n]_+ = \delta_{m+n} I \quad \text{and} \quad \psi_n^* = \psi_{-n}$$

**Definition 3.2.** *(Verma module) Let  $H = \mathcal{F}_{NS}$  freely generated by:*

- (a)  $\Omega \in H$  is called the vacuum vector,  $\Omega \neq 0$ .
- (b)  $\psi_m \Omega = 0 \quad \forall m > 0$
- (c)  $I \Omega = \Omega$

**Lemma 3.3.** *A set of generators of  $H$  is given by:*  
 $\psi_{-m_1} \dots \psi_{-m_r} \Omega \quad m_1 < \dots < m_r \quad r \in \mathbb{N}, \quad m_i \in \mathbb{N} + \frac{1}{2}$

*Proof.* It's clear. □

**Lemma 3.4.**  *$H$  admits the sesquilinear form  $(\cdot, \cdot)$  completely defined by :*

- (a)  $(\Omega, \Omega) = 1$
  - (b)  $(u, v) = \overline{(v, u)} \quad \forall u, v \in H$
  - (c)  $(\psi_n u, v) = (u, \psi_{-n} v) \quad \forall u, v \in H \quad \text{ie } \pi(\psi_n)^* = \pi(\psi_n^*)$
- $(\cdot, \cdot)$  is a scalar product and  $H$  is a prehilbert space.*

*Proof.* It's clear. □

**Remark 3.5.**  *$H$  is an irreducible representation of the fermion algebra. It is its unique unitary highest weight representation.*

**Remark 3.6.**  $\psi_n^2 = \frac{1}{2}[\psi_n, \psi_n]_+ = 0$  if  $n \neq 0$

**Definition 3.7.** (Operator  $D$ ) Let  $D \in \text{End}(H)$  inductively defined by :

(a)  $D\Omega = 0$

(b)  $D\psi_{-m}a = \psi_{-m}Da + m\psi_{-m}a \quad \forall m \in \mathbb{N} + \frac{1}{2} \text{ and } \forall a \in H$

**Lemma 3.8.**  $D$  decomposes  $H$  into  $\bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  with  $D\xi = n\xi$   
 $\forall \xi \in H_n$ ,  $\dim(H_n) < \infty$  and  $H_n \perp H_m$  if  $n \neq m$

*Proof.* Let  $a = \psi_{-m_1} \dots \psi_{-m_r} \Omega$  be a generic element of the base of  $H$ , then  $D.a = (\sum m_i)a$ . □

**Remark 3.9.**  $[D, \psi_m] = -m\psi_m$  and  $\Omega \in H_0$ , so  $\psi_m : H_{m+n} \rightarrow H_n$ .

**Definition 3.10.** (Operator  $T$ ) Let  $T \in \text{End}(H)$  inductively defined by :

(a)  $T\Omega = 0$

(b)  $T\psi_{-m}a = \psi_{-m}Ta + (m - \frac{1}{2})\psi_{-m-1}a \quad \forall m \in \mathbb{N} + \frac{1}{2} \text{ and } \forall a \in H$

**Remark 3.11.**  $[T, \psi_m] = -(m - \frac{1}{2})\psi_{m-1}$ .

**Definition 3.12.** Let  $\psi(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}} \cdot z^{-n-1}$  the fermion operator.

**Remark 3.13.**  $\psi \in (\text{End}H)[[z, z^{-1}]]$  is a formal power series.

**Lemma 3.14.** (Relations with  $\psi_n$ ,  $D$  and  $T$ )

(a)  $[\psi_{m+\frac{1}{2}}, \psi]_+ = z^m$

(b)  $[D, \psi] = z.\psi' + \frac{1}{2}\psi$

(c)  $[T, \psi] = \psi'$

*Proof.*  $[\psi_{m+\frac{1}{2}}, \psi(z)]_+ = \sum [\psi_{m+\frac{1}{2}}, \psi_{n+\frac{1}{2}}]_+ \cdot z^{-n-1} = z^m$

$[D, \psi(z)] = \sum (-n - \frac{1}{2})\psi_{n+\frac{1}{2}} \cdot z^{-n-1} = z.\psi'(z) + \frac{1}{2}\psi(z)$

$[T, \psi(z)] = \sum (-n)\psi_{n-\frac{1}{2}} \cdot z^{-n-1} = \sum (-n-1)\psi_{n+\frac{1}{2}} \cdot z^{-n-2} = \psi'(z)$  □

**Remark 3.15.**  $(.,.)$  induces  $(\psi(z_1) \dots \psi(z_n)c, d) \in \mathbb{C}[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$ ,  $\forall c, d \in H$ .

**Lemma 3.16.**  $(\psi(z)\Omega, \Omega) = 0$  and  $(\psi(z)\psi(w)\Omega, \Omega) = \frac{1}{z-w}$  if  $|z| > |w|$ .

*Proof.*  $(\psi(z)\Omega, \Omega) = \sum_{n \in \mathbb{Z}} (\psi_{n+\frac{1}{2}}\Omega, \Omega) \cdot z^{-n-1} = 0$   
 $(\psi(z)\psi(w)\Omega, \Omega) = \sum_{m, n \in \mathbb{Z}} (\psi_{m+\frac{1}{2}}\Omega, \psi_{-n-\frac{1}{2}}\Omega) \cdot z^{-n-1} w^{-m-1}$   
 $= \sum_{m, n \in \mathbb{Z}} (\psi_{m-\frac{1}{2}}\Omega, \psi_{-n-\frac{1}{2}}\Omega) \cdot z^{-n-1} w^{-m} = \sum_{n \in \mathbb{N}} (\psi_{-n-\frac{1}{2}}\Omega, \psi_{-n-\frac{1}{2}}\Omega) \cdot z^{-n-1} w^n$   
 $= z^{-1} \sum_{n \in \mathbb{N}} \left(\frac{w}{z}\right)^n = \frac{1}{z-w}$  if  $|z| > |w|$   $\square$

**Lemma 3.17.**  $\forall c, d \in H, (\psi(z)c, d) \in \mathbb{C}[z, z^{-1}]$ .

*Proof.*  $(\psi(z)\psi_{-n-\frac{1}{2}}c, d) = (c, d) \cdot z^{-n-1} - (\psi(z)c, \psi_{n+\frac{1}{2}}d)$   
 $(\psi(z)c, \psi_{-n-\frac{1}{2}}d) = (c, d) \cdot z^n - (\psi(z)\psi_{n+\frac{1}{2}}c, d)$   
Then, the result follows by lemma 3.16 and induction.  $\square$

**Proposition 3.18.**  $\forall c, d \in H, \exists X(c, d) \in (z-w)^{-1}\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$  such that:

$$X(c, d)(z, w) = \begin{cases} (\psi(z)\psi(w)c, d) & \text{if } |z| > |w| \\ -(\psi(w)\psi(z)c, d) & \text{if } |w| > |z| \end{cases}$$

*Proof.*  $(\psi(z)\psi(w)\psi_{-n-\frac{1}{2}}c, d) = (\psi(z)c, d)w^{-n-1} - (\psi(w)c, d)z^{-n-1} + (\psi(z)\psi(w)c, \psi_{n+\frac{1}{2}}d)$   
 $(\psi(z)\psi(w)c, \psi_{-n-\frac{1}{2}}d) = (\psi(w)c, d)z^n - (\psi(z)c, d)w^n + (\psi(z)\psi(w)\psi_{n+\frac{1}{2}}c, d)$   
Then, the result follows by lemma 3.16, 3.17, symmetry and induction.  $\square$

## 3.2 General framework

**Definition 3.19.** Let  $H$  prehilbert and  $A \in (\text{End}H)[[z, z^{-1}]]$  a formal power series defined as  $A(z) = \sum_{n \in \mathbb{Z}} A(n)z^{-n-1}$  with  $A(n) \in \text{End}(H)$ .

**Definition 3.20.** Let  $A, B \in (\text{End}H)[[z, z^{-1}]]$

$A$  and  $B$  are **local** if  $\exists \varepsilon \in \mathbb{Z}_2, \exists N \in \mathbb{N}$  such that  $\forall c, d \in H$ :  
 $\exists X(A, B, c, d) \in (z-w)^{-N}\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$  such that:

$$X(A, B, c, d)(z, w) = \begin{cases} (A(z)B(w)c, d) & \text{if } |z| > |w| \\ (-1)^\varepsilon (B(w)A(z)c, d) & \text{if } |w| > |z| \end{cases}$$

**Example 3.21.**  $\psi$  is local with itself, with  $N = 1$  and  $\varepsilon = \bar{1}$

**Notation 3.22.**  $[X, Y]_\varepsilon = \begin{cases} XY - YX & \text{if } \varepsilon = \bar{0} \\ XY + YX & \text{if } \varepsilon = \bar{1} \end{cases}$

**Remark 3.23.** Let  $n \in \mathbb{N}$ , then,  $(z-w)^n = \sum_{p=0}^n C_n^p (-1)^p w^p z^{n-p}$  and,

$$(z-w)^{-n} = \begin{cases} \sum_{p \in \mathbb{N}} C_{p+n-1}^p w^p z^{-p-n} & \text{if } |z| > |w| \\ (-1)^n \sum_{p \in \mathbb{N}} C_{p+n-1}^p z^p w^{-p-n} & \text{if } |w| > |z| \end{cases}$$



**Proposition 3.24.** *Let  $A, B$  local and  $c, d \in H$  then:*

$$X(A, B, c, d)(z, w) = \sum_{n \in \mathbb{Z}} X_n(A, B, c, d)(w)(z - w)^{-n-1},$$

$$X_n(A, B, c, d)(w) = (A_n B(w)c, d),$$

$$A_n B(w) = \sum_{m \in \mathbb{Z}} (A_n B)(m)w^{-m-1} \text{ and } (A_n B)(m) =$$

$$\begin{cases} \sum_{p=0}^n (-1)^p C_n^p [A(n-p), B(m+p)]_\varepsilon & \text{if } n \geq 0 \\ \sum_{p \in \mathbb{N}} C_{p-n-1}^p (A(n-p)B(m+p) - (-1)^{\varepsilon+n} B(m+n-p)A(p)) & \text{if } n < 0 \end{cases}$$

*Proof.*  $X(A, B, c, d) \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}, (z-w)^{-1}]$ , we develop it around  $z = w$ :

$$X(A, B, c, d)(z, w) = \sum_{n \in \mathbb{Z}} X_n(A, B, c, d)(w)(z - w)^{-n-1}$$

$$\text{with } X_n(A, B, c, d)(w) = \frac{1}{2\pi i} \oint_w (z - w)^n X(A, B, c, d)(z, w) dz.$$

