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**Extremal functions in some interpolation inequalities:
Symmetry, symmetry breaking and estimates of the best constants**

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This contribution is devoted to a review of some recent results on existence, symmetry and symmetry breaking of optimal functions for Caffarelli-Kohn-Nirenberg (CKN) and weighted logarithmic Hardy (WLH) inequalities. These results have been obtained in a series of papers¹⁻⁵ in collaboration with M. del Pino, S. Filippas, M. Loss, G. Tarantello and A. Tertikas and are presented from a new viewpoint.

Keywords: Caffarelli-Kohn-Nirenberg inequality; Gagliardo-Nirenberg inequality; logarithmic Hardy inequality; logarithmic Sobolev inequality; extremal functions; radial symmetry; symmetry breaking; Emden-Fowler transformation; linearization; existence; compactness; optimal constants

1. Two families of interpolation inequalities

Let $d \in \mathbb{N}^*$, $\theta \in [0, 1]$, consider the set \mathcal{D} of all smooth functions which are compactly supported in $\mathbb{R}^d \setminus \{0\}$ and define $\vartheta(d, p) := d \frac{p-2}{2p}$, $a_c := \frac{d-2}{2}$, $\Lambda(a) := (a - a_c)^2$ and $p(a, b) := \frac{2d}{d-2+2(b-a)}$. We shall also set $2^* := \frac{2d}{d-2}$ if $d \geq 3$ and $2^* := \infty$ if $d = 1$ or 2 . For any $a < a_c$, we consider the two families of interpolation inequalities:

(CKN) *Caffarelli-Kohn-Nirenberg inequalities*^{3,4,6} – Let $b \in (a + 1/2, a + 1]$ and $\theta \in (1/2, 1]$ if $d = 1$, $b \in (a, a + 1]$ if $d = 2$ and $b \in [a, a + 1]$ if $d \geq 3$. Assume that $p = p(a, b)$, and $\theta \in [\vartheta(d, p), 1]$ if $d \geq 2$. There exists a finite positive constant $C_{\text{CKN}}(\theta, p, a)$ such that, for any $u \in \mathcal{D}$,

$$\| |x|^{-b} u \|_{\mathbf{L}^p(\mathbb{R}^d)}^2 \leq C_{\text{CKN}}(\theta, p, a) \| |x|^{-a} \nabla u \|_{\mathbf{L}^2(\mathbb{R}^d)}^{2\theta} \| |x|^{-(a+1)} u \|_{\mathbf{L}^2(\mathbb{R}^d)}^{2(1-\theta)}.$$

(WLH) *Weighted logarithmic Hardy inequalities*^{3,4} – Let $\gamma \geq d/4$ and $\gamma > 1/2$ if $d = 2$. There exists a positive constant $C_{\text{WLH}}(\gamma, a)$ such that, for any $u \in \mathcal{D}$, normalized by $\| |x|^{-(a+1)} u \|_{\mathbf{L}^2(\mathbb{R}^d)} = 1$,

$$\int_{\mathbb{R}^d} \frac{|u|^2 \log(|x|^{d-2-2a} |u|^2)}{|x|^{2(a+1)}} dx \leq 2\gamma \log \left[C_{\text{WLH}}(\gamma, a) \| |x|^{-a} \nabla u \|_{\mathbf{L}^2(\mathbb{R}^d)}^2 \right].$$

(WLH) appears as a limiting case^{3,4} of (CKN) with $\theta = \gamma(p - 2)$ as $p \rightarrow 2_+$. By a standard completion argument, these inequalities can be extended to the set

$\mathcal{D}_a^{1,2}(\mathbb{R}^d) := \{u \in \mathbf{L}_{\text{loc}}^1(\mathbb{R}^d) : |x|^{-a} \nabla u \in \mathbf{L}^2(\mathbb{R}^d) \text{ and } |x|^{-(a+1)} u \in \mathbf{L}^2(\mathbb{R}^d)\}$. We shall assume that all constants in the inequalities are taken with their optimal values. For brevity, we shall call *extremals* the functions which realize equality in (CKN) or in (WLH).

