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Offsprings of a Point Vortex

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Abstract. The distribution engendered by successive splitting of one point vortex are considered. The process of splitting a vortex in three using a reverse three-point vortex collapse course is analysed in great details and shown to be dissipative. A simple process of successive splitting is then defined and the resulting vorticity distribution and vortex populations are analysed.

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1 Introduction

One of the striking features of two-dimensional turbulent flows is the presence of the inverse energy cascade, which results in the emergence of coherent vortices, dominating the flow dynamics [1,2,3,4,5]. In order to tackle these problems, point vortices have been used with some success to approximate the dynamics of finite-sized vortices [6,7,8], as for instance in punctuated Hamiltonian models [5,9]. In these models, the advection of well-separated vortices is approximated via Hamiltonian point-vortex dynamics; to account for the change in the vortex population toward smaller number of bigger vortices, dissipative merging processes are included for vortices which have approached each other closer than a certain critical distance. In fact it has been shown that high-dimensional point vortex systems have both the features of extremely high-dimensional as well as low-dimensional systems [10]; moreover, the merging processes observed in decaying two-dimensional turbulence results from the interaction of a few number of close vortices [11] and make the understanding of low dimensional vortex dynamics an essential ingredient of the whole picture [12].

Ever since the pioneering work of Onsager on two dimensional turbulence [13], an statistical approach to turbulence using point vortices has been developed. These vortex systems display negative temperature, corresponding to states where same-sign vortices bound to form larger vortices [14,13], but special care in the definition of the thermodynamic limit has to be done [15]. In the same spirit, work has been done as well in order to specify stationary flows resulting from point vortices [16,17,18] as well as kinetic theories derived (see for instance [19] and references therein).

Regarding the dynamical aspect, point vortex seen as exact solutions of the Euler equation are still debated, indeed finite time singularities such as three vortex collapse can arise [20,21,22], and can be seen as the consequences of an ill posed problem. In this paper we take a different perspective on the existence of this finite time singularity. Indeed we think of it as a potential source of vorticity and vortices and in this perspective investigate the consequences of such possibility. One of the perspective is to offer the possibility of a statistical mechanics approach with a varying number of vortices. To be more precise, in this paper our main goal is to lay the ground for further statistical studies. We consider a simple mechanism of generating distributions of vortices and vortex strength resulting from the successive splitting according to a reverse collapse course of one point vortex. We shall refer to this result as the offsprings of a point vortex. The paper is organised as follows, first we recall briefly the notion of point vortices, and how they naturally appear as a solution of Euler equation. We then consider the process of splitting of one vortex in three, introduce the relevant parameters and analyse briefly preliminary consequences, namely exhibit that the splitting process is a dissipative one. Finally we define simple rules for successive splitting, compute the offsprings of a point vortex with these rules and perform some analysis.

2 Basic equations

Point vortices are singular solutions of some bidimensional physical systems described by a conservation equation of

what we shall call a generalised vorticity Ω given by

$$\frac{\partial \Omega}{\partial t} + \{\Omega, \psi\} = 0, \quad (1)$$

where $\{\cdot, \cdot\}$ denotes the usual Poisson bracket, and ψ is a stream function. The actual relation $\Omega = F(\psi)$ may depend on the considered physical system. For instance for the Euler equation it is simply given by $\Omega = -\nabla^2 \psi$. When $\Omega = -\nabla^2 \psi + \psi/\rho_s^2$ where ψ is, in this context, related to the electric potential (in suitable units) in a plasma, ρ_s is the hybrid Larmor radius. Point vortices are defined by a vorticity distribution given by a superposition of Dirac functions,

$$\Omega(\mathbf{r}, t) = \frac{1}{2\pi} \sum_{i=1}^N k_i \delta(\mathbf{r} - \mathbf{r}_i(t)), \quad (2)$$

where $\mathbf{r} = (x, y)$ is a vector in the plane of the flow, k_i is the strength of vortex i (circulation), N is the total number of vortices, and $\mathbf{r}_i(t)$ is the vortex position at time t . Using this expression of the vorticity and solving the Poisson equation, in the Euler case, or the Helmholtz equation in the more general case, one obtains the current function associated to the point vortices. By Helmholtz theorem, the motion of the vortices is determined by the value of the velocity field created by the other vortices at the position of the vortex. The point vortex motion is Hamiltonian and given by