By contour integration argument ( $\oint_w = \int_{|z|=R>|w|} - \int_{|z|=r<|w|}$ ), we obtain:

$$\begin{aligned} X_n(A, B, c, d)(w) &= \frac{1}{2\pi i} \left( \int_{|z|=R>|w|} - \int_{|z|=r<|w|} \right) (z - w)^n X(A, B, c, d)(z, w) dz \\ &= \frac{1}{2\pi i} \int_{|z|=R>|w|} (z - w)^n (A(z)B(w)c, d) dz - \frac{(-1)^\varepsilon}{2\pi i} \int_{|z|=r<|w|} (z - w)^n (B(w)A(z)c, d) dz \\ &= \frac{1}{2\pi i} \sum_{q \in \mathbb{Z}, p=0}^n \left( \int_{|z|=R>|w|} C_n^p (-1)^p z^{n-p} w^p (A(q)B(w)c, d) z^{-q-1} dz \right. \\ &\quad \left. - (-1)^\varepsilon \int_{|z|=r<|w|} C_n^p (-1)^p z^{n-p} w^p (B(w)A(q)c, d) z^{-q-1} dz \right) \\ &= \left( \sum_{p=0}^n (-1)^p w^p C_n^p [A(n-p), B(w)]_\varepsilon c, d \right), \text{ with } n \in \mathbb{N}. \end{aligned}$$

$$\begin{aligned} X_{-n}(A, B, c, d)(w) &= \frac{1}{2\pi i} \sum_{q \in \mathbb{Z}, p \in \mathbb{N}} \left( \int_{|z|=R>|w|} C_{p+n-1}^p z^{-n-p} w^p (A(q)B(w)c, d) z^{-q-1} dz \right. \\ &\quad \left. - (-1)^\varepsilon \int_{|z|=r<|w|} C_{p+n-1}^p (-1)^n w^{-n-p} z^p (B(w)A(q)c, d) z^{-q-1} dz \right) \\ &= \left( \sum_{p \in \mathbb{N}} C_{p+n-1}^p (w^p A(-n-p)B(w) - (-1)^{\varepsilon+n} w^{-n-p} B(w)A(p)) c, d \right) \quad \square \end{aligned}$$

**Definition 3.25.** *Let the operation  $(A, B) \rightarrow A_n B$  as for proposition 3.24.*

**Formula 3.26.** *The formula of  $(A_n B)(m)$  on proposition 3.24.*

**Corollary 3.27.** *(Operator product expansion) Let  $A, B$  local, and  $c, d \in H$ :*

$$(A(z)B(w)c, d) \sim \left( \sum_{n=0}^{N-1} \frac{(A_n B)(w)}{(z-w)^{n+1}} c, d \right) \text{ near } z = w$$

*Proof.*  $X(A, B, c, d)(z, w) = \sum_{n \in \mathbb{Z}} (A_n B)(w)c, d)(z - w)^{-n-1}$

$\in (z - w)^{-N} \mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ , so  $A_n B = 0$  for  $-n - 1 < -N$  ie  $n \geq N$ .  $\square$

**Remark 3.28.** We write OPE as:  $A(z)B(w) \sim \sum_{n=0}^{N-1} \frac{(A_n B)(w)}{(z-w)^{n+1}}$ .

**Remark 3.29.**  $z^m = \begin{cases} \sum_{k=0}^m C_m^k (z-w)^k w^{m-k} & \text{if } m \geq 0 \\ \sum_{k \in \mathbb{N}} (-1)^k C_{k-m-1}^k (z-w)^k w^{m-k} & \text{if } m < 0 \end{cases}$

**Proposition 3.30.** (Lie bracket) Let  $A, B$  local, with  $\varepsilon \in \mathbb{Z}_2$  then:

$$[A(m), B(n)]_\varepsilon = \begin{cases} \sum_{p=0}^{N-1} C_m^p (A_p B)(m+n-p) & \text{if } m \geq 0 \\ \sum_{p=0}^{N-1} (-1)^p C_{p-m-1}^p (A_p B)(m+n-p) & \text{if } m < 0 \end{cases}$$

*Proof.*  $\forall c, d \in H, ([A(m), B(n)]_\varepsilon c, d) = \frac{1}{(2\pi i)^2} (\int \int_{|z|=R > |w|} - \int \int_{|z|=r < |w|}) z^m w^n X(A, B, c, d)(z, w) dz dw$

By contour integration argument ( $\int \int_{|z|=R > |w|} - \int \int_{|z|=r < |w|} = \oint_0 \oint_w$ ):

$$([A(m), B(n)]_\varepsilon c, d) = \frac{1}{2\pi i} \oint_0 w^n \frac{1}{2\pi i} \oint_w z^m (\sum_{p=0}^{N-1} \frac{(A_p B)(w)}{(z-w)^{p+1}} c, d) dz dw$$

$$\begin{aligned} & \text{We suppose } m \geq 0, \text{ then by previous remark, } ([A(m), B(n)]_\varepsilon c, d) = \\ &= \frac{1}{2\pi i} \oint_0 w^n \frac{1}{2\pi i} \oint_w \sum_{k=0}^m C_m^k w^{m-k} (\sum_{p=0}^{N-1} \frac{(A_p B)(w)}{(z-w)^{p+1-k}} c, d) dz dw \\ &= \frac{1}{2\pi i} \oint_0 (\sum_{p=0}^{N-1} w^{n+m-p} C_m^p (A_p B)(w) c, d) dw \\ &= \frac{1}{2\pi i} \oint_0 (\sum_{r \in \mathbb{Z}, p=0}^{N-1} w^{n+m-p-r-1} C_m^p (A_p B)(r) c, d) dw \\ &= (\sum_{p=0}^{N-1} C_m^p (A_p B)(m+n-p) c, d) \quad (\text{we take } C_m^p = 0 \text{ if } p > m). \end{aligned}$$

Similar for  $m < 0 \dots$ , and the result follows. □

**Formula 3.31.** The formula of  $[A(m), B(n)]_\varepsilon$  on proposition 3.30.

**Definition 3.32.** (Operator  $D$ ) Let  $D \in \text{End}(H)$  decomposing  $H$  into  $\bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  with  $D\xi = n\xi \forall \xi \in H_n, \dim(H_n) < \infty$  and  $H_n \perp H_m$  if  $n \neq m$ .

**Notation 3.33.** Let  $A'(z) = \frac{d}{dz} A(z) = \sum_{n \in \mathbb{Z}} (-n) A(n-1) z^{-n-1}$ .

**Definition 3.34.**  $A \in (\text{End} H)[[z, z^{-1}]]$  is graded if:  
 $\exists \alpha \in \frac{1}{2}\mathbb{N}$  such that  $[D, A(z)] = zA'(z) + \alpha A(z)$

**Lemma 3.35.**  $A$  is graded with  $\alpha \iff$   
 $A(n) : H_m \rightarrow H_{m-n+\alpha-1} \quad \forall n \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{N}$

*Proof.*  $[D, A(z)] = zA'(z) + \alpha A(z) = \sum_{n \in \mathbb{Z}} (\alpha - 1 - n)A(n)z^{-n-1}$   
 $\iff [D, A(n)] = (\alpha - 1 - n)A(n) \quad \forall n \in \mathbb{Z}$   
 $\iff \forall n \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{N}, \forall \xi \in H_m$   
 $DA(n)\xi = A(n)D\xi + [D, A(n)]\xi = (m - n + \alpha - 1)A(n)\xi$   
 $\iff A(n) : H_m \rightarrow H_{m-n+\alpha-1} \quad \forall n \in \mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{N}. \quad \square$

**Lemma 3.36.** *Let  $A, B$  local and graded with  $\alpha$  and  $\beta$  then:*  
 $[D, A_n B(z)] = z(A_n B)'(z) + (\alpha + \beta - n - 1)A_n B(z).$

*Proof.*  $A(n) : H_m \rightarrow H_{m-n+\alpha-1}$  and  $B(n) : H_m \rightarrow H_{m-n+\beta-1}$   
Now, by formula 3.26,  $A_p B(n) : H_m \rightarrow H_{m-n+(\alpha+\beta-p-1)-1}$   
The result follows by the previous lemma.  $\square$

**Lemma 3.37.** *Let  $A, B \in (\text{End}H)[[z, z^{-1}]]$ , graded with  $\alpha$  and  $\beta$ , then:*  
 *$A$  and  $B$  are local  $\iff \exists \varepsilon \in \mathbb{Z}_2, \exists N \in \mathbb{N}$  such that  $\forall c, d \in H$ :*  
 $(z-w)^N (A(z)B(w)c, d) = (-1)^\varepsilon (z-w)^N (B(w)A(z)c, d)$  *as formal series.*

*Proof.*  $(\Rightarrow)$  True by definition.  
 $(\Leftarrow)$  Let  $c \in H_p, d \in H_q$   
 $A(n)c \in H_{p-n+\alpha-1} = 0$  for  $n > p + \alpha - 1$ ,  
 $B(m)c \in H_{p-m+\beta-1} = 0$  for  $m > p + \beta - 1$ ,  
 $A(n)B(m)c, B(m)A(n)c \in H_{p-(m+n)+\alpha+\beta-2}, d \in H_q$  and  $H_r \perp H_q$  if  $q \neq r$ .

Let  $S = \{(m, n) \in \mathbb{Z}^2; m + n = p - q + \alpha + \beta - 2, m \leq p + \beta - 1\}$   
and  $S' = \{(m, n) \in \mathbb{Z}^2; m + n = p - q + \alpha + \beta - 2, n \leq p + \alpha - 1\}$

$$(z-w)^N (A(z)B(w)c, d) = \sum_{S, k=0}^N C_N^k (A(n)B(m)c, d) z^{-n-1-k} w^{-m-1+N-k}$$

$$(z-w)^N (B(w)A(z)c, d) = (-1)^\varepsilon \sum_{S', k=0}^N C_N^k (B(m)A(n)c, d) z^{-n-1-k} w^{-m-1+N-k}$$

But,  $S \cap S'$  is a finite subset of  $\mathbb{Z}^2$ , so the formal series is a polynomial:  
 $P(A, B, c, d) \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ ; now, using remark 3.23, and the fact that  
 $A(n)c = 0$  for  $n > p + \alpha - 1$  and  $B(m)c = 0$  for  $m > p + \beta - 1$ , then:

$$(z, w)^{-N} P(A, B, c, d)(z, w) = \begin{cases} (A(z)B(w)c, d) & \text{if } |z| > |w| \\ (-1)^\varepsilon (B(w)A(z)c, d) & \text{if } |w| > |z| \end{cases} \quad \square$$

**Remark 3.38.** (*associativity*)  $(A_n B)_m C = A_n (B_m C) = A_n B_m C$

**Lemma 3.39.** *Let  $A_1, \dots, A_R$  graded,  $A_i$  and  $A_j$  local with  $N = N_{ij} \in \mathbb{N}$ . Then,  $\forall c, d \in H$ :*

$$\prod_{i < j} (z_i - z_j)^{N_{ij}} (A_1(z_1) \dots A_R(z_R) c, d) \in \mathbb{C}[z_1^{\pm 1}, \dots, z_R^{\pm 1}]$$

*Proof.* It is exactly as the previous lemma:

We can put each  $A_i(z_i)$  on the first place by commutations.

We obtain equalities between  $R$  series with support  $S_i \cup T$ , with  $T$  the support due to  $\prod_{i < j} (z_i - z_j)^{N_{ij}}$  (finite), and as the previous lemma:

$$S_i = \{(m_1, \dots, m_R) \in \mathbb{Z}^R; m_1 + \dots + m_R = K, m_i \leq k_i\}$$

So,  $\bigcap S_i$  is a finite subset of  $\mathbb{Z}^R$ , and the result follows.  $\square$

**Lemma 3.40.** (*Dong's lemma*) *Let  $A, B, C$  graded and pairwise local, then  $A_n B$  and  $C$  are local.*

*Proof.* Let  $Q(z_1, z_2, z_3) = \prod_{i < j} (z_i - z_j)^{N_{ij}}$ , by lemma 3.39,  $\forall d, e \in H$ :

$$Q.(A(z_1)B(z_2)C(z_3)d, e) = Q.(-1)^{\varepsilon_1 + \varepsilon_2} (C(z_3)A(z_1)B(z_2)d, e) \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$$

Now, we divide this polynomial by  $Q$ , we fix  $z_2$  and we develop around  $z_1 = z_2$ .

Then  $\exists N \in \mathbb{N}$  such that  $\forall n \in \mathbb{Z}$  if  $P_n$  is the coefficient of  $(z_1 - z_2)^{-n-1}$  then  $S_n = (z_2 - z_3)^N P_n \in \mathbb{C}[z_2^{\pm 1}, z_3^{\pm 1}]$ .