Let $\mathbf{C}_{\text{CKN}}^*(\theta, p, a)$ and $\mathbf{C}_{\text{WLH}}^*(\gamma, a)$ denote the optimal constants when admissible functions are restricted to the radial ones. *Radial extremals* are explicit and the values of the constants, $\mathbf{C}_{\text{CKN}}^*(\theta, p, a)$ and $\mathbf{C}_{\text{WLH}}^*(\gamma, a)$, are known.³ Moreover, we have

$$\begin{aligned} \mathbf{C}_{\text{CKN}}(\theta, p, a) &\geq \mathbf{C}_{\text{CKN}}^*(\theta, p, a) = \mathbf{C}_{\text{CKN}}^*(\theta, p, a_c - 1) \Lambda(a)^{\frac{p-2}{2p} - \theta}, \\ \mathbf{C}_{\text{WLH}}(\gamma, a) &\geq \mathbf{C}_{\text{WLH}}^*(\gamma, a) = \mathbf{C}_{\text{WLH}}^*(\gamma, a_c - 1) \Lambda(a)^{-1 + \frac{1}{4\gamma}}. \end{aligned} \quad (1)$$

Radial symmetry for the extremals of (CKN) and (WLH) implies that $\mathbf{C}_{\text{CKN}}(\theta, p, a) = \mathbf{C}_{\text{CKN}}^*(\theta, p, a)$ and $\mathbf{C}_{\text{WLH}}(\gamma, a) = \mathbf{C}_{\text{WLH}}^*(\gamma, a)$, while *symmetry breaking* only means that inequalities in (1) are strict.

2. Existence of extremals

Theorem 2.1. *Equality⁴ in (CKN) is attained for any $p \in (2, 2^*)$ and $\theta \in (\vartheta(p, d), 1)$ or $\theta = \vartheta(p, d)$ and $a \in (a_\star^{\text{CKN}}, a_c)$, for some $a_\star^{\text{CKN}} < a_c$. It is not attained if $p = 2$, or $a < 0$, $p = 2^*$, $\theta = 1$ and $d \geq 3$, or $d = 1$ and $\theta = \vartheta(p, 1)$.*

Equality⁴ in (WLH) is attained if $\gamma \geq 1/4$ and $d = 1$, or $\gamma > 1/2$ if $d = 2$, or for $d \geq 3$ and either $\gamma > d/4$ or $\gamma = d/4$ and $a \in (a_\star^{\text{WLH}}, a_c)$, where $a_\star^{\text{WLH}} := a_c - \sqrt{\Lambda_\star^{\text{WLH}}}$ and $\Lambda_\star^{\text{WLH}} := (d-1)e(2^{d+1}\pi)^{-1/(d-1)}\Gamma(d/2)^{2/(d-1)}$.

Let us give some hints on how to prove such a result. Consider first Gross' logarithmic Sobolev inequality in Weissler's form⁷

$$\int_{\mathbb{R}^d} |u|^2 \log |u|^2 dx \leq \frac{d}{2} \log \left(\mathbf{C}_{\text{LS}} \|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^d)}^2 \right) \quad \forall u \in \mathbf{H}^1(\mathbb{R}^d) \text{ s.t. } \|u\|_{\mathbf{L}^2(\mathbb{R}^d)} = 1.$$

The function $u(x) = (2\pi)^{-d/4} \exp(-|x|^2/4)$ is an extremal for such an inequality. By taking $u_n(x) := u(x + n\mathbf{e})$ for some $\mathbf{e} \in \mathbb{S}^{d-1}$ and any $n \in \mathbb{N}$ as test functions for (WLH), and letting $n \rightarrow +\infty$, we find that $\mathbf{C}_{\text{LS}} \leq \mathbf{C}_{\text{WLH}}(d/4, a)$. If equality holds, this is a mechanism of loss of compactness for minimizing sequences. On the opposite, if $\mathbf{C}_{\text{LS}} < \mathbf{C}_{\text{WLH}}(d/4, a)$, which is the case if $a \in (a_\star^{\text{WLH}}, a_c)$ where $a_\star^{\text{WLH}} = a$ is given by the condition $\mathbf{C}_{\text{LS}} = \mathbf{C}_{\text{WLH}}^*(d/4, a)$, we can establish a compactness result which proves that equality is attained in (WLH) in the critical case $\gamma = d/4$.

A similar analysis for (CKN) shows that $\mathbf{C}_{\text{GN}}(p) \leq \mathbf{C}_{\text{CKN}}(\theta, p, a)$ in the critical case $\theta = \vartheta(p, d)$, where $\mathbf{C}_{\text{GN}}(p)$ is the optimal constant in the Gagliardo-Nirenberg-Sobolev interpolation inequalities

$$\|u\|_{\mathbf{L}^p(\mathbb{R}^d)}^2 \leq \mathbf{C}_{\text{GN}}(p) \|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^d)}^{2\vartheta(p,d)} \|u\|_{\mathbf{L}^2(\mathbb{R}^d)}^{2(1-\vartheta(p,d))} \quad \forall u \in \mathbf{H}^1(\mathbb{R}^d)$$

and $p \in (2, 2^*)$ if $d = 2$ or $p \in (2, 2^*]$ if $d \geq 3$. However, extremals are not known explicitly in such inequalities if $d \geq 2$, so we cannot get an explicit interval of existence in terms of a , even if we also know that compactness of minimizing sequences

for (CKN) holds when $C_{GN}(p) < C_{CKN}(\vartheta(p, d), p, a)$. This is the case if $a > a_\star^{\text{CKN}}$ where $a = a_\star^{\text{CKN}}$ is defined by the condition $C_{GN}(p) = C_{CKN}^*(\vartheta(p, d), p, a)$.

