



HAL
open science

Study of the recurrent set for planar Markov Random Walks

Loïc Hervé, Françoise Pene

► **To cite this version:**

Loïc Hervé, Françoise Pene. Study of the recurrent set for planar Markov Random Walks. 2010. hal-00522041v1

HAL Id: hal-00522041

<https://hal.science/hal-00522041v1>

Preprint submitted on 29 Sep 2010 (v1), last revised 2 Mar 2012 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

STUDY OF THE RECURRENT SET FOR PLANAR MARKOV RANDOM WALKS

LOÏC HERVÉ AND FRANÇOISE PÈNE

ABSTRACT. In this paper, we investigate the properties of recurrent planar Markov random walks. More precisely, we study the set of recurrent points with the use of local limit theorems. The Nagaev-Guivarc'h spectral method provides several examples for which these local limit theorems are satisfied as soon as the (standard or non-standard) central limit theorem holds.

MSC (2010) : 60F05

Keywords : recurrence, Markov chain, spectral method

INTRODUCTION

Let \mathbb{X} be a measurable space endowed with σ -algebra \mathcal{X} , and let $(X_n, S_n)_{n \in \mathbb{N}}$ be a Markov random walk (MRW) with state space $\mathbb{X} \times \mathbb{R}^2$. Namely: $(X_n, S_n)_{n \in \mathbb{N}}$ is a Markov chain with transition probability P satisfying, for any set $A \in \mathcal{X}$ and any Borel subset S of \mathbb{R}^2 , the following additive property (in the second component):

$$\forall (x, s) \in \mathbb{X} \times \mathbb{R}^2, \quad P((x, s); A \times S) = P((x, 0); A \times (S - s)). \quad (1)$$

This means that $(X_{n+1}, S_{n+1} - S_n)$ depends on the past only through X_n . From the definition, it can be easily seen that the first component $(X_n)_{n \in \mathbb{N}}$ is a Markov chain, called the driving Markov chain of the MRW. We suppose that $S_0 = 0$, and given any initial distribution μ of $(X_n)_{n \in \mathbb{N}}$, we use the notation $\mathbb{P}_{(\mu, 0)}$ to refer to the initial distribution of $(X_n, S_n)_{n \in \mathbb{N}}$. Of course, such a notation takes the usual sense when $(X_n, S_n)_{n \in \mathbb{N}}$ is the canonical version defined on $(\mathbb{X} \times \mathbb{R}^2)^{\mathbb{N}}$. Anyway, in this work, the last assumption may be assumed without loss of generality. The transition probability of $(X_n)_{n \in \mathbb{N}}$ is denoted by Q . Throughout the paper, we assume that there exists a Q -invariant probability measure on \mathbb{X} , called π .

The recurrence/transience property of random walks (with \mathbb{R}^d -valued second component) is investigated in many papers. Of course, this study is well-known for i.i.d. increments, see for instance [9]. In the dependent case, let us mention in particular [3, 7, 6, 27, 26, 28] for random walks with stationary increments, [13] for MRWs (case $d \geq 3$) associated with uniformly ergodic Markov chains, [12] for MRWs associated with strongly ergodic Markov chains, and [16] for additive functionals of Harris recurrent Markov chains. The results of the i.i.d. case have been extended in [1] to the MRWs with real-valued second component.

In this work, we only deal with the case when the second component S_n is \mathbb{R}^2 -valued, and we are interested in its recurrence properties. More specifically, throughout the paper we assume that S_1 is centered (i.e. S_1 is $\mathbb{P}_{(\pi, 0)}$ -integrable and $\mathbb{E}_{(\pi, 0)}[S_1] = 0$) and that there exists a two-dimensional closed subgroup \mathbb{S} in \mathbb{R}^2 such that we have

$$\forall x \in \mathbb{X}, \quad \forall n \in \mathbb{N}, \quad \mathbb{P}_{(x, 0)}(S_n \in \mathbb{S}) = 1. \quad (2)$$

Let us recall that, contrarily to the one-dimensional case, the strong law of large numbers (i.e. $S_n/n \rightarrow 0$ a.s.) is not sufficient in dimension 2 to obtain the recurrence property for $(S_n)_n$ (i.e. $\forall \varepsilon > 0, \mathbb{P}_\pi(|S_n| < \varepsilon \text{ i.o.}) = 1$)¹. This is true even in the independent case: if $(X_n)_n$ is a sequence of \mathbb{R}^2 -valued independent and identically distributed (i.i.d.) centered random variables (r.v.) and if $S_n = X_1 + \dots + X_n$, we have recurrence if and only if $\sum_n \mathbb{P}(|S_n| < \varepsilon) = \infty$ (for every $\varepsilon > 0$). Hence, in this case, if $(X_n)_n$ is in the domain of attraction of a stable distribution of index α , then $\alpha = 2$ is required. In other words, in the i.i.d. case, a central limit theorem (CLT) with a good normalization is needed for the simple random walk $S_n = X_1 + \dots + X_n$ to be recurrent, see [9, Sect. 3.2]. This fact has been extended to general stationary random walks in [6, 27] but only in situation when we have the CLT with the standard normalization in \sqrt{n} . We will see that our approach based on local limit theorems (LLT) applies to situations where we have another normalization.

In fact the purpose of this work goes beyond the question of recurrence since we want to investigate the set of recurrent points $\mathcal{R}_{(\mu,0)}$, defined by

$$\mathcal{R}_{(\mu,0)} := \left\{ s \in \mathbb{S} : \forall \varepsilon > 0, \mathbb{P}_{(\mu,0)}(|S_n - s| < \varepsilon \text{ i.o.}) = 1 \right\},$$

where $|\cdot|$ stands for the euclidean norm on \mathbb{R}^2 . We simply write $\mathcal{R}_{(x,0)}$ when μ is the Dirac distribution δ_x at $x \in \mathbb{X}$.

The set of recurrent points is well-known in the i.i.d. case (e.g. see [9, Sect. 3.2]), and it has been fully investigated in [1] for one-dimensional MRW (i.e. S_n is real-valued). However, to the best of our knowledge, the set of recurrent points has not been investigated for planar MRWs. Note that this study cannot be only based on the CLT (as for the question of recurrence): indeed it naturally requires some assumption ensuring (roughly speaking) that \mathbb{S} is the smallest lattice in \mathbb{R}^2 such that we have (2). It is worth noticing (again roughly speaking) that this type of assumption (usually called lattice or nonlattice in the i.i.d. case) is the only one we have to specify in order to extend the CLT to the LLT. Therefore, to study the set $\mathcal{R}_{(\mu,0)}$ associated to a MRW, it is not surprising that the LLT plays an important role.

Anyway mention that the transience/recurrence properties of MRWs (with \mathbb{R}^d -valued second component) have been already investigated in [12] on the basis of the LLT. Here we extend this approach to the study of the recurrent points of planar MRWs.

The first section gives a way to prove that $\mathcal{R}_{(\pi,0)} = \mathbb{S}$. To that effect, we appeal to classical arguments used to study the set of recurrence in the independent case (see [9, Sect. 3.2]). This approach holds when the following local limit theorem is valid for every $s \in \mathbb{S}$, every $\varepsilon > 0$ and every bounded measurable function $f : \mathbb{X} \rightarrow \mathbb{R}$ (see Prop. 1.1):

$$\mathbb{E}_{(\pi,0)}[f(X_n)\mathbf{1}_{\{|S_n-s|<\varepsilon\}}] \sim a_n \pi(f) m_{\mathbb{S}}(B),$$

where the a_n 's are some positive real numbers such that $\sum_{n \geq 1} a_n = \infty$ and $m_{\mathbb{S}}$ denotes the Haar measure on \mathbb{S} . When $(X_n)_n$ is Harris recurrent, this result also gives $\mathcal{R}_{(x,0)} = \mathbb{R}^2$ for every $x \in \mathbb{X}$ (see Prop. 1.4).

The second section describes another approach based on the Kochen and Stone adaptation of the Borel-Cantelli lemma. This gives a strategy to prove that $\mathcal{R}_{(x,0)} = \mathbb{R}^2$ for every $x \in \mathbb{X}$, even for non Harris recurrent driving Markov chain $(X_n)_n$. The idea is to establish that the following local limit theorems hold for every $x \in \mathbb{X}$ and every open ball $B := B(s, \varepsilon)$ of \mathbb{R}^2 centered at

¹The notation " A_n i.o." for given events A_n ($n \in \mathbb{N}$) means that "the A_n 's occur infinitely often", namely "the A_n 's occur for infinitely many times n ". The precise definition is: $[A_n \text{ i.o.}] := \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k$.

$s \in \mathbb{S}$ with (sufficiently small) radius $\varepsilon > 0$:

- $\mathbb{P}_{(x,0)}(S_n \in B) \sim a_n m_{\mathbb{S}}(B)$,
- $\mathbb{P}_{(x,0)}((S_n, S_{n+m}) \in B^2) \sim a_n a_m m_{\mathbb{S}}(B)^2$,

with as above $\sum_{n \geq 1} a_n = \infty$. This idea (combined with the spectral method) has already been used by Szász and Varjú in [29, 30].

In the third and fourth sections, we explain how the previous local limit theorems can be established with the use of the Nagaev-Guivarc'h spectral method. In this part, the fact that \mathbb{S} is the smallest lattice in \mathbb{R}^2 satisfying (2) is expressed as a spectral condition. Simple reductions of this condition are presented in Subsection 4.3. Furthermore, we point out in Subsection 4.2 the link between our hypotheses and the (standard or non-standard) central limit theorem. The key result, to put in order the spectral method and to obtain the previous reductions, is the operator perturbation theorem of Keller and Liverani, which is presented in Subsection 4.1.

In the fifth section and at the end of Section 4, we illustrate our general strategies with examples. Assuming that \mathbb{S} is the smallest lattice in \mathbb{R}^2 satisfying (2) and that $(S_n)_n$ verifies a (standard or non-standard) CLT, applications of our general results are for instance:

- If $(X_n)_n$ is ρ -mixing, then we have $\mathcal{R}_{(\pi,0)} = \mathbb{S}$.
- If $(X_n)_n$ is V -geometrically ergodic and $S_n = \sum_{k=1}^n \xi(X_k)$ with $\xi : \mathbb{X} \rightarrow \mathbb{R}^2$, then for every $x \in \mathbb{X}$, we have $\mathcal{R}_{(x,0)} = \mathbb{S}$.
- If $(X_n)_n$ is a stable stochastic affine recursion and $S_n = \sum_{k=1}^n (X_k - \mathbb{E}_{\pi}[X_1])$, then for every $x \in \mathbb{X}$, we have $\mathcal{R}_{(x,0)} = \mathbb{S}$.

Sufficient conditions for $(S_n)_n$ to satisfy a central limit theorem are recalled for each of our examples.

We end this introduction with some comments on the closed subgroup \mathbb{S} involved in (2). In fact the general statements of Sections 1-2 are valid for every subgroup \mathbb{S} of \mathbb{R}^d (for any $d \geq 1$). However, in practice, the above condition $\sum_{n \geq 1} a_n = \infty$ is only fulfilled in dimension $d = 1$ or 2 . The one-dimensional cases $\mathbb{S} = \mathbb{R}\vec{u}$ and $\mathbb{S} = \mathbb{Z}\vec{u}$ ($\vec{u} \in \mathbb{R}^2$) are not investigated here since the (expected) set of recurrent points of S_n can be then deduced from [1] (thanks to the strong law of large numbers). Consequently the subgroups \mathbb{S} of interest in our work are the two-dimensional closed subgroups of \mathbb{R}^2 (see Cases (H1) (H2) (H3) in Section 3). Anyway the (above mentioned) minimality assumption of \mathbb{S} with respect to (2) is not explicitly stated in Sections 1-2, but it is actually hidden behind the hypotheses. When the spectral method is applied in Section 3 to check these hypotheses, then a minimality assumption will be imposed on \mathbb{S} .

Given any subgroup H of \mathbb{R}^2 , we define its dual subgroup as

$$H^* := \{s \in \mathbb{R}^2 : \forall t \in H, \langle s, t \rangle \in 2\pi\mathbb{Z}\}. \quad (4)$$

Recall that H^* is a subgroup of \mathbb{R}^2 , and that the dual subgroup of H^* (i.e. the bidual of H) coincides with H . These properties are classical, anyway they can be easily proved in our three cases (H1) (H2) (H3) of interest given in Section 3

1. A FIRST APPROACH

Let $(X_n, S_n)_{n \in \mathbb{N}}$ be a Markov random walk with state space $\mathbb{X} \times \mathbb{R}^2$, and let $m_{\mathbb{S}}$ denote the Haar measure on the closed subgroup \mathbb{S} of \mathbb{R}^2 given in (2). We denote by $\mathcal{B}(\mathbb{R}^2)$ the Borel σ -algebra of \mathbb{R}^2 . For $(s, \varepsilon) \in \mathbb{R}^2 \times (0, +\infty)$, we denote by $B(s, \varepsilon)$ the open ball of \mathbb{R}^2 centered at s , with radius ε .