Now, on one hand  $S_n = (z_2 - z_3)^N (A_n B(z_2) C(z_3) d, e)$  and on the other hand  $S_n = (-1)^\varepsilon (z_2 - z_3)^N (C(z_3) A_n B(z_2) d, e)$ , with  $\varepsilon = \varepsilon_1 + \varepsilon_2$ .

Then, the result follows by lemmas 3.36 and 3.37.  $\square$

**Proof's corollary 3.41.** *If in addition,  $A$  and  $C$  are local with  $\varepsilon_1 \in \mathbb{Z}_2$ , and,  $B$  and  $C$ , local with  $\varepsilon_2$ , then,  $A_n B$  and  $C$  are local with  $\varepsilon = \varepsilon_1 + \varepsilon_2$ .*

**Lemma 3.42.** *If  $A$  and  $B$  are local with  $\varepsilon \in \mathbb{Z}_2$ , so is  $A'$  and  $B$*

$$\text{Proof. } (z - w)^N (A(z)B(w)c, d) = (-1)^\varepsilon (z - w)^N (B(w)A(z)c, d)$$

Then, applying  $\frac{d}{dz}$  and the lemma 3.37, the result follows.  $\square$

**Definition 3.43.** (*Operator  $T$* ) *Let  $T \in \text{End}(H)$ .*

**Lemma 3.44.** *Let  $A, B$  local such that  $[T, A] = A'$  and  $[T, B] = B'$ .*

*Then,  $[T, A_n B] = (A_n B)' = A'_n B + A_n B'$  and  $[T, A'] = A''$*

*Proof.*  $(z-w)^N([T, A(z)B(w)]c, d) = (z-w)^N((A'(z)B(w) + A(z)B'(w))c, d)$

$$\begin{aligned} &= (z-w)^N \sum_{n \in \mathbb{Z}} ((A'_n B + A_n B')(w)c, d)(z-w)^{-n-1} \quad \text{on one hand} \\ &= (z-w)^N \left( \frac{d}{dz} + \frac{d}{dw} \right) \left( \sum_{n \in \mathbb{Z}} A_n B(w)(z-w)^{-n-1}c, d \right) \quad \text{on the other hand} \end{aligned}$$

$$\begin{aligned} &= (z-w)^N \left[ \left( \sum_{n \in \mathbb{Z}} (-n-1)A_n B(w)(z-w)^{-n-2}c, d \right) + \right. \\ &\quad \left. \left( \sum_{n \in \mathbb{Z}} (A_n B)'(w)(z-w)^{-n-1}c, d \right) + \left( \sum_{n \in \mathbb{Z}} (n+1)A_n B(w)(z-w)^{-n-2}c, d \right) \right] \\ &= (z-w)^N \sum_{n \in \mathbb{Z}} ((A_n B)'(w)c, d)(z-w)^{-n-1} \end{aligned}$$

By identification:  $[T, A_n B] = (A_n B)' = A'_n B + A_n B'$

Now,  $[T, A] = A' \Rightarrow [T, A(n)] = -nA(n-1)$ , so  $[T, A'] = A''$  □

**Lemma 3.45.** *Let  $\Omega \in H$ ;  $A, B$  local with  $A(m)\Omega = B(m)\Omega = 0 \forall m \in \mathbb{N}$ , then  $A'(m)\Omega = A_n B(m)\Omega = 0 \forall m \in \mathbb{N}, \forall n \in \mathbb{Z}$ .*

*Proof.*  $A'(m) = -mA(m-1)$ , so  $A'(m)\Omega = 0 \forall m \in \mathbb{N}$

On the formula 3.26,  $A(n-p)\Omega = B(m+p)\Omega = A(p)\Omega = 0$  because  $n-p, m+p, p \in \mathbb{N}$ , then,  $A_n B(m)\Omega = 0 \forall m \in \mathbb{N}, \forall n \in \mathbb{Z}$ . □

### 3.3 System of generators

**Definition 3.46.** *Let  $H$  prehilbert space;  $\{A_1, \dots, A_r\} \subset (\text{End}H)[[z, z^{-1}]]$  is a system of generators if  $\exists D, T \in \text{End}(H), \Omega \in H$  such that:*

- (a)  $\forall i, j$   $A_i$  and  $A_j$  are local with  $N = N_{ij}$  and  $\varepsilon = \varepsilon_{ij} = \varepsilon_{ii} \cdot \varepsilon_{jj}$
- (b)  $\forall i$   $[T, A_i] = A'_i$
- (c)  $D$  decomposes  $H = \bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  with  $D\xi = n\xi \forall \xi \in H_n, \dim(H_n) < \infty$ ,  $H_n \perp H_m$  if  $n \neq m$  and  $\forall i$   $A_i$  is graded with  $\alpha_i \in \mathbb{N} + \frac{\varepsilon_{ii}}{2}$
- (d)  $\Omega \in H_0, \|\Omega\| = 1$ , and  $\forall i \forall m \in \mathbb{N}, A_i(m)\Omega = D\Omega = T\Omega = 0$
- (e)  $\mathcal{A} = \{A_i(m), \forall i \forall m \in \mathbb{Z}\}$  acts irreducibly on  $H$ , so that  $H$  is the minimal space containing  $\Omega$  and stable by the action of  $\mathcal{A}$

**Definition 3.47.** *Let  $S \subset (\text{End}H)[[z, z^{-1}]]$ , the minimal subset containing  $Id, A_1, \dots, A_r$ , stable by the operations:*

$$(A, B) \mapsto (A_n B) \quad (\forall n \in \mathbb{Z}) \quad , \quad A \mapsto A'$$

Let  $\mathcal{S}_\varepsilon = \{A \in \mathcal{S} \mid A \text{ is local with itself with } \varepsilon \in \mathbb{Z}_2\}$ , so that  $\mathcal{S} = \mathcal{S}_0 \amalg \mathcal{S}_1$ .

Let  $\mathcal{S}_\varepsilon = \text{lin} \langle \mathcal{S}_\varepsilon \rangle$  and  $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$ .

**Remark 3.48.** All is well defined by previous lemmas.

**Lemma 3.49.**  $\forall A, B \in \mathcal{S}$ , they are local,  $A_n B \in \mathcal{S}$  and  $[T, A] = A' \in \mathcal{S}$

*Proof.* By previous lemmas and linearizing Dong's lemma.  $\square$

**Lemma 3.50.** Let  $E \in \mathcal{S}_{\varepsilon_1}$  and  $F \in \mathcal{S}_{\varepsilon_2}$  then:

(a)  $E_n F \in \mathcal{S}_{\varepsilon_1 + \varepsilon_2}$

(b)  $E$  and  $F$  are local with  $\varepsilon = \varepsilon_1 \cdot \varepsilon_2$

*Proof.* (a)  $E$  and  $F$  are local with an  $\varepsilon \in \mathbb{Z}_2$ .

We use the corollary 3.41 with  $A = E, B = F, C = E$ , with  $A = E, B = F, C = F$  and finally with  $A = E, B = F, C = E_n F$ . Then we see that  $E_n F$  is local with itself with  $\varepsilon' = \varepsilon_1 + \varepsilon + \varepsilon_2 + \varepsilon = \varepsilon_1 + \varepsilon_2$ , so,  $E_n F \in \mathcal{S}_{\varepsilon_1 + \varepsilon_2}$

(b) By induction:

Base case:  $\forall i, j A_i \in \mathcal{S}_{\varepsilon_{ii}}, A_j \in \mathcal{S}_{\varepsilon_{jj}}$  and are local with  $\varepsilon = \varepsilon_{ij} = \varepsilon_{ii} \cdot \varepsilon_{jj}$  by definition 3.46.

Inductive step: We suppose the property for  $E \in \mathcal{S}_{\varepsilon_1}, F \in \mathcal{S}_{\varepsilon_2}$  and  $G \in \mathcal{S}_{\varepsilon_3}$ .

We prove it for  $E_n F$  and  $G$ :

$E$  and  $G$  are local with  $\varepsilon = \varepsilon_1 \cdot \varepsilon_3$

$F$  and  $G$  are local with  $\varepsilon = \varepsilon_2 \cdot \varepsilon_3$

Now,  $E_n F \in \mathcal{S}_{\varepsilon_1 + \varepsilon_2}, G \in \mathcal{S}_{\varepsilon_3}$  and by corollary 3.41 with  $A = E, B = F, C = G, E_n F$  and  $G$  are local with  $\varepsilon = \varepsilon_1 \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 = (\varepsilon_1 + \varepsilon_2) \cdot \varepsilon_3$

The following lemma completes the proof.  $\square$

**Lemma 3.51.**  $A \in \mathcal{S}_\varepsilon \Rightarrow A' \in \mathcal{S}_\varepsilon$

*Proof.* By lemma 3.42, if  $A$  and  $B$  are local with  $\varepsilon \in \mathbb{Z}_2$ , so is  $A'$  and  $B$ .

The result follows by taking  $B = A$  and then  $B = A'$ .  $\square$

**Definition 3.52.** (well defined by lemma 3.45)

$$\begin{array}{ccc} R: \mathcal{S} & \longrightarrow & H \\ A & \longmapsto & a := A(z)\Omega|_{z=0} \end{array} \quad \text{linear.}$$

**Examples 3.53.**

- (a)  $R(Id) = \Omega, \quad R(A) = A(-1)\Omega$
- (b)  $R(A') = A(-2)\Omega = T.R(A)$
- (c)  $R(A_n B) = A(n)R(B)$  (by formula 3.26)
- (d)  $R(A_n Id) = A(n)\Omega$

**Lemma 3.54.**  $A$  is graded with  $\alpha \iff R(A) \in H_\alpha$

*Proof.* By lemma 3.35 and 3.36, inductions and linear combinations. □

**State-Field correspondence:**

**Lemma 3.55.** (Existence)  $\forall a \in H, \exists A \in \mathcal{S}$  such that  $R(A) = a$ .

*Proof.*  $R((A_{i_1})_{m_1}(A_{i_2})_{m_2}\dots(A_{i_k})_{m_k} Id) = A_{i_1}(m_1)R((A_{i_2})_{m_2}\dots(A_{i_k})_{m_k} Id)$   
 $= \dots = A_{i_1}(m_1)\dots A_{i_k}(m_k)\Omega$

Now, the action of the  $A_i(m)$  on  $\Omega$  generates  $H$  by definition 3.46. □

**Lemma 3.56.** Let  $A \in \mathcal{S}$ , then  $A(z)\Omega = e^{zT}R(A)$ .

*Proof.* Let  $F_A(z) = A(z)\Omega = \sum_{n \in \mathbb{N}} A(-n-1)\Omega z^n$ ,

Then,  $\forall b \in H, (F_A(z), b) \in \mathbb{C}[z]$

Now,  $\frac{d}{dz}(F_A(z), b) = (\frac{d}{dz}F_A(z), b) = (A'(z)\Omega, b)$

$= ([T, A(z)]\Omega, b) = (T.A(z)\Omega, b) = (T.F_A(z)\Omega, b)$

But,  $F_A(0) = R(A)$ , so we see that:  $(F_A(z), b) = (e^{zT}R(A), b) \forall b \in H$

Finally,  $F_A(z) = e^{zT}R(A)$  □

**Lemma 3.57.** (Unicity)  $R(A) = R(B) \Rightarrow A = B$ .

*Proof.* Let  $C = A - B$ , then  $R(C) = R(A) - R(B) = 0$

and  $F_C(z) = e^{zT}R(C) = 0$

Now,  $\forall e \in H, \exists E \in \mathcal{S}$  such that  $R(E) = e$ .