$$k_i y_i = -\frac{\partial H}{\partial x_i} \quad ; \quad \dot{x}_i = \frac{\partial H}{\partial(k_i y_i)} \quad ; \quad (i = 1, \dots, N) \quad (3)$$

where the Hamiltonian H is given by

$$H = \frac{1}{2\pi} \sum_{i>j} k_i k_j U(|\mathbf{r}_i - \mathbf{r}_j|) \quad (4)$$

with for an unbounded plane $U(x) = -\log(x)$ in the Euler case, and $U(x) = K_0(x)$ in the more general case (when $\rho_s \rightarrow \infty$ the modified Bessel function tends to the logarithm). The Hamiltonian (4) exhibits clearly that a system of point vortices is invariant by translation and rotation, which implies both the conservation of the centre of vorticity and the total angular momentum, the motion of three vortices is integrable, while for four or more vortices Hamiltonian chaos come into play. When the distance between the vortices is smaller than the typical interaction length (ρ_s) the behaviour of the two systems is similar, in the opposite case, the K_0 interaction decreases exponentially and the vortices are almost free. In the following, analytical computations will be made using the logarithmic interaction, which corresponds to the Euler flow, and we can expect the results to be qualitatively valid for the Bessel interaction as long as the vortices are not ‘‘too far’’ from each other.

Let us now focus on a situation with only three vortices. In this restricted situation, it is easier to tackle the motion of the vortices by studying in fact their relative motion. Namely the three vortices form a triangle, and the

relative motion describes the deformation of this triangle [20,23,24,25,22]. The invariance by translation of (4), allows us a free choice of the origin of the plane, which we choose to be the centre of vorticity (when it exists). The other constants of motion written in a frame independent form become,

$$\begin{cases} H = -\frac{1}{2\pi} [k_1 k_2 \ln R_3 + k_1 k_3 \ln R_2 + k_3 k_2 \ln R_1] \\ K = k_1 k_2 R_3^2 + k_1 k_3 R_2^2 + k_3 k_2 R_1^2, \end{cases} \quad (5)$$

where $R_i = |\mathbf{r}_j - \mathbf{r}_k|$, with $i \neq j \neq k$. In fact it has been known for a long time that the motion of vortices can lead to singular solutions and finite time singularities, the most striking one occurring for with a system of three vortices resulting in the collapse of the vortices in a finite time [20,21,22] or by time reversal, to an infinite expansion of the triangle formed by the vortices. The collapse or infinite expansion of the three point vortices are obtained when the following conditions are satisfied

$$K = 0 \quad (6)$$

$$\sum_i \frac{1}{k_i} = 0, \quad (7)$$

the harmonic mean of the vortex strengths (7) and the total angular momentum in its frame free form (6), are both equal to zero. [24,26].

In this paper we shall use this specific singularity and consider vorticity distribution arising from successive splitting of a point vortex according to the collapse conditions. Note that other type of singularities involving more vortices are effectively possible, however for the sake of simplicity, we restrict ourselves to reverse three-vortex collapse rules. In what follows we shall refer to the successive splitting process as the point vortex offsprings.

3 Splitting of a Point Vortex

3.1 Splitting rules

In order to be consistent with the physical properties of vortex collapse, we successively divide vortices according to the collapses rules and keep the total vorticity constant. The splitting rules from one generation n of a vortex i of strength $k_{i,n}$ to the next generation $n+1$ read

$$\sum_{i=1}^3 k_{i,n+1} = k_{i,n} \quad (8)$$

$$\sum_{i=1}^3 \frac{1}{k_{i,n+1}} = 0. \quad (9)$$

These equations are equivalent to

$$\sum_{i=1}^3 k_{i,n+1} = k_{i,n} \quad (10)$$

$$\sum_{i=1}^3 k_{i,n+1}^2 = k_{i,n}^2, \quad (11)$$

the $k_{i,n+1}$ lie on the circle at the intersection of the sphere of radius $|k_{i,n}|$ defined by Eq. (11) and the plane defined by Eq. (10). We therefore discuss the splitting in terms of the vector $\mathbf{k}_{n+1} = (k_{1,n+1}, k_{2,n+1}, k_{3,n+1})$. The tip of the vector lies on a circle, hence we parametrise it with an angle θ_n as:

$$\mathbf{k}_{n+1} = k_n \left(\mathbf{a} + \sqrt{\frac{2}{3}} (\cos \theta_n \mathbf{u} + \sin \theta_n \mathbf{v}) \right), \quad (12)$$

where $\mathbf{a} = (1, 1, 1)/3$, and for instance $\mathbf{u} = (1, -1, 0)/\sqrt{2}$ and $\mathbf{v} = \sqrt{3} \mathbf{a} \wedge \mathbf{u}$. Given these rules and the fact that $k_{3,n}(\theta + 2/3\pi) = k_{1,n}, k_{3,n}(\theta - 2/3\pi) = k_{2,n}$, θ_n can be restricted to the segment $[0, 2\pi/3]$.

3.2 Is the splitting always possible

We imagine that we are dealing with a system with many vortices ($2n + 1$ after n splittings) and consider the splitting of one point vortex according to the rules (6) and (7) in this system. The total number of vortices changes, see Fig 1 for an illustration of the process. Regarding the energy we have

$$\delta H_n = H_{n+1} - H_n = -\frac{1}{4\pi} \ln \Lambda_n, \quad (13)$$

$$= -\frac{1}{4\pi} [k_1 k_2 \ln R_3 + k_1 k_3 \ln R_2 + k_3 k_2 \ln R_1], \quad (14)$$

where for instance the vortex $k_N = k_1 + k_2 + k_3$ was split in three. In order to be dynamically compatible, we first neglect the influence of the other vortices (which are considered far enough to not interfere locally), but still we need to make sure that there is at least one triangle satisfying the conditions and define the value of δH_n associated with the splitting. For that purpose after the splitting we name 2 and 3 the vortices whose strengths have the same sign with $|k_2| < |k_3|$, exponentiating Eq. (13) gives

$$R_3^{k_1 k_2} R_2^{k_1 k_3} R_1^{k_2 k_3} = \Lambda_n, \quad (15)$$

then we divide by one noticing that $R_1^{\sum k_i k_j} = 1$, and obtain

$$\left(\frac{R_3}{R_1} \right)^{k_1 k_2} \left(\frac{R_2}{R_1} \right)^{k_1 k_3} = \Lambda_n. \quad (16)$$

Since the collapse is scale free (self-similar), we choose R_1 as our length units, note $r_2 = R_2/R_1$, $r_3 = R_3/R_1$ and arrive at

$$r_2 = \Lambda_n^{1/k_3 k_1} r_3^{-k_2/k_3}. \quad (17)$$

The condition (6) becomes once rescaled

$$k_1 k_2 r_3^2 + k_1 k_3 r_2^2 + k_3 k_2 = 0, \quad (18)$$

combining this last expression with (7) we have

$$k_2(r_3^2 - 1) + k_3(r_2^2 - 1) = 0, \quad (19)$$

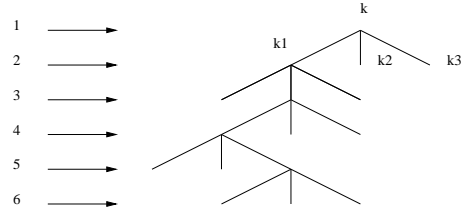


Fig. 1. Simple 6th order vortex lineage

and noting $\alpha = k_2/k_3 (< 1)$, we finally obtain

$$\alpha r_3^{2(1+\alpha)} + \Lambda_n^{2/k_1 k_3} = (1 + \alpha) r_3^{2\alpha}. \quad (20)$$

Λ_n being positive, $\alpha X^{1+\alpha} - (1 + \alpha) X^\alpha$ shows that $r_3^2 \in]0; 1 + 1/\alpha[$ and that $0 < \Lambda_n^{2/k_1 k_2} \leq 1$. The equality $\Lambda_n = 1$ being reached for $r_3 = 1$ which implies an equilateral triangle. This last configuration has to be excluded as it is dynamically a fix point, i.e the triangle does not expand or shrink hence can not be a starting point for a vortex splitting. So, since $k_1 k_3 < 0$, this means that $\Lambda_n > 1$ and that $\delta H_n = -\frac{1}{4\pi} \ln \Lambda_n < 0$. The splitting of a vortex in three is a dissipative process.