It is very convenient to reformulate (CKN) and (WLH) inequalities in cylindrical variables.⁸ By means of the Emden-Fowler transformation

$$s = \log |x| \in \mathbb{R}, \quad \omega = x/|x| \in \mathbb{S}^{d-1}, \quad y = (s, \omega), \quad v(y) = |x|^{a_c - a} u(x),$$

(CKN) for u is equivalent to a Gagliardo-Nirenberg-Sobolev inequality on the cylinder $\mathcal{C} := \mathbb{R} \times \mathbb{S}^{d-1}$ for v , namely

$$\|v\|_{\mathbf{L}^p(\mathcal{C})}^2 \leq C_{CKN}(\theta, p, a) \left(\|\nabla v\|_{\mathbf{L}^2(\mathcal{C})}^2 + \Lambda \|v\|_{\mathbf{L}^2(\mathcal{C})}^2 \right)^\theta \|v\|_{\mathbf{L}^2(\mathcal{C})}^{2(1-\theta)} \quad \forall v \in H^1(\mathcal{C})$$

with $\Lambda = \Lambda(a)$. Similarly, with $w(y) = |x|^{a_c - a} u(x)$, (WLH) is equivalent to

$$\int_{\mathcal{C}} |w|^2 \log |w|^2 dy \leq 2\gamma \log \left[C_{WLH}(\gamma, a) \left(\|\nabla w\|_{\mathbf{L}^2(\mathcal{C})}^2 + \Lambda \right) \right]$$

for any $w \in H^1(\mathcal{C})$ such that $\|w\|_{\mathbf{L}^2(\mathcal{C})} = 1$. Notice that radial symmetry for u means that v and w depend only on s .

Consider a sequence $(v_n)_n$ of functions in $H^1(\mathcal{C})$, which minimizes the functional

$$\mathcal{E}_{\theta, \Lambda}^p[v] := \left(\|\nabla v\|_{\mathbf{L}^2(\mathcal{C})}^2 + \Lambda \|v\|_{\mathbf{L}^2(\mathcal{C})}^2 \right)^\theta \|v\|_{\mathbf{L}^2(\mathcal{C})}^{2(1-\theta)}$$

under the constraint $\|v_n\|_{\mathbf{L}^p(\mathcal{C})} = 1$ for any $n \in \mathbb{N}$. As quickly explained below, if bounded, such a sequence is relatively compact and converges up to translations and the extraction of a subsequence towards a minimizer of $\mathcal{E}_{\theta, \Lambda}^p$.

Assume that $d \geq 3$, let $t := \|\nabla v\|_{\mathbf{L}^2(\mathcal{C})}^2 / \|v\|_{\mathbf{L}^2(\mathcal{C})}^2$ and $\Lambda = \Lambda(a)$. If v is a minimizer of $\mathcal{E}_{\theta, \Lambda}^p[v]$ such that $\|v\|_{\mathbf{L}^p(\mathcal{C})} = 1$, then we have

$$(t + \Lambda)^\theta = \mathcal{E}_{\theta, \Lambda}^p[v] \frac{\|v\|_{\mathbf{L}^p(\mathcal{C})}^2}{\|v\|_{\mathbf{L}^2(\mathcal{C})}^2} = \frac{\|v\|_{\mathbf{L}^p(\mathcal{C})}^2}{C_{CKN}(\theta, p, a) \|v\|_{\mathbf{L}^2(\mathcal{C})}^2} \leq \frac{S_d^{\vartheta(d, p)}}{C_{CKN}(\theta, p, a)} (t + a_c^2)^{\vartheta(d, p)}$$

where $S_d = C_{CKN}(1, 2^*, 0)$ is the optimal Sobolev constant, while we know from (1) that $\lim_{a \rightarrow a_c} C_{CKN}(\theta, p, a) = \infty$ if $d \geq 2$. This provides a bound on t if $\theta > \vartheta(p, d)$. An estimate can be obtained also for v_n , for n large enough, and standard tools of the concentration-compactness method allow to conclude that $(v_n)_n$ converges towards an extremal. A similar approach holds for (CKN) if $d = 2$, or for (WLH).