Then  $\forall f \in H, \exists N \in \mathbb{N} \exists \varepsilon \in \mathbb{Z}_2$  such that :

$(z-w)^N(C(z)E(w)\Omega, f) = (-1)^\varepsilon(z-w)^N(E(w)C(z)\Omega, f)$

Now,  $(E(w)C(z)\Omega, f) = (E(w)F_C(z), f) = 0 = (C(z)E(w)\Omega, f)$

So,  $(C(z)E(w)\Omega, f)|_{w=0} = (C(z)e, f) = 0 \forall e, f \in H$

Finally,  $C = 0$  and  $A = B$  □

Now, we can well defined:

**Definition 3.58.** (*State-Field correspondence map*)

$$V : \begin{array}{ccc} H & \longrightarrow & \mathcal{S} \\ a & \longmapsto & V(a) \end{array} \quad \text{linear.}$$

$$\text{such that : } \begin{cases} \forall a \in H & R(V(a)) = a \\ \forall A \in \mathcal{S} & V(R(A)) = A \end{cases}$$

**Notation 3.59.**  $V(a)(z)$  is noted  $V(a, z)$  and  $A(z) = V(R(A), z)$

**Examples 3.60.**

(a)  $V(0, z) = 0, \quad V(\Omega, z) = Id$

(b)  $V'(a, z) = V(T.a, z)$

(c)  $(A_n B)(z) = V(A(n)R(B), z)$

**Definition 3.61.** Let  $H_\varepsilon = \bigoplus_{n \in \mathbb{N} + \frac{\varepsilon}{2}} H_n$  so that  $H = H_0 \oplus H_1$ .

**Lemma 3.62.**  $R(\mathcal{S}_\varepsilon) = H_\varepsilon \quad (\varepsilon \in \mathbb{Z}_2)$

*Proof.* Base step: by definition 3.46 and lemma 3.54,

$\forall i \ A_i \in \mathcal{S}_{\varepsilon_{ii}}$  and  $R(A_i) \in H_{\alpha_i}$  with  $\alpha_i \in \mathbb{N} + \frac{\varepsilon_{ii}}{2}$

Inductive step: by lemma 3.50 □

**Corollary 3.63.** (*Relation with T and D*) Let  $a \in H_\alpha$ , we have that:

(a)  $[T, V(a, z)] = V'(a, z) = V(T.a, z) \in \mathcal{S}$

(b)  $[D, V(a, z)] = z.V'(a, z) + \alpha.V(a, z) \quad (\notin \mathcal{S} \text{ in general})$

### 3.4 Application to fermion algebra

$$H = \mathcal{F}_{NS}, \quad \psi(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1} \text{ with } [\psi_m, \psi_n]_+ = \delta_{m+n} Id.$$

**Proposition 3.64.**  $\{\psi\}$  is a system of generator.



*Proof.*  $\psi$  is local with itself with  $N = 1$  and  $\varepsilon = \bar{1} = \bar{1}.\bar{1}$  (see definition 3.46)  
 We have construct  $D$  and  $T$  (p 14),  $\Omega \in H_0$ ,  $\|\Omega\| = 1$ ,  $D\Omega = T\Omega = 0$ .  
 $[T, \psi(z)] = \psi'(z)$ ,  $[D, \psi(z)] = z.\psi'(z) + \frac{1}{2}\psi(z)$  and  $\frac{1}{2} \in \mathbb{N} + \frac{1}{2}$   
 Finally,  $\{\psi_n, n \in \frac{1}{2}\mathbb{N}\}$  acts irreducibly on  $H$   $\square$

**Corollary 3.65.**  $\{\psi\}$  generates an  $\mathcal{S}$  with a state-field correspondence with:

$$R(\psi) = \psi_{-\frac{1}{2}}\Omega \text{ and } \psi(z) = V(\psi_{-\frac{1}{2}}\Omega, z)$$

**Lemma 3.66.** (OPE)  $\psi(z)\psi(w) \sim \frac{Id}{z-w}$

*Proof.*  $\psi_n\psi(w) = V(\psi_{n+\frac{1}{2}}\psi_{-\frac{1}{2}}\Omega, w) = 0$  if  $n \geq 1$  ( here  $N = 1$  )

Now, for  $0 \leq n \leq N - 1$  i.e  $n = 0$  :

$$\psi_{\frac{1}{2}}\psi_{-\frac{1}{2}}\Omega = ([\psi_{\frac{1}{2}}, \psi_{-\frac{1}{2}}]_+ - \psi_{-\frac{1}{2}}\psi_{\frac{1}{2}})\Omega = \Omega, \text{ so } \psi_0\psi(w) = Id \quad \square$$

**Remark 3.67.** (Next operator)  $\psi_{-\frac{1}{2}}\psi_{-\frac{1}{2}}\Omega = 0$ , so  $\psi_{-1}\psi = 0$ ; and the next

operator of the expansion is  $2L(w) := \psi_{-2}\psi(w) = 2 \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$

Now,  $R(L) = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$ , then  $L(w) = V(\frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega, w)$ .

**Remark 3.68.**  $L(n) = L_{n-1}$  so,  $L_0\Omega = L_{-1}\Omega = 0$  by lemma 3.45.

**Lemma 3.69.** (OPE)  $\psi(z)L(w) \sim \frac{1/2\psi(w)}{(z-w)^2} - \frac{1/2\psi'(w)}{(z-w)}$

*Proof.*  $\psi_n L(w) = \frac{1}{2}V(\psi_{n+\frac{1}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega, w) = 0$  if  $n \geq 2$  ( here  $N = 2$  )

Now,  $\psi_{\frac{1}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega = -\psi_{-\frac{3}{2}}\Omega = R(\psi')$  ,  $\psi_{\frac{3}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega = \psi_{-\frac{1}{2}}\Omega = R(\psi')$   $\square$

**Lemma 3.70.** (Lie bracket)  $[L_m, \psi_n] = -(n + \frac{1}{2}m)\psi_{m+n}$

*Proof.* By lemma 3.50,  $\psi$  and  $L$  are local with  $\varepsilon = \bar{0}$ , and by formula 3.31:

$$\begin{aligned} [\psi(m), L(n+1)] &= -\frac{1}{2}C_m^0\psi'(m+n+1) + \frac{1}{2}C_m^1\psi(m+n+1-1) \\ &= \frac{1}{2}(m+n+1)\psi(m+n) + \frac{1}{2}m\psi(m+n) = (m + \frac{1}{2} + \frac{1}{2}n)\psi(m+n) \end{aligned}$$

We have computed for  $m \geq 0$ , we find the same result for  $m < 0$ .

Now,  $\psi(m) = \psi_{m+\frac{1}{2}}$  and  $L(n+1) = L_n$ , so the result follows.  $\square$

**Lemma 3.71.**  $D = L_0$  and  $T = L_{-1}$

*Proof.*  $[L_0, \psi_n] = -n\psi_n = [D, \psi_n]$  ,  $[L_{-1}, \psi_n] = -(n - \frac{1}{2})\psi_{n-1} = [T, \psi_n]$

So, by irreducibility and Schur's lemma,  $L_0 - D$  and  $L_{-1} - T \in \mathbb{C}Id$

Now,  $L_0\Omega = D\Omega = L_{-1}\Omega = T\Omega = 0$ , then,  $D = L_0$  and  $T = L_{-1}$   $\square$

**Corollary 3.72.**  $\forall a \in H_s$ :

(a)  $[L_{-1}, V(a, z)] = V'(a, z) = V(L_{-1}.a, z) \in \mathcal{S}$

(b)  $[L_0, V(a, z)] = z.V'(a, z) + s.V(a, z)$

**Remark 3.73.**  $\forall A \in \mathcal{S}$ ,  $A' = (L_0A)$ , so, by Dong's lemma, we finally don't need here to  $A \mapsto A'$  for the construction of  $\mathcal{S}$ .

**Lemma 3.74.** (OPE)  $L(z)L(w) \sim \frac{(c/2)Id}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{(z-w)}$

*Proof.*  $L_n L(w) = V(L(n)L(-1)\Omega, w) = V(L_{n-1}L_{-2}\Omega, w) = 0$  if  $n \geq 4$ .

Then, here,  $N = 4$ , so, for  $0 \leq n \leq N - 1$ :

(a)  $V(L_{-1}L_{-2}\Omega, w) = L'(w)$

(b)  $L_0L_{-2}\Omega = 2L_{-2}\Omega = 2R(L)$  because  $L_{-2}\Omega \in H_2$

(c)  $L_1L_{-2}\Omega = \frac{1}{2}L_1\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega = \frac{1}{2}[L_1, \psi_{-\frac{3}{2}}]\psi_{-\frac{1}{2}}\Omega = \frac{1}{2}\psi_{-\frac{1}{2}}^2\Omega = 0$

(d)  $L_2L_{-2}\Omega \in H_0 = \mathbb{C}\Omega$ , so,  $L_2L_{-2}\Omega = K\Omega$  with  $K = \|L_{-2}\Omega\|^2$

□

**Notation 3.75.**  $c := 2\|L_{-2}\Omega\|^2$ , the central charge.

(here  $c = \frac{1}{2}(\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega, \psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega) = \frac{1}{2}$ )

**Notation 3.76.** Let  $\delta_k = \begin{cases} 0 & \text{if } k \neq 0 \\ Id & \text{if } k = 0 \end{cases}$

**Lemma 3.77.** (Lie bracket)  $[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n}$ .

*Proof.* By lemma 3.50,  $L \in \mathcal{S}_0$ , and by formula 3.31:

If  $m + 1 \geq 0$ , then:  $[L(m + 1), L(n + 1)] =$

$$C_{m+1}^0 L'(m + n + 2) + 2C_{m+1}^1 L(m + n + 2 - 1) + \frac{c}{2}C_{m+1}^3 Id(m + n + 2 - 3)$$

$$= -(m + n + 2)L(m + n + 2) + 2(m + 1)L(m + n + 1) + \frac{c}{2}\frac{m(m^2-1)}{6}\delta_{m+n}$$

$$= (m - n)L(m + n + 1) + \frac{c}{12}m(m^2 - 1)\delta_{m+n}$$

We find the same result for  $m + 1 < 0$

□

**Remark 3.78.**  $L_m^* = L_{-m}$

*Proof.*  $[\psi_{-n}, L_m^*] = [L_m, \psi_n]^* = -(n + \frac{1}{2}m)\psi_{-m-n} = [\psi_{-n}, L_{-m}]$ , then the result follows by irreducibility, Schur's lemma and grading. □

**Remark 3.79.** The  $(L_n)$  generate a Virasoro algebra  $\mathfrak{Vir}$ .

**Corollary 3.80.**  $\mathfrak{Vir}$  acts on  $H = \mathcal{F}_{NS}$ , and admits  $L(c, h) = L(\frac{1}{2}, 0)$  as minimal submodule containing  $\Omega$ .

**Definition 3.81.** Let call  $L$  the Virasoro operator, and  $\omega = R(L) = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$ , the Virasoro vector.

### 3.5 Vertex operator superalgebra

**Definition 3.82.** A vertex operator superalgebra is an  $(H, V, \Omega, \omega)$  with:

- (a)  $H = H_{\bar{0}} \oplus H_{\bar{1}}$  a prehilbert superspace.
- (b)  $V : H \rightarrow (\text{End}H)[[z, z^{-1}]]$  a linear map.
- (c)  $\Omega, \omega \in H$  the vacuum and Virasoro vectors.