We now enforce that the solution is a triangle, using the rescaled variables this means that

$$1 = r_2^2 + r_3^2 - 2r_3 r_2 \cos \varphi, \quad (21)$$

which implies (using Eq. (19))

$$\cos \varphi = \frac{r_3^2(1 - \alpha) + \alpha}{2r_3 \sqrt{1 + \alpha - \alpha r_3^2}}. \quad (22)$$

The right hand side of Eq. (22) is always positive, so $-\pi/2 < \varphi < \pi/2$. The minimum is obtained for $r_3^2 = \alpha(1 + \alpha)/(1 + \alpha^2)$ and equal to $\alpha^{1/2}/(1 + \alpha)$, which is always smaller than one. There is therefore always a range of possibilities available for r_3 and thus splitting is always possible. Note that for a given value of r_3 the two mirroring shapes of the triangle are possible, one giving rise to expansion (splitting) the other one to collapse.

4 Vorticity distributions

We are now interested in the vorticity distribution we obtain from such process. For this purpose we compute different trees originating from one vortex of strength $k = 1$ as depicted in Fig. 1. The distributions are computed by successive vortex splitting. After each division, a vortex is chosen randomly among the global population and is split according to the reverse collapse rules, the division is the result of the uniformly random choice of an angle θ_n (see Eq. (12)). Other possible rules made by assigning different probabilities on vortices will be explored in future work. Results of the obtained distributions are depicted in Fig. 2. We shall notice that the chosen rules gives rise to a large spectrum of vorticities (see the logarithmic scale in Fig.2). We notice as well that as the number of division increases the distribution spreads and the location of

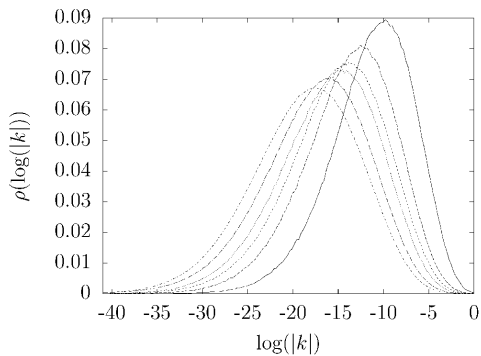


Fig. 2. Probability density function (PDF) of the absolute values of vortex strengths after respectively (from right to left) 10^3 , $5 \cdot 10^3$, 10^4 , $2 \cdot 10^4$, $5 \cdot 10^4$, 10^5 vortex splits. The PDF have been averaged over 64 trees, for the large values and up to 512 trees for the smallest ones. The shape appears as self-similar.

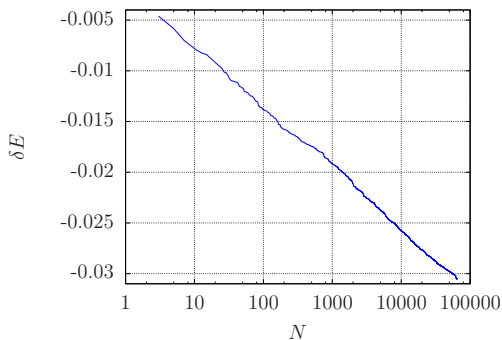


Fig. 3. Variation of energy versus number of created vortices. The curve has been averaged over 100 trees. Splittings have been performed up to $N = 65537$ vortices starting from one vortex of strength $k_0 = 1$. One can notice a small logarithmic decay of energy versus the number of vortices, of the type $\delta E \approx -\lambda \log_{10}(N)$ with $\lambda \approx 5.910^{-3}$.

its maximum is slowly moving towards smaller values of the vorticity, moreover, the evolution of the distribution appears to have some kind of self-similar behaviour. In the same spirit the variation of energy as a function of generated vortices can be monitored. Results are shown in Fig. 3 and show a slow logarithmic decay of the total energy.