The above variational approach also provides an existence result of extremals for (CKN) in the critical case $\theta = \vartheta(p, d)$, if $a \in (a_1, a_c)$ where $a_1 := a_c - \sqrt{\Lambda_1}$ and $\Lambda_1 = \min\{(C_{CKN}^*(\theta, p, a_c - 1))^{1/\theta} / S_d\}^{d/(d-1)}, (a_c^2 C_{CKN}^*(\theta, p, a_c - 1))^{1/\theta} / S_d\}^d$.

If symmetry is known, then there are (radially symmetric) extremals.³ Anticipating on the results of the next section, we can state the following result which arises as a consequence of Schwarz' symmetrization method (see Theorem 3.2, below).

Proposition 2.1. *Let $d \geq 3$. Then (CKN) with $\theta = \vartheta(p, d)$ admits a radial extremal if⁵ $a \in [a_0, a_c)$ where $a_0 := a_c - \sqrt{\Lambda_0}$ and $\Lambda = \Lambda_0$ is defined by the condition $\Lambda^{(d-1)/d} = \vartheta(p, d) C_{CKN}^*(\theta, p, a_c - 1)^{1/\vartheta(d, p)} / S_d$.*

A similar estimate also holds if $\theta > \vartheta(d, p)$, with less explicit computations.⁵

3. Symmetry and symmetry breaking

Define

$$\begin{aligned}\underline{a}(\theta, p) &:= a_c - \frac{2\sqrt{d-1}}{p+2} \sqrt{\frac{2p\theta}{p-2} - 1}, \quad \tilde{a}(\gamma) := a_c - \frac{1}{2}\sqrt{(d-1)(4\gamma-1)}, \\ \Lambda_{\text{SB}}(\gamma) &:= \frac{1}{8}(4\gamma-1) e \left(\frac{\pi^4 \gamma^{-d-1}}{16} \right)^{\frac{1}{4\gamma-1}} \left(\frac{d}{\gamma} \right)^{\frac{4\gamma}{4\gamma-1}} \Gamma \left(\frac{d}{2} \right)^{\frac{2}{4\gamma-1}}.\end{aligned}$$

Theorem 3.1. *Let $d \geq 2$ and $p \in (2, 2^*)$. Symmetry breaking holds in (CKN) if either^{3,5} $a < \underline{a}(\theta, p)$ and $\theta \in [\vartheta(p, d), 1]$, or⁵ $a < a_{\star}^{\text{CKN}}$ and $\theta = \vartheta(p, d)$.*

Assume that $\gamma > 1/2$ if $d = 2$ and $\gamma \geq d/4$ if $d \geq 3$. Symmetry breaking holds in (WLH) if^{3,5} $a < \max\{\tilde{a}(\gamma), a_c - \sqrt{\Lambda_{\text{SB}}(\gamma)}\}$.

When $\gamma = d/4$, $d \geq 3$, we observe that $\Lambda_{\star}^{\text{WLH}} = \Lambda_{\text{SB}}(d/4) < \Lambda(\tilde{a}(d/4))$ with the notations of Theorem 2.1 and there is symmetry breaking if $a \in (-\infty, a_{\star}^{\text{WLH}})$, in the sense that $\mathcal{C}_{\text{WLH}}(d/4, a) > \mathcal{C}_{\text{WLH}}^*(d/4, a)$, although we do not know if extremals for (WLH) exist when $\gamma = d/4$.

Results of symmetry breaking for (CKN) with $a < \underline{a}(\theta, p)$ have been established first^{1,8,9} when $\theta = 1$ and later³ extended to $\theta < 1$. The main idea in case of (CKN) is consider the quadratic form associated to the second variation of $\mathcal{E}_{\theta, \Lambda}^p$ around a minimizer among functions depending on s only and observe that the linear operator $\mathcal{L}_{\theta, \Lambda}^p$ associated to the quadratic form has a negative eigenvalue if $a < \underline{a}$. Results³ for (WLH), $a < \tilde{a}(\gamma)$, are based on the same method.

For any $a < a_{\star}^{\text{CKN}}$, we have $\mathcal{C}_{\text{CKN}}^*(\vartheta(p, d), p, a) < \mathcal{C}_{\text{GN}}(p) \leq \mathcal{C}_{\text{CKN}}(\vartheta(p, d), p, a)$, which proves symmetry breaking. Using well-chosen test functions, it has been proved⁵ that $\underline{a}(\vartheta(p, d), p) < a_{\star}^{\text{CKN}}$ for $p - 2 > 0$, small enough, thus also proving symmetry breaking for $a - \underline{a}(\vartheta(p, d), p) > 0$, small, and $\theta - \vartheta(p, d) > 0$, small.