Let  $\mathcal{S}_\varepsilon = V(H_\varepsilon)$ ,  $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$  and  $A(z) = V(a, z) = \sum_{n \in \mathbb{Z}} A(n)z^{-n-1}$ , then  $(H, V, \Omega, \omega)$  satisfies the followings axioms:

1. (vacuum axioms):  $\forall A \in \mathcal{S}$  and  $\forall n \in \mathbb{N}$ ,  $A(n)\Omega = 0$ ,  
 $V(a, z)\Omega|_{z=0} = a$  and  $V(\Omega, z) = Id$
2. (irreducibility axiom): Let  $\mathcal{A} = \{A(n) | A \in \mathcal{S}, n \in \mathbb{Z}\}$  then,  
 $\mathcal{A}.$  acts irreducibly on  $H$ , so that  $\mathcal{A}.\Omega = H$
3. (locality axiom):  $\forall A \in \mathcal{S}_{\varepsilon_1}$ ,  $\forall B \in \mathcal{S}_{\varepsilon_2}$ ,  $A$  and  $B$  are local  
 (see definition 3.20 and lemma 3.37), with  $\varepsilon = \varepsilon_1.\varepsilon_2$  and  $A_n B \in \mathcal{S}_{\varepsilon_1 + \varepsilon_2}$
4. (Virasoro axiom):  $V(\omega, z) = L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$  Virasoro operator  
 $(L_0\Omega = L_{-1}\Omega = 0$  and  $\omega = L_{-2}\Omega)$ . Let  $c = 2\|\omega\|^2$  the central charge:  
 $[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n}$
5. ( $L_0$  axioms)  $L_0$  decomposes  $H$  into  $\bigoplus_{n \in \mathbb{N} + \frac{1}{2}} H_n$  with  $\dim(H_n) < \infty$ ,  
 $H_n \perp H_m$  if  $n \neq m$ ,  $H_\varepsilon = \bigoplus_{n \in \mathbb{N} + \frac{\varepsilon}{2}} H_n$ ,  $\Omega \in H_0$ ,  $\omega \in H_2$ , and  
 $\forall a \in H_\alpha$ ,  $[L_0, V(a, z)] = z.V'(a, z) + \alpha.V(a, z)$
6. ( $L_{-1}$  axioms):  $[L_{-1}, V(a, z)] = V'(a, z) = V(L_{-1}.a, z) \in \mathcal{S}$

**Corollary 3.83.** *A system of generators, generating a Virasoro operator  $L \in \mathcal{S}$ , with  $D = L_0$  and  $T = L_{-1}$ , generates a vertex operator superalgebra.*

**Corollary 3.84.** *The fermion operator  $\psi$  generates a vertex operator superalgebra, with Virasoro vector  $\omega = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\Omega$ .*

**Remark 3.85.** *The Virasoro operator  $L$  alone, generates the minimal vertex operator (super)algebra.*

**Remark 3.86.** *Let  $A(z) = V(a, z)$  and  $B(w) = V(b, w)$ ; the formula 3.26 is general, so similarly, by vacuum axioms,  $A_n B(w) = V(A(n)b, w)$ .*

**Proposition 3.87.** *(Borcherds associativity)  $\exists N \in \mathbb{N}$  such that  $\forall c, d \in H$ :  $(z - w)^N (V(a, z)V(b, w)c, d) = (z - w)^N (V(V(a, z - w)b, w)c, d)$*

*Proof.* To simplify the proof, we don't write:  
 $\exists N \in \mathbb{N}$  such that  $\forall c, d \in H (z - w)^N (\cdot, c, d)$ , but it is implicit.

$$\begin{aligned} V(a, z)V(b, w) &= A(z)B(w) = \sum A_n B(w)(z - w)^{-n-1} \\ &= \sum V(A(n)b, w)(z - w)^{-n-1} = V(\sum A(n)b(z - w)^{-n-1}, w) \\ &= V(\sum A(n)(z - w)^{-n-1}b, w) = V(V(a, z - w)b, w). \end{aligned} \quad \square$$

## 4 Vertex $\mathfrak{g}$ -superalgebras and modules

### 4.1 Preliminaries

#### 4.1.1 Simple Lie algebra $\mathfrak{g}$

Let  $\mathfrak{g}$  be a simple Lie algebra of dimension  $N$ , a basis  $(X_a)$  with  $[X_a, X_b] = i \sum_c \Gamma_{ab}^c X_c$  with  $\Gamma_{ab}^c \in \mathbb{R}$  totally antisymmetric.

**Lemma 4.1.** *Let  $\mathcal{C} = \sum_b X_b^2$ , then  $[\mathfrak{g}, \mathcal{C}] = 0$*

*Proof.* It suffices to prove  $[X_a, \mathcal{C}] = 0$  for each  $X_a$ .

$$[X_a, \mathcal{C}] = \sum_b [X_a, X_b^2] = \sum_b ([X_a, X_b]X_b + X_b[X_a, X_b]) = i \sum_{b,c} \Gamma_{ab}^c X_c X_b + i \sum_{b,c} \Gamma_{ab}^c X_b X_c = i \sum_{b,c} (\Gamma_{ab}^c + \Gamma_{ac}^b) X_c X_b = 0 \quad \text{by antisymmetry.} \quad \square$$

**Remark 4.2.**  $\mathcal{C}$  is a multiple of the **Casimir** of  $\mathfrak{g}$ . We suppose to have well normalized the basis such that  $\mathcal{C}$  is exactly the Casimir.

**Corollary 4.3.** *By Schur's lemma,  $\mathcal{C}$  acts as multiplicative constant  $c_V$  on each irreducible representation  $V$ .*

**Example 4.4.**  $\mathfrak{g}$  is simple, it acts irreducibly on  $V = \mathfrak{g}$  with  $ad$ .

**Lemma 4.5.**  $\sum_{a,c} \Gamma_{ac}^b \cdot \Gamma_{ac}^d = \delta_{bd} c_{\mathfrak{g}}$

$$\begin{aligned} \text{Proof. } (\sum_a ad_{X_a}^2)(X_b) &= c_{\mathfrak{g}} X_b = \sum_a [X_a, [X_a, X_b]] \\ &= i^2 \sum_{a,c,d} \Gamma_{ab}^c \cdot \Gamma_{ac}^d X_d = \sum_{a,c,d} \Gamma_{ac}^b \cdot \Gamma_{ac}^d X_d. \end{aligned}$$

$$\text{Then, } \sum_{a,c} \Gamma_{ac}^b \cdot \Gamma_{ac}^d = \delta_{bd} c_{\mathfrak{g}} \quad \square$$

**Definition 4.6.**  $g = \frac{c_{\mathfrak{g}}}{2}$  is called the dual Coxeter number.

**Example 4.7.**  $\mathfrak{g} = A_1 = \mathfrak{sl}_2$ ,  $dim(\mathfrak{g}) = 3$

$$[E, F] = H, [H, E] = 2E, [H, F] = -2F, \text{ with Casimir } EF + FE + \frac{1}{2}H^2$$

We choose the basis:  $X_1 = \frac{i\sqrt{2}}{2}(E - F)$ ,  $X_2 = \frac{\sqrt{2}}{2}(E + F)$ ,  $X_3 = \frac{\sqrt{2}}{2}H$ ,  
with relations:  $[X_1, X_2] = i\sqrt{2}X_3$ ,  $[X_3, X_1] = i\sqrt{2}X_2$ ,  $[X_2, X_3] = i\sqrt{2}X_1$   
 $\mathcal{C} = \sum_a X_a^2 = EF + FE + \frac{1}{2}H^2$  and  $g = \frac{1}{2} \sum_{a,b} (\Gamma_{ab}^c)^2 = 2$

Table (see [7] p 111)

$\mathfrak{g}$	$A_n$	$B_n$	$C_n$	$D_n$	$E_6$	$E_7$	$E_8$	$F_4$	$G_2$
$dim(\mathfrak{g})$	$n^2 + 2n$	$2n^2 + n$	$2n^2 + n$	$2n^2 - n$	78	133	248	52	14
$g$	$n + 1$	$2n - 1$	$n + 1$	$2n - 2$	12	18	30	9	4

### 4.1.2 Loop algebra $L\mathfrak{g}$

**Definition 4.8.** Let  $L\mathfrak{g} = C^\infty(\mathbb{S}^1, \mathfrak{g})$  the loop algebra of  $\mathfrak{g}$ . It's an infinite dimensional Lie  $\star$ -algebra, admitting the  $X_n^a = X_a e^{in\theta}$  as basis, with  $n \in \mathbb{Z}$  and  $(X_a)$  the base of  $\mathfrak{g}$ ; so:

$$[X_m^a, X_n^b] = [X_a, X_b]_{m+n} \quad \text{and} \quad (X_n^a)^\star = X_{-n}^a$$

**Proposition 4.9.** (Boson cocycle)  $L\mathfrak{g}$  has a unique central extension, up to equivalent, i.e.  $H_2(L\mathfrak{g}, \mathbb{C})$  is 1-dimensional.  $H_2(L\mathfrak{g}, \mathbb{C})$  is 1-dimensional. Let  $\mathcal{L}$  the central element and  $\widehat{\mathfrak{g}}_+ = L\mathfrak{g} \oplus \mathbb{C}\mathcal{L}$  called  $\mathfrak{g}$ -boson algebra, then:

$$[X_m^a, X_n^b] = [X_a, X_b]_{m+n} + m\delta_{ab}\delta_{m+n}\mathcal{L}$$

*Proof.* See [13] or [18] p 46. □

**Theorem 4.10.** The unitary highest weight representations of  $\widehat{\mathfrak{g}}_+$  are  $H = L(V_\lambda, \ell)$  with:

- (a)  $\ell \in \mathbb{N}$  such that  $\mathcal{L}\Omega = \ell\Omega$  (the level of  $H$ ).
- (b)  $H_0 = V_\lambda$  irreducible representation of  $\mathfrak{g}$  such that:  
 $(\lambda, \theta) \leq \ell$  with  $\lambda$  the highest weight and  $\theta$  the highest root.

*Proof.* See [13] or [18] p 48. □

**Remark 4.11.** Let  $\mathcal{C}_\ell$  the category of such representations for  $\ell$  fixed.  $\mathcal{C}_\ell$  is a finite set and  $\mathcal{C}_\ell \subset \mathcal{C}_{\ell+1}$

**Remark 4.12.** The irreducible unitary projective positive energy representations of  $L\mathfrak{g}$  are given by the unitary highest weight representation of  $\widehat{\mathfrak{g}}_+$ .

**Example 4.13.** We take  $\mathfrak{g} = \mathfrak{sl}_2$ , then  $H = L(j, \ell)$  with:

- $\mathcal{L}\Omega = \ell\Omega, \quad \ell \in \mathbb{N}$
- $H_0 = V_j$  with  $j \in \frac{1}{2}\mathbb{N}$  the spin and  $j \leq \frac{\ell}{2}$ , such that  
 $\mathcal{C}\Omega = c_{V_j}\Omega$  with  $\mathcal{C} = \sum_a (X_0^a)^2$  the Casimir and  $c_{V_j} = 2j^2 + 2j$

## 4.2 $\mathfrak{g}$ -vertex operator superalgebras

### 4.2.1 $\mathfrak{g}$ -fermion

**Definition 4.14.** Let  $\widehat{\mathfrak{g}}_-$  be the  $\mathfrak{g}$ -fermion algebra, generated by  $(\psi_m^a)$  with  $a \in \{1, \dots, N\}$ ,  $N = \dim(\mathfrak{g})$ ,  $m \in \mathbb{Z} + \frac{1}{2}$  and relations:

$$[\psi_m^a, \psi_n^b]_+ = \delta_{ab} \delta_{m+n} \quad \text{and} \quad (\psi_m^a)^* = \psi_{-m}^a$$

**Remark 4.15.** As for the fermion algebra of section 3.1, we generate the Verma module  $H = \mathcal{F}_{NS}^{\mathfrak{g}}$ , and the sesquilinear form  $(\cdot, \cdot)$  which is a scalar product;  $\pi(\psi_n^a)^* = \pi((\psi_n^a)^*)$ ,  $\mathcal{F}_{NS}^{\mathfrak{g}}$  is a prehilbert space, an irreducible representation of  $\widehat{\mathfrak{g}}_-$  and its unique unitary highest weight representation.