In order to analyse this in more details we need to characterise the lineage (see Fig. 1) after a given number of splitting. We will note the total number of splittings that occurred n . These successive divisions generate a tree (the phase space of the process) with 3^n leaves at the extremities. Each division results in the choice of an angle θ_i . Now let us consider a particular “lineage” of order n , it gives rise to a family of $N = 2n + 1$ vortices. At each step of the division process (from n to $n + 1$) we choose any already existing vortex and split it with the rules (12). The trajectory in phase space corresponds to a connected

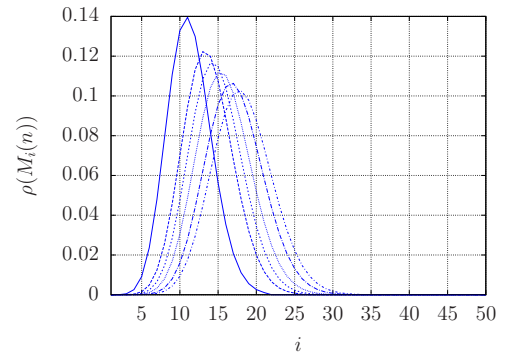


Fig. 4. Probability density function (PDF) of the occupation of each generation M after respectively (from left to right) 10^3 , $5 \cdot 10^3$, 10^4 , $2 \cdot 10^4$, $5 \cdot 10^4$, 10^5 vortex splits. The PDF have been computed using the mean field equations (23). The shape appears as self-similar and is reminiscent of what is observed in Fig. 2.

graph with $2n + 1$ leaves starting from the top of the tree and of total length $3n + 1$. The number of possible graphs on the tree after n splittings is: $(2n + 1)! / (2^n n!)$. To move further on, and due to the large amount of possible graphs, we consider the global occupation M_i of the level i (see Fig. 1). We note $M_i(n)$ the average number of leaves at level i at time n . Then we have

$$M_i(n + 1) = M_i(n) + 3 \frac{M_{i-1}(n)}{2n + 1} - \frac{M_i(n)}{2n + 1}, \quad (23)$$

with initial conditions $M_0(0) = 1$, $M_{n>0}(0) = 0$. It is easy to integrate numerically this equation in order to have an idea of the solution. We find that the form

$$M_i(n) = a_i \frac{(\log(2n + 1))^{i-1}}{\sqrt{2n + 1}} \quad (24)$$

is solution, with $a_i = 3a_{i-1} / (2(i - 1))$, i.e.

$$a_i = \left(\frac{3}{2}\right)^{i-1} \frac{a_1}{(i-1)!}. \quad (25)$$

This can be checked easily by induction, we assume that M_{i-1} is of this form, and then we can solve Eq. (23) for M_i in the continuous time limit in which it becomes

$$\frac{dM_i}{dn} = 3 \frac{M_{i-1}}{2n + 1} - \frac{M_i}{2n + 1}. \quad (26)$$

In this way we obtain that at a fixed level i , the occupancy M_i first increases with time, then decreases, with a maximum at $n^*(i) \sim \exp(2i)$. At fixed time n on the other hand, $M_i(n)$ has a maximum at $i^* \sim 3 \log(2n + 1) / 2$. A numerical integration of the global populations given by Eq. (23) is displayed in Fig. 4. One can notice similarities with the distributions of vorticity although the distributions are more peaked. In order to test as well our analysis, the location of the maximum of the distribution is displayed in Fig. 5 and a good agreement with the logarithmic law is found.