1.1. Recurrence set in the stationary case. Let us assume that there exist $\varepsilon_{\mathbb{S}} \in (0; +\infty)$ and a sequence $(a_n)_{n \geq 1}$ of strictly positive numbers such that, for every $(s, \varepsilon) \in \mathbb{S} \times (0; \varepsilon_{\mathbb{S}})$, for every ball $B := B(s, \varepsilon)$, and for every nonnegative bounded measurable function $f : \mathbb{X} \rightarrow \mathbb{R}$, we have the following property as $n \rightarrow +\infty$:

$$\mathbb{E}_{(\pi, 0)}[f(X_n) \mathbf{1}_B(S_n)] \sim a_n \pi(f) m_{\mathbb{S}}(B). \quad (5)$$

Proposition 1.1. *If $\sum_{n \geq 1} a_n = \infty$, then we have $\mathcal{R}_{(\pi, 0)} = \mathbb{S}$.*

The LLT-type property (5) is investigated in Subsection 3.2. To prove Proposition 1.1, define the r.v. $\xi_0 = 0$ and $\xi_k = S_k - S_{k-1}$ for $k \geq 1$. From the additive property (1), it can be easily seen that the distribution of $((X_{n+k}, \xi_{n+k}))_{k \geq 1}$ given $\{X_n = x, S_n = s\}$ is equal to the distribution of $((X_k, \xi_k))_{k \geq 1}$ under $\mathbb{P}_{(x, 0)}$. We can assume (without any loss of generality) that $((X_n, \xi_n))_{n \geq 0}$ is the canonical Markov chain (associated with the transition probability $\tilde{P}((x, s); \cdot) := P((x, 0); \cdot)$). Hence, defining the σ -algebra $\mathcal{F}_n = \sigma(\xi_k, 0 \leq k \leq n)$ and writing θ for the usual shift operator on $\Omega = (\mathbb{X} \times \mathbb{R}^2)^{\mathbb{N}}$, we obtain for every bounded measurable function $F : \Omega \rightarrow \mathbb{R}$ and for every $x \in \mathbb{X}$: $\mathbb{E}_{(x, 0)}[F \circ \theta^n | \mathcal{F}_n] = \mathbb{E}_{(X_n, 0)}[F]$.

Lemma 1.2. *For $A \in \mathcal{B}(\mathbb{R}^2)$, $k \in \mathbb{N}^*$, set $Y_k = \prod_{j=k}^{+\infty} \mathbf{1}_A(S_j)$, and $f_k(x) = \mathbb{E}_{(x, 0)}[Y_k]$ ($x \in \mathbb{X}$). Then, for any $B \in \mathcal{B}(\mathbb{R}^2)$ and $n \in \mathbb{N}^*$, we have*

$$\mathbb{P}_{(\pi, 0)}\left(S_n \in B, S_{n+j} - S_n \in A, \forall j \geq k\right) = \mathbb{E}_{(\pi, 0)}[\mathbf{1}_B(S_n) f_k(X_n)].$$

Proof. Writing $S_{n+j} - S_n = \xi_{n+1} + \dots + \xi_{n+j}$, we clearly have $Y_k \circ \theta^n = \prod_{j=k}^{+\infty} \mathbf{1}_A(S_{n+j} - S_n)$. Thus

$$\begin{aligned} \mathbb{P}_{(\pi, 0)}(S_n \in B, S_{n+j} - S_n \in A, \forall j \geq k) &= \mathbb{E}_{(\pi, 0)}[\mathbf{1}_B(S_n) Y_k \circ \theta^n] \\ &= \mathbb{E}_{(\pi, 0)}[\mathbf{1}_B(S_n) \mathbb{E}[Y_k \circ \theta^n | \mathcal{F}_n]] \\ &= \mathbb{E}_{(\pi, 0)}[\mathbf{1}_B(S_n) \mathbb{E}_{(X_n, 0)}[Y_k]]. \end{aligned}$$

□

Note that, for any $A \in \mathcal{B}(\mathbb{R}^2)$, the corresponding function f_k in Lemma 1.2 is nonnegative, bounded and measurable. We start by proving the recurrence of $(S_n)_n$.

Lemma 1.3. *We have: $0 \in \mathcal{R}_{(\pi, 0)}$.*

Proof. Let $\varepsilon > 0$, $k \geq 1$. Let us prove that $\mathbb{P}_{\pi}(\exists j \geq k : |S_j| < 2\varepsilon) = 1$. For any $n \geq 1$, set

$$A_n^{(k)} = \left[|S_n| < \varepsilon, |S_{n+j}| \geq \varepsilon, \forall j \geq k \right].$$

If $|n - n'| \geq k$, then $A_n^{(k)} \cap A_{n'}^{(k)} = \emptyset$. Hence we have $\sum_{n \geq 1} \mathbb{P}_{(\pi, 0)}(A_n^{(k)}) \leq k$. Besides we have

$$\mathbb{P}_{(\pi, 0)}(A_n^{(k)}) \geq \mathbb{P}_{(\pi, 0)}\left(|S_n| < \varepsilon, |S_{n+j} - S_n| \geq 2\varepsilon, \forall j \geq k\right).$$

Then, applying Lemma 1.2 with $B = \{z \in \mathbb{R}^2 : |z| < \varepsilon\}$ and $A = \{z \in \mathbb{R}^2 : |z| \geq 2\varepsilon\}$, we get $\sum_{n \geq 1} \mathbb{E}_{(\pi, 0)}[\mathbf{1}_B(S_n) f_k(X_n)] \leq k$. But (5) gives as $n \rightarrow +\infty$:

$$\mathbb{E}_{(\pi, 0)}[\mathbf{1}_B(S_n) f_k(X_n)] \sim a_n \pi(f_k) m_{\mathbb{S}}(B).$$

Since $0 \in \mathbb{S}$ and B is centered at 0, we have $m_{\mathbb{S}}(B) > 0$, and finally the fact that $\sum_{n \geq 1} a_n = \infty$ implies $\pi(f_k) = \mathbb{P}_{(\pi, 0)}(Y_k = 1) = 0$. □

Proof of Proposition 1.1. Let $s \in \mathbb{S}$. Let us show that

$$\forall \varepsilon > 0, \forall k \geq 1, \mathbb{P}_{(\pi,0)}(|S_j - s| \geq 2\varepsilon, \forall j \geq k) = 0.$$

Let $\varepsilon > 0$ and $k \geq 1$ be fixed. Set $B' = \{z \in \mathbb{R}^2 : |z + s| < \varepsilon\}$ and $A' = \{z \in \mathbb{R}^2 : |z - s| \geq 2\varepsilon\}$, and denote by Y'_k and f'_k the elements associated to A' as in Lemma 1.2. Then, according to lemmas 1.3 and 1.2, we have for $n \geq 1$

$$\begin{aligned} 0 = \mathbb{P}_{(\pi,0)}(|S_{n+j}| \geq \varepsilon, \forall j \geq k) &\geq \mathbb{P}_{(\pi,0)}(S_n \in B', S_{n+j} - S_n \in A', \forall j \geq k) \\ &= \mathbb{E}_{(\pi,0)}[\mathbf{1}_{B'}(S_n) f'_k(X_n)]. \end{aligned}$$

Hence $\mathbb{E}_{(\pi,0)}[\mathbf{1}_{B'}(S_n) f'_k(X_n)] = 0$. From (5) and $m_{\mathbb{S}}(B') > 0$ (since B' is centered at $-s \in \mathbb{S}$), it then follows that we have $\pi(f'_k) = \mathbb{P}_{\pi}(Y'_k = 1) = 0$. \square

1.2. From stationarity to non-stationarity. Let us define the following subset of \mathbb{X} :

$$\mathcal{A} := \{x \in \mathbb{X} : \mathcal{R}_{(x,0)} = \mathbb{S}\}.$$

The property $\mathcal{R}_{(\pi,0)} = \mathbb{S}$ implies that $\pi(\mathcal{A}) = 1$ (since \mathbb{R}^2 is separable). Of course, if $\mathcal{A} = \mathbb{X}$, then we obtain $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for any initial distribution μ of the driving Markov chain $(X_n)_{n \in \mathbb{N}}$. Let us recall that $(X_n)_{n \in \mathbb{N}}$ is said to be Harris recurrent if, for any set $B \in \mathcal{X}$ such that $\pi(B) > 0$, we have for every $x \in \mathbb{X}$: $\mathbb{P}_{(x,0)}(X_k \in B \text{ i.o.}) = 1$.

Proposition 1.4. *We have the following properties:*

- (i) *If $x \in \mathbb{X}$ is such that $\mathbb{P}_{(x,0)}(X_k \in \mathcal{A} \text{ i.o.}) = 1$, then we have $x \in \mathcal{A}$ (i.e. $\mathcal{R}_{(x,0)} = \mathbb{S}$).*
- (ii) *If the driving Markov chain $(X_n)_{n \in \mathbb{N}}$ is Harris recurrent and $\mathcal{R}_{(\pi,0)} = \mathbb{S}$, then $\mathcal{A} = \mathbb{X}$. In this case, we have $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for any initial distribution μ of $(X_n)_{n \in \mathbb{N}}$.*

Proof of Proposition 1.4. We suppose that $(X_n, S_n)_n$ is the canonical version defined on the set $\Omega := (\mathbb{X} \times \mathbb{R}^2)^{\mathbb{N}}$. Let us fix any $(s, \varepsilon) \in \mathbb{S} \times (0, 1)$, and set $S := [|S_n - s| < \varepsilon \text{ i.o.}]$.

Lemma 1.5. *For all $x \in \mathbb{X}$, we have for $\mathbb{P}_{(x,0)}$ -almost every $\omega \in \Omega$:*

$$\lim_{k \rightarrow +\infty} \mathbb{P}_{(X_k(\omega),0)}(|S_n - (s - S_k(\omega))| < \varepsilon \text{ i.o.}) = \mathbf{1}_S(\omega).$$

Proof of lemma 1.5. Let $x \in \mathbb{X}$. According to a classical argument due to Doob (see [24, Prop. V-2.4]), we have for $\mathbb{P}_{(x,0)}$ -almost every $\omega \in \Omega$: $\lim_{k \rightarrow +\infty} \mathbb{P}_{(X_k(\omega), S_k(\omega))}(S) = \mathbf{1}_S(\omega)$. Then the desired property easily follows from the additive property (1). \square

End of the proof of Proposition 1.4. Using the assumption in (i), Lemma 1.5 and Lebesgue's theorem, and using finally the definition of \mathcal{A} and the fact that $S_k - s \in \mathbb{S}$ $\mathbb{P}_{(x,0)}$ -a.s. (use (2)), we obtain the following property

$$\mathbb{P}_{(x,0)}(S) = \lim_k \int_{\{\omega: X_k(\omega) \in \mathcal{A} \text{ i.o.}\}} \mathbb{P}_{(X_k(\omega),0)}(|S_n - (s - S_k(\omega))| < \varepsilon \text{ i.o.}) d\mathbb{P}_{(x,0)}(\omega) = 1,$$

from which we deduce $\mathcal{R}_{(x,0)} = \mathbb{S}$. Now, if $\mathcal{R}_{(\pi,0)} = \mathbb{S}$, then $\pi(\mathcal{A}) = 1$, so that the Harris recurrence of $(X_n)_{n \in \mathbb{N}}$ gives $\mathbb{P}_{(x,0)}(X_k \in \mathcal{A} \text{ i.o.}) = 1$ for all $x \in \mathbb{X}$. Thus (ii) follows from (i). \square

Lemma 1.5, which is based on both Markov and additive properties of $(X_n, S_n)_n$, plays an important role in the previous proof, as well as in the main statement (Prop. 2.3) of the next section.

2. BOREL-CANTELLI ADAPTATION OF KOCHEN AND STONE. APPLICATIONS TO MRWS

We present now a general strategy to obtain $\mathcal{R}_{(x,0)} = \mathbb{S}$ for every $x \in \mathbb{X}$, even when the driving Markov chain is not Harris-recurrent. It is based on the following statement which, as well as its corollary, is true for any sequence $(Y_n)_{n \in \mathbb{N}}$ of r.v. defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking their values in \mathbb{R}^2 .

Proposition 2.1 ([21]). *Let $(s, \varepsilon) \in \mathbb{R}^2 \times (0, 1]$, and assume that there exists $c \in (0, +\infty)$ such that*

$$- \sum_{n \geq 1} \mathbb{P}(|Y_n - s| < \varepsilon) = \infty \quad (6a)$$

$$- \liminf_{N \rightarrow +\infty} \frac{\sum_{n,m=1}^N \mathbb{P}(|Y_n - s| < \varepsilon, |Y_m - s| < \varepsilon)}{\left(\sum_{n=1}^N \mathbb{P}(|Y_n - s| < \varepsilon)\right)^2} \leq c. \quad (6b)$$

Then we have: $\mathbb{P}(|Y_n - s| < \varepsilon \text{ i.o.}) \geq \frac{1}{c}$.

In the sequel, we shall use the following alternative statement.

Corollary 2.2. *Let $(s, \varepsilon) \in \mathbb{R}^2 \times (0, 1]$, and assume that Condition (6a) is fulfilled and that there exists $d \in (0, +\infty)$ such that*

$$\liminf_{N \rightarrow +\infty} \frac{\sum_{n,m=1}^N \mathbb{P}(|Y_n - s| < \varepsilon, |Y_{n+m} - s| < \varepsilon)}{\left(\sum_{n=1}^N \mathbb{P}(|Y_n - s| < \varepsilon)\right)^2} \leq d. \quad (7)$$

Then we have: $\mathbb{P}(|Y_n - s| < \varepsilon \text{ i.o.}) \geq \frac{1}{2d}$.

Proof. Set $p_{n,m} := \mathbb{P}(|Y_n - s| < \varepsilon, |Y_{n+m} - s| < \varepsilon)$. Note that $p_{n,0} := \mathbb{P}(|Y_n - s| < \varepsilon)$. We have

$$\sum_{n,m=1}^N \mathbb{P}(|Y_n - s| < \varepsilon, |Y_m - s| < \varepsilon) \leq 2 \sum_{n=1}^N \sum_{m=n}^N p_{n,m-n} \leq 2 \sum_{n=1}^N \sum_{m=0}^N p_{n,m} = 2 \left(\sum_{n,m=1}^N p_{n,m} + \sum_{n=1}^N p_{n,0} \right).$$

From (6a) and the previous inequality, we obtain (6b) with $c = 2d$. \square

Let us notice that, if $(Y_n - Y_{n-1})_{n \geq 1}$ is a sequence of non-lattice square integrable i.i.d. random variables, then for every $s \in \mathbb{R}^2$ the quantity $\mathbb{P}(|Y_n - s| < \varepsilon, |Y_{n+m} - s| < \varepsilon)$ behaves as $\mathbb{P}(|Y_n - s| < \varepsilon) \mathbb{P}(|Y_m| < \varepsilon)$ (thanks to independence and to the proof of the local limit theorem). Hence Corollary 2.2 does not give $p_n := \mathbb{P}(|Y_n - s| < \varepsilon \text{ i.o.}) = 1$ as expected, but only $p_n \geq 1/2$. Therefore, further arguments (here based on the additive property (1)) must be exploited to deduce the recurrent set from Corollary 2.2. The next proposition gives such a result for general Markov random walks.