**Definition 4.16.** Let  $\psi^a(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}}^a \cdot z^{-n-1}$  the fermion operators.

**Remark 4.17.**  $\psi^a(z)\psi^b(w) \sim \frac{\delta_{ab}}{(z-w)}$

**Remark 4.18.** As for the single fermion operator  $\psi$ , of section 3.4,  $\{\psi^a, a \in \{1, \dots, N\}\}$  generates a vertex operator superalgebra with:

$$\omega = \frac{1}{2} \sum_a \psi_{-\frac{3}{2}}^a \psi_{-\frac{1}{2}}^a \Omega \quad \text{and} \quad c = 2\|\omega\|^2 = \frac{\dim(\mathfrak{g})}{2}$$

**Definition 4.19.** Let  $S^c(z) = V(s^c, z) = \sum_{n \in \mathbb{Z}} S_n^c z^{-n-1}$  with:

$$s^c = -\frac{i}{2} \sum_{a,b} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \Omega \in H_1 \subset H_{\bar{0}}$$

**Lemma 4.20.** (OPE and Lie bracket)

$$\psi^a(z)S^b(w) \sim \frac{i \sum_c \Gamma_{ab}^c \psi^c(w)}{(z-w)} \quad \text{and} \quad [\psi_m^a, S_n^b] = i \sum_c \Gamma_{ab}^c \psi_{m+n}^c = [S_m^a, \psi_n^b]$$

*Proof.*  $\psi_{n+\frac{1}{2}}^d \cdot s^c = 0$  if  $n \geq 1$  and  $\psi_{\frac{1}{2}}^d \cdot s^c = i \sum_a \Gamma_{dc}^a \psi_{-\frac{1}{2}}^a \Omega$ . □

**Remark 4.21.**  $[S_m^a, \psi_n^a] = 0$

**Lemma 4.22.**  $(S_m^b)^* = S_{-m}^b$

*Proof.*  $[(S_n^b)^*, \psi_{-m}^a] = [\psi_m^a, S_n^b]^* = -i \sum_c \Gamma_{ab}^c \psi_{-m-n}^c = [S_{-n}^b, \psi_{-m}^a]$   
The result follows by irreducibility, Schur's lemma and grading. □

**Remark 4.23.** (Jacobi)  $[X_a, [X_b, X_c]] = [[X_a, X_b], X_c] + [X_b, [X_a, X_c]]$   
 $\Leftrightarrow \sum_d \Gamma_{bc}^d \Gamma_{ad}^e = \sum_d (\Gamma_{ab}^d \Gamma_{dc}^e + \Gamma_{ac}^d \Gamma_{bd}^e) \Leftrightarrow \sum_e (\Gamma_{ab}^e \Gamma_{cd}^e + \Gamma_{da}^e \Gamma_{ce}^e + \Gamma_{db}^e \Gamma_{ac}^e) = 0$

**Notation 4.24.**  $[S^a, S^b] := i \sum_c \Gamma_{ab}^c S^c$

**Lemma 4.25.** (OPE and Lie bracket)

$$S^a(z)S^b(w) \sim \frac{[S^a, S^b](w)}{(z-w)} + \frac{g \cdot \delta_{ab}}{(z-w)^2}$$

$$\text{and } [S_m^a, S_n^b] = [S^a, S^b](m+n) + \ell \cdot m \delta_{ab} \delta_{m+n} \quad (\text{with } \ell = g \in \mathbb{N})$$

*Proof.*  $S_n^d S^c = 0$  if  $n \geq 2$  and:

$$\begin{aligned} \text{(a)} \quad S_0^d S^c &= -\frac{i}{2} \sum_{a,b} \Gamma_{ab}^c S_0^d \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \Omega \\ &= -\frac{i}{2} (i \sum_{a,b,e} \Gamma_{ab}^c \Gamma_{da}^e \psi_{-\frac{1}{2}}^e \psi_{-\frac{1}{2}}^b \Omega + i \sum_{a,b,e} \Gamma_{ab}^c \Gamma_{db}^e \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^e \Omega) \\ &= -\frac{i}{2} (i \sum_{a,b,e} (\Gamma_{eb}^c \Gamma_{de}^a + \Gamma_{ae}^c \Gamma_{de}^b) \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \Omega) \\ &= i \sum_e \Gamma_{dc}^e \frac{-i}{2} \sum_{a,b} \Gamma_{ab}^e \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \Omega = i \sum_e \Gamma_{dc}^e S^e = [S^d, S^c](-1) \end{aligned}$$

$$\text{(b)} \quad S_1^d S^c = -\frac{i}{2} i \sum_{a,b,e} \Gamma_{ab}^c \Gamma_{da}^e \psi_{\frac{1}{2}}^e \psi_{-\frac{1}{2}}^b \Omega = \frac{1}{2} \sum_{a,b} \Gamma_{ab}^c \Gamma_{ab}^d = g \cdot \delta_{cd}$$

□

**Corollary 4.26.**  $(S_m^a)$  is the basis of a  $\mathfrak{g}$ -boson algebra.

It admits  $L(V_0, g)$  as minimal submodule of  $\mathcal{F}_{NS}^{\mathfrak{g}}$  containing  $\Omega$  (with  $V_0 = \mathbb{C}$  the trivial representation of  $\mathfrak{g}$ ).

**Lemma 4.27.**  $\sum_a (S_{-1}^a)^2 \Omega = 4g\omega$

$$\begin{aligned} \text{Proof.} \quad \sum_e (S_{-1}^e)^2 \Omega &= -\frac{i}{2} \sum_{a,b,e} \Gamma_{ab}^c S_{-1}^e \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \Omega \\ &= -\frac{1}{4} \sum_{a,b,c,d,e} \Gamma_{ab}^e \Gamma_{cd}^e \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \psi_{-\frac{1}{2}}^c \psi_{-\frac{1}{2}}^d \Omega - \frac{i}{2} \sum_{a,b,c} \Gamma_{ab}^e [S_{-1}^e, \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b] \Omega \\ &= -\frac{1}{12} \sum_{a,b,c,d} (\sum_e (\Gamma_{ab}^e \Gamma_{cd}^e + \Gamma_{da}^e \Gamma_{cb}^e + \Gamma_{db}^e \Gamma_{ac}^e) \psi_{-\frac{1}{2}}^a \psi_{-\frac{1}{2}}^b \psi_{-\frac{1}{2}}^c \psi_{-\frac{1}{2}}^d \Omega) \\ &+ \sum_{a,b,c,e} \Gamma_{ea}^b \Gamma_{ea}^c \psi_{-\frac{3}{2}}^c \psi_{-\frac{1}{2}}^b \Omega = 4g\omega \end{aligned}$$

□

**Lemma 4.28.** (OPE and Lie bracket)

$$S^a(z)L(w) \sim \frac{S^a(w)}{(z-w)^2} \quad \text{and} \quad [L_m, S_n^a] = -n S_{m+n}^a$$



*Proof.*  $S_n^a \cdot \omega = 0$  for  $n \geq 3$  and:

$$(a) \quad S_0^a \cdot \omega = \frac{1}{4g} \sum_b S_0^a (S_{-1}^b)^2 \Omega = \frac{1}{4g} \sum_b ([S_0^a, S_{-1}^b] S_{-1}^b \Omega + S_{-1}^b [S_0^a, S_{-1}^b] \Omega) \\ = \frac{i}{4g} \sum_{b,c} (\Gamma_{ab}^c + \Gamma_{ac}^b) S_{-1}^c S_{-1}^b \Omega = 0$$

$$(b) \quad S_2^a \cdot \omega = \frac{1}{4g} \sum_b ([S_2^a, S_{-1}^b] S_{-1}^b \Omega + S_{-1}^b [S_2^a, S_{-1}^b] \Omega) \\ = \frac{i}{4g} \sum_{b,c} \Gamma_{ab}^c S_1^c S_{-1}^b \Omega = \frac{i}{4g} \sum_{b,c} \Gamma_{ab}^c \delta_{bc} \ell = 0$$

$$(c) \quad S_1^a \cdot \omega = \frac{1}{4g} \sum_b ([S_1^a, S_{-1}^b] S_{-1}^b \Omega + S_{-1}^b [S_1^a, S_{-1}^b] \Omega) \\ = \frac{i}{4g} (2\ell + i \sum_{b,c} \Gamma_{ab}^c S_0^c S_{-1}^b \Omega) = \frac{2(\ell+g)}{4g} S_{-1}^a \Omega = S_{-1}^a \Omega \quad (\star)$$

□

**Corollary 4.29.**  $(S^a)$  generate a vertex operator (super)algebra with  $\omega = \frac{1}{4g} \sum_a (S_{-1}^a)^2 \Omega$  as Virasoro vector.

#### 4.2.2 $\mathfrak{g}$ -boson

**Definition 4.30.** Let  $X^a(z) = \sum_{n \in \mathbb{Z}} X_n^a z^{-n-1}$  the boson operators with  $[X_m^a, X_n^b] = [X^a, X^b]_{m+n} + m \delta_{ab} \delta_{m+n} \cdot \mathcal{L}$

**Corollary 4.31.** The  $\mathfrak{g}$ -boson algebra  $\widehat{\mathfrak{g}}_+$  generates a vertex operator (super)algebra on  $H = L(V_0, g)$ , and also on  $H = L(V_0, \ell)$  for any  $\ell \in \mathbb{N}$ , with  $\omega = \frac{1}{2(\ell+g)} \sum_a (X_{-1}^a)^2 \Omega$  as Virasoro vector; and:

$$X^a(z) X^b(w) \sim \frac{[X^a, X^b](w)}{(z-w)} + \frac{g \cdot \delta_{ab}}{(z-w)^2}$$

$$X^a(z) L(w) \sim \frac{X^a(w)}{(z-w)^2} \quad \text{and} \quad [L_m, X_n^a] = -n X_{m+n}^a$$

*Proof.* By the previous work on  $(S^a)$  and  $(\star)$ . □

**Lemma 4.32.**  $c = 2\|\omega\|^2 = \frac{\ell \dim(\mathfrak{g})}{\ell+g}$

$$\begin{aligned} \text{Proof.} \quad & 4(\ell+g)^2 \|\omega\|^2 = \sum_{a,b} ((X_{-1}^a)^2 \Omega, (X_{-1}^b)^2 \Omega) = \sum_{a,b} (\Omega, (X_1^a)^2 (X_{-1}^b)^2 \Omega) \\ & = \sum_{a,b} (\Omega, X_1^a X_{-1}^b [X_1^a, X_{-1}^b] \Omega + X_1^a [X_1^a, X_{-1}^b] X_{-1}^b \Omega) \\ & = (\sum_{a,b,c} i \Gamma_{ab}^c (\Omega, X_1^a X_0^c X_{-1}^b \Omega)) + 2\ell \sum_a (\Omega, X_1^a X_{-1}^a \Omega) \\ & = (\sum_{a,b,c,d} (-1) \Gamma_{ab}^c \Gamma_{cb}^d (\Omega, X_1^a X_{-1}^d \Omega) + 2\ell^2 \dim(\mathfrak{g})) \\ & = (2g \ell \dim(\mathfrak{g}) + 2\ell^2 \dim(\mathfrak{g})) = 2\ell \dim(\mathfrak{g}) (\ell+g) \end{aligned} \quad \square$$