Theorem 3.2. *For all $d \geq 2$, there exists^{2,5} a continuous function a^* defined on the set $\{(\theta, p) \in (0, 1] \times (2, 2^*) : \theta > \vartheta(p, d)\}$ such that $\lim_{p \rightarrow 2^+} a^*(\theta, p) = -\infty$ with the property that (CKN) has only radially symmetric extremals if $(a, p) \in (a^*(\theta, p), a_c) \times (2, 2^*)$, and none of the extremals is radially symmetric if $(a, p) \in (-\infty, a^*(\theta, p)) \times (2, 2^*)$.*

*Similarly, for all $d \geq 2$, there exists⁵ a continuous function $a^{**} : (d/4, \infty) \rightarrow (-\infty, a_c)$ such that, for any $\gamma > d/4$ and $a \in [a^{**}(\gamma), a_c)$, there is a radially symmetric extremal for (WLH), while for $a < a^{**}(\gamma)$ no extremal is radially symmetric.*

Schwarz' symmetrization allows to characterize⁵ a subdomain of $(0, a_c) \times (0, 1) \ni (a, \theta)$ in which symmetry holds for extremals of (CKN), when $d \geq 3$. If $\theta = \vartheta(p, d)$ and $p > 2$, there are radially symmetric extremals⁵ if $a \in [a_0, a_c)$ where a_0 is given in Propositions 2.1.

Symmetry also holds if $a - a_c$ is small enough, for (CKN) as well as for (WLH), or when $p \rightarrow 2_+$ in (CKN), for any $d \geq 2$, as a consequence of the existence of the spectral gap of $\mathcal{L}_{\theta, \Lambda}^p$ when $a > \underline{a}(\theta, p)$.

For given θ and p , there is^{2,5} a unique $a^* \in (-\infty, a_c)$ for which there is symmetry breaking in $(-\infty, a^*)$ and for which all extremals are radially symmetric when $a \in$

(a^*, a_c) . This follows from the observation that, if $v_\sigma(s, \omega) := v(\sigma s, \omega)$ for $\sigma > 0$, then $(\mathcal{E}_{\theta, \sigma^2 \Lambda}^p[v_\sigma])^{1/\theta} - \sigma^{(2\theta-1+2/p)/\theta^2} (\mathcal{E}_{\theta, \Lambda}^p[v])^{1/\theta}$ is equal to 0 if v depends only on s , while it has the sign of $\sigma - 1$ otherwise.

From Theorem 3.1, we can infer that radial and non-radial extremals for (CKN) with $\theta > \vartheta(p, d)$ coexist on the threshold, in some cases.

Numerical results illustrating our results on existence and on symmetry / symmetry breaking have been collected in Fig. 1 below in the critical case for (CKN).

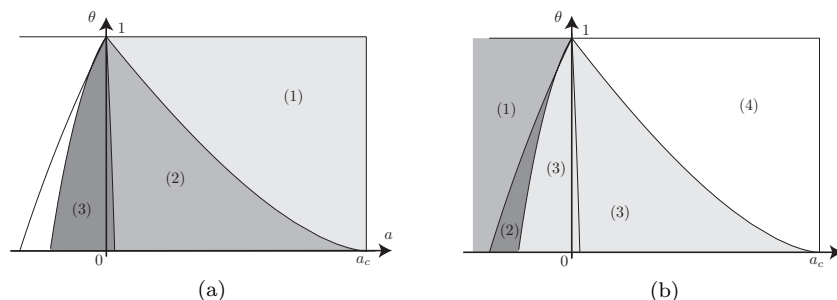


Fig. 1. Critical case for (CKN): $\theta = \vartheta(p, d)$. Here we assume that $d = 5$.

(a) The zones in which existence is known are (1) in which $a \geq a_0$, because extremals are achieved among radial functions, (2) using the *a priori* estimates: $a > a_1$, and (3) by comparison with the Gagliardo-Nirenberg inequality: $a > a_*^{\text{CKN}}$.

(b) The zone of symmetry breaking contains (1) by linearization around radial extremals: $a < \underline{a}(\theta, p)$, and (2) by comparison with the Gagliardo-Nirenberg inequality: $a < a_*^{\text{CKN}}$; in (3) it is not known whether symmetry holds or if there is symmetry breaking, while in (4) symmetry holds by Schwarz' symmetrization: $a_0 \leq a < a_c$.

Numerically, we observe that \underline{a} and a_*^{CKN} intersect for some $\theta \approx 0.85$.

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