Again $(X_n, S_n)_{n \in \mathbb{N}}$ denotes a MRW with state space $\mathbb{X} \times \mathbb{R}^2$, and \mathbb{S} is given in (2).

Proposition 2.3. *Let $\varepsilon > 0$. Assume that there exists a real number $e_\varepsilon > 0$ such that*

$$\forall (x', s') \in \mathbb{X} \times \mathbb{S}, \quad \mathbb{P}_{(x',0)}(|S_n - s'| < \varepsilon \text{ i.o.}) \geq e_\varepsilon. \quad (8)$$

Then, for every $(x, s) \in \mathbb{X} \times \mathbb{S}$, we have: $\mathbb{P}_{(x,0)}(|S_n - s| < \varepsilon \text{ i.o.}) = 1$. In particular, if (8) is fulfilled for every $\varepsilon \in (0, 1)$, then we have for every $x \in \mathbb{X}$: $\mathcal{R}_{(x,0)} = \mathbb{S}$.

Proof. Suppose that $(X_n, S_n)_n$ is the canonical version defined on $\Omega := (\mathbb{X} \times \mathbb{R}^2)^\mathbb{N}$. Let us fix $(x, s) \in \mathbb{X} \times \mathbb{S}$, and set $S := [|S_n - s| < \varepsilon \text{ i.o.}]$. Then, from Lemma 1.5, (2) and (8), it follows that, for $\mathbb{P}_{(x,0)}$ -almost every $\omega \in \Omega$, we have: $\mathbf{1}_S(\omega) \geq e_\varepsilon$. Hence: $\mathbf{1}_S = 1$ $\mathbb{P}_{(x,0)}$ -a.s.. \square

Finally, from Corollary 2.2 and Proposition 2.3, we deduce the following.

Corollary 2.4. *Assume that there exists $\varepsilon_\mathbb{S} > 0$ and $d > 0$ such that, for every $\varepsilon \in (0; \varepsilon_\mathbb{S})$ and for every $(x, s) \in \mathbb{X} \times \mathbb{S}$, we have*

$$- \sum_{n \geq 1} \mathbb{P}_{(x,0)}(|S_n - s| < \varepsilon) = \infty \quad (9a)$$

$$- \liminf_{N \rightarrow +\infty} \frac{\sum_{n,m=1}^N \mathbb{P}_{(x,0)}(|S_n - s| < \varepsilon, |S_{n+m} - s| < \varepsilon)}{\left(\sum_{n=1}^N \mathbb{P}_{(x,0)}(|S_n - s| < \varepsilon)\right)^2} \leq d. \quad (9b)$$

Then, for every $x \in \mathbb{X}$, we have $\mathcal{R}_{(x,0)} = \mathbb{S}$.

3. USE OF THE SPECTRAL METHOD

In this section, we introduce the "taylor-made" operator-type assumptions in order to check, with the help of Fourier techniques, Condition (5) of Proposition 1.1 and Conditions (9a) (9b) of Corollary 2.4.

Let $(X_n, S_n)_{n \in \mathbb{N}}$ be a Markov random walk with state space $\mathbb{X} \times \mathbb{R}^2$, and let \mathbb{S} be a closed subgroup of \mathbb{R}^2 satisfying (2). Recall that $m_\mathbb{S}$ denotes the Haar measure on \mathbb{S} . As explained at the end of the introduction, in this work the subgroups \mathbb{S} of interest in (2) are the two-dimensional subgroups of \mathbb{R}^2 , which correspond to the three following cases:

(H1) $\mathbb{S} = \mathbb{R}^2$ (also called the non-lattice case). We have $\mathbb{S}^* = \{0\}$. We set $\varepsilon_\mathbb{S} := 1$.

(H2) There exists $(b, \vec{u}, \vec{v}) \in \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R}^2$ such that: $\mathbb{S} = b\mathbb{Z}\vec{u} \oplus \mathbb{R}\vec{v}$. We suppose, without any loss of generality, that (\vec{u}, \vec{v}) is an orthonormal basis of \mathbb{R}^2 . We set $\varepsilon_\mathbb{S} := b$. Note that, for every $(s, \varepsilon) \in \mathbb{S} \times (0; \varepsilon_\mathbb{S})$, we have $B(s, \varepsilon) \cap \mathbb{S} = \{s + w\vec{v} : w \in \mathbb{R}, |w| < \varepsilon\}$, and that $\mathbb{S}^* = a\mathbb{Z}\vec{u}$ with $a = 2\pi/b$.

(H3) There exists some real-valued invertible 2×2 -matrix B such that: $\mathbb{S} = B\mathbb{Z}^2$. We set $\varepsilon_\mathbb{S} := \min\{|s|; s \in \mathbb{S} \setminus \{0\}\}$. Note that we have, for every $(s, \varepsilon) \in \mathbb{S} \times (0; \varepsilon_\mathbb{S})$, $B(s, \varepsilon) \cap \mathbb{S} = \{s\}$, and that $\mathbb{S}^* = A\mathbb{Z}^2$ with $A = 2\pi(B^*)^{-1}$, where B^* is the transpose matrix of B .

Recall that the transition probability of the driving Markov chain $(X_n)_{n \in \mathbb{N}}$ is denoted by Q , that π is a Q -invariant probability measure on \mathbb{X} , and that μ stands for the law of X_0 . The results of this section concern the MRWs $(X_n, S_n)_{n \in \mathbb{N}}$, the driving Markov chain $(X_n)_{n \in \mathbb{N}}$ of which is assumed to be strongly ergodic on some Banach space \mathcal{B} . This property, defined in (15) below, will be illustrated in several examples, see Section 5 and the end of Section 4.

3.1. Functional setting. We denote by $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ a \mathbb{C} -Banach space composed of π -integrable \mathbb{C} -valued functions on \mathbb{X} (or of classes modulo π of such functions).

Hypothesis 3.1. *The space \mathcal{B} satisfies the following assumptions:*

(3.1.a) *The function $\mathbf{1}_\mathbb{X}$ (or its class) is in \mathcal{B} ,*

(3.1.b) *There exists $C_0 > 0$ such that, for every $f \in \mathcal{B}$, we have $\pi(|f|) \leq C_0 \|f\|_\mathcal{B}$.*

We denote by $\mathcal{L}(\mathcal{B})$ the space of linear continuous endomorphisms of \mathcal{B} . The associated operator norm on $\mathcal{L}(\mathcal{B})$ is also denoted by $\|\cdot\|_{\mathcal{B}}$. The dual space of \mathcal{B} is denoted by \mathcal{B}' , and the property $\mu \in \mathcal{B}'$ means that every $f \in \mathcal{B}$ is μ -integrable and that the map $f \mapsto \mu(f) := \int_{\mathbb{X}} f d\mu$ is in \mathcal{B}' . Thanks to the above condition (3.1.b), the following rank-one projection can be defined in $\mathcal{L}(\mathcal{B})$:

$$\Pi f = \pi(f) \mathbf{1}_{\mathbb{X}} \quad (f \in \mathcal{B}). \quad (10)$$

We consider the Fourier operators $Q(t)$ ($t \in \mathbb{R}^2$), associated with the MRW, acting (in a first step) on the space of the bounded measurable functions $f : \mathbb{X} \rightarrow \mathbb{C}$, as follows:

$$\forall t \in \mathbb{R}^2, \forall x \in \mathbb{X}, \quad (Q(t)f)(x) := \mathbb{E}_{(x,0)} [e^{i\langle t, S_1 \rangle} f(X_1)]. \quad (11)$$

We clearly have $Q(0) = Q$, and $Q(t+g) = Q(t)$ for every $(t, g) \in \mathbb{R}^2 \times \mathbb{S}^*$.

Hypothesis 3.2. *For every $t \in \mathbb{R}^2$, $Q(t) \in \mathcal{L}(\mathcal{B})$ and we have the following property:*

(3.2.a) *There exist two real numbers $\alpha > 0$ and $\kappa \in [0, 1)$, a function $\lambda : B(0, \alpha) \rightarrow \mathbb{C}$, and a bounded map $\Pi : B(0, \alpha) \rightarrow \mathcal{L}(\mathcal{B})$ such that $\lambda(0) = 1$, $\Pi(0) = \Pi$, and*

$$\sup_{t \in B(0, \alpha)} \|Q(t)^n - \lambda(t)^n \Pi(t)\|_{\mathcal{B}} = O(\kappa^n), \quad (12)$$

(3.2.b) *There exists a definite positive symmetric 2×2 -matrix Γ such that*

$$\lambda(t) = 1 - \frac{1}{2} \langle t, \Gamma t \rangle L(|t|^{-1}) (1 + \varepsilon(t)), \quad (13)$$

with $\lim_{t \rightarrow 0} \varepsilon(t) = 0$, where $L : (0; +\infty) \rightarrow (0; +\infty)$ is a slowly varying function at infinity in the sense of Karamata, i.e. $\forall k > 0, \lim_{x \rightarrow +\infty} \frac{L(kx)}{L(x)} = 1$.

(3.2.c) *For any compact subset K of $\mathbb{R}^2 \setminus \mathbb{S}^*$, there exists $\rho = \rho(K) \in [0, 1)$ such that*

$$\sup_{t \in K} \|Q(t)^n\|_{\mathcal{B}} = O(\rho^n). \quad (14)$$

For $t = 0$, (12) is nothing else but the above mentioned strong ergodicity property on \mathcal{B} :

$$\|Q^n - \Pi\|_{\mathcal{B}} \leq C \kappa^n. \quad (15)$$

Recall that this property is equivalent to : $\lim_n \|Q^n - \Pi\|_{\mathcal{B}} = 0$. The property $Q(t) \in \mathcal{L}(\mathcal{B})$ means that for every $f \in \mathcal{B}$, the function $\mathbb{X} \ni x \mapsto (Q(t)f)(x)$ (or its class mod. π) belongs to \mathcal{B} , and that the map $f \mapsto Q(t)f$ is in $\mathcal{L}(\mathcal{B})$. Under the assumption (15), Condition (3.2.a) can be checked thanks to an operator perturbation theorem due to G. Keller and C. Liverani [20]. This is fully described and illustrated in [15, Cond. (K)] for the Markov context. Condition (3.2.b) is closely related to the centered assumption $\mathbb{E}_{(\pi,0)}[S_1] = 0$ and to the convergence in distribution to the Gaussian law of the (properly normalized) second component S_n of the MRW. This part is detailed in Subsection 4.2. As expected (in comparison with the i.i.d. case), Condition (3.2.c) can be reduced under general assumptions as a simple hypothesis on the subgroup \mathbb{S} , saying (in substance) that \mathbb{S} is the smallest lattice in \mathbb{R}^2 satisfying (2). On the basis of what is done in [15, Sect. 5, 12] for additive functionals, the reductions of (3.2.c) are presented in Subsection 4.3. Finally, each one of the next statements will also involve a functional assumption on the initial distribution μ .

3.2. Study of Conditions (5) and (9a) (9b). Applications to recurrence. Under Hypothesis 3.2, we consider a sequence $(A_n)_n$ of positive real numbers such that

$$A_n^2 \sim nL(A_n), \quad (16)$$

where $L(\cdot)$ is the slowly varying function in (13). We also define the following positive constant: $D_{\mathbb{S}} := (2\pi)^{-1} c_{\mathbb{S}} (\det \Gamma)^{-1/2}$, where $c_{\mathbb{S}} = \varepsilon_{\mathbb{S}}$ for (H1) (H2), and $c_{\mathbb{S}} = |\det B|$ for (H3).

Proposition 3.3. *Assume that Hypotheses 3.1 and 3.2 hold true, and that the initial distribution μ is in \mathcal{B}' and satisfies the following condition*

$$\forall f \in \mathcal{B}, \quad \lim_{t \rightarrow 0} \mu(\Pi(t)f) = \pi(f). \quad (17)$$

Then, for every $(s, \varepsilon) \in \mathbb{S} \times (0; \varepsilon_{\mathbb{S}})$ and for every bounded nonnegative $f \in \mathcal{B}$, we have:

$$\mathbb{E}_{(\mu, 0)}[f(X_n) \mathbf{1}_{B(s, \varepsilon)}(S_n)] \sim D_{\mathbb{S}} \pi(f) A_n^{-2} m_{\mathbb{S}}(B(s, \varepsilon)). \quad (18)$$

In the next proposition, for every $(s, \varepsilon) \in \mathbb{S} \times (0; \varepsilon_{\mathbb{S}})$, and for every $(n, m) \in \mathbb{N}^2$, we set

$$p_{n,m}(\mu, s, \varepsilon) := \mathbb{P}_{(\mu, 0)}((S_n, S_{n+m}) \in B(s, \varepsilon)^2).$$

Proposition 3.4. *Assume that Hypotheses 3.1 and 3.2 hold true, and that the initial distribution μ is in \mathcal{B}' and satisfies the following condition*

$$\lim_{(u,v) \rightarrow 0} \mu(\Pi(u)\Pi(v)\mathbf{1}_{\mathbb{X}}) = 1. \quad (19)$$

Then, for every $(s, \varepsilon) \in \mathbb{S} \times (0; \varepsilon_{\mathbb{S}})$, we have the following results:

$$- \quad p_{n,0}(\mu, s, \varepsilon) \sim D_{\mathbb{S}} A_n^{-2} m_{\mathbb{S}}(B(s, \varepsilon)), \quad (20a)$$

$$- \quad p_{n,m}(\mu, s, \varepsilon) \sim D_{\mathbb{S}}^2 A_n^{-2} A_m^{-2} m_{\mathbb{S}}(B(s, \varepsilon))^2, \quad (20b)$$

The equivalence relation in (20b) means that $p_{n,m}(\mu, s, \varepsilon) = D_{\varepsilon}^2 A_n^{-2} A_m^{-2} (1 + \eta_{n,m})$ with $D_{\varepsilon} := D_{\mathbb{S}} m_{\mathbb{S}}(B(s, \varepsilon))$ and for some bounded $(\eta_{n,m})_{n,m}$ such that $\eta_{n,m} \rightarrow 0$ when $\min(n, m) \rightarrow +\infty$.