**Remark 4.33.** *By vacuum axiom of vertex operator superalgebra,  $X_0^a \Omega = 0$ , then, the representation  $H_0 = V_\lambda$  of  $\mathfrak{g}$  is necessary the trivial one  $V_0$ . At section 4.3, we see that general  $L(V_\lambda, \ell)$  admits the structure of vertex module over  $L(V_0, \ell)$ .*

### 4.2.3 $\mathfrak{g}$ -supersymmetry

By lemma 4.20, the  $\mathfrak{g}$ -boson algebra  $\widehat{\mathfrak{g}}_+$  acts on the  $\mathfrak{g}$ -fermion algebra  $\widehat{\mathfrak{g}}_-$ , then, we can build their semi-direct product:

**Definition 4.34.** *Let  $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \ltimes \widehat{\mathfrak{g}}_-$  the  $\mathfrak{g}$ -supersymmetric algebra.*

**Proposition 4.35.** *The unitary highest weight representations (irreducible) of  $\widehat{\mathfrak{g}}$  are  $H = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  (see [6]).*

*Proof.* Let  $H$  be such a representation of  $\widehat{\mathfrak{g}}$ , then,  $\widehat{\mathfrak{g}}_-$  acts on, but it admits a unique irreducible representation:  $\mathcal{F}_{NS}^{\mathfrak{g}}$ , so  $H = M \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ , with  $M$  a multiplicity space. Now,  $\widehat{\mathfrak{g}}_+$  acts on  $H$  and on  $\mathcal{F}_{NS}^{\mathfrak{g}}$  (corollary 4.26), and the difference commutes with  $\widehat{\mathfrak{g}}_-$ ; but  $\widehat{\mathfrak{g}}_-$  acts irreducibly on  $\mathcal{F}_{NS}^{\mathfrak{g}}$ , so, the commutant of  $\widehat{\mathfrak{g}}_-$  is  $End(M) \otimes \mathbb{C}$  by Schur's lemma. So,  $\widehat{\mathfrak{g}}_+$  acts on  $M$ , and this action is necessarily irreducible. Finally, by unitary highest weight context,  $\exists \lambda$  such that  $M = L(V_\lambda, \ell)$ .  $\square$

**Remark 4.36.** *Using the previous notations,  $\widehat{\mathfrak{g}}_+$  acts on  $L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  as  $B_n^a = X_n^a + S_n^a$ , bosons of level  $d = \ell + g$ .*

**Corollary 4.37.** *From  $(\psi^a(z))$  and  $(B^a(z))$ , we generate  $S^a(z)$  and  $X^a = B^a - S^a$  a vertex operator superalgebra on  $H = L(V_0, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  with the Virasoro vector:*

$$\omega = \frac{1}{2} \sum_a \psi_{-\frac{3}{2}}^a \psi_{-\frac{1}{2}}^a \Omega + \frac{1}{2(\ell + g)} \sum_a (X_{-1}^a)^2 \Omega \quad \text{and :}$$

$$c = 2\|\omega\|^2 = \frac{dim(\mathfrak{g})}{2} + \frac{\ell dim(\mathfrak{g})}{\ell + g} = \frac{3}{2} \cdot \frac{\ell + \frac{1}{3}g}{\ell + g} dim(\mathfrak{g})$$

**Definition 4.38.** *(SuperVirasoro operator)*

*Let  $\tau_1 = \sum_a \psi_{-\frac{1}{2}}^a X_{-1}^a \Omega$ ,  $\tau_2 = \frac{1}{3} \sum_a \psi_{-\frac{1}{2}}^a S_{-1}^a \Omega$  and  $\tau = (\ell + g)^{-\frac{1}{2}} (\tau_1 + \tau_2)$ . Let  $G(z) = V(\tau, z) = \sum_{n \in \mathbb{Z}} G_{n-\frac{1}{2}} z^{-n-1} = \sum_{n \in \mathbb{Z} + \frac{1}{2}} G_n z^{-n-\frac{3}{2}}$*

**Proposition 4.39.** (*Supersymmetry boson-fermion*)

$$B^a(z)G(w) \sim d^{\frac{1}{2}} \frac{\psi^a(w)}{(z-w)^2} \quad \text{and} \quad \psi^a(z)G(w) \sim d^{-\frac{1}{2}} \frac{B^a(w)}{(z-w)}$$

$$[G_m, B_n^a] = -nd^{\frac{1}{2}}\psi_{m+n}^a \quad \text{and} \quad [G_m, \psi_n^a]_+ = d^{-\frac{1}{2}}B_{m+n}^a$$

*Proof.*  $\psi_{n+\frac{1}{2}}^a \tau_i = 0$  for  $n \geq 2$  and:

(a)  $\psi_{\frac{1}{2}}^a \tau_1 = X_{-1}^a \Omega$

(b)  $\psi_{\frac{1}{2}}^a \tau_2 = \frac{1}{3}(S_{-1}^a \Omega - \sum_b \psi_{-\frac{1}{2}}^b \psi_{\frac{1}{2}}^a S_{-1}^b \Omega) = \frac{1}{3}(S_{-1}^a \Omega - i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^b \psi_{-\frac{1}{2}}^c \Omega)$   
 $= S_{-1}^a \Omega$

(c)  $\psi_{\frac{3}{2}}^a \tau_1 = \psi_{\frac{3}{2}}^a \tau_2 = 0.$

$S_n^a \tau_i, X_n^a \tau_i = 0$  for  $n \geq 2$  and:

(a)  $S_0^a \tau_1 = \sum_b S_0^a \psi_{-\frac{1}{2}}^b X_{-1}^b \Omega = i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^c X_{-1}^b \Omega$

(b)  $S_0^a \tau_2 = \frac{1}{3} \sum_b S_0^a \psi_{-\frac{1}{2}}^b S_{-1}^b \Omega = \frac{1}{3}(i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^c S_{-1}^b \Omega + \sum_b \psi_{-\frac{1}{2}}^b S_0^a S_{-1}^b \Omega)$   
 $= \frac{1}{3}(i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^c S_{-1}^b \Omega + i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^b S_{-1}^c \Omega)$   
 $= \frac{i}{3} \sum_{b,c} (\Gamma_{ab}^c + \Gamma_{ac}^b) \psi_{-\frac{1}{2}}^c S_{-1}^b \Omega = 0$

(c)  $X_0^a \tau_1 = \sum_b \psi_{-\frac{1}{2}}^b X_0^a X_{-1}^b \Omega = i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^b X_{-1}^c \Omega = -S_0^a \tau_1$

(d)  $X_0^a \tau_2 = X_1^a \tau_2 = S_1^a \tau_1 = 0$

(e)  $X_1^a \tau_1 = \ell \psi_{-\frac{1}{2}}^a \Omega$

(f)  $S_1^a \tau_2 = \frac{1}{3} \sum_b S_1^a \psi_{-\frac{1}{2}}^b S_{-1}^b \Omega = \frac{1}{3}(i \sum_{b,c} \Gamma_{ab}^c \psi_{-\frac{1}{2}}^c S_{-1}^b \Omega + \sum_b \psi_{-\frac{1}{2}}^b S_1^a S_{-1}^b \Omega)$   
 $= \frac{1}{3}(\sum_{b,c,d} \Gamma_{ab}^c \Gamma_{bc}^d \psi_{-\frac{1}{2}}^d \Omega + g \psi_{-\frac{1}{2}}^a \Omega) = g \psi_{-\frac{1}{2}}^a \Omega$

□

**Remark 4.40.**  $G_m^* = G_{-m}$  (as lemma 4.22)

**Lemma 4.41.** (*OPE and Lie bracket*)

$$L(z)G(w) \sim \frac{G'(w)}{(z-w)} + \frac{\frac{3}{2}G(w)}{(z-w)^2} \quad \text{and} \quad [G_m, L_n] = (m - \frac{1}{2}n)G_{m+n}$$

*Proof.*  $L(n)\tau = L_{n-1}\tau = 0$  for  $n \geq 3$  and:

(a)  $L_{-1}\tau = R(G')$  (see  $L_{-1}$  axioms and definition 3.52)

(b)  $L_0\tau = \frac{3}{2}R(G)$  (see  $L_0$  axioms)

(c)  $L_1(\tau_1 + \tau_2) = \sum_a L_1\psi_{-\frac{1}{2}}^a(X_{-1}^a + \frac{1}{3}S_{-1}^a)\Omega = \sum_a \psi_{-\frac{1}{2}}^a L_1(X_{-1}^a + \frac{1}{3}S_{-1}^a)\Omega$   
 $= \sum_a \psi_{-\frac{1}{2}}^a(X_0^a + \frac{1}{3}S_0^a)\Omega = 0$

□

**Remark 4.42.**  $[[A, B]_+, C] = [A, [B, C]_+] + [B, [A, C]_+]$   
 $= [A, [B, C]]_+ + [B, [A, C]]_+$

**Lemma 4.43.** (*OPE and Lie bracket*)

$$G(z)G(w) \sim \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2L(w)}{(z-w)} \quad \text{and} \quad [G_m, G_n]_+ = 2L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n}$$

*Proof.* By supersymmetry:

(a)  $[[G_m, G_n]_+, B_r^a] = -2rB_{m+n+r}^a = [2L_{m+n}, B_r^a]$

(b)  $[[G_m, G_n]_+, \psi_r^a] = -2(r + \frac{1}{2}(m+n))\psi_{m+n+r}^a = [2L_{m+n}, \psi_r^a]$

Then,  $[[G_m, G_n]_+ - 2L_{m+n}, B_r^a] = [[G_m, G_n]_+ - 2L_{m+n}, \psi_r^a] = 0$ .

Now,  $(B_r^a), (\psi_r^a)$  act irreducibly on  $H$ , so by Schur's lemma:

$$[G_m, G_n]_+ - 2L_{m+n} = k_{m,n}I$$

Now, among the  $G_n\tau$ ,  $G_{\frac{3}{2}}\tau$  is the only to give a constant term and:

$$G_{\frac{3}{2}}\tau = (\ell + g)^{-1} \sum_a G_{\frac{3}{2}}\psi_{-\frac{1}{2}}^a(X_{-1}^a + \frac{1}{3}S_{-1}^a)\Omega$$

$$= (\ell + g)^{-1} \sum_a (X_1^a + S_1^a)(X_{-1}^a + \frac{1}{3}S_{-1}^a)\Omega$$

$$= (\ell + g)^{-1} \dim(\mathfrak{g})(\ell + \frac{1}{3}g)\Omega = \frac{2}{3}c\Omega.$$

Finally, by formulas 3.26 and 3.31,  $k_{m,n} = \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n}$ .

□

**Summary 4.44.**

$$\begin{cases} L(z)L(w) \sim \frac{(c/2)}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L'(w)}{(z-w)} \\ L(z)G(w) \sim \frac{G'(w)}{(z-w)} + \frac{\frac{3}{2}G(w)}{(z-w)^2} \\ G(z)G(w) \sim \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2L(w)}{(z-w)} \end{cases}$$

and:

$$\begin{cases} [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} \\ [G_m, L_n] = (m - \frac{n}{2})G_{m+n} \\ [G_m, G_n]_+ = 2L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n} \end{cases}$$

$$L_n^* = L_{-n}, \quad G_m^* = G_{-m}, \quad \text{and } c = \frac{3}{2} \cdot \frac{\ell + \frac{1}{3}g}{\ell + g} \dim(\mathfrak{g})$$

the SuperVirasoro algebra of sector (NS), or Neveu-Schwarz algebra  $\mathfrak{Vir}_{1/2}$ .

**Corollary 4.45.**  $\mathfrak{Vir}_{\frac{1}{2}}$  acts unitarily on  $H = L(V_0, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  and admits  $L(c, 0)$  as minimal submodule containing  $\Omega$  (see definition 2.21).