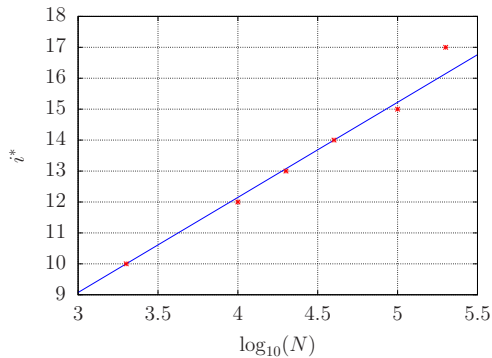


Fig. 5. Maximum i^* versus number of vortices N . i^* is obtained from the distributions displayed in Fig. 4. A linear scaling is observed as expected, with a measured slope of ~ 3.1 , “close to the expected value $3/2 \times \ln 10 \approx 3.45$ ”

Let us now compute the distribution of vorticity $\rho(k, n)$ assuming we know the occupancy $M_i(n)$. Hence let us consider a vortex living in the generation i . It has been the result of i splitting. Since the splitting rules (12) have no preferred order (as mentioned earlier they permute if we add $2\pi/3$ to the random angle), we assume that the obtained vortex is always the third vortex hence its absolute vorticity k will end up being

$$k = \frac{1}{3^i} \prod_{j=1}^i (1 - 2 \sin \theta_j), \quad (27)$$

and consequently its logarithm is

$$\log |k| = \sum_{j=1}^i \log |1 - 2 \sin \theta_j| - i \log 3, \quad (28)$$

with θ_j being uniformly distributed random variables in $[0, 2\pi[$. We can then gather the probability distribution of vortex strengths at generation i , which we note $\rho_i(k)$. And thus the vorticity distribution after N division writes

$$\rho(k, n) = \sum_i \rho_i(k) \rho(M_i, n), \quad (29)$$

with $\rho(M_i, n) = M_i / (2n + 1)$.

In order to check these results, we compute $\rho(k, n)$ using Eq.(29) and compare the results to those displayed in Fig. 2. In fact, given the shape of the occupancy $M_i(n)$, we can compute $\rho(k, n)$ using only a “few” distributions $\rho_i(k)$. For instance in Fig. 6 we computed the $\rho_i(k)$ taking into account in the tree the vortices only up to generation $i = 40$. We notice as well a very similar behaviour as the one displayed in Fig. 2.

5 Conclusion

This paper is a first attempt at analysing the distribution of vorticities originating from one point vortex using

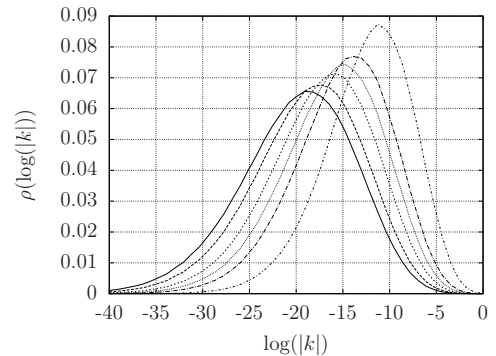


Fig. 6. Probability density function (PDF) of the absolute values of vortex strengths after respectively (from right to left) 10^3 , $5 \cdot 10^3$, 10^4 , $2 \cdot 10^4$, $5 \cdot 10^4$, 10^5 vortex splits. The PDF have been computed using Eq.(29), where we took into account only up to generation $i = 40$. The results appear to be coherent with what is displayed in Fig. 2.

a dynamically compatible process, namely a reverse collapse route with the conservation of total vorticity. The splitting process has been analysed in great details and shown to be dissipative. Afterwards a simple process consisting of randomly successive splittings is proposed and the resulting distributions have been analysed. Analytical computation of the proposed process have been made, resulting for instance in the computation of the vorticity distribution after n consecutive splitting of vortices and show very good agreement with the numerical simulation of the process. This paper is a first step for further work. One could for instance modify the splitting rules in order to obtain a conservative process, but also could try to pick vortices and how the splitting is done, with a non uniform probabilities, such as a Gibbsian one and analyse the resulting distributions. In other words, we could perform statistical physics of point vortices allowing a varying number of vortices according to the collapse rules, and see if this possibility changes the equilibrium features. Last but not least, it will be important to compare the obtained distribution with real data and see if the proposed mechanism or its variants could describe results obtained on two-dimensional physical flows. For instance, one has to notice that if we have started with a positive vortex, the reverse collapse course induces naturally the creation of negative vortices, thus the engendered distributions will consist of both positive and negative vortices. Work is currently under way to analyse these different possibilities.

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