Propositions 3.3 and 3.4 are established in Section 3.3. Now, using these propositions, one applies the results of Sections 1-2 to investigate the recurrence properties of S_n .

Stationary case. Condition (5) in Proposition 1.1 is nothing else but Property (18) stated with $\mu = \pi$ and imposed for every nonnegative bounded measurable function $f : \mathbb{X} \rightarrow \mathbb{R}$. Consequently, from Propositions 1.1 and 1.4, we deduce the following.

Corollary 3.5. *Assume that Hypotheses 3.1 and 3.2 hold true with \mathcal{B} containing all the nonnegative bounded measurable functions on \mathbb{X} , that $\mu = \pi$ satisfies Condition (17), and finally that $\sum_{n \geq 1} A_n^{-2} = \infty$. Then we have $\mathcal{R}_{(\pi, 0)} = \mathbb{S}$. If in addition $(X_n)_{n \in \mathbb{N}}$ is Harris recurrent, then we have $\mathcal{R}_{(\mu, 0)} = \mathbb{S}$ for any initial distribution μ of $(X_n)_{n \in \mathbb{N}}$.*

The above assumption on \mathcal{B} is not fulfilled if it is defined as some space of regular functions. In this case, the two next corollaries (of Proposition 3.4) are relevant, and it is worth noticing that they do not require the Harris-recurrence hypothesis.

Corollary 3.6. *Assume that Hypotheses 3.1 and 3.2 hold, that μ is in \mathcal{B}' and satisfies (19), and that $\sum_{n \geq 1} A_n^{-2} = \infty$. Then we have for every $(s, \varepsilon) \in \mathbb{S} \times (0; \varepsilon_{\mathbb{S}})$: $\mathbb{P}_{(\mu, 0)}(|S_n - s| < \varepsilon \text{ i.o.}) \geq 1/2$.*

Proof. Let $(s, \varepsilon) \in \mathbb{S} \times (0, \varepsilon_{\mathbb{S}})$. To simplify, set $p_{n,m} = p_{n,m}(\mu, s, \varepsilon)$. From (20a) and (20b), we have when $N \rightarrow +\infty$:

$$\sum_{n=1}^N p_{n,0} \sim D_{\varepsilon} \sum_{n=1}^N A_n^{-2} \quad \text{and} \quad \sum_{n,m=1}^N p_{n,m} \sim D_{\varepsilon}^2 \left(\sum_{n=1}^N A_n^{-2} \right) \left(\sum_{m=1}^N A_m^{-2} \right), \quad (21)$$

hence:

$$\sum_{n \geq 1} p_{n,0} = \infty \quad \text{and} \quad \lim_{N \rightarrow +\infty} \left(\sum_{n,m=1}^N p_{n,m} \right) \left(\sum_{n=1}^N p_{n,0} \right)^{-2} = 1. \quad (22)$$

So Corollary 2.2, applied with $\mathbb{P} = \mathbb{P}_{(\mu, 0)}$ and $Y_n = S_n$, gives the desired statement. \square

In the next application, we suppose that the space \mathcal{B} is composed of functions (not of classes), so that the Dirac distribution at any $x \in \mathbb{X}$, called δ_x , is defined on \mathcal{B} (i.e. $\delta_x(f) := f(x)$). Note that the condition $\delta_x \in \mathcal{B}'$ means that there exists $c_x > 0$ such that: $\forall f \in \mathcal{B}, |f(x)| \leq c_x \|f\|_{\mathcal{B}}$.

Corollary 3.7. *Assume that Hypotheses 3.1 and 3.2 hold true, that for every $x \in \mathbb{X}$ the Dirac distribution δ_x is in \mathcal{B}' and satisfies (19), and that $\sum_{n \geq 1} A_n^{-2} = \infty$. Then, for every $x \in \mathbb{X}$, we have $\mathcal{R}_{(x,0)} = \mathbb{S}$ (thus $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for every initial distribution μ).*

Proof. Since (20a) and (20b) hold with $\mu = \delta_x$ for each $x \in \mathbb{X}$, the numbers $p_{n,m} = p_{n,m}(\delta_x, s, \varepsilon)$ satisfy (21), hence (22). Then Corollary 2.4 gives the desired statement. \square

3.3. Proof of Propositions 3.3 and 3.4.

3.3.1. *Preliminary lemmas.* Let f be a \mathbb{C} -valued bounded measurable function on \mathbb{X} .

Lemma 3.8. *We have for every $x \in \mathbb{X}$, every $(u, v) \in \mathbb{R}^2$ and every $(n, m) \in \mathbb{N}^2$:*

$$\mathbb{E}_{(x,0)} [e^{i\langle u, S_n \rangle} e^{i\langle v, S_{n+m} - S_n \rangle} f(X_{n+m})] = (Q(u)^n Q(v)^m f)(x).$$

Consequently, we have for any initial distribution μ on \mathbb{X} :

$$\mathbb{E}_{(\mu,0)} [e^{i\langle u, S_n \rangle} e^{i\langle v, S_{n+m} - S_n \rangle} f(X_{n+m})] = \mu(Q(u)^n Q(v)^m f).$$

Proof of lemma 3.8. Using the additivity property (1) (see Subsection 1.1), we obtain

$$\begin{aligned} \mathbb{E}_{(x,0)} [e^{i\langle u, S_n \rangle} e^{i\langle v, S_{n+m} - S_n \rangle} f(X_{n+m})] &= \mathbb{E}_{(x,0)} \left[e^{i\langle u, S_n \rangle} \mathbb{E}_{(x,0)} [e^{i\langle v, S_{n+m} - S_n \rangle} f(X_{n+m}) \mid \mathcal{F}_n] \right] \\ &= \mathbb{E}_{(x,0)} \left[e^{i\langle u, S_n \rangle} \mathbb{E}_{(X_n,0)} [f(X_m) e^{i\langle v, S_m \rangle}] \right]. \end{aligned} \quad (23)$$

Now, applying (23) with $m = 1$ and $u = v$, and according to the definition (11) of the Fourier maps, we get for any $n \geq 0$,

$$\mathbb{E}_{(x,0)} [e^{i\langle v, S_{n+1} \rangle} f(X_{n+1})] = \mathbb{E}_{(x,0)} \left[e^{i\langle v, S_n \rangle} \mathbb{E}_{(X_n,0)} [f(X_1) e^{i\langle v, S_1 \rangle}] \right] = \mathbb{E}_{(x,0)} [e^{i\langle v, S_n \rangle} (Q(v)f)(X_n)].$$

We deduce by induction that we have for all $v \in \mathbb{R}^2$, $k \geq 1$, and for all \mathbb{C} -valued bounded measurable function g on \mathbb{X} :

$$\mathbb{E}_{(x,0)} [e^{i\langle v, S_k \rangle} g(X_k)] = (Q(v)^k g)(x).$$

Next, by applying (23) (with any $m \geq 1$ and $u, v \in \mathbb{R}^2$) and using the previous equality (first with $g = f$, second with $g = Q(v)^m f$), we obtain

$$\mathbb{E}_{(x,0)} [e^{i\langle u, S_n \rangle} e^{i\langle v, S_{n+m} - S_n \rangle} f(X_{n+m})] = \mathbb{E}_{(x,0)} [e^{i\langle u, S_n \rangle} (Q(v)^m f)(X_n)] = (Q(u)^n Q(v)^m f)(x). \quad \square$$

For any Lebesgue-integrable function $h : \mathbb{R}^2 \rightarrow \mathbb{C}$, we define its Fourier transform \hat{h} by $\hat{h}(u) := \int_{\mathbb{R}^2} h(t) e^{-i\langle t, u \rangle} dt$, and we set

$$\forall t \in \mathbb{R}^2, P_h(t) := \sum_{g \in \mathbb{S}^*} \hat{h}(t + g).$$

Let \mathcal{D} be the fundamental domain of $\mathbb{R}^2/\mathbb{S}^*$, namely:

- $\mathcal{D} := \mathbb{R}^2$ in Case (H1),
- $\mathcal{D} := [-\frac{a}{2}, \frac{a}{2}] \times \mathbb{R}$ in Case (H2), with $a = 2\pi/b$,
- $\mathcal{D} := A[-\frac{1}{2}, \frac{1}{2}]^2$ in Case (H3), with $A := 2\pi(B^*)^{-1}$.

Lemma 3.9. *Let h_1 and h_2 be \mathbb{C} -valued Lebesgue-integrable functions on \mathbb{R}^2 such that their Fourier transforms are Lebesgue-integrable on \mathbb{R}^2 . Then we have for any probability measure μ on \mathbb{X} and for every $(n, m) \in \mathbb{N}^2$:*

$$\mathbb{E}_{(\mu,0)}[h_1(S_n) f(X_n)] = \frac{1}{(2\pi)^2} \int_{\mathcal{D}} P_{h_1}(u) \mu(Q(u)^n f) du.$$

$$\mathbb{E}_{(\mu,0)}[h_1(S_n) h_2(S_{n+m} - S_n) f(X_{n+m})] = \frac{1}{(2\pi)^4} \int_{\mathcal{D} \times \mathcal{D}} P_{h_1}(u) P_{h_2}(v) \mu(Q(u)^n Q(v)^m f) dudv.$$

Proof of lemma 3.9. We easily obtain the first formula by applying the inverse Fourier formula to h_1 , Lemma 3.8 (with $m = 0$), and finally the fact that $Q(\cdot)$ and P_{h_1} are \mathbb{S}^* -periodic. The second formula can be proved similarly. \square

Lemma 3.10. *Up to reduce the positive real number α of Hypothesis 3.2, there exists $\tilde{a} > 0$ such that, for every $t \in B(0, \alpha)$, we have $|\lambda(t)| \leq e^{-\tilde{a}|t|^2 L(|t|^{-1})}$, and for all $n \geq 1$*

$$\left| \lambda \left(\frac{u}{A_n} \right) \right|^n \mathbf{1}_{B(0, \alpha A_n)}(u) \leq \mathbf{1}_{B(0,1)}(u) + e^{-\frac{\tilde{a}}{4}|u|} \mathbf{1}_{\{u: 1 \leq |u| \leq \alpha A_n\}}(u). \quad (24)$$

Proof of Lemma 3.10. From (13) and the fact that Γ is positive definite, there exists $\tilde{a} > 0$ such that, for every $t \in B(0, \alpha)$ (with α possibly reduced), we have

$$|\lambda(t)| \leq 1 - \tilde{a}|t|^2 L(|t|^{-1}) \leq e^{-\tilde{a}|t|^2 L(|t|^{-1})}.$$

Next, since L is a slowly varying function, we know (see [19] or [10], p. 282) that there exist two functions $\ell(\cdot)$ and $\tilde{\varepsilon}(\cdot)$ such that $\lim_{x \rightarrow +\infty} \ell(x)$ exists in $(0, +\infty)$ and $\lim_{x \rightarrow +\infty} \tilde{\varepsilon}(x) = 0$, and such that

$$L(x) = \ell(x) \exp \left(\int_1^x \frac{\tilde{\varepsilon}(y)}{y} dy \right). \quad (25)$$

Using this representation of L , it is easy to see that there exists n_0 such that, for any $n \geq n_0$ and any u such that $1 \leq |u| \leq \alpha A_n$, we have :

$$\frac{1}{2}|u|^{-1} \leq \frac{L(A_n|u|^{-1})}{L(A_n)}.$$

From $A_n^2 \sim nL(A_n)$ one can also assume that, for every $n \geq n_0$ (up to a change of n_0), we have $n/A_n^2 \geq \frac{1}{2L(A_n)}$. Therefore we have: $\forall u \in B(0, \alpha A_n), \forall n \geq n_0$,

$$\begin{aligned} \left| \lambda \left(\frac{u}{A_n} \right) \right|^n \mathbf{1}_{B(0, \alpha A_n)}(u) &\leq e^{-n \tilde{a} \frac{|u|^2}{A_n^2} L(A_n|u|^{-1})} \mathbf{1}_{B(0, \alpha A_n)}(u) \\ &\leq \mathbf{1}_{B(0,1)}(u) + e^{-\frac{\tilde{a}}{4}|u|} \mathbf{1}_{B(0, \alpha A_n) \setminus B(0,1)}(u). \end{aligned}$$

\square

3.3.2. Proof of Proposition 3.3. Let \mathcal{H}_2 denote the space of all the Lebesgue-integrable continuous functions on \mathbb{R}^2 having a compactly supported Fourier transform. Let $f \in \mathcal{B}$, $f \geq 0$ be fixed. Property (18) will be proved if we establish that we have for every $h \in \mathcal{H}_2$:

$$\lim_n 2\pi A_n^2 \mathbb{E}_{(\mu,0)}[f(X_n) h(S_n)] = c_{\mathbb{S}} (\det \Gamma)^{-1/2} \pi(f) m_{\mathbb{S}}(h). \quad (26)$$

Indeed, (26) ensures that the sequence $(\nu_n)_n$ of positive measures defined by

$$\forall C \in \mathcal{B}(\mathbb{R}^2), \quad \nu_n(C) := 2\pi A_n^2 \mathbb{E}_{(\mu,0)}[f(X_n) \mathbf{1}_C(S_n)],$$

converges weakly to the measure $\nu(\cdot) := c_{\mathbb{S}} (\det \Gamma)^{-1/2} \pi(f) m_{\mathbb{S}}(\cdot)$, see [4]. Since the boundary of the ball $B = B(s, \varepsilon)$ has zero ν -measure when $\varepsilon \in (0, \varepsilon_{\mathbb{S}})$, we have: $\lim_n \nu_n(B) = \nu(B)$, which is (18).