### 4.3 Vertex modules

**Remark 4.46.** If  $\ell = 0$ , then  $\lambda = 0$  and  $L(V_0, 0) = \mathbb{C}$  trivial, and what we will show is ever proved by the previous section. So, we suppose  $\ell \in \mathbb{N}^*$  fixed.

#### 4.3.1 Summary

Let  $H = L(V_0, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$ , the vacuum representation of the  $\mathfrak{g}$ -supersymmetric algebra  $\widehat{\mathfrak{g}}$ , with  $\pi : \widehat{\mathfrak{g}} \rightarrow \text{End}(H)$ .

We have construct the vertex operator superalgebra  $(H, \Omega, \omega, V)$  with  $V : H \rightarrow (\text{End}H)[[z, z^{-1}]]$  the state-field correspondance map.

$\mathcal{S} = V(H)$  is generated by  $(V(\psi_{-\frac{1}{2}}^a \Omega))_a$ ,  $(V(X_{-1}^b \Omega))_b$ , and  $V(\mathcal{L}\Omega)$ , pairwise local, with the operations,  $(A, B) \mapsto A_n B$  and linear combinations.

We write  $V(\psi_{-\frac{1}{2}}^a \Omega, z) = \sum_{n \in \mathbb{Z}} \pi(\psi_{n+\frac{1}{2}}^a) z^{-n-1}$ ,

$V(X_{-1}^b \Omega, z) = \sum_{n \in \mathbb{Z}} \pi(X_n^b) z^{-n-1}$  and  $V(\mathcal{L}\Omega, z) = \pi(\mathcal{L}) (= \ell Id_H)$ .

### 4.3.2 Modules

Let  $H^\lambda = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  a unitary highest weight representation of  $\widehat{\mathfrak{g}}$  and  $\pi^\lambda : \widehat{\mathfrak{g}} \rightarrow \text{End}(H^\lambda)$

**Remark 4.47.**  $H^\lambda$  is itself the minimal subspace containing  $\Omega^\lambda$  and stable by the action of  $\widehat{\mathfrak{g}}$ :  $\Omega^\lambda$  is the cyclic vector of  $H^\lambda$ .

On the vacuum representation,  $\Omega$  is called the vacuum vector.

**Lemma 4.48.**  $(\sum_{n \in \mathbb{Z}} \pi^\lambda(\psi_{n+\frac{1}{2}}^a) z^{-n-1})_a$ ,  $(\sum_{n \in \mathbb{Z}} \pi^\lambda(X_n^b) z^{-n-1})_b$  and  $\pi^\lambda(\mathcal{L})$  are pairwise local (definition 3.20).

*Proof.* Let  $A, B \in \widehat{\mathfrak{g}}[[z, z^{-1}]]$ ;  $\pi$  and  $\pi^\lambda$  are faithful representations of  $\widehat{\mathfrak{g}}$ .

Then, as formal power series, with  $N \in \mathbb{N}$  and  $\varepsilon \in \mathbb{Z}_2$ :

$$\begin{aligned} & (z-w)^N \pi(A(z)) \pi(B(w)) c, d \\ &= (-1)^\varepsilon (z-w)^N (\pi(B(w)) \pi(A(z)) c, d) \quad \forall c, d \in H \quad \text{if and only if} \\ & (z-w)^N (\pi^\lambda(A(z)) \pi^\lambda(B(w)) e, f) \\ &= (-1)^\varepsilon (z-w)^N (\pi^\lambda(B(w)) \pi^\lambda(A(z)) e, f) \quad \forall e, f \in H^\lambda \end{aligned}$$

We generate inductively an operator  $D$  decomposing  $H^\lambda$  into  $\bigoplus H_n^\lambda$  by:  
 $D\Omega^\lambda = 0$ ,  $D\psi_{-m}^a \xi = \psi_{-m}^a D\xi + m\psi_{-m}^a \xi$ ,  $DX_{-n}^b \xi = X_{-n}^b D\xi + nX_{-n}^b \xi$ ,  $\xi \in H^\lambda$ ,  
clearly well defined; but,  $\psi_m^a : H_p^\lambda \rightarrow H_{p-m}^\lambda$  and  $X_n^b : H_p^\lambda \rightarrow H_{p-n}^\lambda$ , so, by  
lemmas 3.35, 3.36, 3.37, the result follows.  $\square$

**Lemma 4.49.**  $D = L_0 - \frac{c_{V_\lambda}}{2(\ell+g)}$ ,  
with  $c_{V_\lambda}$  the Casimir number of  $V_\lambda$  (see corollary 4.3)

*Proof.*  $[L_0, \psi_n^a] = [D, \psi_n^a]$  and  $[L_0, X_n^a] = [D, X_n^a]$ , so, by irreducibility and Schur's lemma,  $L_0 - D \in \mathbb{C}Id_{H^\lambda}$ . Now,  $D\Omega^\lambda = 0$  and  $L_0\Omega^\lambda = h\Omega^\lambda \neq 0$  in general. Now, writing explicitly  $L_0$  with formula 3.26, we obtain:

$$2(\ell+g)L_0\Omega^\lambda = \sum_a (X_0^a)^2 \Omega^\lambda = \mathcal{C}.\Omega^\lambda = c_{V_\lambda}\Omega^\lambda \quad \square$$

**Theorem 4.50.**  $\mathfrak{Vir}_{\frac{1}{2}}$  acts unitarily on  $H^\lambda = L(V_\lambda, \ell) \otimes \mathcal{F}_{NS}^{\mathfrak{g}}$  and admits  $L(c, h)$  as minimal submodule containing  $\Omega^\lambda$ , with  $c = \frac{3}{2} \cdot \frac{\ell + \frac{1}{3}g}{\ell + g} \dim(\mathfrak{g})$  and  $h = \frac{cV_\lambda}{2(\ell + g)}$ .

*Proof.* We generate  $\mathcal{S}^\lambda$  from generators of previous lemma, with the operations  $(A, B) \mapsto A_n B$  (now available) and linear combinations. The formula 3.26 is independant of the choice between the faithful representations  $\pi$  and  $\pi^\lambda$ . So, we identify  $\mathcal{S}$  and  $\mathcal{S}^\lambda$ , which gives the isomorphism  $i : \mathcal{S} \rightarrow \mathcal{S}^\lambda$ ; we compose it with the state-field correspondence map  $V : H \rightarrow \mathcal{S}$  to give:

$$\begin{aligned} V^\lambda : H &\longrightarrow (End H^\lambda)[[z, z^{-1}]] \\ a &\longmapsto i(V(a)) \end{aligned} \quad (1)$$

Then,  $\sum_{n \in \mathbb{Z}} \pi^\lambda(\psi_{n+\frac{1}{2}}^a) z^{-n-1} = V^\lambda(\psi_{-\frac{1}{2}}^a \Omega, z)$ ,  
 $\sum_{n \in \mathbb{Z}} \pi^\lambda(X_n^b) z^{-n-1} = V^\lambda(X_{-1}^b \Omega, z)$  and  $\pi^\lambda(\mathcal{L}) = V^\lambda(\mathcal{L} \Omega, z)$   
Now,  $V(a)_n V(b) = V(V(a, n)b) \forall a, b \in H$ , so, by construction:

$$V^\lambda(a)_n V^\lambda(b) = V^\lambda(V(a, n)b) \quad (2)$$

Then,  $V^\lambda(\omega, z) = \sum L_n z^{-n-2}$ ,  $V^\lambda(\tau, z) = \sum G_{m-\frac{1}{2}} z^{-m-1}$ ,  $L_n^* = L_{-n}$  and  $G_m^* = G_{-m}$ , with  $(L_n), (G_m)$  verifying superVirasoro relations. (3)  $\square$

**Remark 4.51.**  $[L_m, \psi_n^a] = -(n + \frac{1}{2}m)\psi_{m+n}^a$  and  $[L_m, X_n^a] = -nX_{m+n}^a$ , so:

$$\begin{cases} [L_{-1}, V^\lambda(a, z)] = (V^\lambda)'(a, z) \\ [L_0, V^\lambda(a, z)] = z \cdot (V^\lambda)'(a, z) + rV^\lambda(a, z) \end{cases} \quad (a \in H_r) \quad (4)$$

**Remark 4.52.**  $V^\lambda(\Omega, z) = Id_{H^\lambda}$  because  $\pi$  and  $\pi^\lambda$  are at same level  $\ell$ . (5)

**Definition 4.53.** By (1)...(5),  $(H^\lambda, V^\lambda)$  is called a **vertex module** of  $(H, V, \Omega, \omega)$ .

We now apply the theorem 4.50 to GKO construction with  $\mathfrak{g} = \mathfrak{sl}_2$ .

## References

- [1] R. E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*. Proc. Nat. Acad. Sci. U.S.A. 83 (1986), no. 10, 3068–3071.
- [2] P. Goddard, A. Kent, D. Olive, *Unitary representations of the Virasoro and super-Virasoro algebras*. Comm. Math. Phys. 103 (1986), no. 1, 105–119.
- [3] P. Goddard, *Meromorphic conformal field theory*. Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), 556–587, Adv. Ser. Math. Phys., 7, World Sci. Publ., Teaneck, NJ, 1989.
- [4] L. Guieu, C. Roger, *L’algèbre et le groupe de Virasoro. Aspects géométriques et algébriques, généralisations*. Les Publications CRM, Montreal, QC, 2007.
- [5] V.F.R. Jones, *Fusion en algèbres de von Neumann et groupes de lacets (d’après A. Wassermann)*. Sminaire Bourbaki, Vol. 1994/95. Astérisque No. 237 (1996), Exp. No. 800, 5, 251–273.
- [6] V. G. Kac, I. T. Todorov, *Superconformal current algebras and their unitary representations*. Comm. Math. Phys. 102 (1985), no. 2, 337–347.
- [7] V. G. Kac, A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*. Advanced Series in Mathematical Physics, 2. World Scientific Publishing Co., Inc., Teaneck, NJ, 1987.
- [8] V. G. Kac, J. W. van de Leur, *On classification of superconformal algebras*. Strings ’88 (College Park, MD, 1988), 77–106, World Sci. Publ., Teaneck, NJ, 1989.
- [9] V. G. Kac, *Vertex algebras for beginners*. University Lecture Series, 10. American Mathematical Society, Providence, RI, 1997.
- [10] T. Loke, *Operator algebras and conformal field theory for the discrete series representations of  $\text{Diff}(\mathbb{S}^1)$* , thesis, Cambridge 1994.



- [11] S. Palcoux, *Neveu-Schwarz and operators algebras II : Unitary series and characters*, to appear.
- [12] S. Palcoux, *Neveu-Schwarz and operators algebras III : Subfactors and Connes fusion*, to appear.
- [13] A. Pressley, G. Segal, *Loop groups*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986.
- [14] V. Toledano Laredo, *Fusion of Positive Energy Representations of  $LSpin(2n)$* , thesis, Cambridge 1997, (on the arxiv).
- [15] R. W. Verrill, *Positive energy representations of  $L^\sigma SU(2r)$  and orbifold fusion*. thesis, Cambridge 2001.
- [16] A. J. Wassermann, *Operator algebras and conformal field theory*. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zrich, 1994), 966–979, Birkhuser, Basel, 1995.
- [17] A. J. Wassermann, *Operator algebras and conformal field theory. III. Fusion of positive energy representations of  $LSU(N)$  using bounded operators*. Invent. Math. 133 (1998), no. 3, 467–538.
- [18] A. J. Wassermann, *Kac-Moody and Virasoro algebras*, 1998, arXiv:1004.1287v1
- [19] A. J. Wassermann, *Subfactors and Connes fusion for twisted loop groups*, 2010, arXiv:1003.2292v1