Proof of (26). Note that

- $m_{\mathbb{S}}(h) := \int_{\mathbb{R}^2} h(t) dt$ in Case (H1),
- $m_{\mathbb{S}}(h) := \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} h(bn, y) dy$ in Case (H2),
- $m_{\mathbb{S}}(h) := \sum_{\eta \in \mathbb{S}} h(\eta)$ in Case (H3).

Let $h \in \mathcal{H}_2$. Let β be a positive real number such that $\text{Supp}(\hat{h}) \subset B(0, \beta)$. Without any loss of generality, one can suppose that the positive real numbers β and α (of (12)) are such that $\alpha < a/2 < \beta$ in Case (H2), and $B(0, \alpha) \subset A([-1/2, 1/2]^2) \subset B(0, \beta)$ in Case (H3). We set

$$K := (\overline{B(0, \beta)} \setminus B(0, \alpha)) \cap \mathcal{D}. \quad (27)$$

Observe that K is a compact subset of $\mathbb{R}^2 \setminus \mathbb{S}^*$. Let $\rho \in (0; 1)$ be defined in (14) w.r.t. K , and set $r := \max(\kappa, \rho)$, where κ is defined in Condition (3.2.a). Using (12) and (14), we abuse the notation $O(r^n)$ for $Q(u)^n - \lambda(u)^n \Pi(u)$ when $u \in B(0, \alpha)$, and for $Q(u)^n$ when $u \in K$. So we have:

$$\forall u \in B(0, \beta), \quad Q(u)^n = \mathbf{1}_{B(0, \alpha)}(u) \lambda(u)^n \Pi(u) + O(r^n) \quad \text{in } \mathcal{L}(\mathcal{B}), \quad (28)$$

where $\Pi(\cdot)$ is the $\mathcal{L}(\mathcal{B})$ -valued bounded function of Condition (3.2.a). Recall that, by hypothesis, $f \in \mathcal{B}$, $\mu \in \mathcal{B}'$. Since \hat{h} is integrable, we then deduce from Lemma 3.9 and (28) that

$$\begin{aligned} (2\pi)^2 \mathbb{E}_{(\mu, 0)}[f(X_n) h(S_n)] &= \int_{B(0, \alpha)} P_h(u) \lambda(u)^n \mu(\Pi(u) f) du + O(r^n) \\ &= \frac{1}{A_n^2} \int_{B(0, \alpha A_n)} P_h\left(\frac{u}{A_n}\right) \lambda\left(\frac{u}{A_n}\right)^n \mu\left(\Pi\left(\frac{u}{A_n}\right) f\right) du + O(r^n). \end{aligned}$$

Next, from (13), $A_n^2 \sim nL(A_n)$ and the fact that L is slowly varying, it can be easily seen that $\lim_n \lambda\left(\frac{u}{A_n}\right)^n = e^{-\langle u, \Gamma u \rangle / 2}$. Besides we know by (17) that $\lim_n \mu(\Pi(u/A_n) f) = \pi(f)$. By using (24), Lebesgue's theorem gives:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{B(0, \alpha A_n)} P_h\left(\frac{u}{A_n}\right) \lambda\left(\frac{u}{A_n}\right)^n \mu\left(\Pi\left(\frac{u}{A_n}\right) f\right) du &= \pi(f) P_h(0) \int_{\mathbb{R}^2} e^{-\langle u, \Gamma u \rangle / 2} du \\ &= 2\pi(\det \Gamma)^{-1/2} \pi(f) P_h(0). \end{aligned}$$

Finally, the Poisson summation formula yields $P_h(0) := \sum_{g \in \mathbb{S}^*} \hat{h}(g) = c_{\mathbb{S}} m_{\mathbb{S}}(h)$. \square

3.3.3. Proof of Proposition 3.4. Note that (19) (with $v = 0$) implies that (17) holds with $f = \mathbf{1}_{\mathbb{X}}$. So we have (18) with $f = \mathbf{1}_{\mathbb{X}}$, which is nothing else but (20a). Now, let us prove (20b). Let us first state a lemma concerning the sequence $(S_n, S_{n+m} - S_n)_{n,m}$.

Lemma 3.11. *The sequence $(\nu_{n,m})_{n,m}$ of positive measures on \mathbb{R}^4 defined by*

$$\forall C \in \mathcal{B}(\mathbb{R}^4), \quad \nu_{n,m}(C) := (2\pi)^2 A_n^2 A_m^2 \mathbb{E}_{(\mu, 0)}[\mathbf{1}_C(S_n, S_{n+m} - S_n)],$$

converges weakly, when $\min(n, m) \rightarrow +\infty$, to the measure: $\nu(C) := c_{\mathbb{S}}^2 (\det(\Gamma))^{-1} m_{\mathbb{S}} \otimes m_{\mathbb{S}}(C)$.

Let us admit for the moment this lemma and prove (20b). Let T be the linear (invertible) endomorphism on \mathbb{R}^4 defined by: $Tw := (u, u + v)$, where we write $w = (u, v) \in \mathbb{R}^4$, with u and v in \mathbb{R}^2 . From Lemma 3.11, the measures $\tilde{\nu}_{n,m}$ on \mathbb{R}^4 defined by

$$\tilde{\nu}_{n,m}(C) := \nu_{n,m}(\mathbf{1}_C \circ T) = (2\pi)^2 A_n^2 A_m^2 \mathbb{P}_{(\mu, 0)}((S_n, S_{n+m}) \in C)$$

converges weakly to $\tilde{\nu}(C) := \nu(\mathbf{1}_C \circ T)$ when $\min(n, m) \rightarrow +\infty$. But, from Fubini's theorem and since $m_{\mathbb{S}}$ is the Haar measure, we have $\tilde{\nu} = \nu$. Now set $B = B(s, \varepsilon)$ for $(s, \varepsilon) \in \mathbb{S} \times (0; \varepsilon_{\mathbb{S}})$. Since the boundary of $B \times B$ has zero ν -measure, we obtain the following convergence when $\min(n, m) \rightarrow +\infty$: $\lim \tilde{\nu}_{n,m}(B \times B) = \nu(B \times B) = c_{\mathbb{S}}^2 (\det(\Gamma))^{-1} m_{\mathbb{S}}(B)^2$, which is (20b).

Proof of Lemma 3.11. Recall that there exists a continuous Lebesgue-integrable function $h > 0$ on \mathbb{R}^2 having a compactly supported Fourier transform. Define the following function on \mathbb{R}^4 : $G(w) := h(u)h(v)$, where $w = (u, v) \in \mathbb{R}^4$, with u and v in \mathbb{R}^2 . Then the Fourier transform of G is compactly supported on \mathbb{R}^4 , and we have $G(w) e^{i\langle w, c \rangle} = h(u) e^{i\langle u, a \rangle} h(v) e^{i\langle v, b \rangle}$ for any $c = (a, b) \in \mathbb{R}^4$, with a and b in \mathbb{R}^2 . Therefore, using again the classical properties on convergence of positive measures [4], Lemma 3.11 will be established provided that we prove the following: $\forall (h_1, h_2) \in \mathcal{H}_2 \times \mathcal{H}_2$,

$$\lim_{n \rightarrow +\infty} (2\pi)^2 A_n^2 A_m^2 \mathbb{E}_{(\mu, 0)} [h_1(S_n) h_2(S_{n+m} - S_n)] = c_{\mathbb{S}}^2 (\det(\Gamma))^{-1} m_{\mathbb{S}}(h_1) m_{\mathbb{S}}(h_2). \quad (29)$$

Let $h_1, h_2 \in \mathcal{H}_2$ be fixed, and let $\beta > 0$ be such that both $\text{Supp}(\hat{h}_1)$ and $\text{Supp}(\hat{h}_2)$ are contained in $B(0, \beta)$. The real numbers α, κ in (12), and ρ in (14), are chosen as in the previous proof, and again we set $r := \max(\kappa, \rho)$. We get by using (28): $\forall (u, v) \in B(0, \beta)^2$,

$$Q(u)^n Q(v)^m = \mathbf{1}_{B(0, \alpha)}(u) \mathbf{1}_{B(0, \alpha)}(v) \lambda(u)^n \lambda(v)^m \Pi(u) \Pi(v) + O(r^{\min(n, m)}) \quad \text{in } \mathcal{L}(\mathcal{B}).$$

Using the second formula of Lemma 3.9 (with $f = \mathbf{1}_{\mathbb{X}}$) and Assumption (19), the arguments used to prove (26) can be easily extended to prove (29). \square

4. REDUCTION OF HYPOTHESIS 3.2. A FIRST APPLICATION TO THE ρ -MIXING CASE

In this section, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ denotes a \mathbb{C} -Banach space composed of π -integrable \mathbb{C} -valued functions on \mathbb{X} (or of classes of such functions) satisfying Hypothesis 3.1.

4.1. Reduction of Condition (3.2.a). Recall that π is a stationary distribution of the driving Markov chain $(X_n)_{n \in \mathbb{N}}$ of the MRW. We denote by $(\mathbb{L}^1(\pi), \|\cdot\|_{\mathbb{L}^1(\pi)})$ the usual associated Lebesgue space ($\|f\|_{\mathbb{L}^1(\pi)} = \pi(|f|)$), and by $\|\cdot\|_{\mathcal{B}, \mathbb{L}^1(\pi)}$ the operator norm of bounded linear operators from \mathcal{B} to $\mathbb{L}^1(\pi)$. The one-dimensional projection Π associated with π is defined in (10).

Hypothesis 4.1. *We have the following properties:*

(4.1.a) *Q is strongly ergodic on \mathcal{B} , namely (see also (15)):*

$$\lim_n \|Q^n - \Pi\|_{\mathcal{B}} = 0,$$

(4.1.b) *There exists $\alpha \in (0; +\infty)$ such that, for every $t \in B(0, \alpha)$, we have $Q(t) \in \mathcal{L}(\mathcal{B})$ and:*

$$\forall t \in B(0, \alpha), \lim_{h \rightarrow 0} \|Q(t+h) - Q(t)\|_{\mathcal{B}, \mathbb{L}^1(\pi)} = 0, \quad (30)$$

(4.1.c) *There exist $\kappa_1 \in [0, 1)$ and $C \in (0; +\infty)$ such that*

$$\forall n \geq 1, \forall t \in B(0, \alpha), \forall f \in \mathcal{B}, \|Q(t)^n f\|_{\mathcal{B}} \leq C \kappa_1^n \|f\|_{\mathcal{B}} + C \|f\|_{\mathbb{L}^1(\pi)}.$$

The next statement follows from the perturbation theorem due to G. Keller and C. Liverani [20].

Proposition 4.2. *Hypothesis 4.1 implies Condition (3.2.a).*

Also notice that, under Hypothesis 4.1, we have for every $f \in \mathcal{B}$: $\lim_{t \rightarrow 0} \pi(\Pi(t)f) = \pi(f)$, which is Condition (17) with $\mu = \pi$ (this condition is used in Corollary 3.5). Proposition 4.2 and the previous property still hold if (30) is replaced with: $\lim_{h \rightarrow 0} \|Q(h) - Q\|_{\mathcal{B}, \mathbb{L}^1(\pi)} = 0$. However (30) gives the following additional property (used in Section 4.2): the function $\lambda(\cdot)$ of (12) is continuous on $B(0, \alpha)$ (see [14]). Anyway, in our examples, Condition (30) is fulfilled. Further comments on Hypothesis 4.1 are presented in [15, Sect. 4].

4.2. Reduction of (13) in Condition (3.2.b) (to a CLT). Condition (13) plays a central role in the proof of the key Propositions 3.3 and 3.4. In this subsection, we prove that this condition is closely related to the fact that S_n (properly normalized) converges in distribution to a non-degenerate Gaussian law on \mathbb{R}^2 . More precisely let us introduce the following assumption.

Hypothesis 4.3. *Under $\mathbb{P}_{(\pi,0)}$ (stationary case), the sequence $(S_n/A_n)_n$ converges in distribution to a Gaussian distribution $\mathcal{N}(0, \Gamma)$, with some definite positive symmetric 2×2 -matrix Γ and with some positive real numbers A_n such that $A_n^2 \sim nL(A_n)$ where $L : (0; +\infty) \rightarrow (0; +\infty)$ is a slowly varying function at infinity in the sense of Karamata.*

The standard case in Hypothesis 4.3 is $A_n = \sqrt{n}$, but as in the i.i.d. case, examples with A_n defined by non constant slowly varying function $L(\cdot)$ can be met in classic Markov models or dynamical system (see Section 5).

Proposition 4.4. *Assume that Hypothesis 4.1 is fulfilled. Then Condition (3.2.b) is equivalent to Hypothesis 4.3 (with the same covariance matrix Γ and the same slowly varying function $L(\cdot)$).*

Proof. This result is established in [14, lem. 4.2] in dimension $d = 1$ and with $A_n = \sqrt{n}$ (i.e. in the case of the standard one-dimensional CLT). The proof is similar under Hypothesis 4.3, we just outline the main arguments. Without any loss of generality, we suppose that Γ is the identity matrix. First observe that we have by Lemma 3.8 (applied with $f = \mathbf{1}_{\mathbb{X}}$ and $m = 0$):

$$\forall u \in B(0, \alpha), \quad \mathbb{E}_{(\pi,0)}[e^{i\langle u, S_n \rangle}] = \pi(Q(u)^n \mathbf{1}_{\mathbb{X}}). \quad (31)$$

The proof of the part "if" in Proposition 4.4 is easy: indeed, assume that (13) holds, and let $(A_n)_n$ be a sequence of positive real numbers such that $A_n^2 \sim nL(A_n)$. From (31) and (12), we obtain for any fixed $t \in \mathbb{R}^2$ and for n sufficiently large

$$\mathbb{E}_{(\pi,0)}[e^{i\langle t, S_n/A_n \rangle}] = \lambda(t/A_n)^n \pi(\Pi(t/A_n) \mathbf{1}_{\mathbb{X}}) + O(\kappa^n),$$

and by using (13), $A_n^2 \sim nL(A_n)$ and the fact that L is slowly varying, one can easily see that $\lim_n \lambda(t/A_n)^n = e^{-|t|^2/2}$. Hence the desired CLT in Hypothesis 4.3 holds true.

Conversely, assume that Hypothesis 4.3 is fulfilled. We want to prove that the function $\lambda(\cdot)$ in (12) satisfies the following property:

$$\psi(u) := \frac{\lambda(u) - 1}{|u|^2 L(|u|^{-1})} + 1/2 \rightarrow 0 \quad \text{when } u \rightarrow 0.$$

From Levy's theorem, we have: $\forall t \in B(0, \alpha)$, $\lim_n \mathbb{E}_{(\pi,0)}[e^{i\langle t, S_n/A_n \rangle}] = \exp(-|t|^2/2)$. Thus, by using (31), (12) and the complex logarithm function $\log(\cdot)$, this gives $\lim_n n \log \lambda(t/A_n) = -|t|^2/2$, from which we easily deduce: $\lim_n \psi(t/A_n) = 0$ (use the properties $\log(z) \sim (z - 1)$ when $z \rightarrow 1$, $n \sim A_n^2/L(A_n)$ and $L(A_n) \sim L(A_n/|t|)$). More precisely, by using the classic refinement of Levy's theorem in terms of uniform convergence on compact sets and the fact that the property $\lim_{x \rightarrow +\infty} \frac{L(kx)}{L(x)} = 1$ is uniform in k on each compact subset of $(0; +\infty)$ (according to formula (25)), one can see that the limit $\lim_n \psi(t/A_n) = 0$ is uniform on $C := \{t \in \mathbb{R}^2 : \alpha/2 \leq |t| < \alpha\}$ (see [14, lem. 4.2] for details). So, given $\varepsilon > 0$, one can choose $N_0 = N_0(\varepsilon)$ such that: $n \geq N_0, t \in C \Rightarrow |\psi(t/A_n)| < \varepsilon$. Next, since $\lim_n A_{n+1}/A_n = 1$, one can suppose that N_0 is such that: $\forall n \geq N_0, 1/(2A_n) < 1/A_{n+1}$. From that, we easily deduce that $\cup_{n \geq N_0} C/A_n = \{t \in \mathbb{R}^2 : 0 < |u| < \alpha/A_{N_0}\}$. Therefore we have: $0 < |u| < \alpha/A_{N_0} \Rightarrow |\psi(u)| < \varepsilon$. \square

Remark 4.5. *The expansion (13) in Hypothesis 3.2 may be adapted to cover the convergence in distribution of the (properly normalized) second component S_n of the MRW to a stable distribution of index $0 < p < 2$. In this case, Propositions 3.3 and 3.4 extends (with A_n such that $A_n^p \sim$*

$nL(A_n)$), but we have $\sum_{n \geq 1} A_n^{-2} < \infty$.² Thus: $\sum_{n \geq 1} \mathbb{P}_{(\mu,0)}(|S_n - s| < \varepsilon) < \infty$. This gives the expected transience property.

4.3. Reduction of Condition (3.2.c). Without any loss of generality, we suppose that the MRW $(X_n, S_n)_{n \in \mathbb{N}}$ is the canonical version defined on $\Omega = (\mathbb{X} \times \mathbb{R}^2)^{\mathbb{N}}$. We recall that, in the sense given in (2), S_1 takes its values in a closed two-dimensional subgroup \mathbb{S} of \mathbb{R}^2 : this corresponds to the cases (H1) (H2) (H3) described at the beginning of Section 3. We want to give natural hypotheses ensuring that Condition (3.2.c) holds true. More precisely, we want to make a link between Condition (3.2.c) and the following conditions.

Definition 4.6. *Under Hypothesis (2) we shall say that:*

- $(S_n)_n$ is **arithmetic** in \mathbb{S} w.r.t. \mathcal{B} if there exist $t \in \mathbb{R}^2 \setminus \mathbb{S}^*$, $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and $w \in \mathcal{B}$ such that, for π -almost every $x \in \mathbb{X}$, we have $|w(x)| = 1$ and the following property:

$$\forall n \geq 1, \quad e^{i\langle t, S_n \rangle} w(X_n) = \lambda^n w(x) \quad \mathbb{P}_{(x,0)} - a.s.. \quad (32)$$

- $(S_n)_n$ is **sub-lattice** in \mathbb{S} if there exist a closed proper subgroup \mathbb{S}_0 of \mathbb{S} , a bounded measurable function $\chi : \mathbb{X} \rightarrow \mathbb{R}^2$ and a family $(\beta_t)_{t \in \mathbb{S}_0^*}$ of real numbers such that, for π -almost every $x \in \mathbb{X}$, we have

$$\forall t \in \mathbb{S}_0^*, \forall n \geq 1, \quad \langle t, S_n + \chi(X_n) - \chi(x) \rangle \in n\beta_t + 2\pi\mathbb{Z} \quad \mathbb{P}_{(x,0)} - a.s.. \quad (33)$$

Remark 1. *Using a simple argument of class modulo \mathbb{S}_0 , Property (33) is equivalent to the existence of a sequence $(a_n)_n$ of vectors in \mathbb{R}^2 and of a bounded measurable function $\chi : \mathbb{X} \rightarrow \mathbb{R}$ such that:*

$$\forall t \in \mathbb{S}_0^*, \exists \beta_t \in \mathbb{R}, \forall n \geq 1, \quad \langle t, a_n \rangle \in n\beta_t + 2\pi\mathbb{Z}. \quad (34)$$

and, for π -almost every $x \in \mathbb{X}$, we have

$$\forall n \geq 1, \quad S_n + \chi(X_n) - \chi(x) \in a_n + \mathbb{S}_0 \quad \mathbb{P}_{(x,0)} - a.s.. \quad (35)$$

During the paper, we will be interested in establishing the non-sub-lattice condition. To this end, it will be enough to prove the following simpler condition : there exists no $(a_1, \mathbb{S}_0, \chi(\cdot))$ with $a_1 \in \mathbb{R}^2$, \mathbb{S}_0 a proper subgroup of \mathbb{S} and χ a bounded measurable function from \mathbb{X} to \mathbb{R} satisfying, for π -almost every $x \in \mathbb{X}$,

$$S_1 + \chi(X_1) - \chi(x) \in a_1 + \mathbb{S}_0 \quad \mathbb{P}_{(x,0)} - a.s..$$

Actually, in some cases (such as additive functionals, or general MRW with \mathcal{B} composed of classes of functions modulo π), the last condition is equivalent to the non-sub-lattice condition.

Now we are going to make a link between Condition (3.2.c), non-arithmeticity and the non-sub-lattice condition under the following functional assumptions.

Hypothesis 4.7. *Q is strongly ergodic on \mathcal{B} (see Cond. (4.1.a)), we have $Q(t) \in \mathcal{L}(\mathcal{B})$ for every $t \in \mathbb{R}^2$, and we have the following property:*

$$(4.7.a) \quad \forall t \in \mathbb{R}^2, \quad \lim_{h \rightarrow 0} \|Q(t+h) - Q(t)\|_{\mathcal{B}, \mathbb{L}^1(\pi)} = 0,$$

(4.7.b) *For every compact K in \mathbb{R}^2 , there exist $\kappa \in (0; 1)$, $C \in (0; +\infty)$ such that, for every $t \in K$, the essential spectral radius of $Q(t)$ satisfies $r_{ess}(Q(t)) \leq \kappa$, and for every $f \in \mathcal{B}$ and every integer $n \geq 1$*

$$\|Q(t)^n f\|_{\mathcal{B}} \leq C\kappa^n \|f\|_{\mathcal{B}} + C\|f\|_{\mathbb{L}^1(\pi)},$$

²For n is large enough, we have $A_n^{p/2-1} \leq L(A_n)$ and $nL(A_n) \leq 2A_n^p$ and so $A_n^{-2} \leq (2/n)^{4/(p+2)}$.

(4.7.c) For every $\lambda \in \mathbb{C}$ such that $|\lambda| \geq 1$ and for every nonzero element $f \in \mathcal{B}$, we have

$$[\exists n_0, \forall n \geq n_0, |\lambda|^n |f| \leq Q^n |f|] \Rightarrow [|\lambda| = 1 \text{ and } |f| \leq \pi(|f|)],$$

where the previous inequality holds true, either everywhere on \mathbb{X} if \mathcal{B} is a space of functions, or π -almost everywhere on \mathbb{X} if \mathcal{B} is a space of classes modulo π .

Note that Hypothesis 4.7 implies Hypothesis 4.1, and so Condition (3.2.a). Further comments on Hypothesis 4.7 can be found in [15, section 5] (in particular Condition (4.7.c) is fulfilled in general when Q is strongly ergodic on \mathcal{B}). Under Hypothesis 4.7, we consider the set

$$G := \{t \in \mathbb{R}^2 : r(Q(t)) = 1\}.$$

Recall that the dual subgroup \mathbb{S}^* of \mathbb{S} is defined in (4). Since $Q(0) = Q$, $r(Q) = 1$ and $Q(\cdot)$ is \mathbb{S}^* -periodic, \mathbb{S}^* is contained in G . The following result is an adaptation of [15, section 12]. We give the scheme of its proof in appendix.

Proposition 4.8. *Under Hypothesis 4.7, we have the following:*

- (i) Condition (3.2.c) $\Leftrightarrow G = \mathbb{S}^* \Leftrightarrow (S_n)_n$ is not arithmetic in \mathbb{S} w.r.t. \mathcal{B} ;
- (ii) If $(S_n)_n$ satisfies Hypothesis 4.3 and is not sub-lattice in \mathbb{S} , then Condition (3.2.c) holds true;
- (iii) If $(S_n)_n$ is sub-lattice in \mathbb{S} and the function $\chi(\cdot)$ in (33) is such that, for every $t \in \mathbb{R}^2$, we have $e^{i\langle t, \chi(\cdot) \rangle} \in \mathcal{B}$, then Condition (3.2.c) does not hold.

4.4. Application: the ρ -mixing case. We assume here that $(X_n)_n$ is ρ -mixing and that, under $\mathbb{P}_{(\pi,0)}$, S_1 is centered square integrable. The ρ -mixing assumption is equivalent to the fact that Q has a spectral gap on $\mathbb{L}^2(\pi)$, see [25]: $\lim_{n \rightarrow +\infty} \|Q^n - \Pi\|_{\mathbb{L}^2(\pi)} = 0$. Next, according to [11, Th. 1], the previous assumption on S_1 implies that, under $\mathbb{P}_{(\pi,0)}$, $(S_n/\sqrt{n})_n$ converges in distribution to a Gaussian distribution $\mathcal{N}(0, \Gamma)$. We suppose in addition that Γ is definite positive.

Theorem 4.9. *Under the previous assumptions, if $(S_n)_n$ is not sub-lattice in \mathbb{S} , then we have $\mathcal{R}_{(\pi,0)} = \mathbb{S}$. If in addition $(X_n)_{n \in \mathbb{N}}$ is Harris recurrent, then we have $\mathcal{R}_{(\mu,0)} = \mathbb{S}$ for any initial distribution μ .*

Proof. By an easy adaptation of [15, Sec. 5.3] (see also [11, Prop. 3]), Hypothesis 4.7 (and so Hypothesis (4.1)) is fulfilled on $\mathcal{B} = \mathbb{L}^2(\pi)$. Hence we have Condition (3.2.a). Note that, since $(S_n/\sqrt{n})_n$ converges in distribution to $\mathcal{N}(0, \Gamma)$, we have $A_n = \sqrt{n}$ in (16), so $\sum_{n \geq 1} A_n^{-2} = \infty$. From Proposition 4.8, Condition (3.2.c) is equivalent to the fact that $(S_n)_n$ is not sub-lattice in \mathbb{S} . Finally, since $\mu = \pi$ satisfies (17) (see Subsection 4.1) and $\mathbb{L}^2(\pi)$ contains all the (classes of) bounded real-valued measurable functions on \mathbb{X} , we conclude by using Corollary 3.5. \square

The convergence to stable laws of additive functionals associated with ρ -mixing Markov chains is investigated in [17], unfortunately the non-standard CLT is not studied in [17].

5. APPLICATION TO ADDITIVE FUNCTIONALS

Let $(X_n)_{n \in \mathbb{N}}$ be a Markov chain with state space \mathbb{X} , transition probability Q , invariant distribution π , and initial distribution μ . Here, given $\xi = (\xi_1, \xi_2) : \mathbb{X} \rightarrow \mathbb{R}^2$ a π -centered function (i.e. for $i = 1, 2$, ξ_i is π -integrable and $\pi(\xi_i) = 0$), we consider the classical MRW $(X_n, S_n)_{n \in \mathbb{N}}$ defined by $S_0 = 0$ and $\forall n \geq 1$:

$$S_n := \sum_{k=1}^n \xi(X_k). \quad (36)$$

The sequence $(S_n)_n$ is called an additive functional of $(X_n)_n$. Throughout the section we assume that the following hypothesis holds true.

Hypothesis 5.1. *We have the following properties:*

- (5.1.a) *The function ξ (so S_n) takes its values in a closed two-dimensional subgroup \mathbb{S} of \mathbb{R}^2 .*
(5.1.b) *$(S_n)_n$ is not sub-lattice in \mathbb{S} .*
(5.1.c) *There exists a function $L : (0; +\infty) \rightarrow (0; +\infty)$ slowly varying at infinity and positive real numbers $A_n, n \geq 1$, satisfying $A_n^2 \sim nL(A_n)$ and $\sum_{n \geq 1} A_n^{-2} = \infty$, such that, under \mathbb{P}_π , $(S_n/A_n)_n$ converges in distribution to a non-degenerate Gaussian distribution $\mathcal{N}(0, \Gamma)$.*

The closed two-dimensional subgroups of \mathbb{R}^2 are described in (H1) (H2) (H3) at the beginning of Section 3. For additive functionals the non-sublattice condition in \mathbb{S} can be reduced to the next condition on ξ , see [15]³. Recall that a set $A \in \mathcal{X}$ is said to be π -full if $\pi(A) = 1$, and Q -absorbing if $Q(z, A) = 1$ for all $z \in A$.

$(S_n)_n$ is not sub-lattice in \mathbb{S} if and only if there exist no $a_0 \in \mathbb{R}^2$, no proper closed subgroup \mathbb{S}_0 in \mathbb{S} , no π -full Q -absorbing set $A \in \mathcal{X}$, and finally no bounded measurable function $\chi : \mathbb{X} \rightarrow \mathbb{R}^2$ such that

$$\forall x \in A, \xi(y) + \chi(y) - \chi(x) \in a_0 + \mathbb{S}_0 \quad Q(x, dy) - a.s..$$

Under the previous condition, we shall also say that ξ is not sub-lattice in \mathbb{S} . Condition (5.1.c) is nothing else but Hypothesis 4.3 with the additional condition $\sum_{n \geq 1} A_n^{-2} = \infty$. In each model below, we shall give precise conditions (and references) for the CLT of Condition (5.1.c) to be true.

If the standard CLT holds true, we have $A_n = \sqrt{n}$, and so $\sum_{n \geq 1} A_n^{-2} = \infty$. If in addition $(S_n)_n$ is not sub-lattice in \mathbb{S} , then the covariance matrix Γ of the CLT is automatically definite positive, e.g. see [15, Sec. 5.2].

Finally, since Hypothesis 4.1 is fulfilled in our examples, Condition (5.1.c) is equivalent to (13).

Example 1. The V -geometrically ergodic Markov chains.

Let $V : \mathbb{X} \rightarrow [1, +\infty)$ be some fixed unbounded function on \mathbb{X} . Let us assume that $\pi(V) < \infty$ and that $(X_n)_{n \geq 0}$ is V -geometrically ergodic, namely there exists $\kappa \in (0, 1)$ such that we have:

$$\sup_{|f| \leq V} \sup_{x \in E} \frac{|\mathbb{E}_x[f(X_n)] - \pi(f)|}{V(x)} = O(\kappa^n).$$

The V -geometrical ergodicity condition can be investigated with the help of the so-called drift conditions. For this fact, and for the classical examples of such models, we refer to [22].

Result 5.1. *Under Hypothesis 5.1, if $|\xi|^\alpha/V$ is bounded for some $\alpha > 0$, then we have $\mathcal{R}_{(\mu, 0)} = \mathbb{S}$ for every initial distribution μ on \mathbb{X} .*

Recall that the (standard) CLT in Condition (5.1.c) is fulfilled when $|\xi|/\sqrt{V}$ is bounded on \mathbb{X} , see [22] (note that the domination assumption on ξ in Result 5.1 is then fulfilled). Alternative conditions for the CLT can be found in [18]. To the best of our knowledge, the non-standard CLT has not been investigated for V -geometrically ergodic Markov chains.

Proof of Result 5.1. For $\theta \in (0, 1]$, let us denote by $(\mathcal{B}_\theta, \|\cdot\|_\theta)$ the space of the measurable \mathbb{C} -valued functions f on \mathbb{X} such that $\|f\|_\theta = \sup_{x \in E} |f(x)|/V(x)^\theta < \infty$. Note that the V -geometrical ergodicity condition is equivalent to the strong ergodicity of Q on \mathcal{B}_1 (see (4.1.a)). Set $\mathcal{B} = \mathcal{B}_{\theta_0}$ for some fixed $\theta_0 \in (0, 1)$. Hypothesis 3.1 is obvious. Using the results of Section 4, let us check that Hypothesis 3.2 holds true. From [15, Lem. 10.1], Hypothesis 4.7 (and so 4.1) is fulfilled on \mathcal{B} , hence we have Condition (3.2.a). From Proposition 4.4 Condition (3.2.b) holds.

³This reduction is stated in [15] in the case $\mathbb{S} = \mathbb{R}^2$, the extension to the cases (H2) (H3) is easy.

Next, Condition (3.2.c) is equivalent to the fact that $(S_n)_n$ is not sub-lattice in \mathbb{S} . Finally, since $|\xi|^\alpha/V$ is bounded, it follows from [15, lemma 10.4] that we have, for some $\eta > 0$

$$\|\Pi(u)\Pi(v)\mathbf{1}_{\mathbb{X}} - \mathbf{1}_{\mathbb{X}}\|_{\mathcal{B}_1} \leq C(|u|^\eta + |v|^\eta)\|\mathbf{1}_{\mathbb{X}}\|_{\mathcal{B}_{\theta_0}},$$

and since each Dirac distribution δ_x ($x \in \mathbb{X}$) is a bounded linear form on \mathcal{B}_1 , we have (19) for $\mu = \delta_x$ ($x \in \mathbb{X}$). Corollary 3.7 then gives the desired conclusion. \square

Example 2. The Lipschitz iterative models.

Here (\mathbb{X}, d) is a non-compact metric space in which every closed ball is compact, and it is endowed with its Borel σ -field \mathcal{X} . Let (V, \mathcal{V}) be a measurable space, let $(\vartheta_n)_{n \geq 1}$ be a i.i.d. sequence of random variables taking values in V , and let $F : \mathbb{X} \times V \rightarrow \mathbb{X}$ be a measurable function. Let X_0 be a \mathbb{X} -valued r.v. independent of $(\vartheta_n)_{n \geq 1}$. Then the random iterative model associated to $(\vartheta_n)_{n \geq 1}$, F and X_0 is defined by (see [8])

$$X_n = F(X_{n-1}, \vartheta_n), \quad n \geq 1.$$

Let us consider the two following random variables which are classical in these models [8] :

$$\mathcal{C} := \sup \left\{ \frac{d(F(x, \vartheta_1), F(y, \vartheta_1))}{d(x, y)}, x, y \in \mathbb{X}, x \neq y \right\} \quad \text{and} \quad \mathcal{M} = 1 + \mathcal{C} + d(F(x_0, \vartheta_1), x_0)$$

where x_0 is some fixed point in \mathbb{X} .

By proceeding as above with here the weighted-Hölder spaces defined in [15, Sect. 11], we can establish the following.

Result 5.2. *Assume that $\mathcal{C} < 1$ a.s., that $\mathbb{E}[\mathcal{M}^\delta] < \infty$ for some arbitrary small $\delta > 0$, and finally that ξ satisfies the following weighted-Lipschitz condition:*

$$\exists \mathcal{C} \geq 0, \exists b \geq 0, \forall (x, y) \in \mathbb{X}^2, \quad |\xi(x) - \xi(y)| \leq \mathcal{C} d(x, y) [1 + d(x, x_0) + d(y, x_0)]^b, \quad (37)$$

Then, under Hypothesis 5.1, we have $\mathcal{R}_{(\mu, 0)} = \mathbb{S}$ for every initial distribution μ on \mathbb{X} .

Proof. The weighted-Hölder spaces $\mathcal{B}_{\alpha, \beta, \gamma}$ defined in [15, Sect. 11] involve three parameters: $\alpha \in (0; 1]$ and $b + 1 \leq \beta \leq \gamma$, where b is given in (37). Since we have assumed $\mathcal{C} < 1$ a.s., the conditions in [15, Prop. 11.4] ensuring that Hypothesis 4.7 (and so 4.1) holds true on $\mathcal{B}_{\alpha, \beta, \gamma}$ are fulfilled if $\mathbb{E}[\mathcal{M}^\delta] < \infty$ for some $\delta > \alpha(\gamma + \beta)$. The same remark is relevant to prove (19) (use [15, Prop. 11.5]) and to establish that the non-sub-lattice assumption implies Condition (3.2.c) (use [15, Prop. 11.8]). Next, to apply Corollary 3.7, we only need the space $\mathcal{B}_{\alpha, \beta, \gamma}$ to contain $\mathbf{1}_{\mathbb{X}}$ and its dual to contain all the Dirac distribution δ_x ($x \in \mathbb{X}$). Since these two properties are true for every (α, β, γ) fixed as above, α can be chosen arbitrarily small so that $\delta > \alpha(\gamma + \beta)$. \square

A more precise use of the results in [15, Sect. 11] allows to obtain the same conclusion under some mean contractive conditions on \mathcal{C} (instead of $\mathcal{C} < 1$ a.s.). Also mention that, for Lipschitz iterative models, the non-sub-lattice assumption is just a sufficient condition for Condition (3.2.c) to hold true on the above mentioned weighted-Hölder spaces (because the condition on $\chi(\cdot)$ in Assertion (iii) of Proposition 4.8 is not automatically fulfilled). However we can use the non-arithmeticity condition of section 4.3 which is equivalent to Condition (3.2.c). Anyway this condition can be simplified in the special case of additional functionals (see [15, section 5]).

To check the (standard) CLT of Hypothesis 5.1, one can use for instance [8, 2, 31] and the references therein. One can also appeal to the statement [15, Prop. 11.3] which is stated under Condition (37): for instance, if $\mathcal{C} < 1$ and $\mathbb{E}[\mathcal{M}^{2(b+1)}] < \infty$, then the desired standard CLT holds true. Observe that the condition $\mathbb{E}[\mathcal{M}^\delta] < \infty$ of Result 5.2 is then automatically fulfilled (actually, it should be implied by any assumptions giving the CLT).

Example 3. Stochastic affine recursion with non-standard CLT.

Such models can be found in [5], see also [23]. Let us specify how our results apply in the context of [5]. Let $(B_n, A_n)_n$ be a sequence of i.i.d. random variables taking their values in $\mathbb{R}^2 \times \text{Sim}(\mathbb{R}^2)$, where $\text{Sim}(\mathbb{R}^2)$ is the similarity group of \mathbb{R}^2 , and having common distribution called ν . Let us consider the affine iterative model

$$X_n = A_n X_{n-1} + B_n.$$

We write $\bar{\nu}$ the projection of ν on $\text{Sim}(\mathbb{R}^2)$ and we denote by $G_{\bar{\nu}}$ the closed sub-group of $\text{Sim}(\mathbb{R}^2)$ generated by the support of $\bar{\nu}$. We recall that $\text{Sim}(\mathbb{R}^2) = \mathbb{R}_+^* \times O(\mathbb{R}^2)$, where $O(\mathbb{R}^2)$ is the orthogonal group. We also denote by $|\cdot|$ the matrix norm associated with the euclidean norm on \mathbb{R}^2 . We suppose that:

- There exists a unique stationary distribution π and its support is unbounded,
- No affine subspace of \mathbb{R}^2 is invariant by the support of ν ,
- We have $\mathbb{E}[|A_1|^2] = 1$, $\mathbb{E}[|A_1|^2 \log |A_1|] < \infty$ and $\mathbb{E}[|B_1|^2] < \infty$,
- The projection of $G_{\bar{\nu}}$ on \mathbb{R}_+^* is equal to \mathbb{R}_+^* .

Then the measure π admits an expectation in \mathbb{R}^2 , called m_0 and there exists a gaussian random variable Z such that, for every $x \in \mathbb{R}^2$, under the probability measure \mathbb{P}_x , the following sequence of random variables

$$\left(\frac{1}{\sqrt{n \log(n)}} \left(\sum_{k=1}^n X_k - nm_0 \right) \right)_{n \geq 1}$$

converges in distribution to Z ([5, Th. 1.5]).

Result 5.3. *If $x \mapsto x$ is not sub-lattice in \mathbb{R}^2 , then, for every $(x, s) \in \mathbb{R}^2 \times \mathbb{R}^2$, we have*

$$\mathbb{P}_x \left(\forall \varepsilon > 0, \left| \sum_{k=1}^n X_k - nm_0 - s \right| < \varepsilon, \text{ i.o.} \right) = 1.$$

Proof. Again we apply Corollary 3.7. Hypothesis 3.1 follows from [5, (3.1)], Hypothesis 4.7 (and so 4.1) follows from Lemmas 3.9 and 3.12 of [5], in particular Condition (3.2.a) is fulfilled. Next Condition (3.2.b) follows from [5, Prop 3.18]), and finally we have Condition (3.2.c) since $\xi(x) = x$ is not sub-lattice in \mathbb{R}^2 . \square

APPENDIX A. PROOF OF PROPOSITION 4.8

Such a result is established in [15, section 12] in the special case of additive functionals. In fact, [15, section 12] essentially investigates the case $G = \{0\}$. Here we give the adaptation to general MRWs and subsets G . First [15, Lem. 12.1] gives⁴

$$\forall t \in \mathbb{R}^2 \setminus G, \quad r(Q(t)) < 1. \quad (38)$$

Second, an easy adaptation of [15, Lem. 12.3] gives⁵:

For any compact subset K of $\mathbb{R}^2 \setminus G$, there exists $\rho = \rho(K) \in [0, 1)$ such that

$$\sup_{t \in K} \|Q(t)^n\|_{\mathcal{B}} = O(\rho^n). \quad (39)$$

Next, since \mathbb{S}^* is contained in G , the previous property yields the first equivalence in (i): indeed, if $G = \mathbb{S}^*$, then (39) obviously gives Condition (3.2.c). Conversely, if Condition (3.2.c) is true,

⁴This part is based on the functional Hypothesis 4.7, and it exactly borrows the arguments of [15, Sect. 12.1].

⁵Consider compact subsets K of $\mathbb{R}^2 \setminus G$ instead of compact subsets of $\mathbb{R}^2 \setminus \{0\}$ in the proof of [15, Lem. 12.3].

then for every $t \in \mathbb{R}^2 \setminus \mathbb{S}^*$ we have $r(Q(t)) < 1$, i.e. $t \in \mathbb{R}^2 \setminus G$. Therefore Condition (3.2.c) gives $G \subset \mathbb{S}^*$, hence $G = \mathbb{S}^*$.

In addition [15, Lem. 12.1] gives the following equivalence:

Property (A): we have $t \in G$ if and only if there exist $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and $w \in \mathcal{B}$, $w \neq 0$, such that we have the following equality: $Q(t)w = \lambda w$ in \mathcal{B} . Moreover the previous function $w(\cdot)$ is such that $|w| = \pi(|w|)$ π -a.s..

The fact that $Q(t)w = \lambda w$ implies $|w| = \pi(|w|)$ π -a.s. is easy to obtain. Indeed, we have $|w| = |\lambda^n w| = |Q(t)^n w| \leq Q^n |w|$ for every $n \geq 1$, thus we deduce from (4.7.c) that $|w| \leq \pi(|w|)$ π -a.s. So $g := \pi(|w|) - |w|$ is nonnegative, and $\pi(g) = 0$, hence $|w| = \pi(|w|)$ π -a.s..

Finally, from the previous property, we can deduce the following.

Property (B): we have $\mathbb{S}^ \neq G$ if and only if there exist $t \in \mathbb{R}^2 \setminus \mathbb{S}^*$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and $w \in \mathcal{B}$, $w \neq 0$, such that $|w| = 1$ π -a.s. and $Q(t)w = \lambda w$ in \mathcal{B} .*

To prove the second equivalence in (i), one needs the following.

Lemma A.1. *We have $\mathbb{S}^* \neq G$ if and only if there exist $t \in \mathbb{R}^2 \setminus \mathbb{S}^*$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and $w \in \mathcal{B}$, $w \neq 0$, such that for π -a.e. $x \in \mathbb{X}$ we have $|w(x)| = 1$ and*

$$\forall n \geq 1, \quad \mathbb{E}_{(x,0)}[e^{i\langle t, S_n \rangle} w(X_n)] = \lambda^n w(x). \quad (40)$$

Proof. Assume that $\mathbb{S}^* \neq G$, and let (t, λ, w) be as stated in Property (B). Then we have: $\forall n \geq 1$, $Q(t)^n w = \lambda^n w$ in \mathcal{B} . From (3.1.b) it follows that $Q(t)^n w = \lambda^n w$ in $\mathbb{L}^1(\pi)$, hence we have (40) for π -a.e. $x \in \mathbb{X}$ (use Lemma 3.8 with $m = 0$). Conversely, let $t \in \mathbb{R}^2 \setminus \mathbb{S}^*$ and (λ, w) as stated in Lemma A.1. Then we have for π -a.e. $x \in \mathbb{X}$: $\forall n \geq 1$, $Q(t)^n w(x) = \lambda^n w(x)$. This implies that $t \in G$. Indeed, if $t \notin G$, then by (38) we would have $r(Q(t)) < 1$, thus $\lim_n Q(t)^n w = 0$ in \mathcal{B} , and so in $\mathbb{L}^1(\pi)$: this would give the property: $w = 0$ π -a.s., which is impossible since by hypothesis $|w| = 1$ π -a.s.. \square

Using the facts that $\mathbb{P}_{(x,0)}$ is a probability measure and $|w| = 1$ π -a.s., the property stated in Lemma A.1 is equivalent to the arithmeticity of $(S_n)_n$ in \mathbb{S} w.r.t. \mathcal{B} , which proves the second equivalence in (i).

Now we prove Assertion (ii) of Proposition 4.8. Under Hypothesis 4.7, G is a closed subgroup of \mathbb{R}^2 , and under the additional Hypothesis 4.3, G is discrete, see [15, Prop. 12.4]. Observe that, since $\mathbb{S}^* \subset G$, we have $G^* \subset \mathbb{S}$. To prove Assertion (ii) of Proposition 4.8, one needs to use the following statement, which is an easy adaptation of the proof of [15, Prop. 12.4]:

Property (C): there exists a bounded measurable function $\chi : \mathbb{X} \rightarrow \mathbb{R}^2$ and a family $(\beta_t)_{t \in G}$ of real numbers such that, for π -almost every $x \in \mathbb{X}$, we have

$$\forall t \in G, \forall n \geq 1, \quad \langle t, S_n + \chi(X_n) - \chi(x) \rangle \in n\beta_t + 2\pi\mathbb{Z} \quad \mathbb{P}_{(x,0)} - a.s.. \quad (41)$$

(The fact that G is discrete plays an important role in Property (C) for obtaining the existence of the above function χ , which does not depend on t .)

Assume that Condition (3.2.c) is not fulfilled. Then from Assertion (i) of Proposition 4.8, \mathbb{S}^* is a proper subgroup of G . Hence G^* is a proper sub-group of \mathbb{S} . Consequently, from Property (C), $(S_n)_n$ is sub-lattice in \mathbb{S} . This proves (ii).

Finally we establish Assertion (iii) of Proposition 4.8. Suppose that $(S_n)_n$ is sub-lattice in \mathbb{S} , with \mathbb{S}_0 , $\chi(\cdot)$ and $(\beta_t)_{t \in \mathbb{S}_0^*}$ as indicated in Definition 4.6, and with the additional condition: $\forall t \in \mathbb{R}^2$, $e^{i\langle t, \chi(\cdot) \rangle} \in \mathcal{B}$. Since by hypothesis \mathbb{S}_0 is strictly contained in \mathbb{S} , there exists $t_0 \in \mathbb{S}_0^* \setminus \mathbb{S}^*$.

We deduce from (33) that, for π -almost every $x \in \mathbb{X}$, we have

$$\forall n \geq 1, \quad e^{i\langle t_0, S_n \rangle} e^{i\langle t_0, \chi(X_n) \rangle} = e^{in\beta_{t_0}} e^{i\langle t_0, \chi(x) \rangle} \quad \mathbb{P}_{(x,0)} - a.s..$$

So we obtain (32) with $\lambda := e^{i\beta_{t_0}}$ and $w := e^{i\langle t_0, \chi(\cdot) \rangle} \in \mathcal{B}$. Hence, from Assertion (i) of Proposition 4.8, Condition (3.2.c) does not hold. \square

Remark A.2. When $(S_n)_n$ is an additive functional (see (36)), Conditions (32) and (33) in Definition 4.6 may be stated only for $n = 1$ and specified in terms of conditions involving an absorbing set (in place of properties fulfilled π -a.s.), see [15] and Section 5. In the same way, for general MRW, if \mathcal{B} is composed of classes of functions modulo π (for instance $\mathcal{B} = \mathbb{L}^2(\pi)$), then one can prove that the equivalence in Lemma A.1 is valid when (40) is required only for $n = 1$. Indeed this condition says that we have for π -a.e. $x \in \mathbb{X}$: $Q(t)w(x) = \lambda w(x)$, so $Q(t)w = \lambda w$ in \mathcal{B} and the proof of Lemma A.1 can be then repeated. Consequently, under the previous condition on \mathcal{B} , the conditions (32) and (33) may be also stated only for $n = 1$. Of course, in both cases above, Condition (34) is not relevant.

REFERENCES

- [1] ALSMEYER G. *Recurrence theorems for Markov random walks*. Prob. Math. Statist. **21**, 123-134 (2001).
- [2] BENDA M. *A central limit theorem for contractive stochastic dynamical systems*. J. App. Prob. **35** (1998) 200-205.
- [3] BERBEE H.. *Recurrence and transience for random walks with stationary increments*. Z. Wahrscheinlichkeitstheorie verw. Gebiete, **56**, 531–536 (1981).
- [4] BREIMAN L. *Probability* Classic in Applied Mathematics, SIAM, 1993.
- [5] D. BURACZEWSKI, E. DAMEK, Y. GUIVARC'H. *Convergence to stable laws for a class of multidimensional stochastic recursions*. Accepted for the publication in Probability Theory and Related Fields (2009).
- [6] CONZE J-P. *Sur un critère de récurrence en dimension 2 pour les marches stationnaires, applications*. Ergod. Th. Dynam. Sys. **19**(5), 1233-1245 (1999).
- [7] DEKKING F.M.. *On transience and recurrence of generalized random walks*. Z. Wahrscheinlichkeitstheorie verw. Gebiete, **61**, p. 459-465 (1982).
- [8] DUFLO M. *Random Iterative Models*. Applications of Mathematics, Springer-Verlag Berlin Heidelberg (1997).
- [9] DURRETT R. *Probability: theory and examples*. Wadsworth Brooks (1991).
- [10] FELLER W., *An introduction to probability theory and its applications*, vol. 2, John Wiley and Sons, Inc., New York-London-Sydney, second edn. (1971).
- [11] FERRÉ D., HERVÉ L., LEDOUX J. *Limit theorems for stationary Markov processes with \mathbb{L}^2 -spectral gap*. Preprint (2010).
- [12] GUIVARC'H Y. *Application d'un théorème limite local à la transience et à la récurrence de marches aléatoires*. Lecture Notes in Math. Springer, 301-332 (1984).
- [13] HENNION H. *Transience de certaines chaînes semi-markoviennes*. Ann. Inst. Henri Poincaré, Vol. XVIII, No 3, 1982, p. 277-291.
- [14] HERVÉ L. *Théorème local pour chaînes de Markov de probabilité de transition quasi-compacte. Applications aux chaînes V -géométriquement ergodiques et aux modèles itératifs*. Ann. I. H. Poincaré - PR **41** (2005) 179-196.
- [15] Hervé L., Pène F. *The Nagaeu-Guivarc'h method via the Keller-Liverani theorem*. Bull. Soc. Math. France, **138** (2010) 415-489.
- [16] HÖGNAS G. *Markov random walks on groups*. Math. Scand. **58** (1986), 35–45.
- [17] JARA M., KOMOROWSKI T., OLLA S. *Limit theorems for additive functionals of a Markov chain*, to appear in Ann. Applied Probab (2009).
- [18] JONES G.L. *On the Markov chain central limit theorem*. Probability surveys, Vol. 1 (2004) 299-320.
- [19] KARAMATA J. *Sur un mode de croissance régulière. Théorèmes fondamentaux*. Bull. Soc. Math. France, **61** (1933), 55-62.
- [20] KELLER G., LIVERANI C. *Stability of the Spectrum for Transfer Operators*, Ann. Scuola Norm. Sup. Pisa. CI. Sci. **28** No 4 (1999) 141–152.
- [21] KOCHEN S. P., STONE C. J. *A note on the Borel-Cantelli Lemma*, Illinois J. Math. **8** (1964), 248–251.
- [22] S.P. MEYN AND R.L. TWEEDIE. *Markov chains and stochastic stability*. Springer Verlag, New York, Heidelberg, Berlin (1993).

- [23] MIREK M. *Heavy tail phenomenon and convergence to stable laws iterated Lipschitz maps*. arXiv:0907.2261v1
- [24] NEVEU J. *Bases mathématiques du calcul des probabilités* Masso et Cie editeurs (1964).
- [25] ROSENBLATT M. *Markov processes. Structure and asymptotic behavior*. Springer-Verlag. New York (1971).
- [26] SCHMIDT K. *On recurrence*. Z. Wahrsch. verw. Gebiete **68** (1984) 75-95.
- [27] SCHMIDT K. *On joint recurrence*. C. R. Acad. Sci. Paris, t. 327, Série I, p. 837-842 (1998).
- [28] SCHMIDT K. *Recurrence of cocycles and stationary random walks*. IMS Lecture Notes-Monograph Series, Dynamical and Stochastics, Vol. 48 (2006), 78-84.
- [29] SZÁSZ D.; VARJÚ T. *Local limit theorem for the Lorentz process and its recurrence in the plane*. Ergod. Th. Dynam. Sys. **24**, 257-278 (2004).
- [30] SZÁSZ D.; VARJÚ T. *Limit Laws and Recurrence for the Planar Lorentz Process with Infinite Horizon*. J. Statist. Phys. **129**, 59-80 (2007).
- [31] WU, WEI BIAO; SHAO, XIAOFENG. *Limit theorems for iterated random functions*. J. Appl. Probab. 41 (2004), no. 2, 425-436.

UNIVERSITÉ EUROPÉENNE DE BRETAGNE, I.R.M.A.R. (UMR-CNRS 6625), INSTITUT NATIONAL DES SCIENCES APPLIQUÉES DE RENNES, FRANCE

E-mail address: `Loic.Herve@insa-rennes.fr`

UNIVERSITÉ EUROPÉENNE DE BRETAGNE, UNIVERSITÉ DE BREST, LABORATOIRE DE MATHÉMATIQUES, UMR CNRS 6205, 29238 BREST CEDEX, FRANCE

E-mail address: `francoise.pene@univ-brest.fr`